A note on a bivariate gamma distribution

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Abstract

Vere-Jones (1967) introduced a bivariate generalization of gamma distributions and proved its infinite divisibility. Maejima and Ueda (2009) and others studied α-selfdecomposability, which is a generalization of selfdecomposability. In this paper, the (-2)-selfdecomposability of bivariate gamma distributions is shown.

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1. Introduction and preliminaries

According to Vere-Jones (1967), one multivariate generalization of gamma distributions is the joint distribution of $W_d := (X_1^2, X_2^2, \dots, X_d^2)$, where $(X_1, X_2, \dots, X_d)$ is a d-dimensional Gaussian random variable whose components $X_j, j = 1, 2, \dots, d$, are 1-dimensional standard normal random variables. Vere-Jones (1967) proved that when $d = 2$, $W := W_2 = (X_1^2, X_2^2)$ is infinitely divisible. He actually gave the Lévy measure of W. Let σ be the correlation coefficient of X_1 and X_2. He treated the problem in terms of moment generating functions, but if we read it in terms of characteristic functions, his result is turned out to be the following. Let $S_n := (W_1 + W_2 + \cdots + W_n)/2$, where $W_1, W_2, \dots, W_n$ are independently identically distributed with the same law as that of W.

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(1) By (4) of Vere-Jones (1967), the density function $f(x_1, x_2)$ of S_4 is expressed as

$$f(x_1,x_2) = f(x_1)f(x_2)\exp\{-\sigma^2(x_1+x_2)/(1-\sigma^2)\} \times (1-\sigma^2)^{-1}(\sigma^2 x_1 x_2)^{-1/2} I_1\left(2\sqrt{\sigma^2 x_1 x_2}(1-\sigma^2)^{-1}\right), \tag{1.1}$$

for $x_1 > 0$ and $x_2 > 0$, where $f(x) = xe^{-x}, x > 0$. Here and in what follows, I_ν's are modified Bessel functions. (For modified Bessel functions, see, e.g., 8.4–8.5 of Gradshteyn and Ryzhik (2007).)

(2) For $z \in \mathbb{R}^2$,

$$C(z) := \log E\left[e^{i\langle z,S_2\rangle}\right] = \int_0^\infty \int_0^\infty \left(e^{i\langle z,(x_1,x_2)\rangle} - 1\right) \frac{M(dx_1dx_2)}{x_1^2+x_2^2},$$

where

$$M(dx_1dx_2) = \begin{cases} \sigma^2(x_1^2+x_2^2)(x_1x_2)^{-1}f(x_1,x_2)dx_1dx_2 =: g(x_1,x_2)dx_1dx_2, & \text{for } x_1>0, x_2>0,\\ x_1e^{-x_1/(1-\sigma^2)}dx_1\delta_0(dx_2), & \text{for } x_1>0, x_2=0,\\ \delta_0(dx_1)x_2e^{-x_2/(1-\sigma^2)}dx_2, & \text{for } x_1=0, x_2>0. \end{cases} \tag{1.2}$$

(See p. 422 of Vere-Jones (1967).) We divide the integral into three parts:

$$C(z) = \iint_{D_1} + \iint_{D_2} + \iint_{D_3},$$

where

$$\begin{cases} D_1 = \{(x_1,x_2)\colon x_1>0, x_2>0\},\\ D_2 = \{(x_1,x_2)\colon x_1>0, x_2=0\},\\ D_3 = \{(x_1,x_2)\colon x_1=0, x_2>0\}. \end{cases}$$

For the integral $\iint_{D_1}$, let us change variables as $(x_1, x_2) = (r\cos\theta, r\sin\theta)$. Write $\xi = (\cos\theta, \sin\theta)$ on the unit circle $\mathbb{S}$ in $\mathbb{R}^2$. Then we have

$$\begin{aligned}\iint_{D_1} &= \int_0^{\pi/2} d\theta \int_0^\infty \left(e^{i\langle z,(r\cos\theta,r\sin\theta)\rangle} - 1\right)\frac{1}{r^2} g(r\cos\theta, r\sin\theta)rdr\\ &= \int_{\mathbb{S}} \lambda(d\xi)\int_0^\infty \left(e^{i\langle z,r\xi\rangle}-1\right)\frac{g(r\xi)}{r}dr, \end{aligned} \tag{1.3}$$

where

$$\lambda(B) = \int_{\{\theta\in(0,\pi/2)\colon (\cos\theta,\sin\theta)\in B\}} d\theta, \quad B \in \mathcal{B}(\mathbb{S}).$$

2. Non-selfdecomposability of $\boldsymbol{W}$ unless $\boldsymbol{X_1}$ and $\boldsymbol{X_2}$ are independent

In general, we know the following, (see, e.g., Theorem 15.10 of Sato (1999)). Let μ be an infinitely divisible distribution on $\mathbb{R}^d$ with the Lévy measure ν. Then, μ is selfdecomposable if and only if

$$\nu(B) = \int_{\mathbb{S}} \lambda(d\xi) \int_0^\infty \mathbb{1}_B(r\xi) \frac{k_\xi(r)}{r} dr,$$

with a finite measure λ on the unit sphere $\mathbb{S}$ in $\mathbb{R}^d$ and a nonnegative function $k_\xi(r)$ measurable in $\xi \in \mathbb{S}$ and decreasing in $r > 0$. Thus, for checking the selfdecomposability, it is enough to check the behavior of $k_\xi(r)$ as a function of $r > 0$ for each ξ. Let $\widehat{\mu}(z), z \in \mathbb{R}^d$, be the characteristic function of a infinitely divisible distribution μ. For $t \geq 0$, we write μ^t for the distribution with characteristic function $\widehat{\mu}(z)^t$ and call μ^t the t-fold convolution of μ. Then, note that selfdecomposability is closed under multiplying selfdecomposable random variables by constants and under t-fold convolution for $t > 0$. Hence the selfdecomposability of W is equivalent to that of S_2. By (1.3), if $\xi \neq (1, 0)$ or $(0, 1)$, equivalently if $0 < \theta < \pi/2$, then $k_\xi(r) = g(r\xi)$, which is not nonincreasing in $r > 0$ unless $\sigma = 0$. Thus, if $\sigma \neq 0$, then W is not selfdecomposable. If $\sigma = 0$, namely if the components of W are independent, then by (1.2), $k_\xi(r)$ is nonincreasing, and hence W is selfdecomposable.

3. Which class does the distribution of $\boldsymbol{W}$ belong to?

In Maejima et al. (2010) and Maejima and Ueda (2010), they defined wider classes than the class of selfdecomposable distributions as follows. Let $\alpha \in \mathbb{R}$. An infinitely divisible distribution μ on $\mathbb{R}^d$ is said to be α-selfdecomposable, if any $b > 1$, there exists another infinitely divisible distribution ρ_b on $\mathbb{R}^d$ satisfying

$$\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)^{b^\alpha} \widehat{\rho}_b(z), \quad z \in \mathbb{R}^d. \tag{3.1}$$

Denote the totality of α-selfdecomposable distributions on $\mathbb{R}^d$ by $L^{\langle\alpha\rangle}(\mathbb{R}^d)$. Then by the definition, $L^{\langle 0\rangle}(\mathbb{R}^d)$ is the class of selfdecomposable distributions on $\mathbb{R}^d$ and $L^{\langle -1\rangle}(\mathbb{R}^d)$ is the so-called Jurek class of s-selfdecomposable distributions on $\mathbb{R}^d$. Jurek (1988, 1989, 1992), and Jurek and Schreiber (1992) studied the classes $\mathscr{U}_\beta(Q), \beta \in \mathbb{R}$, of distributions on a real separable Banach space E, where Q is a linear operator on E with certain properties. These classes are equal to $L^{\langle\alpha\rangle}(\mathbb{R}^d)$ if $\beta = -\alpha$, $E = \mathbb{R}^d$ and Q is the identity operator. As to these classes, they studied the decomposability and stochastic integral characterizations, although some results are only for the case that Q is the identity operator. However, since, for $0 < \alpha < 2$, $L^{\langle\alpha\rangle}(\mathbb{R}^d)$ contains all α-stable distributions and any $\mu \in L^{\langle\alpha\rangle}(\mathbb{R}^d)$ belongs to the normal domain of attraction of some α-stable distribution, we adopt the parametrization in (3.1).

By the observations in Sections 1 and 2, the distribution of W is not only non-selfdecomposable but also not in a bigger class, the Jurek class. However, the following proposition hold. Note that, by (3.1), α-selfdecomposability is closed under multiplying α-selfdecomposable random variables by constants and under t-fold convolution for $t > 0$. Hence the α-selfdecomposability of W is equivalent to that of S_2.

Proposition 3.1. *Let $\sigma \neq 0$. Then*

$$\mathcal{L}(W) \begin{cases} \in L^{\langle\alpha\rangle}(\mathbb{R}^2) & \text{for all } \alpha \le -2, \\ \notin L^{\langle\alpha\rangle}(\mathbb{R}^2) & \text{for all } \alpha > -2. \end{cases}$$

Proof. In Maejima et al. (2010), it was shown that when $\alpha < 0$, $\mu \in L^{\langle\alpha\rangle}(\mathbb{R}^d)$ if and only if the Lévy measure ν of μ has the form

$$\nu(B) = \int_{\mathbb{S}} \lambda(d\xi) \int_0^\infty \mathbb{1}_B(r\xi) r^{-\alpha-1} \ell_\xi(r) dr,$$

where λ is a measure on $\mathbb{S}$ and $\ell_\xi(r)$ is a nonnegative function measurable in $\xi \in \mathbb{S}$ and nonincreasing and right-continuous in $r > 0$. From the fact observed in Section 2,

$$\begin{aligned} f_\xi(r) &:= \frac{k_\xi(r)}{r} = \frac{g(r\xi)}{r} \\ &= \frac{|\sigma|}{1-\sigma^2}(\cos\theta\sin\theta)^{-1/2} e^{-r(1-\sigma^2)^{-1}(\cos\theta+\sin\theta)} I_1\left(\frac{2(\sigma^2\cos\theta\sin\theta)^{1/2}}{1-\sigma^2} r\right), \end{aligned}$$

where $\xi = (\cos\theta, \sin\theta)$. Note that for all $r > 0$ and all $\theta \in (0, \pi/2)$,

$$
\begin{aligned}
r f_\xi'(r)/f_\xi(r) &= -\frac{r}{1-\sigma^2}(\cos\theta + \sin\theta) \\
&\quad + \left(\frac{2(\sigma^2\cos\theta\sin\theta)^{1/2}}{1-\sigma^2} r\right) I_1'\left(\frac{2(\sigma^2\cos\theta\sin\theta)^{1/2}}{1-\sigma^2} r\right) \Big/ I_1\left(\frac{2(\sigma^2\cos\theta\sin\theta)^{1/2}}{1-\sigma^2} r\right) \\
&\le 1 + \frac{r}{1-\sigma^2}\left\{2(\sigma^2\cos\theta\sin\theta)^{1/2} - \cos\theta - \sin\theta\right\} \\
&\le 1 - \frac{r}{1-\sigma^2}\left(\sqrt{\cos\theta} - \sqrt{\sin\theta}\right)^2 \le 1,
\end{aligned}
$$

where we have used the inequality

$$
u I_1'(u)/I_1(u) \le u + 1, \quad \text{for all } u > 0, \tag{3.2}
$$

which will be proved later. Then $r f_\xi'(r) - f_\xi(r) \le 0$, namely, $\frac{\partial}{\partial r}\{r^{-1} f_\xi(r)\} \le 0$. Hence $\ell_\xi(r) := r^{-1} f_\xi(r)$ is nonincreasing in $r > 0$ and

$$
\nu(B) = \int_{\mathbb{S}} \lambda(d\xi) \int_0^\infty \mathbb{1}_B(r\xi) f_\xi(r) dr = \int_{\mathbb{S}} \lambda(d\xi) \int_0^\infty \mathbb{1}_B(r\xi) r^{-(-2)-1} \ell_\xi(r) dr,
$$

which yields $\mathcal{L}(W) \in L^{\langle -2 \rangle}(\mathbb{R}^2)$. Since

$$
L^{\langle \alpha_1 \rangle}(\mathbb{R}^d) \supset L^{\langle \alpha_2 \rangle}(\mathbb{R}^d) \quad \text{for } \alpha_1 < \alpha_2
$$

(Corollary 1.1(b) of Jurek (1988) or Proposition 3.1 of Maejima and Ueda (2010)), we have $\mathcal{L}(W) \in L^{\langle \alpha \rangle}(\mathbb{R}^2)$ for all $\alpha \le -2$. Also take into account that

$$
\begin{aligned}
r^{\alpha+1} f_\xi(r) &= \frac{r^{\alpha+1}|\sigma|}{1-\sigma^2}(\cos\theta\sin\theta)^{-1/2} e^{-r(1-\sigma^2)^{-1}(\cos\theta+\sin\theta)} I_1\left(\frac{2(\sigma^2\cos\theta\sin\theta)^{1/2}}{1-\sigma^2} r\right) \\
&= \frac{r^{\alpha+2}\sigma^2}{(1-\sigma^2)^2} e^{-r(1-\sigma^2)^{-1}(\cos\theta+\sin\theta)} \sum_{m=0}^\infty \frac{(\cos\theta\sin\theta)^m}{m!(m+1)!}\left(\frac{\sigma}{1-\sigma^2}\right)^{2m} r^{2m},
\end{aligned}
$$

where we have used the series representation of I_1 given in 8.445 of Gradshteyn and Ryzhik (2007). If $\alpha > -2$, then the right-hand side of the equation above tends to 0 as $r \downarrow 0$. Hence $r^{\alpha+1} f_\xi(r)$ with $\alpha > -2$ is not nonincreasing and thus $\mathcal{L}(W) \notin L^{\langle \alpha \rangle}(\mathbb{R}^2)$.

We finally prove (3.2). Since $u I_1'(u) - I_1(u) = u I_2(u)$ (8.486[4] of Gradshteyn and Ryzhik (2007)), it is enough to show $I_1(u) \ge I_2(u)$ for all $u > 0$.

Also it follows from the two recurrence formulae $uI_1'(u) - I_1(u) = uI_2(u)$ and $uI_2'(u) + 2I_2(u) = uI_1(u)$ (8.486^3 of Gradshteyn and Ryzhik (2007)) that $u\,(I_1'(u) - I_2'(u)) + (u+2)\,(I_1(u) - I_2(u)) = 3I_1(u)$. Then

$$\begin{aligned}\frac{d}{du}\left\{u^2 e^u\,(I_1(u) - I_2(u))\right\} &= ue^u\,\{u\,(I_1'(u) - I_2'(u)) + (u+2)\,(I_1(u) - I_2(u))\}\\ &= 3ue^u I_1(u) \geq 0.\end{aligned}$$

Hence we have $u^2 e^u\,(I_1(u) - I_2(u)) \geq 0$ for all $u > 0$, that is, $I_1(u) \geq I_2(u)$ for all $u > 0$. □

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