A note on a bivariate gamma distribution

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Abstract

Vere-Jones (1967) introduced a bivariate generalization of gamma distributions and proved its infinite divisibility. Maejima and Ueda (2009) and others studied $\alpha$-selfdecomposability, which is a generalization of selfdecomposability. In this paper, the $(-2)$-selfdecomposability of bivariate gamma distributions is shown.

Keywords: infinitely divisible distribution, bivariate gamma distribution, $\alpha$-selfdecomposable distribution

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1. Introduction and preliminaries

According to Vere-Jones (1967), one multivariate generalization of gamma distributions is the joint distribution of $W_d := (X_1^2, X_2^2, \ldots, X_d^2)$, where $(X_1, X_2, \ldots, X_d)$ is a $d$-dimensional Gaussian random variable whose components $X_j, j = 1, 2, \ldots, d$, are 1-dimensional standard normal random variables. Vere-Jones (1967) proved that when $d = 2$, $W := W_2 = (X_1^2, X_2^2)$ is infinitely divisible. He actually gave the Lévy measure of $W$. Let $\sigma$ be the correlation coefficient of $X_1$ and $X_2$. He treated the problem in terms of moment generating functions, but if we read it in terms of characteristic functions, his result is turned out to be the following. Let $S_n := (W_1 + W_2 + \cdots + W_n)/2$, where $W_1, W_2, \ldots, W_n$ are independently identically distributed with the same law as that of $W$.

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(1) By (4) of Vere-Jones [1967], the density function \( f(x_1, x_2) \) of \( S_4 \) is expressed as

\[
f(x_1, x_2) = f(x_1) f(x_2) \exp\left\{ -\sigma^2(x_1 + x_2)/(1 - \sigma^2) \right\} \times (1 - \sigma^2)^{-1}(\sigma^2 x_1 x_2)^{-1/2} I_1 \left( 2\sqrt{\sigma^2 x_1 x_2(1 - \sigma^2)} \right)
\]

for \( x_1 > 0 \) and \( x_2 > 0 \), where \( f(x) = xe^{-x}, x > 0 \). Here and in what follows, \( I_\nu \)'s are modified Bessel functions. (For modified Bessel functions, see, e.g., 8.4–8.5 of Gradshteyn and Ryzhik [2007].)

(2) For \( z \in \mathbb{R}^2 \),

\[
C(z) := \log E \left[ e^{i\langle z, S_2 \rangle} \right] = \int_0^\infty \int_0^\infty \left( e^{i\langle z, (x_1, x_2) \rangle} - 1 \right) \frac{M(dx_1 dx_2)}{x_1^2 + x_2^2},
\]

where \( M(dx_1 dx_2) \)

\[
= \begin{cases} 
\frac{\sigma^2(x_1^2 + x_2^2)}{(x_1 x_2)^{-1}} f(x_1, x_2) dx_1 dx_2 =: g(x_1, x_2) dx_1 dx_2, & \text{for } x_1 > 0, x_2 > 0, \\
 x_1 e^{-x_1/(1 - \sigma^2)} dx_1 \delta_0(dx_2), & \text{for } x_1 > 0, x_2 = 0, \\
 \delta_0(dx_1) x_2 e^{-x_2/(1 - \sigma^2)} dx_2, & \text{for } x_1 = 0, x_2 > 0.
\end{cases}
\]

(See p. 422 of Vere-Jones [1967].) We divide the integral into three parts:

\[
C(z) = \int_D + \int_{D_2} + \int_{D_3},
\]

where

\[
D_1 = \{(x_1, x_2): x_1 > 0, x_2 > 0\}, \\
D_2 = \{(x_1, x_2): x_1 > 0, x_2 = 0\}, \\
D_3 = \{(x_1, x_2): x_1 = 0, x_2 > 0\}.
\]

For the integral \( \int_{D_1} \), let us change variables as \((x_1, x_2) = (r \cos \theta, r \sin \theta)\). Write \( \xi = (\cos \theta, \sin \theta) \) on the unit circle \( \mathbb{S} \) in \( \mathbb{R}^2 \). Then we have

\[
\frac{\pi}{2} \int_{D_1} d\theta \int_0^\infty \left( e^{i\langle z, (r \cos \theta, r \sin \theta) \rangle} - 1 \right) \frac{1}{r^2} g(r \cos \theta, r \sin \theta) r dr \\
= \int_\mathbb{S} \lambda(d\xi) \int_0^\infty \left( e^{i\langle z, r \xi \rangle} - 1 \right) \frac{g(r \xi)}{r} dr,
\]

where

\[
\lambda(B) = \int_{\{\theta \in (0, \pi/2): (\cos \theta, \sin \theta) \in B\}} d\theta, \quad B \in \mathcal{B}(\mathbb{S}).
\]

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2. Non-selfdecomposability of $W$ unless $X_1$ and $X_2$ are independent

In general, we know the following, (see, e.g., Theorem 15.10 of [Sato (1999)]). Let $\mu$ be an infinitely divisible distribution on $\mathbb{R}^d$ with the Lévy measure $\nu$. Then, $\mu$ is selfdecomposable if and only if

$$\nu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi) \frac{k_{\xi}(r)}{r} dr,$$

with a finite measure $\lambda$ on the unit sphere $S$ in $\mathbb{R}^d$ and a nonnegative function $k_{\xi}(r)$ measurable in $\xi \in S$ and decreasing in $r > 0$. Thus, for checking the selfdecomposability, it is enough to check the behavior of $k_{\xi}(r)$ as a function of $r > 0$ for each $\xi$. Let $\hat{\mu}(z)$, $z \in \mathbb{R}^d$, be the characteristic function of a infinitely divisible distribution $\mu$. For $t \geq 0$, we write $\mu^t$ for the distribution with characteristic function $\hat{\mu}(z)^t$ and call $\mu^t$ the $t$-fold convolution of $\mu$. Then, note that selfdecomposability is closed under multiplying selfdecomposable random variables by constants and under $t$-fold convolution for $t > 0$. Hence the selfdecomposability of $W$ is equivalent to that of $S_2$. By (1.3), if $\xi \neq (1,0)$ or $(0,1)$, equivalently if $0 < \theta < \pi/2$, then $k_{\xi}(r) = g(r\xi)$, which is not nonincreasing in $r > 0$ unless $\sigma = 0$. Thus, if $\sigma \neq 0$, then $W$ is not selfdecomposable. If $\sigma = 0$, namely if the components of $W$ are independent, then by (1.2), $k_{\xi}(r)$ is nonincreasing, and hence $W$ is selfdecomposable.

3. Which class does the distribution of $W$ belong to?

In [Maejima et al. (2010)] and [Maejima and Ueda (2010)], they defined wider classes than the class of selfdecomposable distributions as follows. Let $\alpha \in \mathbb{R}$. An infinitely divisible distribution $\mu$ on $\mathbb{R}^d$ is said to be $\alpha$-selfdecomposable, if any $b > 1$, there exists another infinitely divisible distribution $\rho_b$ on $\mathbb{R}^d$ satisfying

$$\hat{\mu}(z) = \hat{\mu}(b^{-1}z)^{b^\alpha} \hat{\rho}_b(z), \quad z \in \mathbb{R}^d. \quad (3.1)$$

Denote the totality of $\alpha$-selfdecomposable distributions on $\mathbb{R}^d$ by $L^{(\alpha)}(\mathbb{R}^d)$. Then by the definition, $L^{(0)}(\mathbb{R}^d)$ is the class of selfdecomposable distributions on $\mathbb{R}^d$ and $L^{(-1)}(\mathbb{R}^d)$ is the so-called Jurek class of $s$-selfdecomposable distributions on $\mathbb{R}^d$. [Jurek (1988, 1989, 1992)] and [Jurek and Schreiber (1992)] studied the classes $\mathcal{U}_\beta(Q)$, $\beta \in \mathbb{R}$, of distributions on a real separable Banach
space \( E \), where \( Q \) is a linear operator on \( E \) with certain properties. These classes are equal to \( L(\alpha)(\mathbb{R}^d) \) if \( \beta = -\alpha, \ E = \mathbb{R}^d \) and \( Q \) is the identity operator. As to these classes, they studied the decomposability and stochastic integral characterizations, although some results are only for the case that \( Q \) is the identity operator. However, since, for \( 0 < \alpha < 2 \), \( L(\alpha)(\mathbb{R}^d) \) contains all \( \alpha \)-stable distributions and any \( \mu \in L(\alpha)(\mathbb{R}^d) \) belongs to the normal domain of attraction of some \( \alpha \)-stable distribution, we adopt the parametrization in (3.1).

By the observations in Sections 1 and 2, the distribution of \( W \) is not only non-selfdecomposable but also not in a bigger class, the Jurek class. However, the following proposition hold. Note that, by (3.1), \( \alpha \)-selfdecomposability is closed under multiplying \( \alpha \)-selfdecomposable random variables by constants and under \( t \)-fold convolution for \( t > 0 \). Hence the \( \alpha \)-selfdecomposability of \( W \) is equivalent to that of \( S_2 \).

**Proposition 3.1.** Let \( \sigma \neq 0 \). Then

\[
\mathcal{L}(W) \begin{cases} 
\in L(\alpha)(\mathbb{R}^2) & \text{for all } \alpha \leq -2, \\
\notin L(\alpha)(\mathbb{R}^2) & \text{for all } \alpha > -2.
\end{cases}
\]

**Proof.** In [Maejima et al. (2010)](cite), it was shown that when \( \alpha < 0, \mu \in L(\alpha)(\mathbb{R}^d) \) if and only if the Lévy measure \( \nu \) of \( \mu \) has the form

\[
\nu(B) = \int_{\mathbb{S}} \lambda(d\xi) \int_0^\infty 1_B(r\xi) r^{-\alpha-1} \ell_\xi(r) dr,
\]

where \( \lambda \) is a measure on \( \mathbb{S} \) and \( \ell_\xi(r) \) is a nonnegative function measurable in \( \xi \in \mathbb{S} \) and nonincreasing and right-continuous in \( r > 0 \). From the fact observed in Section 2,

\[
f_\xi(r) := \frac{k_\xi(r)}{r} = \frac{g(r\xi)}{r} = \frac{|\sigma|}{1 - \sigma^2} (\cos \theta \sin \theta)^{-1/2} e^{-r(1-\sigma^2)^{-1}(\cos \theta \sin \theta)} I_1 \left( \frac{2(\sigma^2 \cos \theta \sin \theta)^{1/2}}{1 - \sigma^2} r \right),
\]
where \( \xi = (\cos \theta, \sin \theta) \). Note that for all \( r > 0 \) and all \( \theta \in (0, \pi/2) \),

\[
rf'(r)/f_r(r) = -\frac{r}{1-\sigma^2}(\cos \theta + \sin \theta)
\]
\[
+ \left(\frac{2(\sigma^2 \cos \theta \sin \theta)^{1/2}}{1-\sigma^2}r\right) I_1'(\frac{2(\sigma^2 \cos \theta \sin \theta)^{1/2}}{1-\sigma^2}r) / I_1\left(\frac{2(\sigma^2 \cos \theta \sin \theta)^{1/2}}{1-\sigma^2}r\right)
\]
\[
\leq 1 + \frac{r}{1-\sigma^2} \left\{ 2(\sigma^2 \cos \theta \sin \theta)^{1/2} - \cos \theta - \sin \theta \right\}
\]
\[
\leq 1 - \frac{r}{1-\sigma^2} \left( \sqrt{\cos \theta} - \sqrt{\sin \theta} \right)^2 \leq 1,
\]

where we have used the inequality

\[
u I_1'(u)/I_1(u) \leq u + 1, \quad \text{for all } u > 0, \quad (3.2)
\]

which will be proved later. Then \( rf'_r(r) = f_r(r) \leq 0 \), namely, \( \frac{d}{dr} \{ r^{-1}f_r(r) \} \leq 0 \). Hence \( \ell_r(\xi) := r^{-1}f_r(r) \) is nonincreasing in \( r > 0 \) and

\[
\nu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi)f_r(r)dr = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi)r^{-(\alpha-2)}\ell_r(\xi)dr,
\]

which yields \( L(W) \in L^{(\alpha-2)}(\mathbb{R}^2) \). Since

\[
L^{(\alpha_1)}(\mathbb{R}^d) \supset L^{(\alpha_2)}(\mathbb{R}^d) \quad \text{for } \alpha_1 < \alpha_2
\]

(Corollary 1.1(b) of [Jurek, 1988] or Proposition 3.1 of [Maejima and Ueda, 2010]), we have \( L(W) \in L^{(\alpha)}(\mathbb{R}^2) \) for all \( \alpha \leq -2 \). Also take into account that

\[
r^{\alpha+1}f_r(r) = \frac{r^{\alpha+1}|\sigma|}{1-\sigma^2} (\cos \theta \sin \theta)^{-1/2} e^{-r(1-\sigma^2)^{-1}(\cos \theta + \sin \theta)} I_1\left(\frac{2(\sigma^2 \cos \theta \sin \theta)^{1/2}}{1-\sigma^2}r\right)
\]
\[
= \frac{r^{\alpha+2} \sigma^2}{(1-\sigma^2)^2} e^{-r(1-\sigma^2)^{-1}(\cos \theta + \sin \theta)} \sum_{m=0}^\infty \frac{(\cos \theta \sin \theta)^m}{m!(m+1)!} \left(\frac{\sigma}{1-\sigma^2}\right)^{2m} r^{2m},
\]

where we have used the series representation of \( I_1 \) given in 8.445 of [Gradshiteyn and Ryzhik, 2007]. If \( \alpha > -2 \), then the right-hand side of the equation above tends to 0 as \( r \to 0 \). Hence \( r^{\alpha+1}f_r(r) \) with \( \alpha > -2 \) is not nonincreasing and thus \( L(W) \notin L^{(\alpha)}(\mathbb{R}^2) \).

We finally prove \( (3.2) \). Since \( uI_1'(u) - I_1(u) = uI_2(u) \) (8.484 of [Gradshiteyn and Ryzhik, 2007]), it is enough to show \( I_1(u) \geq I_2(u) \) for all \( u > 0 \).
Also it follows from the two recurrence formulae \( uI'_1(u) - I_1(u) = uI_2(u) \) and \( uI'_2(u) + 2I_2(u) = uI_1(u) \) (8.486 of [Gradshteyn and Ryzhik (2007)]) that 

\[
\begin{align*}
u(I'_1(u) - I'_2(u)) + (u + 2)(I_1(u) - I_2(u)) &= 3I_1(u). \\
\end{align*}
\]

Then

\[
\frac{d}{du}\{u^2e^u(I_1(u) - I_2(u))\} = ue^u\{u(I'_1(u) - I'_2(u)) + (u + 2)(I_1(u) - I_2(u))\}
\]

\[= 3ue^uI_1(u) \geq 0.\]

Hence we have \( u^2e^u(I_1(u) - I_2(u)) \geq 0 \) for all \( u > 0 \), that is, \( I_1(u) \geq I_2(u) \) for all \( u > 0 \).

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**References**


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