On the distribution of the Rosenblatt process

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Abstract

We prove that the multivariate Rosenblatt distribution belongs to the Thorin class which is a subset of the class of selfdecomposable distributions. Using this fact we derive new properties of the Rosenblatt distribution.

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1. Introduction and theorems

The Rosenblatt process appears as the limit of long-range dependent stationary series. It is a selfsimilar process with stationary increments and lives in the so-called second Wiener chaos. Consequently, it is not a Gaussian process. In the last few years, this stochastic process has been the object of several research papers. (See Pipiras and Taqqu (2010), Tudor (2008), Tudor and Viens (2009), Veillette and Taqqu (in press) among others.)

The Rosenblatt distribution is the law of the Rosenblatt process evaluated at time $t = 1$. Very few things are known concerning this probability distribution. The first results related to this law have been given by Albin

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(1998). Recently, in the work by Veillette and Taqqu (in press), the authors computed, among others, its Lévy measure, its cumulants and moments. They also derived numerically the shape of the density.

Our purpose is to bring new lights to the Rosenblatt distribution. Actually, we will show that the multivariate Rosenblatt distribution belongs to the so-called Thorin class, which is a subset of the class of selfdecomposable distributions. From this perspective, we derive new results related to this distribution. For example, we show that a random variable that follows the Rosenblatt distribution can be represented in law as a Wiener integral with respect to some Lévy process. We also obtain new properties of the density of the Rosenblatt distribution.

Let $0 < D < \frac{1}{2}$. We consider the Rosenblatt process given, for $t \geq 0$,

$$Z_D(t) = C(D) \int_{\mathbb{R}^2}' \left(\int_0^t (u - s_1)_+^{-(1+D)/2} (u - s_2)_+^{-(1+D)/2} du \right) dB(s_1) dB(s_2),$$

where $\{B(s), s \in \mathbb{R}\}$ is a standard Brownian motion, $\int_{\mathbb{R}^2}'$ is the integral over $\mathbb{R}^2$ except the hyperplane $s_1 = s_2$,

$$C(D) = \frac{\sqrt{(1-D)(1-2D)/2}}{b((1-D)/2, D)},$$

with $b(\cdot, \cdot)$ being the beta function and the symbol a_+^c is read as 0 when $a \leq 0$ and $c < 0$. The stochastic process $\{Z_D(t), t \geq 0\}$ is $H(= 1 - D)$-selfsimilar and has stationary increments. The distribution of $Z_D(1)$ is called the Rosenblatt distribution, which is the first non-Gaussian limiting distribution of the normalized partial sums of some strongly dependent stationary random variables discovered by Rosenblatt (1961).

Let W be a complex-valued Gaussian random measure on $\mathbb{R}$ such that for Borel sets in $\mathbb{R}$, A, B, A_j, $E[W(A)] = 0, E[W(A)\overline{W(B)}] =$ the Lebesgue measure of $A \cap B$, $W\left(\bigcup_{j=1}^n A_j\right) = \sum_{j=1}^n W(A_j)$ for mutually disjoint sets $A_1, ..., A_n$ and $W(A) = \overline{W(-A)}$.

Let

$$\mathcal{H}_D = \Big\{ h : h \text{ is a complex-valued function on } \mathbb{R}, h(x) = \overline{h(-x)}, \\ \int_{\mathbb{R}} h(x)^2 |x|^{D-1} dx < \infty \Big\}$$

and for every $t \geq 0$ define an integral operator A_t by

$$A_t h(x) = C(D) \int_{-\infty}^{\infty} \frac{e^{it(x-y)-1}}{i(x-y)} h(y)|y|^{D-1} dy, \quad h \in \mathcal{H}_D. \tag{1.1}$$

Since A_t is a self-adjoint Hilbert-Schmidt operator (see Dobrushin and Major (1979)), all eigenvalues $\lambda_n(t), n = 1, 2, ...,$ are real and satisfy $\sum_{n=1}^{\infty} \lambda_n^2(t) < \infty$.

Our first theorem is as follows.

Theorem 1.1. *For every $t_1, ..., t_d \geq 0$,*

$$(Z_D(t_1), ..., Z_D(t_d)) \stackrel{\mathrm{d}}{=} \left(\sum_{n=1}^{\infty} \lambda_n(t_1)(\varepsilon_n^2 - 1), ..., \sum_{n=1}^{\infty} \lambda_n(t_d)(\varepsilon_n^2 - 1) \right),$$

where $\{\varepsilon_n\}$ are i.i.d. $N(0, 1)$ random variables.

The case $d = 1$ was shown by Taqqu (see Proposition 2 of Dobrushin and Major (1979)).

We next define the Thorin class of probability distributions on $\mathbb{R}_+$. Originally this class was studied by Thorin (1977a,b) and the Thorin class on $\mathbb{R}_+$, denoted by $T(\mathbb{R}_+)$, is the smallest class of distributions on $\mathbb{R}_+$ that contains all gamma distributions and is closed under convolution and weak convergence. A probability distribution in $T(\mathbb{R}_+)$ is called generalized gamma convolution. See also Bondesson (1992). This class was extended to $\mathbb{R}^d$ by Barndorff-Nielsen et al. (2006) as follows: call Γx an elementary gamma random variable in $\mathbb{R}^d$ if x is a non-random non-zero vector in $\mathbb{R}^d$ and Γ is a gamma random variable on $\mathbb{R}_+$. Then the Thorin class on $\mathbb{R}^d$, denoted by $T(\mathbb{R}^d)$, is defined as the smallest class of distributions on $\mathbb{R}^d$ that contains all elementary gamma distributions on $\mathbb{R}^d$ and is closed under convolution and weak convergence. (The Thorin class on $\mathbb{R}$ is already defined in Bondesson (1992) as the name of the extended generalized gamma convolutions.)

Our second theorem is as follows.

Theorem 1.2. *For every $t_1, ..., t_d \geq 0$, the law of $(Z_D(t_1), ..., Z_D(t_d))$ belongs to $T(\mathbb{R}^d)$.*

The organization of the paper is as follows. Section 2 gives the proofs of Theorems 1.1 and 1.2. In Section 3, we treat the non-symmetric Rosenblatt process, which is an extension of the Rosenblatt process, introduced by Maejima and Tudor (2012), and prove a corresponding result to Theorem 1.2. In Section 4, we go back to the Rosenblatt distribution and give its characterization by a single integral with respect to Lévy processes, and in Section 5, we give a remark on the density function of the Rosenblatt distribution.

2. Proofs of Theorems 1.1 and 1.2

By using Parseval's identity for multiple Wiener-Itô integrals (Lemma 6.2 of Taqqu (1979)), we have the following. Let

$$f_t(s_1, s_2) = C(D) \int_0^t (u - s_1)_+^{-(1+D)/2} (u - s_2)_+^{-(1+D)/2} du.$$

Then

$$\begin{aligned} Z_D(t) &= \int_{\mathbb{R}^2}' f_t(s_1, s_2) dB(s_1) dB(s_2) \\ &\stackrel{\mathrm{d}}{=} \int_{\mathbb{R}^2}'' \frac{e^{it(x_1+x_2)} - 1}{i(x_1 + x_2)} |x_1|^{(D-1)/2} |x_2|^{(D-1)/2} W(dx_1) W(dx_2), \end{aligned} \tag{2.1}$$

where $\int_{\mathbb{R}^2}''$ is the integral over $\mathbb{R}^2$ except the hyperplanes $x_1 \neq \pm x_2$.

We first prove Theorem 1.1.

Proof of Theorem 1.1. Let $\alpha_1, ...\alpha_d \in \mathbb{R}$. It is sufficient to show that

$$\alpha_1 Z_D(t_1) + \cdots + \alpha_d Z_D(t_d)$$

and

$$\alpha_1 \sum_{n=1}^{\infty} \lambda_n(t_1)(\varepsilon_n^2 - 1) + \cdots + \alpha_d \sum_{n=1}^{\infty} \lambda_n(t_d)(\varepsilon_n^2 - 1)$$

have the same distributions. But, by (2.1), we have

$$\begin{aligned} &\alpha_1 Z_D(t_1) + \cdots + \alpha_d Z_D(t_d) \\ &\quad \stackrel{\mathrm{d}}{=} \int_{\mathbb{R}^2}'' \left(\alpha_1 \frac{e^{it_1(x_1+x_2)} - 1}{i(x_1 + x_2)} + \cdots + \alpha_d \frac{e^{it_d(x_1+x_2)} - 1}{i(x_1 + x_2)} \right) \\ &\qquad\qquad |x_1|^{(D-1)/2} |x_2|^{(D-1)/2} W(dx_1) W(dx_2) \qquad (2.2) \\ &\quad =: \int_{\mathbb{R}^2}'' H_{t_1,\dots,t_d}(x_1, x_2) W(dx_1) W(dx_2), \end{aligned}$$

where $H_{t_1,...,t_d}(x_1,x_2) = H_{t_1,...,t_d}(x_2,x_1) = \overline{H_{t_1,...,t_d}(-x_2,-x_1)}, x_1,x_2 \in \mathbb{R}$, and

$$\int_{\mathbb{R}^2} |H_{t_1,...,t_d}(x_1,x_2)|^2 dx_1 dx_2 < \infty.$$

By Proposition 2 of Dobrushin and Major (1979), we see that (2.2) can be represented in law as

$$\sum_{n=1}^{\infty} \lambda_n(t_1,...,t_d)(\varepsilon_n^2 - 1),$$

where $\lambda_n(t_1,...,t_n)$ are the eigenvalues of the integral operator

$$A_{t_1,...,t_d}h(x) = C(D)\int_{-\infty}^{\infty}\left(\alpha_1 \frac{e^{it_1(x-y)}-1}{i(x-y)} + \cdots \alpha_d \frac{e^{it_d(x-y)}-1}{i(x-y)}\right)|y|^{D-1}h(y)dy, \quad h \in \mathcal{H}_D.$$

On the other hand, it is clear that the eigenvalues of $A_{t_1,...,t_d}$ are $\alpha_1\lambda_n(t_1) + \cdots + \alpha_d\lambda_n(t_d)$. This concludes the statement of the theorem. □

We next prove Theorem 1.2.

Proof of Theorem 1.2. By Theorem 1.1,

$$\begin{aligned}(Z_D(t_1),...,Z_D(t_d)) \\ &\stackrel{\mathrm{d}}{=} \left(\sum_{n=1}^{\infty} \lambda_n(t_1)(\varepsilon_n^2-1), ..., \sum_{n=1}^{\infty} \lambda_n(t_d)(\varepsilon_n^2-1)\right) \\ &= \sum_{n=1}^{\infty}(\varepsilon_n^2-1)(\lambda_n(t_1),...,\lambda_n(t_d)),\end{aligned}$$

where $(\varepsilon_n^2 - 1)(\lambda_n(t_1),...,\lambda_n(t_d)), n = 1,2,...,$ are the elementary gamma random variables in $\mathbb{R}^d$. Since they are independent, by the properties of the class $T(\mathbb{R}^d)$ that the class is closed under convolution and weak convergence, we see, by the definition of $T(\mathbb{R}^d)$ defined in Section 1, that $(Z_D(t_1),...,Z_D(t_d))$ belongs to $T(\mathbb{R}^d)$. This completes the proof. □

3. The non-symmetric Rosenblatt process

In Maejima and Tudor (2012), we extended the Rosenblatt process and introduced the non-symmetric Rosenblatt process as follows.

Let $0 < D_1 \neq D_2 < \frac{1}{2}$. The non-symmetric Rosenblatt process is defined as

$$Z_{D_1,D_2}(t) = C(D_1, D_2) \int'_{\mathbb{R}^2} \left(\int_0^t (u - s_1)_+^{-(1+D_1)/2} (u - s_s)_+^{-(1+D_2)/2} du \right) dB(s_1) dB(s_2).$$

This process is selfsimilar with stationary increments in the second Wiener chaos, and it was shown in Maejima and Tudor (2012) that there exist infinitely many such processes. When $D_1 = D_2 = D$, we retrieve the Rosenblatt process $Z_D(t)$ defined in Section 1.

In this section, we show that the distribution of the non-symmetric Rosenblatt process also belongs to the Thorin class. For this purpose, we first extend Lemma 6.2 of Taqqu (1979).

Lemma 3.1. *Let*

$$f_t(s_1, s_2) = C(D_1, D_2) \int_0^t (u - s_1)_+^{-(1+D_1)/2} (u - s_2)_+^{-(1+D_2)/2} du.$$

Then

$$\begin{aligned} Z_{D_1,D_2}(t) &= \int'_{\mathbb{R}^2} f(s_1, s_2) dB(s_1) dB(s_2) \\ &\stackrel{\mathrm{d}}{=} \int''_{\mathbb{R}^2} \frac{e^{it(x_1+x_2)} - 1}{i(x_1 + x_2)} |x_1|^{(D_1-1)/2} |x_2|^{(D_2-1)/2} W(dx_1) W(dx_2) \qquad (3.1) \\ &= \frac{1}{2} \int''_{\mathbb{R}^2} \frac{e^{i(x_1+x_2)} - 1}{i(x_1 + x_2)} \left(|x_1|^{(D_1-1)/2} |x_2|^{(D_2-1)/2} + |x_1|^{(D_2-1)/2} |x_2|^{(D_1-1)/2} \right) \\ &\qquad W(dx_1) W(dx_2). \qquad (3.2) \end{aligned}$$

Proof. The proof of (3.1) can be carried out in the same way as that for Lemma 6.2 of Taqqu (1979). The proof of (3.2) can be completed by the symmetrization of the integrand. □

The main result in this section is the following.

Theorem 3.2. *For every $t_1, ..., t_d \geq 0$, the law of $(Z_{D_1,D_2}(t_1), ..., Z_{D_1,D_2}(t_d))$ belongs to $T(\mathbb{R}^d)$.*

Proof. Let

$$H_t(x_1,x_2) = \frac{e^{it(x_1+x_2)}-1}{2i(x_1+x_2)}\left(|x_1|^{(D_1-1)/2}|x_2|^{(D_2-1)/2} + |x_1|^{(D_2-1)/2}|x_2|^{(D_1-1)/2}\right)$$

Then we have

$$Z_{D_1,D_2}(t) = \int_{\mathbb{R}^2}'' H_t(x_1,x_2)W(dx_1)W(dx_2). \tag{3.3}$$

Observe that $H_t(x_1,x_2) = H_t(x_2,x_1) = \overline{H_t(-x_2,-x_1)}, x_1, x_2 \in \mathbb{R}$, and

$$\int_{\mathbb{R}^2} |H_t(x_1,x_2)|^2 dx_1 dx_2 < \infty.$$

By Proposition 2 of Dobrushin and Major (1979), we see that (3.3) can be represented in law as

$$\sum_{n=1}^{\infty} \lambda_n^{(D_1,D_2)}(t)(\varepsilon_n^2 - 1),$$

where $\lambda_n^{(D_1,D_2)}(t)$ are the eigenvalues of the integral operator

$$A_t^{(D_1,D_2)}h(x) = \int_{-\infty}^{\infty} H_t(x,-y)h(y)dy.$$

Thus the law of $Z_{D_1,D_2}(t)$ belongs to $T(\mathbb{R})$.

The multidimensional extension is the exactly the same as for $\{Z_D(t)\}$. Namely, if we let, for $\alpha_1, ...\alpha_d \in \mathbb{R}$,

$$H_{t_1,\dots,t_d}(x_1,x_2) = \frac{1}{2}\left(\alpha_1 \frac{e^{it_1(x_1+x_2)}-1}{i(x_1+x_2)} + \cdots + \alpha_d \frac{e^{it_d(x_1+x_2)}-1}{i(x_1+x_2)}\right) \left(|x_1|^{(D_1-1)/2}|x_2|^{(D_2-1)/2} + |x_1|^{(D_2-1)/2}|x_2|^{(D_1-1)/2}\right),$$

then

$$\alpha_1 Z_{D_1,D_2}(t_1) + \cdots + \alpha_d Z_{D_1,D_2}(t_d) = \int_{\mathbb{R}^1}'' H_{t_1,\dots,t_d}(x_1,x_2)W(dx_1)W(dx_2)$$

can be represented in law as

$$\sum_{n=1}^{\infty} \lambda_n^{(D_1,D_2)}(t_1,...,t_d)(\varepsilon_n^2 - 1),$$

where $\lambda_n^{(D_1,D_2)}(t_1,...,t_d)$ are eigenvalues of the integral operator

$$A_{t_1,...,t_d}^{(D_1,D_2)}h(x) = \int_{-\infty}^{\infty} H_{t_1,...,t_d}(x,-y)h(y)dxdy.$$

Thus the law of $(Z_{D_1,D_2}(t_1),...,Z_{D_1,D_2}(t_d))$ belongs to $T(\mathbb{R}^d)$. This concludes the statement of the theorem. □

4. Stochastic integral representations with respect to Lévy process of the Rosenblatt distribution

The Rosenblatt distribution (and also the non-symmetric one) is represented by double Wiener-Itô integrals. However, we have seen that their distributions belong to the Thorin class $T(\mathbb{R})$. The distributions in $T(\mathbb{R})$ have several stochastic integral representations with respect to Lévy processes. Here we take one example. We regard them as members of the class of self-decomposable distributions, which is a larger class than the Thorin class. This allows us to obtain a new result related to the Rosenblatt distribution.

The following is known (Aoyama et al. (2011)). If $\{\gamma_{t,\lambda}, t \geq 0\}$ is a gamma process with parameter $\lambda > 0$, $\{N(t), t \geq 0\}$ is a Poisson process with unit rate and they are independent, then for any $c > 0, \lambda > 0$,

$$\gamma_{c,\lambda} \stackrel{\mathrm{d}}{=} \int_0^{\infty} e^{-t} d\gamma_{N(ct),\lambda}.$$

Let

$$Y_t = \gamma_{N(\frac{1}{2}t),\frac{1}{2}} - t.$$

Note that $\{Y_t, t \geq 0\}$ is a Lévy process. Then we have

$$\varepsilon_n^2 - 1 \stackrel{\mathrm{d}}{=} \gamma_{\frac{1}{2},\frac{1}{2}}^{(n)} - 1 \stackrel{\mathrm{d}}{=} \int_0^{\infty} e^{-t} dY_t^{(n)},$$

where $\gamma_{\frac{1}{2},\frac{1}{2}}^{(n)}$ and $\{Y_t^{(n)}\}$ are independent copies of $\gamma_{\frac{1}{2},\frac{1}{2}}$ and $\{Y_t\}$, respectively. Thus

$$Z_D \stackrel{\mathrm{d}}{=} \int_0^{\infty} e^{-t} d\left(\sum_{n=1}^{\infty} \lambda_n Y_t^{(n)}\right) =: \int_0^{\infty} e^{-t} dZ_t.$$

Remark 4.1. $\sum_{n=1}^{\infty} \lambda_n Y_t^{(n)}$ is convergent a.s. and in L^2 because

$$\sum_{n=1}^{\infty} E\left[\left(\lambda_n Y_t^{(n)}\right)^2\right] = E\left[Y_t^2\right] \sum_{n=1}^{\infty} \lambda_n^2 < \infty.$$

Remark 4.2. Since $\{Y_t^{(n)}\}, n = 1, 2, ...,$ are independent and identically distributed Lévy processes, their infinite weighted sum $\{Z_t\}$ is a Lévy process.

We thus finally have the following theorem.

Theorem 4.3.

$$Z_D \stackrel{\mathrm{d}}{=} \int_0^{\infty} e^{-t} dZ_t,$$

where $\{Z_t\}$ *is a Lévy process defined above.*

5. A concluding remark

We end this paper with some remark on the density function of the Rosenblatt distribution. **??** studied extensively the properties and numerical evaluation of the Rosenblatt distribution. Among others, they obtained its density functions and distribution functions numerically for several values of D. Their shapes are very interesting. By the figures, the Rosenblatt distribution seems unimodal. However, this fact is assured theoretically as follows.

We mentioned in the beginning of Section 4 that the Rosenblatt distribution and the non-symmetric Rosenblatt distribution are selfdecomposable. It is well-known that any selfdecomposable distribution on $\mathbb{R}$ is absolutely continuous (see, e.g., Example 27.8 of Sato (1999)) and is unimodal (by Yamazato (1978): see also Theorem 53.1 of Sato (1999)).

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