# THE GENERALIZED LANGEVIN EQUATION AND AN EXAMPLE OF TYPE $G$ DISTRIBUTIONS

MAKOTO MAEJIMA AND YOHEI NIIYAMA
(DEPARTMENT OF MATHEMATICS, KEIO UNIVERSITY)

## 1. The generalized Langevin equation and generalized Ornstein-Uhlenbeck processes

Kesten [4] and Vervaat [11] studied a stochastic difference equation

$$Y_n = A_n Y_{n-1} + B_n, \qquad n = 1, 2, \cdots, \tag{1.1}$$

where $(A_n, B_n)$ are independent and identically distributed $\mathbb{R}^2$-valued random variables, and obtained conditions under which $Y_n$ converges in law to a random variable $Y$ as $n \to \infty$. This limit, if it exists, is the solution of $Y \overset{\mathrm{d}}{=} A_1 Y + B_1$, where $Y$ and $(A_1, B_1)$ are independent. Here and below $\overset{\mathrm{d}}{=}$ means equaivalence in law.

As a continuous analogue of (1.1), we introduce

$$dY_t = -\lambda Y_{t-} dt + Y_{t-} dL_t^{(1)} + dL_t^{(2)}, \quad t \geq 0, \tag{1.2}$$

where $\lambda > 0$ and $\{(L_t^{(1)}, L_t^{(2)}), t \geq 0\}$ is an $\mathbb{R}^2$-valued Lévy process. The stochastic differential equation (1.2) extends simultaneously the Langevin eqaution and the Black-Scholes equation. When $L_t^{(1)} \equiv 0$, (1.2) is the Langevin equation driven by $\{L_t^{(2)}\}$, and when $L_t^{(2)} \equiv 0$, (1.2) is the Black-Scholes equation driven by $\{L_t^{(1)}\}$. We call (1.2) the generalized Langevin equation.

A similar stochastic differential equation can be found in Carmona et al. [1]. Namely, they mentioned

$$dX_t = (1/2 - r) X_{t-} dt + X_{t-} dB_t + d\eta_t, \tag{1.3}$$

where $\{B_t\}$ is a Brownian motion, $\{\eta_t\}$ is a compound Poisson process and they are independent. Actually, they studied the generalized Ornstein-Uhlenbeck process

$$X_t = e^{-\xi_t} \left( x + \int_0^t e^{\xi_{s-}} d\eta_s \right), \quad t \geq 0, \ x \in \mathbb{R}, \tag{1.4}$$

associated to a given $\mathbb{R}^2$-valued Lévy process $(\xi_t, \eta_t)$, and showed that an example (1.4) with $\xi_t = -B_t + rt$ and $\eta_t$ a compound Poisson process, is the solution of the stochastic differential equation (1.3). Generalized Ornstein-Uhlenbeck processes have recently been studied also by Erickson and Maller [3] and Lindner and Maller [5].

Here we restrict ourselves to the case where $L_t^{(1)} = B_t$ (Brownian motion), $L_t^{(2)} = L_t$ (Lévy process) and they are independent. Namely,

$$dY_t = -\lambda Y_{t-}dt + \sigma Y_{t-}dB_t + dL_t, \quad t \geq 0, \tag{1.5}$$

where $\lambda, \sigma > 0$. Then, an explicit solution of (1.5) is the following.

**Theorem 1.1.** *A unique solution of the generalized Langevin equation* (1.5)*, where* $\{B_t\}$ *and* $\{L_t\}$ *are independent, is*

$$Y_t = e^{-U_t}\left(Y_0 + \int_0^t e^{U_s}dL_s\right),$$

*where*

$$U_t = -\sigma B_t + \left(\lambda + 2^{-1}\sigma^2\right)t,$$

For the proof, it is enough to appply Theorem 52 of Chapter V in Protter [6]. We are interested in the limit of $Y_t$ when $t \to \infty$.

**Theorem 1.2.** *If* $E[\log^+|L_1|] < \infty$*, then* $Y_t$ *converges in law to* $Y = \int_0^\infty e^{-U_s}dL_s$.

For the proof, it is enough to check two conditions in Theorem 3.1 of Carmona et al. [1], namely, (i) $e^{-U_t} = \exp\{\sigma B_t - (\lambda + 2^{-1}\sigma^2)t\} \to 0$ a.s. and (ii) $\int_0^\infty e^{-U_s}dL_s$ is well-defined and almost surely finite. However, both are easy to be seen due to the law of the iterated logarithm for $\{B_t\}$, which is, alomost surely for some $c_1, c_2 > 0, e^{-c_1 t} < e^{U_t} < e^{-c_2 t}$ for large $t$. Thus (i) is trivial. Also, Sato and Yamazato [10] showed that if $E[\log^+|L_1|] < \infty$ then $\int_0^\infty e^{-cs}dL_s$ is well-defined, which proves (ii).

## 2. An example

We are interested in the infinite divisibility of the limiting random variable $Y$ above, and give an example for it.

We start with the notion of "type $S$" of infinitely divisible random variables. We say that a real-valued random variable $Z_\alpha, 0 < \alpha \leq 2$, is symmetric $\alpha$-stable if $E[e^{izZ_\alpha}] = e^{-c|z|^\alpha}, c > 0$. A real-valued random variable $X$ is called a sacling mixture of symmetric $\alpha$-stable, if

$$E[e^{izX}] = \int_0^\infty e^{-|z|^\alpha u}H(du),$$

where $H$ is a probability measure on $[0,\infty)$. An equivalent statement is that

$$X \stackrel{\mathrm{d}}{=} V^{1/\alpha} Z_\alpha, \tag{2.1}$$

where $Z_\alpha$ is symmetric $\alpha$-stable, $V$ is a positive random variable independent of $Z_\alpha$ with $\mathcal{L}(V) = H$. Here $\mathcal{L}(V)$ means the law of $V$

**Definition 2.1.** Let $0 < \alpha \leq 2$. A real-valued random variable $X$ is said to be of type $S_\alpha$, if it satisfies (2.1) with an infinitely divisible $V > 0$. If $X$ is of type $S_\alpha$ for some $\alpha \in (0,2]$, it is called of type $S$. If $\alpha = 2$, it is of type $G$. (For type $G$ distributions, see, e.g., Maejima and Rosiński [6] and the references therin.)

**Proposition 2.2.** *For any $0 < \alpha < 2$, a type $S_\alpha$ random variable is of type $G$ and thus infinitely divisible.*

This statemanet follows from the fact that iteration of subordination is again subordination, (see Sato [9], Theorem 30.4).

**Theorem 2.3.** *Let $Y$ be the random variable given in Theorem 1.2. If $\{L_t\}$ is a symmetric $\alpha$-stable Lévy process $\{S_t\}$, $0 < \alpha \leq 2$, then $Y$ is of type $S_\alpha$, and thus infinitely divisible.*

*Proof.* It is known that for any $a \in \mathbb{R}, a \neq 0, b > 0$,

$$\int_0^\infty e^{aB_t - bt} dt \stackrel{\mathrm{d}}{=} 2\left(a^2 \Gamma_{2ba^{-2}}\right)^{-1},$$

where $\Gamma_\gamma$ is the gamma random variable with parameter $\gamma > 0$. (Dufresne [2].) It is also known that the reciprocal of gamma random variable is infinitely divisible. Now, we have

$$E\left[\exp\left(i\theta Y\right)\right] = E\left[\exp\left(i\theta \int_0^\infty e^{-U_s} dS_s\right)\right] = E_U\left[E_S\left[\exp\left(i\theta \int_0^\infty e^{-U_s} dS_s\right)\right]\right].$$

where $E_U$ and $E_S$ are the expectations with respect to $\{U_t\}$ and $\{S_t\}$, respectively. Since

$$E\left[\exp\left\{i\theta \int_0^\infty f_s dS_s\right\}\right] = \exp\left\{-|\theta|^\alpha \int_0^\infty |f_s|^\alpha ds\right\}$$

(see, e.g., Samorodnitsky-Taqqu [8]), we have

$$
\begin{aligned}
E[e^{i\theta Y}] &= E_U\left[\exp\left(-|\theta|^\alpha \int_0^\infty e^{-\alpha U_s} ds\right)\right] \\
&= E_U\left[\exp\left\{-|\theta|^\alpha \int_0^\infty \exp\left(\alpha\sigma B_u - \alpha(\lambda + 2^{-1}\sigma^2)u\right) du\right\}\right] \\
&= E_U\left[\exp\left\{-|\theta|^\alpha 2\left(\alpha^2\sigma^2\Gamma_{\frac{2\alpha(\lambda+2^{-1}\sigma^2)}{\alpha^2\sigma^2}}\right)^{-1}\right\}\right].
\end{aligned}
$$

If we put

$$
H(dx) = P\left(2\left(\alpha^2\sigma^2\Gamma_{\frac{2\alpha(\lambda+2^{-1}\sigma^2)}{\alpha^2\sigma^2}}\right)^{-1} \in dx\right),
$$

then

$$
E[e^{i\theta Y}] = \int_0^\infty e^{-|\theta|^\alpha u} H(du),
$$

where, as we have seen, $H$ is the distribution function of a positive infinitely divisible random variable. Thus, $Y$ is of type $S_\alpha$, and hence infinitely divisible. □

## References


[1] P. Carmona, F. Petit and M. Yor (2001) : Exponential functionals of Lévy processes, in O. E. Barndorff-Nielsen, T. Mikosch and S. I. Resnick (eds.), *Lévy Processes, Theory and Applications*, Birkhauser, 41-55.

[2] D. Dufresne (1990) : The disiribution of a perpetuity, with applications to risk theory and pension funding, *Scand. Actuarial J.* **1990**, 39-79.

[3] K.B. Erickson and R.A. Maller (2004) : Generalised Ornstein-Uhlenbeck processes and the convergence of Lévy integrals, preprint.

[4] H. Kesten (1973) : Random difference equations and renewal theory for products of random matrices, *Acta Math.* **131**, 207-248.

[5] A. Lindner and R. Maller (2004) : Lévy integrals and the stationarity of generalised Ornstein-Uhlenbeck processes, prerint.

[6] M. Maejima and J. Rosiński (2002) : Type $G$ distributions on $\mathbb{R}^d$, *J. Theor. Probab.* **15**, 323-341.

[7] P. Protter (2004) : *Stochastic Integration and Differential Equations*, 2nd ed. Springer.

[8] G. Samorodnitsky and M. S. Taqqu (1994) : *Stable Non-Gaussian Random Processes*, Chapman & Hall.

[9] K. Sato (1999) : *Lévy Processes and Infinitely Divisible Distributions*, Cambridge Univ. Press.

[10] K. Sato and M. Ymamazato (1983) : Stationary processes of Ornstein-Uhlenbeck type, *Lect. Notes in Math.* No.1021, Springer, 541-551.

[11] W. Vervaat (1979) : On a stochastic equation and a representation of non-negative infinitely divisible random variables, *Adv. Appl. Prob.* **11**, 750-783.