THE GENERALIZED LANGEVIN EQUATION
AND AN EXAMPLE OF TYPE G DISTRIBUTIONS

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1. THE GENERALIZED LANGEVIN EQUATION AND
   GENERALIZED ORNSTEIN-UHLENBECK PROCESSES


\[ Y_n = A_n Y_{n-1} + B_n, \quad n = 1, 2, \ldots, \]

where \((A_n, B_n)\) are independent and identically distributed \(\mathbb{R}^2\)-valued random variables, and obtained conditions under which \(Y_n\) converges in law to a random variable \(Y\) as \(n \to \infty\). This limit, if it exists, is the solution of \(Y = A_1 Y + B_1\), where \(Y\) and \((A_1, B_1)\) are independent. Here and below \(\overset{d}{=}\) means equivalence in law.

As a continuous analogue of (1.1), we introduce

\[ dY_t = -\lambda Y_{t-} dt + Y_{t-} dL^{(1)}_t + dL^{(2)}_t, \quad t \geq 0, \]

where \(\lambda > 0\) and \(\{(L^{(1)}_t, L^{(2)}_t), t \geq 0\}\) is an \(\mathbb{R}^2\)-valued Lévy process. The stochastic differential equation (1.2) extends simultaneously the Langevin equation and the Black-Scholes equation. When \(L^{(1)}_t \equiv 0\), (1.2) is the Langevin equation driven by \(\{L^{(2)}_t\}\), and when \(L^{(2)}_t \equiv 0\), (1.2) is the Black-Scholes equation driven by \(\{L^{(1)}_t\}\). We call (1.2) the generalized Langevin equation.

A similar stochastic differential equation can be found in Carmona et al. [1]. Namely, they mentioned

\[ dX_t = (1/2 - r) X_{t-} dt + X_{t-} dB_t + d\eta_t, \]

where \(\{B_t\}\) is a Brownian motion, \(\{\eta_t\}\) is a compound Poisson process and they are independent. Actually, they studied the generalized Ornstein-Uhlenbeck process

\[ X_t = e^{-\xi_t} \left( x + \int_0^t e^{\xi_s} d\eta_s \right), \quad t \geq 0, \quad x \in \mathbb{R}, \]

associated to a given \(\mathbb{R}^2\)-valued Lévy process \((\xi_t, \eta_t)\), and showed that an example (1.4) with \(\xi_t = -B_t + rt\) and \(\eta_t\) a compound Poisson process, is the solution of the
stochastic differential equation (1.3). Generalized Ornstein-Uhlenbeck processes have recently been studied also by Erickson and Maller [3] and Lindner and Maller [5].

Here we restrict ourselves to the case where \( L_t^{(1)} = B_t \) (Brownian motion), \( L_t^{(2)} = L_t \) (Lévy process) and they are independent. Namely,

\[
dY_t = -\lambda Y_t \, dt + \sigma Y_t \, dB_t + dL_t, \quad t \geq 0,
\]

where \( \lambda, \sigma > 0 \). Then, an explicit solution of (1.5) is the following.

**Theorem 1.1.** A unique solution of the generalized Langevin equation (1.5), where \( \{B_t\} \) and \( \{L_t\} \) are independent, is

\[
Y_t = e^{-U_t} \left( Y_0 + \int_0^t e^{U_s} \, dL_s \right),
\]

where

\[
U_t = -\sigma B_t + \left( \lambda + 2^{-1} \sigma^2 \right) t,
\]

For the proof, it is enough to apply Theorem 52 of Chapter V in Protter [6]. We are interested in the limit of \( Y_t \) when \( t \to \infty \).

**Theorem 1.2.** If \( E[\log^+ |L_1|] < \infty \), then \( Y_t \) converges in law to \( Y = \int_0^\infty e^{-U_s} \, dL_s \).

For the proof, it is enough to check two conditions in Theorem 3.1 of Carmona et al. [1], namely, (i) \( e^{-U_t} = \exp\{\sigma B_t - (\lambda + 2^{-1} \sigma^2) t\} \to 0 \) a.s. and (ii) \( \int_0^\infty e^{-U_s} \, dL_s \) is well-defined and almost surely finite. However, both are easy to be seen due to the law of the iterated logarithm for \( \{B_t\} \), which is, almost surely for some \( c_1, c_2 > 0 \), \( e^{-c_1 t} < e^{U_t} < e^{-c_2 t} \) for large \( t \). Thus (i) is trivial. Also, Sato and Yamazato [10] showed that if \( E[\log^+ |L_1|] < \infty \) then \( \int_0^\infty e^{-c_s} \, dL_s \) is well-defined, which proves (ii).

2. **An example**

We are interested in the infinite divisibility of the limiting random variable \( Y \) above, and give an example for it.

We start with the notion of “type S” of infinitely divisible random variables. We say that a real-valued random variable \( Z_\alpha, 0 < \alpha \leq 2 \), is symmetric \( \alpha \)-stable if \( E[e^{i\alpha Z_\alpha}] = e^{-c|\alpha|^\alpha} \), \( c > 0 \). A real-valued random variable \( X \) is called a scaling mixture of symmetric \( \alpha \)-stable, if

\[
E[e^{i\alpha Z_\alpha}] = \int_0^\infty e^{-1|\alpha|^\alpha u} H(du),
\]

where

\[
H(du) = ||\alpha||^\alpha u^{\alpha-1} |\alpha|^{-\alpha} \varphi(u) du,
\]

\( \varphi(u) \) is a density function.
where $H$ is a probability measure on $[0, \infty)$. An equivalent statement is that

\begin{equation}
X \overset{d}{=} V^{1/\alpha} Z_{\alpha},
\end{equation}

where $Z_{\alpha}$ is symmetric $\alpha$-stable, $V$ is a positive random variable independent of $Z_{\alpha}$ with $\mathcal{L}(V) = H$. Here $\mathcal{L}(V)$ means the law of $V$.

**Definition 2.1.** Let $0 < \alpha \leq 2$. A real-valued random variable $X$ is said to be of type $S_{\alpha}$, if it satisfies (2.1) with an infinitely divisible $V > 0$. If $X$ is of type $S_{\alpha}$ for some $\alpha \in (0, 2]$, it is called of type $S$. If $\alpha = 2$, it is of type $G$. (For type $G$ distributions, see, e.g., Maejima and Rosiński [6] and the references therein.)

**Proposition 2.2.** For any $0 < \alpha < 2$, a type $S_{\alpha}$ random variable is of type $G$ and thus infinitely divisible.

This statement follows from the fact that iteration of subordination is again subordination, (see Sato [9], Theorem 30.4).

**Theorem 2.3.** Let $Y$ be the random variable given in Theorem 1.2. If $\{L_t\}$ is a symmetric $\alpha$-stable Lévy process $\{S_t\}$, $0 < \alpha \leq 2$, then $Y$ is of type $S_{\alpha}$, and thus infinitely divisible.

**Proof.** It is known that for any $a \in \mathbb{R}, a \neq 0, b > 0$, 

\begin{equation}
\int_0^\infty e^{\alpha B_t - bt} \, dt \overset{d}{=} 2 \left( a^2 \Gamma_{2\alpha-2} \right)^{-1},
\end{equation}

where $\Gamma_\gamma$ is the gamma random variable with parameter $\gamma > 0$. (Dufresne [2].) It is also known that the reciprocal of gamma random variable is infinitely divisible. Now, we have

\begin{equation}
E \left[ \exp (i\theta Y) \right] = E \left[ \exp \left( i\theta \int_0^\infty e^{-V_s} \, dS_s \right) \right] = E_U \left[ E_S \left[ \exp \left( i\theta \int_0^\infty e^{-V_s} \, dS_s \right) \right] \right].
\end{equation}

where $E_U$ and $E_S$ are the expectations with respect to $\{U_t\}$ and $\{S_t\}$, respectively. Since

\begin{equation}
E \left[ \exp \left\{ i\theta \int_0^\infty f_s \, dS_s \right\} \right] = \exp \left\{ -|\theta|^\alpha \int_0^\infty |f_s|^\alpha \, ds \right\}
\end{equation}
(see, e.g., Samorodnitsky-Taqqu [8]), we have

\[
E[e^{i\theta Y}] = E_U \left[ \exp \left( -|\theta|^\alpha \int_0^\infty e^{-\alpha u^\alpha} du \right) \right]
\]

\[
= E_U \left[ \exp \left( -|\theta|^\alpha \int_0^\infty \exp \left( \alpha \sigma B_u - \alpha (\lambda + 2^{-1} \sigma^2) u \right) du \right) \right]
\]

\[
= E_U \left[ \exp \left( -|\theta|^\alpha 2 \left( \frac{\alpha^2 \sigma^2 \Gamma \frac{2\lambda + 2 - 1}{\alpha^2 \sigma^2}}{\alpha^2 \sigma^2} \right)^{-1} \right) \right].
\]

If we put

\[
H(dx) = P \left( 2 \left( \frac{\alpha^2 \sigma^2 \Gamma \frac{2\lambda + 2 - 1}{\alpha^2 \sigma^2}}{\alpha^2 \sigma^2} \right)^{-1} \in dx \right),
\]

then

\[
E[e^{i\theta Y}] = \int_0^\infty e^{-|\theta|^\alpha u} H(du),
\]

where, as we have seen, \( H \) is the distribution function of a positive infinitely divisible random variable. Thus, \( Y \) is of type \( S_\alpha \), and hence infinitely divisible.

\[\square\]

**References**


