Fixed points of mappings of infinitely divisible distributions on $\mathbb{R}^d$

Ken Ichifuji[a], Makoto Maejima[a,*], Yohei Ueda[a]

[a] *Department of Mathematics, Keio University, 3-14-1, Hiyoshi, Kohoku-ku, Yokohama 223-8522, Japan.*

Abstract

A fixed point μ under a mapping Φ of infinitely divisible distributions on $\mathbb{R}^d$ is defined by the relation $\Phi(\mu) = \mu^a * \delta_c$ for some $a > 0$ and $c \in \mathbb{R}^d$. We investigate fixed points under some specific mappings, which are related to stable distributions.

Keywords: infinitely divisible distribution, stable distribution, semi-stable distribution, mapping of infinitely divisible distribution, fixed point
2000 MSC: 60E07

1. Introduction

Let $I(\mathbb{R}^d)$ be the class of all infinitely divisible distributions on $\mathbb{R}^d$, $I_{\log^m}(\mathbb{R}^d) = \{\mu \in I(\mathbb{R}^d) \colon \int_{\mathbb{R}^d} (\log^+ |x|)^m \mu(dx) < \infty\}$ for $m \in \mathbb{N}$ and $I_{\log}(\mathbb{R}^d) = I_{\log^1}(\mathbb{R}^d)$. Here $|x|$ is the Euclidean norm of $x \in \mathbb{R}^d$ and $\log^+ |x| = (\log |x|) \vee 0$. Let $\widehat{\mu}(z), z \in \mathbb{R}^d$, be the characteristic function of $\mu \in I(\mathbb{R}^d)$. The Lévy-Khintchine representation of $\widehat{\mu}$ we use in this paper is

$$\widehat{\mu}(z) = \exp\left\{ -\frac{1}{2}\langle z, Az\rangle + i\langle \gamma, z\rangle + \int_{\mathbb{R}^d} \left(e^{i\langle z,x\rangle} - 1 - \frac{i\langle z, x\rangle}{1+|x|^2} \right) \nu(dx) \right\},$$

where $\langle \cdot, \cdot \rangle$ is Euclidean inner product on $\mathbb{R}^d$, A is a nonnegative-definite symmetric $d \times d$ matrix, $\gamma \in \mathbb{R}^d$ and ν is the Lévy measure satisfying $\nu(\{0\}) =$

*Corresponding author.
Email addresses: `maejima@math.keio.ac.jp` (Makoto Maejima), `ueda@2008.jukuin.keio.ac.jp` (Yohei Ueda)

0 and $\int_{\mathbb{R}^d}(|x|^2 \wedge 1)\nu(dx) < \infty$. We call (A, ν, γ) the Lévy-Khintchine triplet of μ and we write $\mu = \mu_{(A,\nu,\gamma)}$ when we want to emphasize the triplet. $C_\mu(z), z \in \mathbb{R}^d$, denotes the cumulant function of $\mu \in I(\mathbb{R}^d)$, that is, $C_\mu(z)$ is the unique continuous function satisfying $\widehat{\mu}(z) = e^{C_\mu(z)}$ and $C_\mu(0) = 0$. For $\mu \in I(\mathbb{R}^d)$ and $t > 0$, we call the distribution with characteristic function $\widehat{\mu}(z)^t := e^{tC_\mu(z)}$ the t-th convolution of μ and write μ^t for it.

Let $S = \{x \in \mathbb{R}^d \colon |x| = 1\}$ and we write, for $E \in \mathcal{B}((0,\infty))$ and $C \in \mathcal{B}(S)$, $EC = \{x \in \mathbb{R}^d \setminus \{0\} \colon |x| \in E \text{ and } x/|x| \in C\}$. The polar decomposition of the Lévy measure ν of $\mu \in I(\mathbb{R}^d)$, with $0 < \nu(\mathbb{R}^d) \le \infty$, is the following: There exist a measure λ on S with $0 < \lambda(S) \le \infty$ and a family $\{\nu_\xi, \xi \in S\}$ of measures on $(0,\infty)$ such that $\nu_\xi(B)$ is measurable in ξ for each $B \in \mathcal{B}((0,\infty))$, $0 < \nu_\xi((0,\infty)) \le \infty$ for each $\xi \in S$ and

$$\nu(B) = \int_S \lambda(d\xi) \int_0^\infty \mathbb{1}_B(r\xi)\nu_\xi(dr), \quad B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}).$$

Here λ and $\{\nu_\xi\}$ are uniquely determined by ν up to multiplication of measurable functions $c(\xi)$ and $c(\xi)^{-1}$ with $0 < c(\xi) < \infty$. We say that ν has the polar decomposition (λ, ν_ξ) and ν_ξ is called the radial component of ν. (See, e.g., Barndorff-Nielsen et al. (2006), Lemma 2.1.)

We use stochastic integrals with respect to Lévy processes $\{X_t, t \ge 0\}$ of nonrandom measurable functions $f \colon [0,\infty) \to \mathbb{R}$, which are $\int_0^t f(s)dX_s, t \in [0,\infty)$. For the definition and the deep study of stochastic integrals with respect to Lévy processes, see Sato (2004, 2006a). The improper stochastic integral $\int_0^\infty f(s)dX_s$ is defined as the limit in probability of $\int_0^t f(s)dX_s$ as $t \to \infty$, provided that the limit exists. Define a mapping $\Phi_f \colon \mathfrak{D}(\Phi_f) \to I(\mathbb{R}^d)$ by

$$\Phi_f(\mu) = \mathcal{L}\left(\int_0^\infty f(t)dX_t^{(\mu)}\right),$$

where $\mathcal{L}$ means "the law of", $\{X_t^{(\mu)}\}$ is a Lévy process with $\mathcal{L}(X_1^{(\mu)}) = \mu \in I(\mathbb{R}^d)$ and $\mathfrak{D}(\Phi_f)$ is the totality of $\mu \in I(\mathbb{R}^d)$ such that $\int_0^\infty f(t)dX_t^{(\mu)}$ is definable in the sense above. For a mapping Φ_f, $\mathfrak{R}(\Phi_f)$ is its range that is $\{\Phi_f(\mu) \colon \mu \in \mathfrak{D}(\Phi_f)\}$. When we consider the composition of two mappings Φ_f and Φ_g, denoted by $\Phi_g \circ \Phi_f$, the domain of $\Phi_g \circ \Phi_f$ is $\mathfrak{D}(\Phi_g \circ \Phi_f) = \{\mu \in I(\mathbb{R}^d) \colon \mu \in \mathfrak{D}(\Phi_f) \text{ and } \Phi_f(\mu) \in \mathfrak{D}(\Phi_g)\}$. For $m \in \mathbb{N}$, Φ_f^m means the m times composition of Φ_f itself.

Following Jurek and Vervaat (1983) and Jurek (1985, 1988), we define a fixed point μ under a mapping Φ_f as follows.

Definition 1.1. $\mu \in \mathfrak{D}(\Phi_f)$ is called a fixed point under the mapping Φ_f, if there exist $a > 0$ and $c \in \mathbb{R}^d$ such that

$$\Phi_f(\mu) = \mu^a * \delta_c,$$

where δ_c is the probability distribution concentrated at $c \in \mathbb{R}^d$, and $*$ means convolution.

The set of all fixed points under the mapping Φ_f is denoted by $\mathrm{FP}(\Phi_f)$ in this paper. The purpose of this paper is to investigate $\mathrm{FP}(\Phi_f)$.

2. A relation among the class of fixed points under a mapping, the class of stable distributions and the limit of ranges of iterated mappings

For $0 < p \le 2$, let $S_p(\mathbb{R}^d)$ be the class of all p-stable distributions on $\mathbb{R}^d$ and $S(\mathbb{R}^d) = \bigcup_{0<p\le 2} S_p(\mathbb{R}^d)$. Furthermore, for $1 < p \le 2$, let $S_p^0(\mathbb{R}^d)$ be the class of p-stable distributions on $\mathbb{R}^d$ with mean 0. We start with the following.

Theorem 2.1. *Let f be a nonnegative nonrandom measurable function. Then*

$$S(\mathbb{R}^d) \cap \mathfrak{D}(\Phi_f) \subset \mathrm{FP}(\Phi_f).$$

For the proof, we need the following.

Proposition 2.2 (Proposition 2.6 of Sato (2006b)). *If $\mu = \mu_{(A,\nu,\gamma)} \in \mathfrak{D}(\Phi_f)$, then $\widetilde{\mu}_{(\widetilde{A},\widetilde{\nu},\widetilde{\gamma})} = \Phi_f(\mu)$ satisfies*

$$C_{\widetilde{\mu}}(z) = \lim_{T\to\infty} \int_0^T C_\mu(f(t)z)dt, \tag{1}$$

$$\widetilde{A} = \int_0^\infty f(t)^2 A dt, \tag{2}$$

$$\widetilde{\nu}(B) = \int_0^\infty dt \int_{\mathbb{R}^d} 1\!\!1_B(f(t)x)\nu(dx), \quad B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}), \tag{3}$$

$$\widetilde{\gamma} = \lim_{T\to\infty} \int_0^T f(t)dt \left(\gamma + \int_{\mathbb{R}^d} x \left(\frac{1}{1+|f(t)x|^2} - \frac{1}{1+|x|^2} \right) \nu(dx) \right).$$

Proof of Theorem 2.1. Let $\mu = \mu_{(A,\nu,\gamma)} \in S(\mathbb{R}^d) \cap \mathfrak{D}(\Phi_f)$. Then $\mu \in S_p(\mathbb{R}^d)$ for some $p \in (0,2]$. Suppose $p = 2$. Then $\nu = 0$ and we have (2). If $\int_0^\infty f(t)^2 dt < \infty$, then $\Phi_f(\mu) = \mu^{\int_0^\infty f(t)^2 dt} * \delta_c$ for some $c \in \mathbb{R}^d$, namely, $\mu \in \mathrm{FP}(\Phi_f)$. If $\int_0^\infty f(t)^2 dt = \infty$, then $A = 0$. Then μ is a δ-distribution, which is a fixed point whenever it belongs to $\mathfrak{D}(\Phi_f)$. Next suppose $p \in (0,2)$. Then $\nu(b^{-1}B) = b^p \nu(B)$ for any $b > 0$ and $B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$. Letting $\widetilde{\nu}$ be the Lévy measure of $\Phi_f(\mu)$ and using (3), we have

$$\widetilde{\nu}(B) = \int_{\{t \geq 0:\, f(t) \neq 0\}} \nu(f(t)^{-1}B)dt = \int_0^\infty f(t)^p \nu(B)dt, \quad B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}).$$

If $\int_0^\infty f(t)^p dt < \infty$, then $\Phi_f(\mu) = \mu^{\int_0^\infty f(t)^p dt} * \delta_c$ for some $c \in \mathbb{R}^d$, namely, $\mu \in \mathrm{FP}(\Phi_f)$. If $\int_0^\infty f(t)^p dt = \infty$, then $\nu = 0$. Indeed, it follows from the equation above that

$$\infty > \int_{|x|>1/n} \widetilde{\nu}(dx) = \int_0^\infty f(t)^p \left(\int_{|x|>1/n} \nu(dx) \right) dt, \quad \text{for all } n \in \mathbb{N},$$

which yields that $\int_{|x|>1/n} \nu(dx) = 0$ for all $n \in \mathbb{N}$. Then μ is a δ-distribution, which is a fixed point whenever it belongs to $\mathfrak{D}(\Phi_f)$. $\square$

Let

$$I_\alpha(\mathbb{R}^d) = \left\{ \mu \in I(\mathbb{R}^d) \colon \int_{\mathbb{R}^d} |x|^\alpha \mu(dx) < \infty \right\}, \quad \text{for } \alpha > 0,$$

$$I_\alpha^0(\mathbb{R}^d) = \left\{ \mu \in I_\alpha(\mathbb{R}^d) \colon \int_{\mathbb{R}^d} x\mu(dx) = 0 \right\}, \quad \text{for } \alpha \geq 1.$$

Then we have the following.

Theorem 2.3. *Let a nonrandom measurable function f satisfy that $\int_0^t f(s)ds$ tends to a finite constant $C \neq 0$ as $t \to \infty$, or that $\mathfrak{D}(\Phi_f) \subset I_1^0(\mathbb{R}^d)$. Then*

$$\mathrm{FP}(\Phi_f) \subset \bigcap_{m=1}^\infty \mathfrak{R}(\Phi_f^m).$$

Remark 2.4. The assumption of Theorem 2.3 seems reasonable as explained below. Actually, most functions appearing in Theorems 2.4 and 2.8 of Sato (2006b) satisfy this assumption. For two functions f and g, we write $f(t) \asymp g(t)$ as $t \to \infty$ if there exist constants $c_1, c_2, t_0 > 0$ such that for all $t \geq t_0$,

$0 < c_1 g(t) \le f(t) \le c_2 g(t)$. Let f be a locally square-integrable function on $[0,\infty)$. If $f(t) \asymp e^{-ct}$ as $t \to \infty$ with some $c > 0$ and $\int_0^\infty f(t)dt \ne 0$, then f satisfies the assumption. Let $0 < \alpha < 1$. If $f(t) \asymp t^{-1/\alpha}$ as $t \to \infty$ and $\int_0^\infty f(t)dt \ne 0$, then f satisfies the assumption. If $f(t) \asymp t^{-1}$ as $t \to \infty$ and there are $t_0, c > 0$ and $g(t)$ such that $f(t) = t^{-1}g(t)$ for $t \ge t_0$ and $\int_{t_0}^\infty t^{-1}|g(t) - c|dt < \infty$, then f satisfies the assumption. Let $\alpha > 1$. If $f(t) \asymp t^{-1/\alpha}$ as $t \to \infty$, then f satisfies the assumption.

Proof of Theorem 2.3. Let $\mu \in \mathrm{FP}(\Phi_f)$. Then $\Phi_f(\mu) = \mu^a * \delta_c$ for some $a > 0$ and $c \in \mathbb{R}^d$.

First suppose that $\int_0^t f(s)ds$ tends to a finite constant $C \ne 0$ as $t \to \infty$. Since $\mu, \delta_c \in \mathfrak{D}(\Phi_f)$, it follows that $\Phi_f(\mu) \in \mathfrak{D}(\Phi_f)$. We thus have

$$\Phi_f^2(\mu) = \Phi_f(\mu^a) * \Phi_f(\delta_c) = \Phi_f(\mu)^a * \delta_{cC} = \mu^{a^2} * \delta_{ac+cC}.$$

Iterating this argument, we have that for all $m \in \mathbb{N}$, there is $c_m \in \mathbb{R}^d$ satisfying

$$\Phi_f^m(\mu) = \mu^{a^m} * \delta_{c_m}.$$

Hence we have

$$\mu = \Phi_f^m(\mu^{a^{-m}}) * \delta_{-a^{-m}c_m} = \Phi_f^m\left(\mu^{a^{-m}} * \delta_{-a^{-m}c_m C^{-m}}\right) \in \mathfrak{R}(\Phi_f^m)$$

for every $m \in \mathbb{N}$. Thus $\mu \in \bigcap_{m=1}^\infty \mathfrak{R}(\Phi_f^m)$.

Next suppose $\mathfrak{D}(\Phi_f) \subset I_1^0(\mathbb{R}^d)$. Then, $\mu \in I_1^0(\mathbb{R}^d)$. Hence $\Phi_f(\mu) = \mu^a * \delta_c \in I_1(\mathbb{R}^d)$. Let $\mu = \mu_{(A,\nu,\gamma)}$ and $\Phi_f(\mu) = \widetilde{\mu} = \widetilde{\mu}_{(\widetilde{A},\widetilde{\nu},\widetilde{\gamma})}$. We have

$$\begin{aligned}
\widetilde{\gamma} &= \lim_{t\to\infty} \int_0^t f(s)ds \left\{\gamma + \int_{\mathbb{R}^d} x\left(\frac{1}{1+|f(s)x|^2} - \frac{1}{1+|x|^2}\right)\nu(dx)\right\} \\
&= \lim_{t\to\infty} \int_0^t f(s)ds \left\{-\int_{\mathbb{R}^d} \frac{x|x|^2}{1+|x|^2}\nu(dx) + \int_{\mathbb{R}^d} x\left(\frac{1}{1+|f(s)x|^2} - \frac{1}{1+|x|^2}\right)\nu(dx)\right\} \\
&= -\lim_{t\to\infty} \int_0^t ds \int_{\mathbb{R}^d} \frac{f(s)x|f(s)x|^2}{1+|f(s)x|^2}\nu(dx) = -\int_{\mathbb{R}^d} \frac{x|x|^2}{1+|x|^2}\widetilde{\nu}(dx),
\end{aligned}$$

where the last equality follows from the condition $\widetilde{\mu} \in I_1(\mathbb{R}^d)$. Then $\Phi_f(\mu) = \widetilde{\mu} \in I_1^0(\mathbb{R}^d)$. Combining this with the condition $\mu^a \in I_1^0(\mathbb{R}^d)$, we have $c = 0$. Hence $\Phi_f(\mu) = \mu^a$. Then it follows that $\mu = \Phi_f^m(\mu^{a^{-m}}) \in \mathfrak{R}(\Phi_f^m)$ for each $m \in \mathbb{N}$. Thus $\mu \in \bigcap_{m=1}^\infty \mathfrak{R}(\Phi_f^m)$. $\square$

As a consequence of Theorems 2.1 and 2.3, we have a relation among the class of fixed points under a mapping, the class of stable distributions and the limit of ranges of iterated mappings as follows.

Theorem 2.5. *Let f be a nonnegative nonrandom measurable function satisfying that $\int_0^\infty f(s)ds$ is finite with nonzero value, or that $\mathfrak{D}(\Phi_f) \subset I_1^0(\mathbb{R}^d)$. Then,*

$$S(\mathbb{R}^d) \cap \mathfrak{D}(\Phi_f) \subset \mathrm{FP}(\Phi_f) \subset \bigcap_{m=1}^{\infty} \mathfrak{R}(\Phi_f^m).$$

Then, it would be natural to ask where $\mathrm{FP}(\Phi_f)$ is located between $S(\mathbb{R}^d) \cap \mathfrak{D}(\Phi_f)$ and $\bigcap_{m=1}^{\infty} \mathfrak{R}(\Phi_f^m)$. In the following sections, we give several examples.

3. Mappings whose fixed points are stable distributions

In this section, we give examples of mappings satisfying $\mathrm{FP}(\Phi_f) = S(\mathbb{R}^d) \cap \mathfrak{D}(\Phi_f)$. We start with defining mappings $\Phi_\alpha, \alpha < 2$, (Jurek (1988), Sato (2006b), and Maejima et al. (2010)). Let

$$t = \varphi_\alpha(s) = \int_s^1 u^{-\alpha-1} du, \quad s \geq 0,$$

and let $s = \varphi_\alpha^*(t)$ be its inverse function. Define a mapping Φ_α by

$$\Phi_\alpha(\mu) = \mathcal{L}\left(\int_0^{\varphi_\alpha(0)} \varphi_\alpha^*(t) dX_t^{(\mu)}\right). \tag{4}$$

Then,

$$\Phi_\alpha(\mu) = \begin{cases} \mathcal{L}\left(\int_0^{-1/\alpha} (1+\alpha t)^{-1/\alpha} dX_t^{(\mu)}\right), & \text{when } \alpha < 0, \\ \mathcal{L}\left(\int_0^{\infty} e^{-t} dX_t^{(\mu)}\right), & \text{when } \alpha = 0, \\ \mathcal{L}\left(\int_0^{\infty} (1+\alpha t)^{-1/\alpha} dX_t^{(\mu)}\right), & \text{when } 0 < \alpha < 2. \end{cases}$$

Sato (2006b) proved that

$$\mathfrak{D}(\Phi_\alpha) = \begin{cases} I(\mathbb{R}^d), & \text{when } \alpha < 0, \\ I_{\log}(\mathbb{R}^d), & \text{when } \alpha = 0, \\ I_\alpha(\mathbb{R}^d), & \text{when } 0 < \alpha < 1, \\ I_1^*(\mathbb{R}^d), & \text{when } \alpha = 1, \\ I_\alpha^0(\mathbb{R}^d), & \text{when } 1 < \alpha < 2, \end{cases}$$

where

$$I_1^*(\mathbb{R}^d) = \left\{ \mu = \mu_{(A,\nu,\gamma)} \in I_1^0(\mathbb{R}^d) \colon \lim_{T\to\infty} \int_1^T t^{-1} dt \int_{|x|>t} x\nu(dx) \text{ exists in } \mathbb{R}^d \right\}.$$

Φ_{-1} and Φ_0 are mappings producing the Jurek class $U(\mathbb{R}^d)$ and the class of selfdecomposable distributions $L(\mathbb{R}^d)$, namely, $\mathfrak{R}(\Phi_{-1}) = U(\mathbb{R}^d)$ and $\mathfrak{R}(\Phi_0) = L(\mathbb{R}^d)$. (For details on $U(\mathbb{R}^d)$, see, e.g., Jurek (1985).) For $H \subset I(\mathbb{R}^d)$, let $\overline{H}$ denote the closure of H under convolution and weak convergence. Maejima and Ueda (2009b) have found the limit $\bigcap_{m=1}^\infty \mathfrak{R}(\Phi_\alpha^m)$ as follows.

$$\bigcap_{m=1}^{\infty} \mathfrak{R}(\Phi_\alpha^m) = \begin{cases} \overline{S(\mathbb{R}^d)}, & \text{when } \alpha \le 0, \\ \overline{\bigcup_{p\in[\alpha,2]} S_p(\mathbb{R}^d)} \cap \mathcal{C}_\alpha(\mathbb{R}^d), & \text{when } 0 < \alpha < 1, \\ \overline{\bigcup_{p\in[1,2]} S_p(\mathbb{R}^d)} \cap \mathcal{C}_1^*(\mathbb{R}^d), & \text{when } \alpha = 1, \\ \overline{\bigcup_{p\in[\alpha,2]} S_p(\mathbb{R}^d)} \cap \mathcal{C}_\alpha^0(\mathbb{R}^d), & \text{when } 1 < \alpha < 2, \end{cases}$$

where

$$\mathcal{C}_\alpha(\mathbb{R}^d) = \left\{ \mu = \mu_{(A,\nu,\gamma)} \in I(\mathbb{R}^d) \colon \lim_{r\to\infty} r^\alpha \int_{|x|>r} \nu(dx) = 0 \right\},$$

$$\mathcal{C}_1^*(\mathbb{R}^d) = \Bigg\{ \widetilde{\mu}_{(\widetilde{A},\widetilde{\nu},\widetilde{\gamma})} \in L^{\langle 1\rangle}(\mathbb{R}^d) \cap \mathcal{C}_1(\mathbb{R}^d) \colon \widetilde{\nu}(B) = \int_S \widetilde{\lambda}(d\xi) \int_0^\infty 1\!\!1_B(r\xi) r^{-2} \widetilde{k}_\xi(r) dr,$$
$$\lim_{\varepsilon\downarrow 0} \int_\varepsilon^1 t dt \int_S \xi \widetilde{\lambda}(d\xi) \int_0^\infty \frac{r^2}{1+t^2r^2} d\widetilde{k}_\xi(r) \text{ exists in } \mathbb{R}^d \text{ and equals } \widetilde{\gamma} \Bigg\},$$

$$\mathcal{C}_\alpha^0(\mathbb{R}^d) = \mathcal{C}_\alpha(\mathbb{R}^d) \cap I_1^0(\mathbb{R}^d), \quad \text{for } 1 < \alpha < 2.$$

Then, applying Theorem 2.5, we have

$$\begin{cases} S(\mathbb{R}^d) \subset \mathrm{FP}(\Phi_\alpha) \subset \overline{S(\mathbb{R}^d)}, & \text{when } \alpha \le 0, \\ \bigcup_{p\in(\alpha,2]} S_p(\mathbb{R}^d) \subset \mathrm{FP}(\Phi_\alpha) \subset \overline{\bigcup_{p\in[\alpha,2]} S_p(\mathbb{R}^d)} \cap \mathcal{C}_\alpha(\mathbb{R}^d), & \text{when } 0 < \alpha < 1, \\ \bigcup_{p\in(1,2]} S_p^0(\mathbb{R}^d) \subset \mathrm{FP}(\Phi_1) \subset \overline{\bigcup_{p\in[1,2]} S_p(\mathbb{R}^d)} \cap \mathcal{C}_1^*(\mathbb{R}^d), & \text{when } \alpha = 1, \\ \bigcup_{p\in(\alpha,2]} S_p^0(\mathbb{R}^d) \subset \mathrm{FP}(\Phi_\alpha) \subset \overline{\bigcup_{p\in[\alpha,2]} S_p(\mathbb{R}^d)} \cap \mathcal{C}_\alpha^0(\mathbb{R}^d), & \text{when } 1 < \alpha < 2. \end{cases}$$

However, a much stronger theorem holds as follows.

Theorem 3.1. *We have*

$$\mathrm{FP}(\Phi_\alpha) = S(\mathbb{R}^d) \cap \mathfrak{D}(\Phi_\alpha) \quad \text{for } \alpha < 2,$$

namely,

$$\mathrm{FP}(\Phi_\alpha) = \begin{cases} S(\mathbb{R}^d), & \text{when } \alpha \le 0, \\ \bigcup_{p\in(\alpha,2]} S_p(\mathbb{R}^d), & \text{when } 0 < \alpha < 1, \\ \bigcup_{p\in(\alpha,2]} S_p^0(\mathbb{R}^d), & \text{when } 1 \le \alpha < 2. \end{cases}$$

Remark 3.2. Theorem 3.1 for $\alpha \le 0$ was already proved in Jurek and Vervaat (1983) and Jurek (1985, 1988) in a general setting of a real separable Banach space. One meaning of this theorem is to give new characterizations of the classes $S(\mathbb{R}^d)$, $\bigcup_{p\in(\alpha,2]} S_p(\mathbb{R}^d)$ with $0 < \alpha < 1$ and $\bigcup_{p\in(\alpha,2]} S_p^0(\mathbb{R}^d)$ with $1 \le \alpha < 2$.

Proof of Theorem 3.1. Let $\alpha < 2$. Due to Theorem 2.1, it suffices to prove that

$$\mathrm{FP}(\Phi_\alpha) \subset S(\mathbb{R}^d) \cap \mathfrak{D}(\Phi_\alpha). \tag{5}$$

Suppose $\mu = \mu_{(A,\nu,\gamma)} \in \mathrm{FP}(\Phi_\alpha)$. Then

$$\widetilde{\mu} = \widetilde{\mu}_{(\widetilde{A},\widetilde{\nu},\widetilde{\gamma})} := \Phi_\alpha(\mu) = \mu^a * \delta_c \quad \text{for some } a > 0 \text{ and } c \in \mathbb{R}^d. \tag{6}$$

If $\nu = 0$, then $\mu \in S(\mathbb{R}^d)$. Let $\nu \neq 0$. It follows from (4) and (3) that

$$\widetilde{\nu}(B) = \int_0^{\varphi_\alpha(0)} \nu\left(\varphi_\alpha^*(t)^{-1}B\right) dt = \int_0^1 \nu\left(s^{-1}B\right) s^{-\alpha-1} ds, \quad B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}).$$

Hence, by (6),

$$\int_0^1 \nu\left(s^{-1}B\right) s^{-\alpha-1} ds = a\nu(B). \tag{7}$$

By the polar decomposition, we have

$$\int_0^1 s^{-\alpha-1} ds \int_S \lambda(d\xi) \int_0^\infty \mathbb{1}_B(sr\xi)\nu_\xi(dr) = a \int_S \lambda(d\xi) \int_0^\infty \mathbb{1}_B(r\xi)\nu_\xi(dr).$$

Letting $B = (u,\infty)C$ with any $u > 0$ and any $C \in \mathcal{B}(S)$ and using Fubini's theorem, we have

$$\int_C \lambda(d\xi) \int_0^1 \nu_\xi((u/s,\infty)) s^{-\alpha-1} ds = a \int_C \nu_\xi((u,\infty))\lambda(d\xi),$$

and thus

$$\int_0^1 \nu_\xi((u/s,\infty))s^{-\alpha-1}ds = a\nu_\xi((u,\infty)) \quad \lambda\text{-a.e. } \xi \in S.$$

Therefore

$$u^{-\alpha}\int_u^\infty \nu_\xi((v,\infty))v^{\alpha-1}dv = a\nu_\xi((u,\infty)) \quad \lambda\text{-a.e. } \xi \in S.$$

Since the both sides of the equation above are right-continuous in u, we have that for λ-a.e. $\xi \in S$,

$$u^{-\alpha}\int_u^\infty \nu_\xi((v,\infty))v^{\alpha-1}dv = a\nu_\xi((u,\infty)), \quad \text{for all } u > 0.$$

This implies that

$$\frac{d}{du}\nu_\xi((u,\infty)) = -(\alpha + a^{-1})\nu_\xi((u,\infty))u^{-1} \quad \text{a.e. } u > 0.$$

Then we have

$$\nu_\xi((u,\infty)) = \nu_\xi((1,\infty))u^{-(\alpha+a^{-1})}, \quad \text{for all } u > 0. \tag{8}$$

Since $\int_0^\infty (r^2 \wedge 1)\nu_\xi(dr) < \infty$, it holds that

$$0 < \alpha + a^{-1} < 2. \tag{9}$$

Then, (8) is the radial component of the Lévy measure of an $(\alpha+a^{-1})$-stable distribution. On the other hand, it follows from (4) and (2) that

$$\widetilde{A} = \int_0^{\varphi_\alpha(0)} \varphi_\alpha^*(t)^2 A dt = \int_0^1 s^{-\alpha+1}Ads = (2-\alpha)^{-1}A.$$

Combining this with (6), we have $(2-\alpha)^{-1}A = aA$. If $(2-\alpha)^{-1} = a$, then $\alpha + a^{-1} = 2$, which contradicts (9). Hence $A = 0$. Namely, $\mu \in S_{\alpha+a^{-1}}(\mathbb{R}^d)$. We thus have $\mathrm{FP}(\Phi_\alpha) \subset S(\mathbb{R}^d)$. This completes the proof of (5). □

4. Mappings which have non-stable distributions as fixed points

In the previous section, we give examples of mappings whose fixed points are stable distributions. However, there also exist mappings which have non-stable distributions as fixed points. They are some special cases of the mappings $\Psi_{\alpha,\beta}, \alpha < 2, \beta > 0$, defined below. Let

$$t = G_{\alpha,\beta}(s) = \int_s^\infty u^{-\alpha-1} e^{-u^\beta} du, \quad s \geq 0,$$

and let $s = G^*_{\alpha,\beta}(t)$ be its inverse function. Define

$$\Psi_{\alpha,\beta}(\mu) = \mathcal{L}\left(\int_0^{G_{\alpha,\beta}(0)} G^*_{\alpha,\beta}(t) dX_t^{(\mu)}\right),$$

where

$$G_{\alpha,\beta}(0) = \begin{cases} \beta^{-1}\Gamma(-\alpha\beta^{-1}), & \text{when } \alpha < 0, \\ \infty, & \text{when } \alpha \geq 0. \end{cases}$$

These mappings are introduced first by Sato (2006b) for $\beta = 1$ and later by Maejima and Nakahara (2009) for general $\beta > 0$. Due to Sato (2006b) and Maejima and Nakahara (2009), we see the domains $\mathfrak{D}(\Psi_{\alpha,\beta})$ as follows, which are independent of the value $\beta > 0$.

$$\mathfrak{D}(\Psi_{\alpha,\beta}) = \begin{cases} I(\mathbb{R}^d), & \text{when } \alpha < 0, \\ I_{\log}(\mathbb{R}^d), & \text{when } \alpha = 0, \\ I_\alpha(\mathbb{R}^d), & \text{when } 0 < \alpha < 1, \\ I_1^*(\mathbb{R}^d), & \text{when } \alpha = 1, \\ I_\alpha^0(\mathbb{R}^d), & \text{when } 1 < \alpha < 2. \end{cases}$$

Note that $\mathfrak{R}(\Psi_{-1,1}) = B(\mathbb{R}^d)$, which is the Goldie-Steutel-Bondesson class on $\mathbb{R}^d$, (see Barndorff-Nielsen et al. (2006)), $\mathfrak{R}(\Psi_{-1,2}) = G(\mathbb{R}^d)$, which is the class of generalized type G distributions on $\mathbb{R}^d$, (see Maejima and Sato (2009)), $\mathfrak{R}(\Psi_{0,1}) = T(\mathbb{R}^d)$, which is the Thorin class on $\mathbb{R}^d$, (see Barndorff-Nielsen et al. (2006)), and $\mathfrak{R}(\Psi_{0,2}) = M(\mathbb{R}^d)$, which is the class M, (see Aoyama et al. (2008) in the symmetric case). By Maejima and Sato (2009), Sato (2007–2009), Aoyama et al. (2009) and Maejima and Ueda (2009a), the limit $\bigcap_{m=1}^{\infty}\mathfrak{R}(\Psi_{\alpha,\beta}^m)$ was found as follows. Let $\beta>0$.

$$\bigcap_{m=1}^{\infty}\mathfrak{R}(\Psi_{\alpha,\beta}^m)=\begin{cases}\overline{S(\mathbb{R}^d)}, & \text{for } \alpha\in(-\infty,0],\\ \overline{\bigcup_{p\in[\alpha,2]}S_p(\mathbb{R}^d)}\cap\mathcal{C}_\alpha(\mathbb{R}^d), & \text{for } \alpha\in(0,1),\\ \overline{\bigcup_{p\in[\alpha,2]}S_p(\mathbb{R}^d)}\cap\mathcal{C}_\alpha^0(\mathbb{R}^d), & \text{for } \alpha\in(1,2)\setminus\{1+n\beta\colon n\in\mathbb{N}\}.\end{cases}$$

Then, applying Theorem 2.5, we have

$$\begin{cases}S(\mathbb{R}^d)\subset \mathrm{FP}(\Psi_{\alpha,\beta})\subset\overline{S(\mathbb{R}^d)}, & \text{for } \alpha\in(-\infty,0],\\ \bigcup_{p\in(\alpha,2]}S_p(\mathbb{R}^d)\subset \mathrm{FP}(\Psi_{\alpha,\beta})\subset\overline{\bigcup_{p\in[\alpha,2]}S_p(\mathbb{R}^d)}\cap\mathcal{C}_\alpha(\mathbb{R}^d), & \text{for } \alpha\in(0,1),\\ \bigcup_{p\in(\alpha,2]}S_p^0(\mathbb{R}^d)\subset \mathrm{FP}(\Psi_{\alpha,\beta})\subset\overline{\bigcup_{p\in[\alpha,2]}S_p(\mathbb{R}^d)}\cap\mathcal{C}_\alpha^0(\mathbb{R}^d), & \text{for } \alpha\in(1,2)\setminus\{1+n\beta\colon n\in\mathbb{N}\}.\end{cases}$$

However, as the following example shows, if $\beta>0$ and $-\beta<\alpha<2-2\beta$, then $S(\mathbb{R}^d)\cap\mathfrak{D}(\Psi_{\alpha,\beta})\subsetneqq \mathrm{FP}(\Psi_{\alpha,\beta})$, which is different from the case of Φ_α.

Example 4.1. Let $\alpha<2$ and $\beta>0$. Assume $\alpha+\beta,\alpha+2\beta\in(0,2)$, namely, $-\beta<\alpha<2-2\beta$. Let μ_1 be a non-trivial $(\alpha+\beta)$-stable distribution and μ_2 a non-trivial $(\alpha+2\beta)$-stable distribution. Note that if $\alpha>0$, then $\mu_1,\mu_2\in I_\alpha(\mathbb{R}^d)$. If $\alpha\geq 1$, we assume that the means of μ_1 and μ_2 are 0. Then, $\mu_1,\mu_2\in\mathfrak{D}(\Psi_{\alpha,\beta})$ and thus $\mu_1*\mu_2\in\mathfrak{D}(\Psi_{\alpha,\beta})$. Denoting by ν_j the Lévy measure of μ_j for $j=1,2$, we have

$$\nu_1(b^{-1}B)=b^{\alpha+\beta}\nu_1(B),\quad \nu_2(b^{-1}B)=b^{\alpha+2\beta}\nu_2(B),\quad B\in\mathcal{B}(\mathbb{R}^d),$$

for all $b>0$, by the stability of μ_1 and μ_2. Then, the Lévy measure of $\Psi_{\alpha,\beta}(\mu_1*\mu_2)$ denoted by $\nu_{\Psi_{\alpha,\beta}(\mu_1*\mu_2)}$ satisfies that

$$\begin{aligned}\nu_{\Psi_{\alpha,\beta}(\mu_1*\mu_2)}(B)&=\int_0^\infty u^{-\alpha-1}e^{-u^\beta}\left\{\nu_1(u^{-1}B)+\nu_2(u^{-1}B)\right\}du\\&=\int_0^\infty u^{\beta-1}e^{-u^\beta}\nu_1(B)du+\int_0^\infty u^{2\beta-1}e^{-u^\beta}\nu_2(B)du\\&=\beta^{-1}\left\{\nu_1(B)+\nu_2(B)\right\}.\end{aligned}$$

Thus $\Psi_{\alpha,\beta}(\mu_1*\mu_2)=(\mu_1*\mu_2)^{\beta^{-1}}*\delta_c$ for some $c\in\mathbb{R}^d$. That is, $\mu_1*\mu_2$ is a fixed point of $\Psi_{\alpha,\beta}$. However, $\mu_1*\mu_2$ is not stable.

More generally, fixed points under $\Psi_{\alpha,\beta}$ are characterized as follows.

Theorem 4.2. *Let $\alpha < 2$, $\beta > 0$, $\mu = \mu_{(A,\nu,\gamma)} \in \mathfrak{D}(\Psi_{\alpha,\beta})$ and $a > 0$. Then*

$$\Psi_{\alpha,\beta}(\mu) = \mu^a * \delta_c \quad \text{for some } c \in \mathbb{R}^d \tag{10}$$

if and only if

$$A = 0 \text{ or } a = \frac{1}{\beta}\Gamma\left(\frac{2-\alpha}{\beta}\right) \tag{11}$$

and

$$\begin{cases} \nu = 0 \text{ or } \nu \neq 0 \text{ and a polar decomposition } (\lambda, \nu_\xi) \text{ of } \nu \text{ fulfills that for} \\ \lambda\text{-a.e. } \xi,\ \nu_\xi(du) = k_\xi(u)du \text{ for some nonnegative function } k_\xi(u) \text{ which} \\ \text{is measurable in } u \text{ and satisfies} \\ \qquad ak_\xi(u) = u^{-\alpha-1}\displaystyle\int_0^\infty e^{-(u/r)^\beta} r^\alpha k_\xi(r)dr \quad \text{a.e. } u > 0. \end{cases} \tag{12}$$

To prove Theorem 4.2, the following lemma is needed.

Lemma 4.3. *Let $\alpha < 2$ and $\beta > 0$. Let $\widetilde{\nu}$ and ν be the Lévy measures of $\widetilde{\mu} = \Psi_{\alpha,\beta}(\mu)$ and μ, respectively. Suppose that $\nu \neq 0$ and that (λ, ν_ξ) is a polar decomposition of ν. Then for $B \in \mathcal{B}((0,\infty))$ and $C \in \mathcal{B}(S)$,*

$$\widetilde{\nu}(BC) = \int_C \lambda(d\xi) \int_B u^{-\alpha-1}du \int_0^\infty e^{-(u/r)^\beta} r^\alpha \nu_\xi(dr). \tag{13}$$

Proof. It follows from (3) that for $B \in \mathcal{B}((0,\infty))$ and $C \in \mathcal{B}(S)$,

$$\begin{aligned} \widetilde{\nu}(BC) &= \int_0^{G_{\alpha,\beta}(0)} dt \int_C \lambda(d\xi) \int_0^\infty 1\!\!1_B(G^*_{\alpha,\beta}(t)r)\nu_\xi(dr) \\ &= \int_C \lambda(d\xi) \int_0^\infty \nu_\xi(dr) \int_0^\infty 1\!\!1_B(sr)s^{-\alpha-1}e^{-s^\beta}ds \\ &= \int_C \lambda(d\xi) \int_0^\infty r^\alpha \nu_\xi(dr) \int_0^\infty 1\!\!1_B(u)u^{-\alpha-1}e^{-(u/r)^\beta}du \\ &= \int_C \lambda(d\xi) \int_B u^{-\alpha-1}du \int_0^\infty e^{-(u/r)^\beta} r^\alpha \nu_\xi(dr). \end{aligned}$$

□

Proof of Theorem 4.2. We first prove the "only if" part. Assume (10). Noting that

$$\int_0^{G_{\alpha,\beta}(0)} G^*_{\alpha,\beta}(t)^2 dt = \int_0^\infty s^{1-\alpha}e^{-s^\beta}ds = \frac{1}{\beta}\Gamma\left(\frac{2-\alpha}{\beta}\right), \tag{14}$$

we have

$$aA = \frac{1}{\beta}\Gamma\left(\frac{2-\alpha}{\beta}\right)A \tag{15}$$

by (2). Therefore (11) holds. Also, if $\nu \neq 0$, then a polar decomposition (λ, ν_ξ) of ν satisfies that for $B \in \mathcal{B}((0,\infty))$ and $C \in \mathcal{B}(S)$,

$$a\int_C \nu_\xi(B)\lambda(d\xi) = \int_C \lambda(d\xi)\int_B u^{-\alpha-1}du\int_0^\infty e^{-(u/r)^\beta} r^\alpha \nu_\xi(dr)$$

by (13). Thus, for $s > 0$,

$$a\nu_\xi((s,\infty)) = \int_s^\infty u^{-\alpha-1}du\int_0^\infty e^{-(u/r)^\beta} r^\alpha \nu_\xi(dr) \quad \lambda\text{-a.e. } \xi.$$

Since the both sides of the equation above are right-continuous in s, it holds that for λ-a.e. ξ,

$$a\nu_\xi((s,\infty)) = \int_s^\infty u^{-\alpha-1}du\int_0^\infty e^{-(u/r)^\beta} r^\alpha \nu_\xi(dr), \quad \text{for } s > 0,$$

and hence

$$a\nu_\xi(du) = u^{-\alpha-1}du\int_0^\infty e^{-(u/r)^\beta} r^\alpha \nu_\xi(dr).$$

This yields the absolute continuity of ν_ξ with respect to the Lebesgue measure. Letting $\nu_\xi(du) = k_\xi(u)du$, we have that for λ-a.e. ξ,

$$ak_\xi(u) = u^{-\alpha-1}\int_0^\infty e^{-(u/r)^\beta} r^\alpha k_\xi(r)dr, \quad \text{a.e. } u > 0.$$

Thus we have (12).

We next show the "if" part. Assume (11) and (12). Let $\widetilde{\mu} = \widetilde{\mu}_{(\widetilde{A},\widetilde{\nu},\widetilde{\gamma})} = \Psi_{\alpha,\beta}(\mu)$. Then we have (15) by (11). Using (14) and (2), we have $aA = \widetilde{A}$. Combining (12) with (13), we have $a\nu = \widetilde{\nu}$. Thus we have (10). □

5. A mapping whose fixed points are semi-stable distributions with a fixed span

A distribution $\mu \in I(\mathbb{R}^d)$ is said to be semi-stable with span $b > 1$ if there exist $\eta > 1$ and $\zeta \in \mathbb{R}^d$ satisfying $\widehat{\mu}(z)^\eta = \widehat{\mu}(bz)e^{i\langle\zeta,z\rangle}$. We denote by $SS(b,\mathbb{R}^d)$ the class of all semi-stable distributions with span b on $\mathbb{R}^d$. In this section, we give examples of mappings satisfying $\mathrm{FP}(\Phi_f) = SS(b, \mathbb{R}^d)$. Fix any $b > 1$. Define a mapping $\Phi_{(b)}$ by

$$\Phi_{(b)}(\mu) = \mathcal{L}\left(\int_0^\infty b^{-[t]} dX_t^{(\mu)}\right), \quad \mu \in \mathfrak{D}(\Phi_{(b)}) = I_{\log}(\mathbb{R}^d),$$

where $[x]$ denotes the largest integer not greater than x. Maejima and Ueda (2009c) introduced this mapping and proved that $\mathfrak{R}(\Phi_{(b)}) = L(b, \mathbb{R}^d)$, which is the class of all semi-selfdecomposable distributions with span b on $\mathbb{R}^d$. They also proved that $\bigcap_{m=1}^\infty \mathfrak{R}(\Phi_{(b)}^m) = \overline{SS(b, \mathbb{R}^d)}$. Then, Theorem 2.5 yields that

$$S(\mathbb{R}^d) \subset \mathrm{FP}(\Phi_{(b)}) \subset \overline{SS(b, \mathbb{R}^d)}.$$

However, more strongly, we can assert the following.

Theorem 5.1. *Let $b > 1$. Then we have*

$$\mathrm{FP}(\Phi_{(b)}) = SS(b, \mathbb{R}^d).$$

Remark 5.2. It holds that $S(\mathbb{R}^d) \subsetneqq \mathrm{FP}(\Phi_{(b)}) \subsetneqq \overline{SS(b, \mathbb{R}^d)}$.

Proof of Theorem 5.1. Let $\mu \in \mathrm{FP}(\Phi_{(b)})$. Then $\mu \in I_{\log}(\mathbb{R}^d)$ and $\Phi_{(b)}(\mu) = \mu^a * \delta_c$ for some $a > 0$ and $c \in \mathbb{R}^d$. Theorem 3.3 of Maejima and Ueda (2009c) yields that $\widehat{\Phi_{(b)}(\mu)}(z) = \widehat{\Phi_{(b)}(\mu)}(b^{-1}z)\widehat{\mu}(z)$. Then $\widehat{\mu}(z)^a e^{i\langle c, z\rangle} = \widehat{\mu}(b^{-1}z)^a e^{i\langle c, b^{-1}z\rangle}\widehat{\mu}(z)$. If $a \leq 1$, then $1 \leq |\widehat{\mu}(z)|^{a-1} = |\widehat{\mu}(b^{-1}z)|^a \leq 1$, which yields $|\widehat{\mu}(z)| = 1$. Then, by Lemma 13.9 of Sato (1999), μ is a δ-distribution and therefore $\mu \in SS(b, \mathbb{R}^d)$. If $a > 1$, then we have $\widehat{\mu}(z)^{a(a-1)^{-1}} = \widehat{\mu}(bz)e^{i\langle c(b-1)(a-1)^{-1}, z\rangle}$, which yields $\mu \in SS(b, \mathbb{R}^d)$.

Conversely, let $\mu \in SS(b, \mathbb{R}^d)$. Then $\widehat{\mu}(z)^\eta = \widehat{\mu}(bz)e^{i\langle \zeta, z\rangle}$ for some $\eta > 1$ and $\zeta \in \mathbb{R}^d$, namely, $C_\mu(z) = \eta^{-1}C_\mu(bz) + i\langle \eta^{-1}\zeta, z\rangle$. Then it follows from (1) that

$$\begin{aligned} C_{\Phi_{(b)}(\mu)}(b^{-1}z) &= \int_0^\infty C_\mu(b^{-[t]-1}z)dt = \eta^{-1}C_{\Phi_{(b)}(\mu)}(z) + i\left\langle \eta^{-1}\zeta \int_0^\infty b^{-[t]-1}dt, z\right\rangle \\ &=: \eta^{-1}C_{\Phi_{(b)}(\mu)}(z) + i\langle \zeta', z\rangle, \end{aligned}$$

say. On the other hand, since $\mu \in I_{\log}(\mathbb{R}^d)$, Theorem 3.3 of Maejima and Ueda (2009c) yields that $\widehat{\Phi_{(b)}(\mu)}(z) = \widehat{\Phi_{(b)}(\mu)}(b^{-1}z)\widehat{\mu}(z)$. Then $\widehat{\Phi_{(b)}(\mu)}(z) = \widehat{\Phi_{(b)}(\mu)}(z)^{\eta^{-1}}e^{i\langle \zeta', z\rangle}\widehat{\mu}(z)$. Noting that $1 - \eta^{-1} > 0$ since $\eta > 1$, we have $\widehat{\Phi_{(b)}(\mu)}(z) = \widehat{\mu}(z)^{(1-\eta^{-1})^{-1}}e^{i\langle (1-\eta^{-1})^{-1}\zeta', z\rangle}$. Thus $\mu \in \mathrm{FP}(\Phi_{(b)})$. $\square$

References

Aoyama, T., Lindner, A., Maejima, M., 2009. A new family of mappings of infinitely divisible distributions related to the Goldie-Steutel-Bondesson class. Preprint.

Aoyama, T., Maejima, M., Rosiński, J., 2008. A subclass of type G selfdecomposable distributions on $\mathbb{R}^d$. J. Theor. Probab. 21, 14–34.

Barndorff-Nielsen, O.E., Maejima, M., Sato, K., 2006. Some classes of multivariate infinitely divisible distributions admitting stochastic integral representations. Bernoulli 12, 1–33.

Jurek, Z.J., 1985. Relations between the s-selfdecomposable and selfdecomposable measures. Ann. Probab. 13, 592–608.

Jurek, Z.J., 1988. Random integral representations for classes of limit distributions similar to Lévy class L_0. Probab. Theory Related Fields 78, 473–490.

Jurek, Z.J., Vervaat, W., 1983. An integral representation for selfdecomposable Banach space valued random variables. Z. Wahrscheinlichkeitstheorie verw. Gebiete 62, 247–262.

Maejima, M., Matsui, M., Suzuki, M., 2010. Classes of infinitely divisible distributions on $\mathbb{R}^d$ related to the class of selfdecomposable distributions. To appear in Tokyo J. Math.

Maejima, M., Nakahara, G., 2009. A note on new classes of infinitely divisible distributions on $\mathbb{R}^d$. Elect. Comm. in Probab. 14, 358–371.

Maejima, M., Sato, K., 2009. The limits of nested subclasses of several classes of infinitely divisible distributions are identical with the closure of the class of stable distributions. Probab. Theory Related Fields 145, 119–142.

Maejima, M., Ueda, Y., 2009a. Compositions of mappings of infinitely divisible distributions with applications to finding the limits of some nested subclasses. Preprint.

Maejima, M., Ueda, Y., 2009b. Nested subclasses of the class of α-selfdecomposable distributions. Preprint.

Maejima, M., Ueda, Y., 2009c. Stochastic integral characterizations of semi-selfdecomposable distributions and related Ornstein-Uhlenbeck type processes. Commun. Stoch. Anal. 3, 349–367.

Sato, K., 1999. Lévy Processes and Infinitely Divisible Distributions. Cambridge University Press, Cambridge.

Sato, K., 2004. Stochastic integrals in additive processes and application to semi-Lévy processes. Osaka J. Math. 41, 211–236.

Sato, K., 2006a. Additive processes and stochastic integrals. Illinois J. Math. 50, 825–851.

Sato, K., 2006b. Two families of improper stochastic integrals with respect to Lévy processes. ALEA, Lat. Am. J. Probab. Math. Stat. 1, 47–87.

Sato, K., 2007–2009. Memos privately communicated.