1. Introduction

Bondesson (1992) said “Since a lot of the standard distributions now as known to be infinitely divisible, the class of infinitely divisible distributions has perhaps partly lost its interest. Smaller classes should be more in focus.”

It was almost 15 years ago, but this sentence seems still alive. Recently, subdivision of the class of infinitely divisible distributions has been developed. In this paper, we try to find the classes that known infinitely divisible distributions belong to, as precisely as possible.

2. A list of classes of infinitely divisible distributions

Throughout the paper, $\mathcal{L}(X)$ denotes the law of a random variable of $X$, $I(\mathbb{R}^d)$ stands for the class of all infinitely divisible distributions on $\mathbb{R}^d$, and let $I_{\text{sym}}(\mathbb{R}^d) = \{\mu \in I(\mathbb{R}^d) : \mu \text{ is symmetric on } \mathbb{R}^d\}$, $I_{\log}(\mathbb{R}^d) = \{\mu \in I(\mathbb{R}^d) : \int_{\mathbb{R}^d} \log^+ |x| \mu(dx) < \infty\}$ and $I_{\log^{m+1}}(\mathbb{R}^d) = \{\mu \in I(\mathbb{R}^d) : \int_{\mathbb{R}^d} (\log^+ |x|)^{m+1} \mu(dx) < \infty\}$. Also, $S(\mathbb{R}^d)$ denotes the class of all stable distributions on $\mathbb{R}^d$.

The following is a basic result on the Lévy measure of the characteristic function of $\mu \in I(\mathbb{R}^d)$.

Proposition 2.1. (Polar decomposition of Lévy measures.) (Rosinski (1990), Barndorff-Nielsen et al. (2006).) Let $\nu$ be the Lévy measure of the characteristic
function of some $\mu \in I(\mathbb{R}^d)$ with $0 < \nu(\mathbb{R}^d) \leq \infty$. Then there exist a measure $\lambda$ on $S = \{\xi \in \mathbb{R}^d : |\xi| = 1\}$ with $0 < \lambda(S) \leq \infty$ and a family $\{\nu_\xi : \xi \in S\}$ of measures on $(0, \infty)$ such that $\nu_\xi(B)$ is measurable in $\xi$ for each $B \in \mathcal{B}((0, \infty))$, $0 < \nu_\xi((0, \infty)) \leq \infty$ for each $\xi \in S$, and

$$
\nu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi)\nu_\xi(dr), \quad B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}).
$$

Here $\lambda$ and $\{\nu_\xi\}$ are uniquely determined by $\nu$ up to multiplication of a measurable function $c(\xi)$ and $c(\xi)^{-1}$ with $0 < c(\xi) < \infty$, and $\nu_\xi$ is called the radial component of $\nu$.

In the following, the classification and characterization are given in term of the radial component $\nu_\xi$ of $\nu$. Classes in $I(\mathbb{R}^d)$ we are concerned with in this paper are the following.

(1) Class $U(\mathbb{R}^d)$ (the Jurek class):

$$
\nu_\xi(dr) = \ell_\xi(r)dr,
$$

where $\ell_\xi(r)$ is measurable in $\xi \in S$ and nonincreasing in $r \in (0, \infty)$.

The class $U(\mathbb{R}^d)$ was introduced by Jurek (1985) and $\mu \in U(\mathbb{R}^d)$ is called $s$-selfdecomposable.

(2) Class $B(\mathbb{R}^d)$ (the Goldie–Steutel–Bondesson class):

$$
\nu_\xi(dr) = \ell_\xi(r)dr,
$$

where $\ell_\xi(r)$ is measurable in $\xi \in S$ and completely monotone on $(0, \infty)$.

Bondesson (1981) studied the class of generalized convolution of mixtures of exponential distributions on $\mathbb{R}^+$, called $\mathcal{T}_2$ in his monograph (Bondesson (1992)). It is the smallest class that contains all mixtures of exponential distributions and that is closed under convolution and weak convergence on $\mathbb{R}^+$. $B(\mathbb{R}^d)$ is its generalization. (Barndorff-Nielsen et al. (2006).) By those definitions, $B(\mathbb{R}^d) \subset U(\mathbb{R}^d)$.

(3) Class $L(\mathbb{R}^d)$ (Class of selfdecomposable distributions):

(2.1)

$$
\nu_\xi(dr) = r^{-1}\ell_\xi(r)dr,
$$

where $\ell_\xi(r)$ is measurable in $\xi \in S$ and nonincreasing on $(0, \infty)$.

It is known that $\mu \in L(\mathbb{R}^d)$ if and only if for any $b > 1$, there exists some $\rho_b \in I(\mathbb{R}^d)$ such that $\hat{\mu}(z) = \hat{\mu}(b^{-1}z)\hat{\rho}_b(z)$. (This statement is used as the definition of the selfdecomposability usually. The equivalence of this statement and (2.1) is proved by Wolfe (1982).) By those definitions, $L(\mathbb{R}^d) \subset U(\mathbb{R}^d)$. 

2
(4) Class $T(\mathbb{R}^d)$ (the Thorin class):

$$\nu_\xi(dr) = k_\xi(r)r^{-1}dr,$$

where $k_\xi(r)$ is measurable in $\xi \in S$ and completely monotone on $(0, \infty)$.

Thorin (1977a, 1977b, 1978) studied generalized gamma-convolution on $\mathbb{R}_+$ and $\mathbb{R}$, called $T$ and $T_e$, respectively, in Bondesson (1992). $T$ is the smallest class that contains all gamma-distributions and that is closed under convolution and weak convergence on $\mathbb{R}_+$. $T_e$ allows the distributions of linear combinations of independent gamma random variables with not necessarily positive coefficient. $T(\mathbb{R}^d)$ is their generalization. (Barndorff-Nielsen et al. (2006).) Again, by those definitions, $T(\mathbb{R}^d) \subset B(\mathbb{R}^d) \cap L(\mathbb{R}^d)$.

(5) Class $G(\mathbb{R}^d)$ (Class of type $G$ distributions):

$$\mu \in I_{sym}(\mathbb{R}^d) \text{ and } \nu_\xi(dr) = g_\xi(r^2)dr,$$

where $g_\xi(r)$ is measurable in $\xi \in S$ and completely monotone on $(0, \infty)$.

When $d = 1$, $\mu \in G(\mathbb{R})$ if and only if $\mu = L(V^{1/2}Z)$, where $V > 0$, $L(V) \in I(\mathbb{R})$, $Z$ is the standard normal random variable, and $V$ and $Z$ are independent. (See Rosinski (1991).) When $d \geq 1$, $\mu \in G(\mathbb{R}^d)$ if and only if $\nu_\mu(B) = E[\nu_0(Z^{-1}B)]$ for some Lévy measure $\nu_0$, where $\nu_\mu$ is the Lévy measure of $\mu$. (Maejima+Rosiński (2001).)

(6) Class $M(\mathbb{R}^d)$:

$$\mu \in I_{sym}(\mathbb{R}^d) \text{ and } \nu_\xi(dr) = g_\xi(r^2)r^{-1}dr,$$

where $g_\xi(r)$ is measurable in $\xi \in S$ and completely monotone on $(0, \infty)$. (This class is introduced in Aoyama+Maejima+Rosiński (2007).) By those definitions, $M(\mathbb{R}^d) \subset G(\mathbb{R}^d)$.

3. Mappings

Let $\{X_t^{(\mu)}\}$ be a Lévy process on $\mathbb{R}^d$ with $L(X_1) = \mu$. We introduce several mapping from $I(\mathbb{R}^d)$ into $I(\mathbb{R}^d)$, in connection with the classes mentioned in the previous section.

Definition 3.1. ($U$-mapping.) (Jurek (1985).) For $\mu \in I(\mathbb{R}^d)$,

$$U(\mu) = L \left( \int_0^1 tdX_t^{(\mu)} \right).$$
Definition 3.2. (Ψ-mapping.) (Barndorff-Nielsen et al. (2006).) For \( \mu \in I(\mathbb{R}^d) \),
\[
\Psi(\mu) = \mathcal{L} \left( \int_0^1 \log \frac{1}{t} dX_t^{(\mu)} \right).
\]
For \( d = 1 \), this \( \Psi \)-mapping was introduced by Barndorff-Nielsen and Thorbjørnsen (2002).

Definition 3.3. (Φ-mapping.) (See Wolfe (1982).) For \( \mu \in I(\log(\mathbb{R}^d)) \),
\[
\Phi(\mu) = \mathcal{L} \left( \int_0^\infty e^{-t} dX_t^{(\mu)} \right).
\]

Definition 3.4. (Ψ-mapping.) (Barndorff-Nielsen+Maejima+Sato (2006).) Let \( e(x) = \int_x^\infty e^{-u} u^{-1} du \) and denote its inverse function by \( e^*(t) \). For \( \mu \in I(\log(\mathbb{R}^d)) \),
\[
\Psi(\mu) = \mathcal{L} \left( \int_0^\infty e^*(t) dX_t^{(\mu)} \right).
\]

Proposition 3.5. (Barndorff-Nielsen+Maejima+Sato (2006).)
\[
\Psi = \Upsilon \circ \Phi = \Phi \circ \Upsilon,
\]
where \( \circ \) means the composition of mappings.

Definition 3.6. (G-mapping.) (Aoyama+Maejima (2007).) Let \( \varphi(u) = (2\pi)^{-1/2} e^{-u^2/2} \) and \( h(x) = \int_x^\infty \varphi(u) du \), \( x \in \mathbb{R} \), and denote its inverse function by \( h^*(t) \). For \( \mu \in I(\mathbb{R}^d) \),
\[
\mathcal{G}(\mu) = \mathcal{L} \left( \int_0^1 h^*(t) dX_t^{(\mu)} \right).
\]

Definition 3.7. (M-mapping.) (Aoyama+Maejima+Rosinski (2007).) Let \( m(x) = \int_x^\infty \varphi(u) u^{-1} du \), \( x > 0 \), and denote its inverse function by \( m^*(t) \). For \( \mu \in I(\log(\mathbb{R}^d)) \),
\[
\mathcal{M}(\mu) = \mathcal{L} \left( \int_0^\infty m^*(t) dX_t^{(\mu)} \right).
\]

The following are characterizations of the classes in the previous section in terms of the mappings above, or equivalently, in terms of stochastic integrals with respect to Lévy processes.

Theorem 3.8. (1) \( U(\mathbb{R}^d) = \mathcal{U}(I(\mathbb{R}^d)) \). (Jurek (1985).)
(2) \( B(\mathbb{R}^d) = \Upsilon(I(\mathbb{R}^d)) \). (Barndorff-Nielsen+Maejima+Sato (2006).)
(3) \( L(\mathbb{R}^d) = \Phi(I(\log(\mathbb{R}^d))) \). (Wolfe (1982) and others.)
(4) \( T(\mathbb{R}^d) = \Psi(I(\log(\mathbb{R}^d))) \). (Barndorff-Nielsen+Maejima+Sato (2006).)
(5) \( G(\mathbb{R}^d) = \mathcal{G}(I(\mathbb{R}^d)) \). (Aoyama+Maejima (2007).)
(6) \( M(\mathbb{R}^d) = \mathcal{M}(I(\log(\mathbb{R}^d))) \cap I_{\text{sym}}(\mathbb{R}^d) \). (Aoyama+Maejima+Rosinski (2007).)
4. Decreasing subclasses

We define decreasing nested classes of the classes in section 2. Let \( U_0(\mathbb{R}^d) = U(\mathbb{R}^d), B_0(\mathbb{R}^d) = B(\mathbb{R}^d), L_0(\mathbb{R}^d) = L(\mathbb{R}^d), T_0(\mathbb{R}^d) = T(\mathbb{R}^d) \) and \( G_0(\mathbb{R}^d) = G(\mathbb{R}^d) \).

In the following, the \( m \)-th power of a mapping denotes \( m \) times compositions of the mapping. For \( m = 0, 1, 2, \ldots \), let

1. \( U_m(\mathbb{R}^d) = U^{m+1}(I(\mathbb{R}^d)) \),
2. \( B_m(\mathbb{R}^d) = \Phi^{m+1}(I(\mathbb{R}^d)) \),
3. \( L_m(\mathbb{R}^d) = \Phi^{m+1}(I_{\log^{m+1}}(\mathbb{R}^d)) \),
4. \( T_m(\mathbb{R}^d) = \Phi^{m+1}(I_{\log^{m+1}}(\mathbb{R}^d)) = \Phi(L_m(\mathbb{R}^d)) \),
5. \( G_m(\mathbb{R}^d) = \Phi^{m+1}(I(\mathbb{R}^d)) \)

and

6. \( M_m(\mathbb{R}^d) = \mathcal{M}^{m+1}(I_{\log^{m+1}}(\mathbb{R}^d)) \cap I_{\text{sym}}(\mathbb{R}^d) \).

and let \( U_\infty(\mathbb{R}^d) = \cap_{m=0}^\infty U_m(\mathbb{R}^d), B_\infty(\mathbb{R}^d) = \cap_{m=0}^\infty B_m(\mathbb{R}^d), L_\infty(\mathbb{R}^d) = \cap_{m=0}^\infty L_m(\mathbb{R}^d), T_\infty(\mathbb{R}^d) = \cap_{m=0}^\infty T_m(\mathbb{R}^d), G_\infty(\mathbb{R}^d) = \cap_{m=0}^\infty G_m(\mathbb{R}^d) \) and \( M_\infty(\mathbb{R}^d) = \cap_{m=0}^\infty M_m(\mathbb{R}^d) \).

Each sequence of the classes is decreasing as \( m \) increases.

5. Known relationships among classes

We now state some relationships among the classes we are talking about.

1. \( B(\mathbb{R}^d) \cup L(\mathbb{R}^d) \subseteq U(\mathbb{R}^d) \) (by definition).
2. \( T(\mathbb{R}^d) \subseteq B(\mathbb{R}^d) \cap L(\mathbb{R}^d) \) (by definition).
3. \( B(\mathbb{R}^d) \cap I_{\text{sym}}(\mathbb{R}^d) \subseteq G(\mathbb{R}^d) \) (Aoyama+Maejima+Rosiński (2007)).
4. \( T(\mathbb{R}^d) \cap I_{\text{sym}}(\mathbb{R}^d) \subseteq M(\mathbb{R}^d) \) (Aoyama+Maejima+Rosiński (2007)).
5. \( M(\mathbb{R}^d) \subseteq L(\mathbb{R}^d) \cap G(\mathbb{R}^d) \) (Aoyama+Maejima+Rosiński (2007)).
6. \( G(L(\mathbb{R}^d)) \subseteq M(\mathbb{R}^d) \) (Aoyama+Maejima+Rosiński (2007)).
7. \( T_m(\mathbb{R}^d) \subseteq L_m(\mathbb{R}^d) \) (Barndorff-Nielsen+Maejima+Sato (2006)).
8. \( S(\mathbb{R}^d) = L_\infty(\mathbb{R}^d) \) (Sato (1980)). \( (S(\mathbb{R}^d) \) is the closure of \( S(\mathbb{R}^d) \) that is closed under convolution and weak convergence.)
9. \( T_\infty(\mathbb{R}^d) = L_\infty(\mathbb{R}^d) \) (Barndorff-Nielsen+Maejima+Sato (2006)).
10. \( L_\infty(\mathbb{R}^d) \cap I_{\text{sym}}(\mathbb{R}^d) \subseteq G_\infty(\mathbb{R}^d) \) (Maejima+Rosiński (2001)).
6. Examples

Below, all examples are restricted to the case \( d = 1 \). So, in the polar decomposition, \( S = \{-1, 1\} \). The order of examples is random.


Let \( \gamma_{c, \lambda} \) be a gamma random variable with parameter \( c > 0 \) and \( \lambda > 0 \). Namely, \( P(\gamma_{c, \lambda} \in B) = \lambda^c \Gamma(c)^{-1} \int_{B \cap (0, \infty)} x^{c-1} e^{-\lambda x} dx. \) (When \( c = 1 \), it is exponential.) Its Lévy measure is \( \nu(dr) = ce^{-\lambda r}r^{-1}1_{(0, \infty)}(r)dr. \) (See Jurek (1997).)

Then \( \mathcal{L}(\gamma_{c, \lambda}) \in T(\mathbb{R}_+) \), (from the form of the Lévy measure of \( \mathcal{L}(\gamma_{c, \lambda}) \)), and \( \mathcal{L}(\gamma_{c, \lambda}) \notin L_1(\mathbb{R}) \), (Barndorff-Nielsen+Pedersen+Sato (2001)).

6.2. Logarithm of gamma random variable \( \gamma_{c, \lambda} \).

Its Lévy measure is \( \nu(dr) = e^{cr}|r|^{-1}(1 - e^{-r})^{-1}1_{(-\infty, 0)}(r)dr. \) (a) \( \mathcal{L}(\log \gamma_{c, \lambda}) \in L(\mathbb{R}) \) (Shanbhag+Sreehari (1977)).
(b) \( \mathcal{L}(\log \gamma_{c, \lambda}) \in L_1(\mathbb{R}) \) if \( c \geq \frac{1}{2} \) (Akita+Maejima (2002)).
(c) \( \mathcal{L}(\log \gamma_{c, \lambda}) \in L_2(\mathbb{R}) \) if \( c \geq 1 \) (Akita+Maejima (2002)).

(a') \( \Upsilon(\mathcal{L}(\log \gamma_{c, \lambda})) \in T(\mathbb{R}) \) (Barndorff-Nielsen+Maejima+Sato (2006)).
(b') \( \Upsilon(\mathcal{L}(\log \gamma_{c, \lambda})) \in T_1(\mathbb{R}) \) if \( c \geq \frac{1}{2} \) (Barndorff-Nielsen+Maejima+Sato (2006)).
(c') \( \Upsilon(\mathcal{L}(\log \gamma_{c, \lambda})) \in T_2(\mathbb{R}) \) if \( c \geq 1 \) (Barndorff-Nielsen+Maejima+Sato (2006)).

6.3. Symmetrized gamma distribution with parameter \( c > 0 \) and \( \lambda > 0 \), written as \( \text{sym-gamma} \ (c, \lambda) \). (See Steutel+van Harn (2004), p.142.)

Its characteristic function is \( \varphi_c(z) = (\lambda^2/(\lambda^2 + z^2))^{c/2} \). Its Lévy measure is \( \nu(dr) = c|r|^{-1}e^{-\lambda|r|}dr, \ (r \neq 0). \) (See Steutel+van Harn (2004), p.279.) (When \( c = 1 \) it is Laplace distribution.)

We have
(a) \( \text{sym-gamma} \ (c, \lambda) \in T(\mathbb{R}) \), (from the form of the Lévy measure above).

Thus
(b) \( \text{sym-gamma} \ (c, \lambda) \in G(\mathbb{R}) \). (See Rosinski (1991), p.29.)

6.4. Tempered stable distribution (defined by Rosiński (2004)).

Let \( 0 < \alpha < 2 \). \( T_\alpha \) is called a tempered \( \alpha \)-stable random variable, if its Lévy measure is

\[
\nu(dr) = c_+ r^{-\alpha-1}q_+(r)1_{(0, \infty)}(r)dr + c_- |r|^{-\alpha-1}q_-(|r|)1_{(-\infty, 0)}(r)dr,
\]

where \( q_\pm(\cdot) \) are completely monotone and \( q_\pm(+\infty) = 0 \), and \( c_\pm \geq 0, c_+ + c_- > 0 \). \( \alpha \)-stable distribution is not tempered \( \alpha \)-stable but tempered \( \beta(< \alpha) \)-stable.
We have the following. (Barndorff-Nielsen+Maejima+Sato (2006).)

(a) \( L(T_\alpha) \in T(\mathbb{R}) \).

(b) \( L(T_\alpha) \in T_1(\mathbb{R}) \) if \( 1 \leq \alpha < 2 \).

(c) \( L(T_\alpha) \in L_1(\mathbb{R}) \) if \( \frac{1}{2} \leq \alpha < 2 \).

(d) \( L(T_\alpha) \in L_2(\mathbb{R}) \) if \( \frac{2}{3} \leq \alpha < 2 \).

(e) \( L(T_\alpha) \notin L_1(\mathbb{R}) \) if \( 0 < \alpha < \frac{1}{4} \) and \( q_\pm(r) = e^{-b_\pm r} \), where \( b_\pm > 0 \).

6.5. Examples in \( T(\mathbb{R}) \).

There are many examples in \( T(\mathbb{R}) \). The following are some of them. (See, e.g. Bondesson (1992).)

(a) \( \chi^2 \)-distribution. \( L(\chi^2(r)) \in T(\mathbb{R}_+) \subset L(\mathbb{R}_+) \), \( r \in \mathbb{N} \), (since \( \chi^2(r) = \gamma_{r/2,1/2} \)).

(b) Generalized inverse Gaussian distributions belong to \( T(\mathbb{R}) \subset L(\mathbb{R}) \). (See, e.g. Bondesson (1992), p.92.)

(c) \( L(\log \gamma_{c,1}) \in T(\mathbb{R}) \). (See Bondesson (1992), p.112.)

(d) Let \( X_\alpha \) be a positive stable random variable with \( 0 < \alpha < 1 \). Then \( L(\log X_\alpha) \in T(\mathbb{R}) \). (See Bondesson (1992), p.114.)

6.6. Examples in \( L(\mathbb{R}) \).

There are many examples in \( L(\mathbb{R}) \). The following are some of them.

(a) Let \( L(Z) \) be the standard normal, \( L(t) \) the student \( t \) distribution and \( L(F) \) the \( F \) distribution. Then \( L(\log |Z|) \in L(\mathbb{R}), L(\log |t|) \in L(\mathbb{R}) \) and \( L(\log F) \in L(\mathbb{R}) \). (Shanbhag+Sreehari (1977).)

(b) Let \( G_1(x) = 1 - e^{-e^x}, x \in \mathbb{R} \), and \( G_2(x) = e^{-e^{-x}}, x \in \mathbb{R} \). \( G_1 \) (resp. \( G_2 \)) is the distribution of the plus (resp. minus) of logarithm of the standard exponential random variable. They are in \( L(\mathbb{R}) \). (See Steutel+van Harn (2004).)

(c) Hyperbolic sine and cosine distributions belong to \( L(\mathbb{R}) \). (See, e.g. Jurek (1998).) The characteristic function of hyperbolic sine distribution is \( \varphi(z) = \pi z (\sinh \pi z)^{-1} \) and that of hyperbolic cosine distribution is \( \varphi(z) = \pi z (\cosh(\pi z/2))^{-1} \).

(d) Generalized hyperbolic distributions belong to \( L(\mathbb{R}) \). (See, e.g. Jurek (1998).)

(e) Let \( Y \) be a beta random variable. Then \( L(\log Y (1-Y)^{-1}) \in L(\mathbb{R}) \). (Barndorff-Nielsen+Kent+Sorensen (1982).)

(f) (The stochastic area of two-dimensional Brownian motion by Lévy.) The density function is

\[
f(x) = \frac{1}{\pi \cosh x} = \frac{2}{\pi (e^x + e^{-x})}
\]
and it belongs to $L(\mathbb{R})$. In this case, $k_{\xi}(r)$ in (2.3) is $|2\sinh r|^{-1}$. (See, e.g. Sato (1999), p. 98, Example 15.15.)

6.7. Limits of generalized Ornstein-Uhlenbeck processes. (Exponential integrals of Lévy processes.)
(a) Let $\{(X_t, Y_t), t \geq 0\}$ be a 2-dimensional Lévy process. Suppose that $\{X_t\}$ does not have positive jumps, $0 < E[X_1] < \infty$ and $L(Y_1) \in I_{\log}(\mathbb{R})$. Then

$$\mathcal{L}\left(\int_0^\infty e^{-X_t}dY_t\right) \in L(\mathbb{R}).$$

(Bertoin+Lindner+Maller (2006).)

(b) Let $\{N_t\}$ be a Poisson process, and let $\{Y_t\}$ be a strictly stable Lévy process or a Brownian motion with drift. Then

$$\mathcal{L}\left(\int_0^\infty e^{-N_t}dY_t\right) \in L(\mathbb{R}).$$

(Kondo+Maejima+Sato (2006).)

(c) Let $X_t = 2t - N_t$, where $\{N_t\}$ is a standard Poisson process and $Y_t = t$. Then

$$\mathcal{L}\left(\int_0^\infty e^{-(2t-N_t)}dt\right) \in L(\mathbb{R}) \cap L_1(\mathbb{R})^c.$$

(Lindner+Maejima (2007).)

6.8. Type $S$ random variables.
Let $0 < \alpha < 1$, $X \overset{d}{=} Y^{1/\alpha}X_\alpha$, where $Y$ and $X_\alpha$ are independent, and where $Y \overset{d}{=} \gamma_{c,1}$ and $X_\alpha \overset{d}{=} \text{strictly } \alpha\text{-stable.}$

(a) $\mathcal{L}(X) \in G(T(\mathbb{R})) \subset T(\mathbb{R})$. (See Bondesson (1992), p.38.)

(b) Suppose $X_\alpha$ is symmetric. $\mathcal{L}(X)$ is of type $S_\alpha$, thus it belongs to $G(\mathbb{R})$. (See Kondo+Maejima+Sato (2006).)

6.9. Convolution of symmetric stable distributions of different indeces.
Its characteristic function is $\varphi(z) = \exp\left\{\int_{(0,2)} -|z|^\alpha m(\alpha)\right\}$, where $m$ is a measure on the interval $(0,2)$.

(a) It belongs to $L_\infty(\mathbb{R})$. (See, e.g. Roch-Arteaga+Sato (2003).)

(b) It belongs to $G_\infty(\mathbb{R}) \subset G(\mathbb{R})$. (Rosinski (1991).)

Let $Z_1$ and $Z_2$ be independent standard normal random variables.

(a) $\mathcal{L}(Z_1Z_2) \in G_1(\mathbb{R})$. (Maejima+Rosinski (2001).)
Since $L(Z_1Z_2) = L(\text{sym-gamma}(\frac{1}{2}, 1))$ (see Steutel+Van Harn (2004), p.504),
(b) $L(Z_1Z_2) \in T(\mathbb{R})$.
(c) $L(Z_1Z_2) \in \mathcal{G}(L(\mathbb{R}))$.

Proof of (c). $Z_1Z_2 \overset{d}{=} (Z_2^2)^{1/2}Z_2$ and $L(Z_2^2)$ is $\chi^2$-distribution, which is known to be selfdecomposable (see, e.g. Jurek (1997), p.98).

6.11 Examples related to gamma random variables.
(a) Product of independent gamma random variables. (Steutel+Van Harn (2004), p.360.)

Let $X_1, X_2, ...X_n$ be independent gamma random variables, and let $q_1, q_2, ..., q_n \in \mathbb{R}$ with $|q_j| \geq 1$. Then

$L(X_q^1X_q^2 \cdots X_q^n) \in L(\mathbb{R}^+)$.

(b) Exponential function of gamma random variable. (Bondesson (1992), p.94.)

Let $X$ be denumerable convolution of gamma random variables $\gamma_{c_j,x_j}$ with $c_j \geq 1$.

Then

$L(e^X) \in T(\mathbb{R}^+)$.

6.12 Log-normal distribution. (Bondesson (1992), p.59.)

Let $Z$ be a standard normal random variable. Then log-normal distribution $L(e^Z)$

$\in T(\mathbb{R}^+)$.

6.13 Random excursion of Bessel processes. (Bertoin+Fujita+Roynette+Yor (2006).)

Let $\{R_t, t \geq 0\}$ be a Bessel process with $R_0 = 0$, with dimension $d = 2(1 - \alpha)$. $(0 < \alpha < 1$, equivalently $0 < d < 2$.) When $\alpha = \frac{1}{2}$, $\{R_t\}$ is a Brownian motion. Let

$g_t^{(\alpha)} := \sup\{s \leq t : R_s = 0\},$

$d_t^{(\alpha)} := \inf\{s \geq t : R_s = 0\}$

and

$\Delta_t^{(\alpha)} := d_t^{(\alpha)} - g_t^{(\alpha)}$,

which is the length of the excursion above 0, straddling $t$, for the process $\{R_u, u \geq 0\}$, and let $\varepsilon$ be a standard exponential variable independent from $\{R_u, u \geq 0\}$. Let $\Delta_\alpha := \Delta_\varepsilon^{(\alpha)}$. Then

$L(\Delta_\alpha) \in T(\mathbb{R}^+)(\subset L(\mathbb{R}^+))$.

In Bertoin+Fujita+Roynette+Yor (2006), only “$\in L(\mathbb{R})$” is mentioned. However, they actually showed that
\[ E[e^{-\lambda \Delta \alpha}] = \exp \left\{ -(1 - \alpha) \int_0^\infty (1 - e^{-\lambda x}) \frac{E[e^{-xG\alpha}]}{x} \, dx \right\}, \quad \lambda > 0, \]

with a random variable \( G\alpha \). (The density function of \( G\alpha \) is explicitly known.) Since \( k(x) := E[e^{-xG\alpha}] \) is completely monotone by Bernstein theorem, \( \mathcal{L}(\Delta \alpha) \) belongs to not only \( L(\mathbb{R}_+) \) but also \( T(\mathbb{R}_+) \).

6.14. Let \( CP(\mathbb{R}) = \{ \mu \in I(\mathbb{R}) : \mu \text{ is compound Poisson distribution} \} \).
(a) \( \mathcal{L}(\Delta \alpha) \in \Phi(CP(\mathbb{R})) \). (Bertoin+Fujita+Roynette+Yor (2006), eq. (1.21).)
(b) \( \mathcal{L}(\gamma_{c,\lambda}) \in \Phi(CP(\mathbb{R})) \). (Jurek (1997).)
(Question) Characterize the class \( \Phi(CP(\mathbb{R})) \).

6.15. From the observation above, we see the following. We know that the compound Poisson distribution is not selfdecomposable. Hence,
\[ \mathcal{L}(\Delta \alpha) \notin \Phi^2(I(\mathbb{R})) = L_1(\mathbb{R}). \]

6.16. Examples in \( B(\mathbb{R}) \).
(a) (Bondesson (1992), p.143.) Compound Poisson \( X = \sum_{j=1}^N Y_j \), where \( \{Y_j\} \) are i.i.d. exponential.
Then \( \mathcal{L}(X) \in B(\mathbb{R}_+) \).
(b) (Bondesson (1992), pp.143-144.) \( X = -\log Y, \ Y = Y(\alpha, \beta) \) is a beta random variable with parameters \( \alpha \) and \( \beta \).
   (b1) \( \mathcal{L}(X) \in B(\mathbb{R}_+) \)
   (b2) \( \mathcal{L}(X) \in L(\mathbb{R}_+) \) iff \( 2\alpha + \beta \geq 1 \)

6.17. Examples in \( T(\mathbb{R}) \cap L_1(\mathbb{R})^c \). (Revisit.)
(a) \( \mathcal{L}(\gamma_{c,\lambda}) \). (See 6.1.)
(b) \( \mathcal{L}(T_\alpha) \) if \( 0 < \alpha < \frac{1}{2} \). (See 6.4, (a) and (e).)
(c) \( \mathcal{L}\left( \int_0^\infty e^{-(2t-N_t)} \, dt \right) \). (See 6.7, (a) and (c).)
(d) \( \mathcal{L}(\Delta \alpha) \). (See 6.12 and 6.14.)

6.18. Examples in \( G(L(\mathbb{R})) \).
Let \( \lambda > 0 \) and \( \{B_t\} \) a standard Brownian motion, and let \( \{Z_t^{(\alpha)}\} \) be a symmetric \( \alpha \)-stable Lévy process. Then \( \mathcal{L}\left( \int_0^\infty e^{-B_s - \lambda s} \, dZ_s^{(\alpha)} \right) \in G(L(\mathbb{R})) \) and also in \( TS_\alpha(\mathbb{R}) \subset G(\mathbb{R}) \), where \( TS_\alpha(\mathbb{R}) \) is the class of type \( S_\alpha \). (See Maejima+Niiyama (2004), Aoyama+Maejima+Rosinski (2006) and Kondo+Maejima+Sato (2006).)
References


