Selfdecomposability of moving average fractional Lévy processes ☆

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Abstract

We study the relationships between the selfdecomposability of marginal distributions or finite dimensional distributions of moving average fractional Lévy processes and distributions of their driving Lévy processes.

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1. Introduction

Let $(X_t)_{t\in\mathbb{R}}$ be a Lévy process with finite variance on $\mathbb{R}^d$. The moving average fractional Lévy process (MAFLP, in short) is defined as

$$Y_t = \int_{\mathbb{R}} \left((t-s)_+^\beta - (-s)_+^\beta \right) dX_s, \quad t \geq 0, \tag{1}$$

where $a_+ = \max(a, 0)$ and $\beta \in (0, 1/2)$. We refer to the process (X_t) as the driving Lévy process. Since the integrand is square integrable on $\mathbb{R}$, this integral is definable. (See Sato and Yamazato (1983).) If (X_t) is a Brownian motion, then (Y_t) is a fractional Brownian motion and is Gaussian

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as a process. In particular, the distribution of Y_t is Gaussian for each t. A natural question about (1) is what we can say about its distribution in the general case. This is our motivation of this paper. We are interested in selfdecomposability. Actually, we show that if the driving Lévy process is selfdecomposable, then the marginal distributions of MAFLPs are also selfdecomposable, but the converse is not necessarily true. However, we show that if the MAFLP is selfdecomposable as a process, then so is the driving Lévy process.

2. A brief summary on infinitely divisible distributions, selfdecomposable distributions and selfdecomposable processes

2.1. Distributions

Let $I(\mathbb{R}^d)$ be the class of all infinitely divisible distributions on $\mathbb{R}^d$. We denote by $\langle\, ,\rangle$ the Euclidean inner product on $\mathbb{R}^d$ and by $|\cdot|$ the associated Euclidean norm. Let $\mu \in I(\mathbb{R}^d)$. Then it is known that the characteristic function $\widehat{\mu}(z)$, $z \in \mathbb{R}^d$, of $\mu \in I(\mathbb{R}^d)$ has the following Lévy-Khintchine representation:

$$\widehat{\mu}(z) = \exp\left\{-2^{-1}\langle z, Az\rangle + i\langle \gamma, z\rangle + \int_{\mathbb{R}^d}\left(e^{i\langle z,x\rangle} - 1 - \frac{i\langle z, x\rangle}{1+|x|^2}\right)\nu(dx)\right\}, \quad z \in \mathbb{R}^d,$$

where A is a nonnegative-definite symmetric $d\times d$ matrix, $\gamma \in \mathbb{R}^d$ and ν is the Lévy measure satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d}(|x|^2 \wedge 1)\nu(dx) < \infty$. The triplet (A, ν, γ) is uniquely determined by μ and we call it the Lévy-Khintchine triplet of $\mu \in I(\mathbb{R}^d)$. We also write $\mu = \mu_{(A,\nu,\gamma)}$, when we want to emphasize the triplet.

When E is a topological space, $\mathcal{B}(E)$ is the set of Borel sets in E. The following is also known. (See, e.g. Rajput and Rosinski (1989), Barndorff-Nielsen et al. (2006b) Lemma 2.1.)

Proposition 2.1. (*Polar decomposition of Lévy measures*) *There exist a measure λ on $S^d = \{\xi \in \mathbb{R}^d : |\xi| = 1\}$ with $0 < \lambda(S^d) < \infty$ and a family $\{\nu_\xi\colon \xi \in S^d\}$ of measures on $(0,\infty)$ such that $\nu_\xi(B)$ is measurable in ξ for each $B \in \mathcal{B}((0,\infty))$, $0 < \nu_\xi((0,\infty)) \le \infty$ for each $\xi \in S^d$ and*

$$\nu(B) = \int_{S^d}\lambda(d\xi)\int_0^\infty \mathbf{1}_B(r\xi)\nu_\xi(dr), \quad B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}).$$

Here λ and $\{\nu_\xi\}$ are uniquely determined by ν up to multiplication of measurable functions $c(\xi)$ and $\frac{1}{c(\xi)}$ with $0 < c(\xi) < \infty$. We call ν_ξ the radial component of ν.

Let us recall the definition of selfdecomposable distributions.

Definition 2.2. $\mu \in I(\mathbb{R}^d)$ is said to be selfdecomposable if for any $b > 1$, there is a distribution μ_b satisfying $\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)\widehat{\mu_b}(z)$.

Any selfdecomposable distribution is infinitely divisible and μ_b is also infinitely divisible. The selfdecomposability of $\mu \in I(\mathbb{R}^d)$ can be characterized as follows. (See, e.g. Sato (1999), Theorem 15.10.)

Proposition 2.3. *$\mu = \mu_{(A,\nu,\gamma)} \in I(\mathbb{R}^d)$ is selfdecomposable if and only if ν can be expressed as*

$$\nu(B) = \int_{S^d} \lambda(d\xi) \int_0^\infty \mathbf{1}_B(r\xi) k_\xi(r) r^{-1} dr, \quad B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}), \tag{2}$$

with a finite measure λ on S^d and a nonnegative function $k_\xi(r)$ measurable in $\xi \in S^d$ and nonincreasing in $r > 0$. The formula (2) is called a canonical decomposition of ν, if $\lambda(\xi \in S^d : \ k_\xi(r) = 0, \ \forall r > 0) = 0$.

It would be helpful to mention Jurek and Verwaat (1983), Jurek (1983, 1984a,b) for readers who are interested in the formula (2) in Banach space setting.

2.2. Selfdecomposable processes.

A stochastic process $(X_t)_{t\in T}$ (where $T = \mathbb{R}$ or $[0,\infty)$) on $\mathbb{R}^d$ is called selfdecomposable (as a process), if all its finite dimensional distributions are selfdecomposable. A selfdecomposable process is infinitely divisible (as a process) in the sense that all finite dimensional distributions are infinitely divisible. In Theorem 3.7 of Barndorff-Nielsen et al. (2006a), it is shown that $(X_t)_{t\in T}$ is selfdecomposable if and only if for any $b > 1$, there exists a stochastic process $(U_t^{(b)})_{t\in T}$ independent of (X_t) such that

$$(X_t)_{t\in\mathbb{R}} \overset{\text{(d)}}{=} \left(b^{-1}X_t\right)_{t\in\mathbb{R}} + \left(U_t^{(b)}\right)_{t\in\mathbb{R}}, \tag{3}$$

where $\overset{\text{(d)}}{=}$ is equality in distribution as processes in the sense of finite dimensional distributions. Finally, a stochastic process $(X_t)_{t\in T}$ on $\mathbb{R}^d$ is called weakly selfdecomposable, if for any $t_1, t_2, ..., t_n \in T$, and for any $a_1, a_2, ..., a_n \in \mathbb{R}$, $\sum_{j=1}^n a_j X_{t_j}$ is selfdecomposable. (See Barndorff-Nielsen et al. (2006a).) Remark that, in general, selfdecomposability of one dimensional marginals does not necessarily imply weak selfdecomposability of the process, which in turn does not imply selfdecomposability of the process. We refer to Barndorff-Nielsen et al. (2006a) for proofs. There is one exception to these general facts : If $(X_t)_{t\in\mathbb{R}}$ is a Lévy process and X_1 is selfdecomposable, then $(X_t)_{t\in\mathbb{R}}$ is selfdecomposable as a process.

3. Selfdecomposability of marginals of moving average fractional Lévy processes

3.1. A sufficient condition

In this subsection, we consider Lévy processes (X_t) with finite variance on $\mathbb{R}^d$. Let ν be the Lévy measure of (X_t).

Proposition 3.1. *Suppose that f is a real-valued square integrable function defined on $\mathbb{R}$. If X_1 is selfdecomposable, then so is $\int_{\mathbb{R}} f(s)dX_s$. Let ν be the Lévy measure of (X_t). Moreover, if ν admits a canonical decomposition* (2), *then the Lévy measure $\widetilde{\nu}$ of $\int_{\mathbb{R}} f(s)dX_s$ admits a decomposition*

$$\widetilde{\nu}(B) = \int_{S^d} (\lambda(d\xi) + \lambda(-d\xi)) \int_0^\infty \mathbf{1}_B(r\xi)\widetilde{k}_\xi(r)r^{-1}dr, \tag{4}$$

where

$$\begin{aligned}\widetilde{k}_\xi(r) = \int_{\mathbb{R}} \Bigg(&\mathbf{1}(f(s) > 0)a(\xi)k_\xi\left(\frac{r}{f(s)}\right) \\ &+\mathbf{1}(f(s) < 0)(1 - a(-\xi))k_{-\xi}\left(\frac{r}{|f(s)|}\right)\Bigg) ds,\end{aligned} \tag{5}$$

where $a(\xi) = \lambda(d\xi)/\left(\lambda(d\xi) + \lambda(-d\xi)\right)$.

Proof of Proposition 3.1. Suppose that X_1 is selfdecomposable. By definition, for any $b > 1$,

$$X_1 \overset{\text{(d)}}{=} b^{-1}X_1 + U^{(b)}, \tag{6}$$

with an infinitely divisible random variable $U^{(b)}$ independent of X_1. Let ν_b denote the Lévy measure associated to $U^{(b)}$. By (6),

$$\nu(du) = \nu(b^{-1}du) + \nu_b(du).$$

If we let $\left(U_t^{(b)}\right)_{t\in\mathbb{R}}$ be a Lévy process with its Lévy measure ν_b, then for any $b > 1$,

$$\int_{\mathbb{R}} f(s)dX_s \overset{\text{(d)}}{=} b^{-1}\int_{\mathbb{R}} f(s)dX_s + \int_{\mathbb{R}} f(s)dU_s^{(b)}.$$

Note that the integrals above are defined because $\int_{\mathbb{R}} |u|^2\nu(du)$, $\int_{\mathbb{R}} |u|^2\nu_b(du)$ are finite, and f is square integrable. See Bebassi et al. (2004) for details. We next note that the Lévy measure $\widetilde{\nu}$ of $\int_{\mathbb{R}} f(s)dX_s$ is expressed as

$$\widetilde{\nu}(B) = \int_{\mathbb{R}} \mathbf{1}(f(s) \neq 0)\nu\left(\frac{B}{f(s)}\right)ds, \quad B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}). \tag{7}$$

Polar decomposition (2) then yields

$$\begin{aligned}\widetilde{\nu}(B) = &\int_{f(s)>0} ds \int_{S^d} \lambda(d\xi) \int_0^\infty k_\xi(r)\mathbf{1}_B(rf(s)\xi)r^{-1}dr \\ &+ \int_{f(s)<0}\int_{S^d} \lambda(d\xi) \int_0^\infty k_\xi(r)\mathbf{1}_B(r \times (-f(s)) \times (-\xi))r^{-1}dr.\end{aligned}$$

Using change of variables, one gets

$$\begin{aligned}\widetilde{\nu}(B) = &\int_{f(s)>0} ds \int_{S^d} \lambda(d\xi) \int_0^\infty k_\xi\left(\frac{u}{f(s)}\right)\mathbf{1}_B(u\xi)u^{-1}du \\ &+ \int_{f(s)<0}\int_{S^d} \lambda(-d\xi) \int_0^\infty k_{-\xi}\left(\frac{u}{|f(s)|}\right)\mathbf{1}_B(u\xi)u^{-1}du.\end{aligned}$$

It can be rewritten

$$\begin{aligned}\widetilde{\nu}(B) = &\int_{f(s)>0} ds \int_{S^d} a(\xi)\left(\lambda(d\xi) + \lambda(-d\xi)\right) \\ &\qquad \int_0^\infty k_\xi\left(\frac{u}{f(s)}\right)\mathbf{1}_B(u\xi)u^{-1}du \\ &+ \int_{f(s)<0}\int_{S^d} (1 - a(-\xi))\left(\lambda(d\xi) + \lambda(-d\xi)\right) \\ &\qquad \int_0^\infty k_{-\xi}\left(\frac{u}{|f(s)|}\right)\mathbf{1}_B(u\xi)u^{-1}du.\end{aligned}$$

Hence we get (4) by applying Fubini's theorem. In turns we have the formula (5) for $\widetilde{k}_\xi$. Since the fact that $r \mapsto k_\xi(r)$ is nonincreasing for every ξ clearly implies the same property for $\widetilde{k}_\xi$, the selfdecomposability of $\int_{\mathbb{R}} f(s)dX_s$ is inherited from the selfdecomposability of X_1. □

Proposition 3.1 can be applied to MAFLPs and shows that the selfdecomposability of the driving Lévy process is inherited by the MAFLPs, as follows.

Corollary 3.2. *Let $\beta \in (0, 1/2)$. If X_1 is selfdecomposable, then for every $t \in \mathbb{R}$,*

$$\int_{\mathbb{R}} \left((t-s)_+^{\beta} - (-s)_+^{\beta} \right) dX_s \tag{8}$$

is selfdecomposable.

Proof of Corollary 3.2. Since, for every $t \in \mathbb{R}$, $f_t(s) = (t-s)_+^{\beta} - (-s)_+^{\beta}$ is in $L^2(\mathbb{R})$, the corollary is a straightforward consequence of Proposition 3.1. □

Note that the selfdecomposability of the driving Lévy process is a sufficient assumption to have the selfdecomposability of the one dimensional marginals of MAFLPs, but it is not a necessary assumption as shown in the following example.

3.2. An example

In this subsection, we will focus on MAFLPs (8) given in Corollary 3.2. First we can rewrite the process in the following form and split it into two parts

$$Y_t = \int_0^{\infty} \left((t-s)^{\beta} - (-s)^{\beta} \right) dX_s + \int_0^t (t-s)^{\beta} dX_s =: U_t + V_t, \quad t > 0,$$

where $\beta \in (0, 1/2)$ and $(X_t)_{t \in \mathbb{R}}$ is a Lévy process on $\mathbb{R}^d$.

As we have seen in Proposition 3.1, if X_1 is selfdecomposable, so is V_1. Here we are going to show that the converse of Proposition 3.1 is not true by giving an example of X_1 such that X_1 is not selfdecomposable but V_1 is selfdecomposable.

Proposition 3.3. *If the radial component ν_ξ of the Lévy measure of X_1 is*

$$\nu_\xi(du) = k_\xi(u) u^{-1} du,$$

where

$$k_\xi(u) = \begin{cases} u^{-p}, & u \in (0,1) \\ Cu^{-p}. & u \in [1, \infty), \end{cases}$$

and $0 < p < 2$, $1 < C < 1 + p\beta$, then V_1 is selfdecomposable, although X_1 is not selfdecomposable.

Proof of Proposition 3.3. Since $k_\xi(u)$ is not nonincreasing due to the assumption that $C > 1$, the distribution of X_1 is *not* selfdecomposable by the uniqueness of the polar decomposition. We next show, however, that V_1 is selfdecomposable.

For notational simplicity, we put $\alpha = -\beta^{-1}$. Let ν and $\widetilde{\nu}$ be the Lévy measures of X_1 and V_1, respectively. Then by (7) we have

$$\widetilde{\nu}(B) = (-\alpha)\int_0^\infty \nu(s^{-1}B)s^{-\alpha-1}ds, \quad B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}).$$

and after a standard calculation, we get

$$\widetilde{\nu}_\xi(dr) = \widetilde{k}_\xi(r)r^{-1}dr,$$

where $\widetilde{k}_\xi(r) = (-\alpha)r^{-\alpha}\int_r^\infty u^\alpha \nu_\xi(du)$. We then have

$$\widetilde{k}_\xi(r) = (-\alpha)r^{-\alpha}\int_{r\wedge 1}^1 u^{\alpha-1-p}du + (-\alpha)r^{-\alpha}\int_{r\vee 1}^\infty Cu^{\alpha-1-p}du.$$

For $0 < r \leq 1$, we have

$$\widetilde{k}_\xi(r) = (-\alpha)(p-\alpha)^{-1}\left[(C-1)r^{-\alpha} + r^{-p}\right],$$

and thus

$$\frac{d}{dr}\widetilde{k}_\xi(r) = (-\alpha)(p-\alpha)^{-1}\left[-(C-1)\alpha r^{-\alpha-1} - pr^{-p-1}\right] < 0,$$

where we have used that $1 < C < 1 + p(-\alpha)^{-1}, -\alpha > 0, -p < 0, r \leq 1$. Thus $\widetilde{k}_\xi(r)$ is nonincreasing in $r \in (0, 1]$. For $1 \leq r < \infty$,

$$\widetilde{k}_\xi(r) = (-\alpha)r^{-\alpha}\int_r^\infty Cu^{\alpha-1-p}du = (-\alpha)C(p-\alpha)^{-1}r^{-p},$$

which is nonincreasing. Thus, $\widetilde{k}_\xi(r)$ is nonincreasing in $r \in (0,\infty)$ and thus V_1 is selfdecomposable. □

Next we consider the random variable U_1.

Proposition 3.4. *The distribution of U_1 is always selfdecomposable.*

Proof of Proposition 3.4. Let $\widetilde{\nu}$ be the Lévy measure of U_1. First note that

$$U_1 = \int_{-\infty}^{0} \left((1-t)^{\beta} - (-t)^{\beta}\right) dX_t \overset{\text{(d)}}{=} \int_{0}^{\infty} \left((1+t)^{\beta} - t^{\beta}\right) d\widehat{X}_t,$$

where $(\widehat{X}_t) \overset{\text{(d)}}{=} (-X_{-t})$ is another Lévy process with the same triplet (A, ν, γ).

Let

$$\varepsilon^*(t) \overset{def}{=} (1+t)^{\beta} - t^{\beta},\ t > 0,$$

and let $\varepsilon(s)$ be the inverse function of $\varepsilon^*(t)$ defined as $\varepsilon^*(t) = s$ if and only if $t = \varepsilon(s)$. Now, we have

$$\widetilde{\nu}_{\xi}(B) = -\int_{0}^{1} \nu_{\xi}(s^{-1}B)\varepsilon'(s)ds = \int_{0}^{\infty} \mathbf{1}_B(r)\widetilde{\ell}_{\xi}(r)r^{-1}dr,$$

where

$$\widetilde{\ell}_{\xi}(r) = -\int_{r}^{\infty} rx^{-1}\varepsilon'(rx^{-1})\nu_{\xi}(dx).$$

An elementary calculation gives us that

$$-s\varepsilon'(s) \text{ is nonincreasing when } s \text{ runs from } 0^{+} \text{ to } 1.$$

Then U_1 is selfdecomposable. which implies that $\widetilde{\ell}_{\xi}(r)$ is nonincreasing. □

Combining Propositions 3.3 and 3.4, we get the following example of Y_1 selfdecomposable with X_1 not selfdecomposable.

Corollary 3.5. *If the Lévy measure of X_1 is the one defined in Proposition 3.3, then it is not selfdecomposable, but V_1 is selfdecomposable. Therefore, in this case Y_1 is selfdecomposable, even if X_1 is not selfdecomposable.*

Proof of Corollary 3.5. The corollary is a straightforward consequence of the independence of V_1 and U_1, which are both selfdecomposable.

4. Selfdecomposability of processes

In this section we consider processes (Y_t) defined by

$$Y_t \overset{def}{=} \int_{\mathbb{R}} f_t(s)dX_s \tag{9}$$

for $(f_t)_{t\in\mathbb{R}}$ a family of functions in $L^2(\mathbb{R})$ and $(X_s)_{s\in\mathbb{R}}$ a Lévy process with finite variance on $\mathbb{R}^d$. In contrast to what happens for one-dimensional marginals, we will prove equivalence between the selfdecomposability of processes (Y_t) and (X_t), when the selfdecomposability for processes is in the sense of Definition 3.1 in Barndorff-Nielsen et al. (2006a), (see also Section 2.2.)

Proposition 4.1. *If (Y_t) defined by (9) is weakly selfdecomposable and for every $n \in \mathbb{N}$ there exists a finite sequence $(a_{i,n}, t_{i,n})_{i\in I(n)}$ such that*

$$\sum_{i\in I(n)} a_{i,n} f_{t_{i,n}}(s) \to \mathbf{1}_{(a,b]}(s)$$

in $L^2(\mathbb{R})$ for some real numbers $a < b$, then (X_t) is a selfdecomposable Lévy process.

Proof of Proposition 4.1. If (Y_t) is weakly selfdecomposable, then for every n, $\sum_{i\in I(n)} a_{i,n} Y_{t_{i,n}}$ is selfdecomposable and it converges in $L^2(\Omega)$ to $\int \mathbf{1}_{(a,b]}(s)dX_s$. Note that the class of selfdecomposable distributions is closed under weak convergence, (see Sato (1980), Corollary 2.2). Hence $\int \mathbf{1}_{(a,b]}(s)dX_s$ is a selfdecomposable random variable. But, since (X_t) is a Lévy process, it is enough to get self-decomposability of X_1 in the sense of Definition 3.1 in Barndorff-Nielsen et al. (2006a). □

In the case of MAFLP we have equivalence of the selfdecomposability of the MAFLP and of the driving Lévy process.

Theorem 4.2. *Let $f_t(s) = (t-s)_+^\beta - (-s)_+^\beta$. Then (Y_t) defined by (9) is selfdecomposable if and only if (X_t) is selfdecomposable.*

Proof of Theorem 4.2.

("If" part.) Suppose that (X_t) is a selfdecomposable process, then by (6), for any $b > 1$ we can find a Lévy process $(U_t^{(b)})$ independent of (X_t) and such that

$$(X_t)_{t\in\mathbb{R}} \overset{(\mathrm{d})}{=} \left(b^{-1}X_t\right)_{t\in\mathbb{R}} + \left(U_t^{(b)}\right)_{t\in\mathbb{R}}.$$

Then

$$\left(\int f_t(s)dX_s\right)_{t\in\mathbb{R}} \overset{(\mathrm{d})}{=} \left(b^{-1}\int f_t(s)dX_s\right)_{t\in\mathbb{R}} + \left(\int f_t(s)dU_s^{(b)}\right)_{t\in\mathbb{R}},$$

which implies that $(Y_t)_{t\in\mathbb{R}}$ is selfdecomposable, due to (3).

("Only if" part.) Let us now assume that (Y_t) is a selfdecomposable process. Because of Proposition 4.1, in order to prove that (X_t) is selfdecomposable, it is enough to prove that $(\sum_{i\in I} a_i f_{t_i},\ I: \text{finite set})$ is a dense set of functions in $L^2(\mathbb{R})$

Actually we have the following lemma.

Lemma 4.3. $(\sum_{i\in I} a_i f_{t_i},\ I:$ *finite set) is a dense set of functions in* $L^2(\mathbb{R})$.

Proof of Lemma 4.3. Let us denote by $\langle f_t,\ t\in\mathbb{R}\rangle$ the vector space spanned by the functions f_t's. We have to prove that $L^2(\mathbb{R}) = \overline{\langle f_t,\ t\in\mathbb{R}\rangle}^{L^2}$, where we denote by $\overline{S}^{L^2}$ the smallest closed subset in $L^2(\mathbb{R})$ containing S. Let us denote by $\widehat{f_t}(\xi)$ the Fourier transform of f_t in $L^2(\mathbb{R})$. Because Fourier transform is an isometry of $L^2(\mathbb{R})$ onto itself, it is equivalent to prove that $L^2(\mathbb{R}) = \overline{\langle \widehat{f_t},\ t\in\mathbb{R}\rangle}^{L^2}$.

If $f_t(s) = (t-s)_+^\beta - (-s)_+^\beta$, then we have

$$\widehat{f_t}(\xi) = \frac{e^{it\xi}-1}{C_\beta^{1/2} i\xi|\xi|^\beta},$$

with $C_\beta > 0$. (See Samorodnitsky and Taqqu (1994), footnote 5 on page 328 in Section 7.2.) Since C^∞ functions with compact support in $\mathbb{R}\setminus\{0\}$ are dense in $L^2(\mathbb{R})$, it is enough to show that for every C^∞ function h with compact support in $\mathbb{R}\setminus\{0\}$, if for all $t\in\mathbb{R}$,

$$\frac{1}{i(2\pi)^{1/2}C_\beta^{1/2}}\int_{\mathbb{R}} \frac{(e^{it\xi}-1)\overline{h(\xi)}}{\xi|\xi|^\beta} d\xi = 0, \tag{10}$$

then $h = 0$. Since the support of h does not include 0, the following integral can be defined

$$\frac{1}{i(2\pi)^{1/2}C_\beta^{1/2}}\int_{\mathbb{R}} \frac{e^{it\xi}\overline{h(\xi)}}{\xi|\xi|^\beta} d\xi.$$

And because of the assumption of (10) it does not depend on t. Moreover, $h(\xi)/C_\beta^{1/2}\xi|\xi|^\beta$ is in $L^1(\mathbb{R})\bigcap L^2(\mathbb{R})$. So its Fourier transform is constant and consequently is vanishing. This implies that $h(\xi)/C_\beta^{1/2}|\xi|^{\beta+1} = 0$ for every ξ, and thus concludes $\varphi = 0$. $\square$.

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