Some classes of multivariate infinitely divisible distributions admitting stochastic integral representation

OLE E. BARNDORFF-NIELSEN[1], MAKOTO MAEJIMA[2] and KEN-ITI SATO[3*]

[1] *Department of Mathematical Sciences, University of Aarhus, Ny Munkegade, DK-8000 Aarhus C, Denmark. E-mail: oebn@imf.au.dk*

[2] *Department of Mathematics, Keio University, Hiyoshi, Yokohama, 223-8522 Japan. E-mail: maejima@math.keio.ac.jp*

[3] *Hachiman-yama 1101-5-103, Tenpaku-ku, Nagoya, 468-0074 Japan. E-mail: ken-iti.sato @nifty.ne.jp*

* Corresponding author

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[2] *Department of Mathematics, Keio University, Hiyoshi, Yokohama, 223-8522 Japan. E-mail: maejima@math.keio.ac.jp*

[3] *Hachiman-yama 1101-5-103, Tenpaku-ku, Nagoya, 468-0074 Japan. E-mail: ken-iti.sato @nifty.ne.jp*

The class of distributions on $\mathbb{R}$ generated by convolutions of Γ-distributions and the one generated by convolutions of mixtures of exponential distributions are generalized to higher dimensions and denoted by $T(\mathbb{R}^d)$ and $B(\mathbb{R}^d)$. From the Lévy process $\{X_t^{(\mu)}\}$ on $\mathbb{R}^d$ with distribution μ at $t = 1$, $\Upsilon(\mu)$ is defined as the distribution of the stochastic integral $\int_0^1 \log(1/t) dX_t^{(\mu)}$. This mapping is a generalization of the mapping Υ introduced by Barndorff-Nielsen and Thorbjørnsen in one dimension. It is proved that $\Upsilon(ID(\mathbb{R}^d)) = B(\mathbb{R}^d)$ and $\Upsilon(L(\mathbb{R}^d)) = T(\mathbb{R}^d)$, where $ID(\mathbb{R}^d)$ and $L(\mathbb{R}^d)$ are the classes of infinitely divisible distributions and of selfdecomposable distributions on $\mathbb{R}^d$, respectively. The relations with the mapping Φ from μ to the distribution at each time of the stationary process of Ornstein–Uhlenbeck type with background driving Lévy process $\{X_t^{(\mu)}\}$ are studied. Developments of these results in the context of the nested sequence $L_m(\mathbb{R}^d)$, $m = 0, 1, \ldots, \infty$, are presented. Other applications and examples are given.
Keywords: infinite divisibility; Lévy process; polar decomposition of Lévy measure; selfdecomposability; stationary process of Ornstein–Uhlenbeck type; stochastic integral.

1. Introduction

For distributions on the positive real line, Thorin (1977a,b) introduced the smallest class that contains all Γ-distributions and that is closed under convolution and convergence, where convergence of distributions means weak convergence. He called distributions of this class generalized Γ-convolutions. This was in connection to his proof of infinite divisibility of Pareto and lognormal distributions. In Bondesson's monograph (1992) the class is denoted by $\mathcal{T}$. Subsequently Thorin (1978) considered the smallest class on the real line $\mathbb{R}$ containing all generalized Γ-convolutions and closed under convolution, convergence, and reflection. We denote this class by $T(\mathbb{R})$. Based on the work of Steutel (1970), Bondesson (1981) studied the smallest class containing all mixtures of exponential distributions and closed under convolution and convergence. He called distributions of this class g.c.m.e.d. (generalized convolutions of mixtures of exponential distributions). It is similarly extended to a class on $\mathbb{R}$ and we denote the extension by $B(\mathbb{R})$. In Bondesson (1992) the class $T(\mathbb{R})$ and the class of g.c.m.e.d. are denoted by $\mathcal{T}_e$ and $\mathcal{T}_2$, respectively; the class $B(\mathbb{R})$ should not be confused with the class $\mathcal{B}$ there.

We study multi-dimensional analogues of the classes $T(\mathbb{R})$ and $B(\mathbb{R})$. We define them as subclasses of the class $ID(\mathbb{R}^d)$ of infinitely divisible distributions on $\mathbb{R}^d$ such that their Lévy measures have radial components having the same property as the part on $\mathbb{R}_+ = [0, \infty)$ of the Lévy measures of distributions in $T(\mathbb{R})$ and $B(\mathbb{R})$, respectively. The class $T(\mathbb{R}^d)$ is included in the class $L(\mathbb{R}^d)$ of selfdecomposable distributions on $\mathbb{R}^d$ but the class $B(\mathbb{R}^d)$ is not. Precise definitions will be given in Section 2. The class $T(\mathbb{R}^d)$ is duly called the Thorin class, as it is the analogue of $T(\mathbb{R}^d)$. Historically, Goldie (1967) proved the infinite divisibility of mixtures of exponential distributions and Steutel (1967) found the description of their Lévy measures. So it would be appropriate to call $B(\mathbb{R}^d)$ the Goldie-Steutel-Bondesson class. We give a probabilistic characterization of these classes on $\mathbb{R}^d$ by using a mapping Υ defined by a stochastic integral; $\Upsilon(\mu)$ is the distribution of $\int_0^1 \log(1/t) dX_t^{(\mu)}$, where $\{X_t^{(\mu)}\}$ is the Lévy process on $\mathbb{R}^d$ with distribution μ at $t = 1$. In one dimension this is the mapping introduced by Barndorff-Nielsen and Thorbjørnsen (2002a,b, 2004a,b) in relation to the Bercovici–Pata bijection between free infinite divisibility and classical infinite divisibility. We will prove that $B(\mathbb{R}^d)$ and $T(\mathbb{R}^d)$ are the images by Υ of $ID(\mathbb{R}^d)$ and $L(\mathbb{R}^d)$, respectively. We will further investigate the relation with the mapping Φ which is defined for $\mu \in ID_{\log}(\mathbb{R}^d)$, the subclass of $ID(\mathbb{R}^d)$ consisting of the ones with finite log-moment, and which gives the distribution $\Phi(\mu)$ of $\int_0^\infty e^{-t} dX_t^{(\mu)}$. Both $\Phi\Upsilon$ and $\Upsilon\Phi$ are defined on $ID_{\log}(\mathbb{R}^d)$; they coincide and give another stochastic integral representation of $T(\mathbb{R}^d)$. In analogy to the construction of the well-known nested sequence of subclasses $L_m(\mathbb{R}^d)$, $m = 0, 1, \ldots, \infty$, of $L(\mathbb{R}^d) = L_0(\mathbb{R}^d)$, we define a new nested sequence of subclasses $T_m(\mathbb{R}^d)$, $m = 0, 1, \ldots, \infty$, of $T(\mathbb{R}^d) = T_0(\mathbb{R}^d)$ by using the property of the innovation parts. Alternatively, the former sequence extended by adding $ID(\mathbb{R}^d)$ at the top and the latter sequence extended by adding $B(\mathbb{R}^d)$ at the top can be generated from the top members by iterating the mapping Φ each time after restriction to $ID_{\log}(\mathbb{R}^d)$. We will show that the latter extended sequence is the image by the mapping Υ of the former extended sequence. Further we will describe $T_m(\mathbb{R}^d)$ by specifying the Lévy measures. A characterization of $T(\mathbb{R}^d)$ and $B(\mathbb{R}^d)$

by using elementary Γ-variables and elementary mixed-exponential variables in $\mathbb{R}^d$, respectively, will also be given.

2. Main results

For any $\mathbb{R}^d$-valued random variable X we denote its distribution by $\mathcal{L}(X)$. The characteristic function and the cumulant function of a distribution μ on $\mathbb{R}^d$ are denoted by $\widehat{\mu}(z)$ and $C_\mu(z)$, respectively. That is, $C_\mu(z)$ is a continuous function with $C_\mu(0) = 0$ such that $\widehat{\mu}(z) = \exp(C_\mu(z))$, $z \in \mathbb{R}^d$; such a function $C_\mu(z)$ exists and is unique if $\widehat{\mu}(z) \neq 0$ for all $z \in \mathbb{R}^d$. If $\mu = \mathcal{L}(X)$, then $C_\mu(z)$ is also written as $C_X(z)$.

Any Lévy process $\{X_t^{(\mu)} \colon t \geqslant 0\}$ on $\mathbb{R}^d$ uniquely induces an $\mathbb{R}^d$-valued independently scattered random measure $\{M^{(\mu)}(B) \colon B \in \mathcal{B}^0_{[0,\infty)}\}$ such that $M^{(\mu)}([0,t]) = X_t^{(\mu)}$ a.s., where $\mathcal{B}^0_{[0,\infty)}$ is the class of bounded Borel sets in $[0,\infty)$. Let $f(t)$ be a real-valued function on $[0,\infty)$, $M^{(\mu)}$-integrable (also called $\{X_t^{(\mu)}\}$-integrable) in the sense of Urbanik and Woyczynski (1967) and Rajput and Rosinski (1989) for $d = 1$ and of Sato (2004) for general d. Then $M^{(f,\mu)}(B) = \int_B f(t) M^{(\mu)}(dt)$ (also written as $\int_B f(t) dX_t^{(\mu)}$) is defined for all $B \in \mathcal{B}^0_{[0,\infty)}$ and $\{M^{(f,\mu)}(B) \colon B \in \mathcal{B}^0_{[0,\infty)}\}$ is again an $\mathbb{R}^d$-valued independently scattered random measure; furthermore, we have $\int_B |C_\mu(f(t)z)| dt < \infty$ and

$$C_{M^{(f,\mu)}(B)}(z) = \int_B C_\mu(f(t)z) dt, \qquad z \in \mathbb{R}^d. \tag{2.1}$$

On $[0,\infty)$ the stochastic integral of f with respect to $X_t^{(\mu)}$ is defined as the limit in probability of the integral on $[0,s]$ as $s \to \infty$ and written as $\int_0^\infty f(t) dX_t^{(\mu)}$, whenever the limit exists. Let

$$\begin{aligned} ID_{\log}(\mathbb{R}^d) &= \left\{ \mu \in ID(\mathbb{R}^d) \colon \int_{|x|>2} \log|x| \, \mu(dx) < \infty \right\} \\ &= \left\{ \mu \in ID(\mathbb{R}^d) \colon \int_{|x|>2} \log|x| \, \nu^{(\mu)}(dx) < \infty \right\}, \end{aligned}$$

where $\nu^{(\mu)}$ is the Lévy measure of μ. It is known (Jurek and Vervaat (1983), Sato and Yamazato (1983), and Sato (1999)) that

$$\int_0^\infty e^{-t} dX_t^{(\mu)}$$

is definable if and only if $\mu \in ID_{\log}(\mathbb{R}^d)$, and that

$$L(\mathbb{R}^d) = \Phi(ID_{\log}(\mathbb{R}^d)), \tag{2.2}$$

where

$$\Phi\mu = \Phi(\mu) = \mathcal{L}\left(\int_0^\infty e^{-t} dX_t^{(\mu)}\right). \tag{2.3}$$

The domain of definition of the mapping Φ is $ID_{\log}(\mathbb{R}^d)$ and Φ is one-to-one. Another characterization of $\Phi\mu$ is given in relation to the Langevin equation

$$dY_t = dX_t^{(\mu)} - Y_t dt. \tag{2.4}$$

The equation (2.4) has a stationary solution $\{Y_t \colon t \geqslant 0\}$ if and only if $\mu \in ID_{\log}(\mathbb{R}^d)$. If $\mu \in ID_{\log}(\mathbb{R}^d)$, then a stationary solution $\{Y_t\}$ is unique, and $\mathcal{L}(Y_t) = \Phi\mu$ for all $t \geqslant 0$. The process $\{Y_t\}$ is called a stationary process of Ornstein–Uhlenbeck type.

For any Borel set E in $\mathbb{R}^d$, the class of Borel subsets of E is denoted by $\mathcal{B}(E)$. A function defined on E is called measurable if it is $\mathcal{B}(E)$-measurable. The unit sphere in $\mathbb{R}^d$ is denoted by $S = \{\xi \in \mathbb{R}^d \colon |\xi| = 1\}$.

We use the Lévy–Khintchine triplet (A, ν, γ) of $\mu \in ID(\mathbb{R}^d)$ in the sense that

$$C_\mu(z) = -\frac{1}{2}\langle z, Az\rangle + \int_{\mathbb{R}^d} \left(e^{i\langle z,x\rangle} - 1 - \frac{i\langle z, x\rangle}{1+|x|^2}\right) \nu(dx) + i\langle \gamma, z\rangle, \tag{2.5}$$

where A is a $d \times d$ symmetric nonnegative-definite matrix, ν is a measure on $\mathbb{R}^d$ called the Lévy measure of μ, and $\gamma \in \mathbb{R}^d$. A measure ν is the Lévy measure of some $\mu \in ID(\mathbb{R}^d)$ if and only if $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d} (|x|^2 \wedge 1)\nu(dx) < \infty$. We sometimes denote an infinitely divisible distribution μ with triplet (A, ν, γ) by $\mu_{(A,\nu,\gamma)}$.

We use the following polar decomposition of Lévy measures.

Lemma 2.1. *Let ν be the Lévy measure of some $\mu \in ID(\mathbb{R}^d)$ with $0 < \nu(\mathbb{R}^d) \leqslant \infty$. Then there exist a measure λ on S with $0 < \lambda(S) \leqslant \infty$ and a family $\{\nu_\xi \colon \xi \in S\}$ of measures on $(0, \infty)$ such that*

$$\nu_\xi(B) \text{ is measurable in } \xi \text{ for each } B \in \mathcal{B}((0, \infty)), \tag{2.6}$$

$$0 < \nu_\xi((0, \infty)) \leqslant \infty \quad \text{for each } \xi \in S, \tag{2.7}$$

$$\nu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi)\nu_\xi(dr) \quad \text{for } B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}). \tag{2.8}$$

Here λ and $\{\nu_\xi\}$ are uniquely determined by ν in the following sense: if λ, $\{\nu_\xi\}$ and λ', $\{\nu'_\xi\}$ both have properties (2.6)–(2.8)*, then there is a measurable function $c(\xi)$ on S such that*

$$0 < c(\xi) < \infty, \tag{2.9}$$

$$\lambda'(d\xi) = c(\xi)\lambda(d\xi), \tag{2.10}$$

$$c(\xi)\nu'_\xi(dr) = \nu_\xi(dr) \quad \text{for } \lambda\text{-a.e. } \xi \in S. \tag{2.11}$$

Rosinski (1990) has the same result, but without the uniqueness. Sometimes we call λ and ν_ξ in Lemma 2.1 *the spherical component* and *the radial component* of ν respectively, as they are uniquely determined in the sense written above. The following description of the Lévy measures of $L(\mathbb{R}^d)$ is well-known (see Sato (1999), Theorem 15.10).

Proposition 2.2. *Let $\mu \in ID(\mathbb{R}^d)$ and let ν be the Lévy measure of μ. Then $\mu \in L(\mathbb{R}^d)$ if and only if either $\nu = 0$ or $\nu \neq 0$ with a polar decomposition (λ, ν_ξ) such that there is a nonnegative function $k_\xi(r)$ measurable in ξ and decreasing, right-continuous in r, satisfying*

$$\nu_\xi(dr) = k_\xi(r) r^{-1} dr \quad \textit{for } \lambda\textit{-a. e. } \xi \in S. \tag{2.12}$$

We call $k_\xi(r)$ *the k-function* of $\mu \in L(\mathbb{R}^d)$ or of its Lévy measure ν, as it is determined by μ λ-a. e. up to multiplication of functions of ξ. The function

$$h_\xi(u) = \lim_{v \downarrow u} k_\xi(e^{-v})$$

is called *the h-function* of $\mu \in L(\mathbb{R}^d)$ or of its Lévy measure ν.

Let us define $T(\mathbb{R}^d)$ and $B(\mathbb{R}^d)$.

Definition 2.3. The class $T(\mathbb{R}^d)$ is the collection of $\mu \in L(\mathbb{R}^d)$ with Lévy measure ν such that either $\nu = 0$ or $\nu \neq 0$ having k-function $k_\xi(r)$ completely monotone in r for λ-a.e. ξ, where λ is the spherical component of ν.

Definition 2.4. The class $B(\mathbb{R}^d)$ is the collection of $\mu \in ID(\mathbb{R}^d)$ with Lévy measure ν such that either $\nu = 0$ or $\nu \neq 0$ having polar decomposition (λ, ν_ξ) such that

$$\nu_\xi(dr) = l_\xi(r) dr \quad \text{for } \lambda\text{-a. e. } \xi \in S, \tag{2.13}$$

where $l_\xi(r)$ is measurable in ξ and completely monotone in r for λ-a.e. ξ.

We call $l_\xi(r)$ *the l-function* of $\mu \in B(\mathbb{R}^d)$ or of its Lévy measure ν. We can prove that

$$B(\mathbb{R}^d) \cap L(\mathbb{R}^d) \supsetneqq T(\mathbb{R}^d). \tag{2.14}$$

Except the strictness, this is clear; the strictness will be proved in Section 3.

We introduce a mapping Υ.

Proposition 2.5. *If $f(t)$ is given by*

$$f(t) = \begin{cases} 0 & \textit{for } t = 0 \\ \log(1/t) & \textit{for } 0 < t \leqslant 1 \\ 0 & \textit{for } t > 1, \end{cases} \tag{2.15}$$

then $f(t)$ is $\{X_t^{(\mu)}\}$-integrable for every $\mu \in ID(\mathbb{R}^d)$.

We write $\int_0^1 \log \frac{1}{t} dX_t^{(\mu)} = \int_{[0,1]} f(t) dX_t^{(\mu)}$ for f of (2.15).

Definition 2.6. For any $\mu \in ID(\mathbb{R}^d)$, define

$$\Upsilon\mu = \Upsilon(\mu) = \mathcal{L}\left(\int_0^1 \log \frac{1}{t} dX_t^{(\mu)}\right). \tag{2.16}$$

Now we state two of our main results.

Theorem A. (i) *The total image of the mapping Υ equals $B(\mathbb{R}^d)$. That is,*

$$B(\mathbb{R}^d) = \Upsilon(ID(\mathbb{R}^d)). \tag{2.17}$$

(ii) *Let $\mu \in ID(\mathbb{R}^d)$ and $\widetilde{\mu} = \Upsilon\mu$ and let ν and $\widetilde{\nu}$ be the Lévy measures of μ and $\widetilde{\mu}$, respectively. Then*

$$\widetilde{\nu}(B) = \int_0^\infty e^{-s}\nu(s^{-1}B)ds \quad \text{for } B \in \mathcal{B}(\mathbb{R}^d). \tag{2.18}$$

If $\nu \neq 0$ and ν has polar decomposition (λ, ν_ξ), then a polar decomposition of $\widetilde{\nu}$ is given by $\widetilde{\lambda} = \lambda$ and $\widetilde{\nu}_\xi(dr) = \widetilde{l}_\xi(r)dr$ with

$$\widetilde{l}_\xi(r) = \int_0^\infty s^{-1}e^{-r/s}\nu_\xi(ds). \tag{2.19}$$

Theorem B. (i) *The image of the class $L(\mathbb{R}^d)$ by the mapping Υ equals $T(\mathbb{R}^d)$. That is,*

$$T(\mathbb{R}^d) = \Upsilon(L(\mathbb{R}^d)). \tag{2.20}$$

(ii) *Let $\mu \in L(\mathbb{R}^d)$ and $\widetilde{\mu} = \Upsilon\mu$ with Lévy measures ν and $\widetilde{\nu}$, respectively. If $\nu \neq 0$ and ν has spherical component λ and k-function $k_\xi(r)$, then $\widetilde{\nu}$ has spherical component $\widetilde{\lambda} = \lambda$ and k-function*

$$\widetilde{k}_\xi(r) = \int_0^\infty k_\xi(rs^{-1})e^{-s}ds = \int_{(0,\infty)} e^{-ru}dk_\xi^\sharp(u). \tag{2.21}$$

Here $k_\xi^\sharp(u)$ is the right-continuous modification of $k_\xi(u^{-1})$.

In the one-dimensional case ($d = 1$), (2.20) was discovered by Barndorff-Nielsen and Thorbjørnsen who also, in effect, noted that $\Upsilon(ID(\mathbb{R})) \subset B(\mathbb{R}^d)$, but without being aware of the connection to the class $B(\mathbb{R})$; see Barndorff-Nielsen and Thorbjørnsen (2004a,b).

In proving Theorems A and B, we will show the following properties of the mapping Υ.

Proposition 2.7. (i) *The mapping Υ is one-to-one from $ID(\mathbb{R}^d)$ into $ID(\mathbb{R}^d)$.*

(ii) *For any $\mu \in ID(\mathbb{R}^d)$,*

$$C_{\Upsilon\mu}(z) = \int_0^1 C_\mu\left(z \log \frac{1}{t}\right) dt, \qquad z \in \mathbb{R}^d, \tag{2.22}$$

with

$$\int_0^1 \left| C_\mu\left(z \log \frac{1}{t}\right)\right| dt < \infty. \tag{2.23}$$

(iii) *$\Upsilon(\mu_1 * \mu_2) = \Upsilon\mu_1 * \Upsilon\mu_2$ for $\mu_1, \mu_2 \in ID(\mathbb{R}^d)$.*

(iv) *Let $\mu_n \in ID(\mathbb{R}^d)$ $(n = 1, 2, \ldots)$. If $\mu_n \to \mu$, then $\mu \in ID(\mathbb{R}^d)$ and $\Upsilon\mu_n \to \Upsilon\mu$. Conversely, if $\Upsilon\mu_n \to \widetilde{\mu}$ for some distribution $\widetilde{\mu}$, then $\widetilde{\mu} = \Upsilon\mu$ for some $\mu \in ID(\mathbb{R}^d)$ and $\mu_n \to \mu$.*

(v) *For $\mu \in ID(\mathbb{R}^d)$ with triplet (A, ν, γ), $\Upsilon\mu$ has triplet $(\widetilde{A}, \widetilde{\nu}, \widetilde{\gamma})$ with expression*

$$\widetilde{A} = 2A, \tag{2.24}$$

$$\widetilde{\nu}(B) = \int_0^\infty \nu(s^{-1}B)e^{-s}ds \quad \text{for } B \in \mathcal{B}(\mathbb{R}^d), \tag{2.25}$$

$$\begin{aligned}\widetilde{\gamma} &= \gamma + \int_0^\infty e^{-s}sds \int_{\mathbb{R}^d} x\left(\frac{1}{1+s^2|x|^2} - \frac{1}{1+|x|^2}\right)\nu(dx) \\ &= \gamma + \int_{\mathbb{R}^d} \frac{x|x|^2}{1+|x|^2}\nu(dx) \int_0^\infty \frac{e^{-s}s(1-s^2)}{1+s^2|x|^2} ds.\end{aligned} \tag{2.26}$$

(vi) *The mapping Υ has the following alternative expressions:*

$$\Upsilon\mu = \mathcal{L}\left(\int_0^1 \log \frac{1}{1-t} dX_t^{(\mu)}\right), \tag{2.27}$$

$$\Upsilon\mu = \mathcal{L}\left(\lim_{s\downarrow 0} \int_s^1 \frac{X_t^{(\mu)}}{t} dt\right). \tag{2.28}$$

For another expression of $T(\mathbb{R}^d)$, we use the function $e_1(u) = \int_u^\infty e^{-s}s^{-1}ds$ and the function $e_1^*(t)$ inverse to $e_1(u)$, that is, $t = e_1(u)$ if and only if $u = e_1^*(t)$.

Theorem C. (i) *let $\mu \in ID(\mathbb{R}^d)$. Then $\Upsilon\mu \in ID_{\log}(\mathbb{R}^d)$ if and only if $\mu \in ID_{\log}(\mathbb{R}^d)$.*

(ii) *The integral $\int_0^\infty e_1^*(t)dX_t^{(\mu)}$ exists if and only if $\mu \in ID_{\log}(\mathbb{R}^d)$. If $\mu \in ID_{\log}(\mathbb{R}^d)$, then*

$$\Phi\Upsilon(\mu) = \Upsilon\Phi(\mu) = \mathcal{L}\left(\int_0^\infty e_1^*(t)dX_t^{(\mu)}\right). \tag{2.29}$$

(iii) *We have*

$$T(\mathbb{R}^d) = \Phi(B(\mathbb{R}^d) \cap ID_{\log}(\mathbb{R}^d)) \tag{2.30}$$

and

$$T(\mathbb{R}^d) = \left\{ \mathcal{L}\left(\int_0^\infty e_1^*(t) dX_t^{(\mu)} \right) : \mu \in ID_{\log}(\mathbb{R}^d) \right\}. \tag{2.31}$$

Let us recall the definition of selfdecomposability. A distribution μ on $\mathbb{R}^d$ is said to be selfdecomposable, or $\mu \in L(\mathbb{R}^d)$, if for each $b > 1$ there is a distribution $\rho_b^{(\mu)}$ such that

$$\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)\widehat{\rho_b^{(\mu)}}(z). \tag{2.32}$$

Note that $\rho_b^{(\mu)}$ is uniquely determined by μ and b and that $\rho_b^{(\mu)} \in ID(\mathbb{R}^d)$. In connection to the relation of $L(\mathbb{R}^d)$ and stationary processes of Ornstein–Uhlenbeck type, the distribution $\rho_b^{(\mu)}$ is sometimes called the innovation part of μ. We define $L_0(\mathbb{R}^d) = L(\mathbb{R}^d)$ and then, for $m = 1, 2, \ldots$, define

$$L_m(\mathbb{R}^d) = \{\mu \in L(\mathbb{R}^d) \colon \rho_b^{(\mu)} \in L_{m-1}(\mathbb{R}^d) \text{ for all } b > 1\}. \tag{2.33}$$

Let $L_\infty(\mathbb{R}^d) = \bigcap_{0 \leqslant m < \infty} L_m(\mathbb{R}^d)$ and let $\mathfrak{S}(\mathbb{R}^d)$ be the class of stable distributions on $\mathbb{R}^d$. Thus we get the nested sequence studied by Urbanik (1972), Sato (1980), and others:

$$ID(\mathbb{R}^d) \supset L_0(\mathbb{R}^d) \supset L_1(\mathbb{R}^d) \supset L_2(\mathbb{R}^d) \supset \cdots \supset L_\infty(\mathbb{R}^d) \supset \mathfrak{S}(\mathbb{R}^d). \tag{2.34}$$

The class $L_\infty(\mathbb{R}^d)$ is the smallest class containing $\mathfrak{S}(\mathbb{R}^d)$ and being closed under convolution and convergence.

Corollary to Theorem C. *We have*

$$T(\mathbb{R}^d) = \{\mu \in L(\mathbb{R}^d) \colon \rho_b^{(\mu)} \in B(\mathbb{R}^d) \text{ for all } b > 1\}. \tag{2.35}$$

Now we define the classes $T_m(\mathbb{R}^d)$, letting $T_0(\mathbb{R}^d) = T(\mathbb{R}^d)$ and, for $m = 1, 2, \ldots$,

$$T_m(\mathbb{R}^d) = \{\mu \in L(\mathbb{R}^d) \colon \rho_b^{(\mu)} \in T_{m-1}(\mathbb{R}^d) \text{ for every } b > 1\}. \tag{2.36}$$

Let $T_\infty(\mathbb{R}^d) = \bigcap_{0 \leqslant m < \infty} T_m(\mathbb{R}^d)$. In this way we get a decreasing sequence

$$B(\mathbb{R}^d) \supset T_0(\mathbb{R}^d) \supset T_1(\mathbb{R}^d) \supset T_2(\mathbb{R}^d) \supset \cdots \supset T_\infty(\mathbb{R}^d) \supset \mathfrak{S}(\mathbb{R}^d). \tag{2.37}$$

The last inclusion is clear because, for any Gaussian distribution μ, $\rho_b^{(\mu)}$ is Gaussian, and because, for any α-stable distribution μ with $0 < \alpha < 2$, $\mu \in L(\mathbb{R}^d)$ with k-function $r^{-\alpha}$ and thus $\mu \in T(\mathbb{R}^d)$ and $\rho_b^{(\mu)}$ is again α-stable.

Theorem D. *The sequence* (2.34) *is transformed to the sequence* (2.37) *by the mapping* Υ*, that is,* (2.17) *and*

$$T_m(\mathbb{R}^d) = \Upsilon(L_m(\mathbb{R}^d)) \qquad \text{for } m = 0, 1, \ldots, \infty, \tag{2.38}$$

$$\mathfrak{S}(\mathbb{R}^d) = \Upsilon(\mathfrak{S}(\mathbb{R}^d)). \tag{2.39}$$

Moreover we have

$$T_m(\mathbb{R}^d) \subsetneqq L_m(\mathbb{R}^d) \qquad \text{for } m = 0, 1, \ldots, \tag{2.40}$$

$$T_\infty(\mathbb{R}^d) = L_\infty(\mathbb{R}^d), \tag{2.41}$$

$$T_{m+1}(\mathbb{R}^d) = \Phi(T_m(\mathbb{R}^d) \cap ID_{\log}(\mathbb{R}^d)) \qquad \text{for } m = 0, 1, \ldots, \infty, \tag{2.42}$$

where we understand $m + 1 = \infty$ *for* $m = \infty$*.*

The relation (2.39) was shown in Barndorff-Nielsen and Thorbjørnsen (2002b) for $d = 1$.

It is known that

$$L_{m+1}(\mathbb{R}^d) = \Phi(L_m(\mathbb{R}^d) \cap ID_{\log}(\mathbb{R}^d)) \qquad \text{for } m = 0, 1, \ldots, \infty. \tag{2.43}$$

The assertion (2.42) is analogous to this. Thus $L_m(\mathbb{R}^d)$ and $T_m(\mathbb{R}^d)$ are the images of $ID(\mathbb{R}^d)$ and $B(\mathbb{R}^d)$, respectively, by Φ^{m+1}, the $(m+1)$ st iteration of Φ. A description of the domain of definition of Φ^{m+1} and a stochastic integral representation of Φ^{m+1} are known. See Jurek (1983), Sato and Yamazato (1983), and also Rocha-Arteaga and Sato (2003) Theorems 46 and 49 and Remark 58[1].

The Lévy measures of $T_m(\mathbb{R}^d)$ may be characterized as follows.

Theorem E. *Let* $m \in \{0, 1, \ldots\}$*. Let* $\mu \in ID(\mathbb{R}^d)$*. Then* $\mu \in T_m(\mathbb{R}^d)$ *if and only if* $\mu \in L(\mathbb{R}^d)$ *and the Lévy measure* ν *of* μ *is either* $\nu = 0$ *or* $\nu \neq 0$ *having infinitely differentiable h-function* $h_\xi(u)$ *such that*

$$\left.\begin{array}{l} h_\xi^{(j)}(u) \geqslant 0 \text{ for } u \in \mathbb{R},\ 0 \leqslant j < m, \text{ and } h_\xi^{(m)}(-\log r) \\ \text{is completely monotone in } r > 0,\ \lambda\text{-a. e. } \xi \end{array}\right\} \tag{2.44}$$

where $h_\xi^{(j)}$ *is the* j*th derivative of* h_ξ *and* λ *is the spherical component of* ν*.*

A characterization of $B(\mathbb{R}^d)$ and $T(\mathbb{R}^d)$ using mixed-exponential distributions and Γ-distributions is as follows.

Definition 2.8. Call Ux an *elementary mixed-exponential variable in* $\mathbb{R}^d$ (resp. *elementary* Γ*-variable in* $\mathbb{R}^d$) if x is a nonrandom nonzero vector in $\mathbb{R}^d$ and U is a real

[1] In Line 4 of this remark, μ_m should be replaced by μ.

random variable whose distribution is a mixture of a finite number of exponential distributions (resp. a real Γ-distributed random variable).

Theorem F. *The class $B(\mathbb{R}^d)$ (resp. $T(\mathbb{R}^d)$) is the smallest class of distributions on $\mathbb{R}^d$ closed under convolution and convergence and containing the distributions of all elementary mixed-exponential variables in $\mathbb{R}^d$ (resp. of all elementary Γ-variables in $\mathbb{R}^d$).*

Many examples of distributions in $T(\mathbb{R})$ supported on $\mathbb{R}_+$ are given in Bondesson (1992) and Steutel and van Harn (2004). As shown by Bondesson (1992), Theorem 7.3.1, all normal variance mixtures where the law of the variance is a generalized Γ-convolution belong to $T(\mathbb{R})$. (Any such mixture equals the law at time 1 of a subordination of Brownian motion by a generalized Γ-convolution subordinator.) We also note that if a distribution μ on $\mathbb{R}^d$ is the direct product of distributions in $T(\mathbb{R})$ (resp. $B(\mathbb{R})$), then $\mu \in T(\mathbb{R}^d)$ (resp. $B(\mathbb{R}^d)$).

We will prove the results above in the sections that follow. In the final section we will discuss several examples.

3. Proof of Theorems A and B

We prove Theorems A and B on the relationship of the classes $B(\mathbb{R}^d)$ and $T(\mathbb{R}^d)$ with the mapping Υ. We also show the relation (2.14) of $B(\mathbb{R}^d)$, $T(\mathbb{R}^d)$, and $L(\mathbb{R}^d)$ and Proposition 2.7 on properties of Υ.

Proof of Lemma 2.1. Let $c = \int_{\mathbb{R}^d}(|x|^2 \wedge 1)\nu(dx)$ and let N be a random variable on $\mathbb{R}^d$ with distribution $c^{-1}(|x|^2 \wedge 1)\nu(dx)$. Let $R = |N|$ and $\Xi = N/|N|$. Define $\lambda^0 = \mathcal{L}(\Xi)$ and $\nu_\xi^0(B) = c\int_B (r^2 \wedge 1)^{-1} P(R \in dr \,|\, \Xi = \xi)$, using the conditional distribution. Then λ^0 and $\{\nu_\xi^0\}$ satisfy (2.6)–(2.8) with the additional properties that $\lambda^0(S) = 1$ and $\int_0^\infty (r^2 \wedge 1)\nu_\xi^0(dr) = 1$ for all $\xi \in S$.

The proof of the uniqueness is as follows. Let λ, $\{\nu_\xi\}$ and λ', $\{\nu'_\xi\}$ both satisfy (2.6)–(2.8). Define $a(\xi) = \int_0^\infty (r^2 \wedge 1)\nu_\xi(dr)$ and $a'(\xi) = \int_0^\infty (r^2 \wedge 1)\nu'_\xi(dr)$. By (2.7), $a(\xi)$ and $a'(\xi)$ are positive for all ξ. We have $a(\xi) < \infty$ for λ-a. e. ξ and $a'(\xi) < \infty$ for λ'-a. e. ξ, since $\int_S a(\xi)\lambda(d\xi) = \int_S a'(\xi)\lambda'(d\xi) = c < \infty$. For any $B \in \mathcal{B}(S)$,

$$c\lambda^0(B) = \int_{\{x\colon |x|^{-1}x \in B\}} (|x|^2 \wedge 1)\nu(dx) = \int_B a(\xi)\lambda(d\xi) = \int_B a'(\xi)\lambda'(d\xi).$$

Hence λ^0, λ, and λ' are mutually absolutely continuous. By the uniqueness of the conditional distribution $P(R \in dr \mid \Xi = \xi)$, we get

$$ca(\xi)^{-1}\nu_\xi(dr) = \nu_\xi^0(dr) \text{ and } ca'(\xi)^{-1}\nu'_\xi(dr) = \nu_\xi^0(dr) \text{ for } \lambda^0\text{-a. e. } \xi.$$

Letting $c(\xi) = a(\xi)/a'(\xi)$ with appropriate modification on a set of λ^0-measure 0, we get (2.9)–(2.11). $\square$

Remark 3.1. By the uniqueness of a polar decomposition of ν in the sense of Lemma 2.1, the properties of μ in Definitions 2.3 and 2.4 of $T(\mathbb{R}^d)$ and $B(\mathbb{R}^d)$ do not depend on the choice of polar decompositions.

Remark 3.2. By an extension of Bernstein's theorem to the case with a parameter, for each $\mu \in B(\mathbb{R}^d)$ there uniquely exists a family $\{Q_\xi \colon \xi \in S\}$ of measures on $(0,\infty)$ such that

$$Q_\xi(B) \text{ is measurable in } \xi \text{ for each } B \in \mathcal{B}((0,\infty)), \tag{3.1}$$

$$l_\xi(r) = \int_{(0,\infty)} e^{-ru} Q_\xi(du); \tag{3.2}$$

see the proof of Lemma 3.3 of Sato (1980) for the details. Here we have used $l_\xi(\infty) = 0$. Since $\int_{\mathbb{R}^d}(|x|^2 \wedge 1)\nu(dx) < \infty$,

$$\int_S \lambda(d\xi) \int_{(0,\infty)} a(u) Q_\xi(du) < \infty, \tag{3.3}$$

where

$$a(u) = u^{-3}\int_0^u r^2 e^{-r} dr + u^{-1}e^{-u}. \tag{3.4}$$

Indeed we have $\int_0^\infty (r^2 \wedge 1) l_\xi(r) dr = \int_{(0,\infty)} a(u) Q_\xi(du)$. Note that

$$a(u) \sim u^{-1} \text{ as } u \downarrow 0 \text{ and } a(u) \sim 2u^{-3} \text{ as } u \uparrow \infty.$$

Thus (3.3) is equivalent to

$$\int_S \lambda(d\xi) \int_{(0,\infty)} (u^{-1} \wedge u^{-3}) Q_\xi(du) < \infty. \tag{3.5}$$

Similarly, for each $\mu \in T(\mathbb{R}^d)$ there uniquely exists a family $\{R_\xi \colon \xi \in S\}$ of measures on $(0,\infty)$ such that

$$R_\xi(B) \text{ is measurable in } \xi \text{ for each } B \in \mathcal{B}((0,\infty)), \tag{3.6}$$

$$k_\xi(r) = \int_{(0,\infty)} e^{-ru} R_\xi(du). \tag{3.7}$$

This time we have

$$\int_S \lambda(d\xi) \int_{(0,\infty)} b(u) R_\xi(du) < \infty, \tag{3.8}$$

with

$$b(u) = u^{-2} \int_0^u r e^{-r} dr + \int_u^\infty r^{-1} e^{-r} dr, \tag{3.9}$$

since $\int_0^\infty (r \wedge r^{-1}) k_\xi(r) dr = \int_{(0,\infty)} b(u) R_\xi(du)$. We have

$$b(u) \sim \log(1/u) \text{ as } u \downarrow 0 \text{ and } b(u) \sim u^{-2} \text{ as } u \uparrow \infty$$

and hence (3.8) is equivalent to

$$\int_S \lambda(d\xi) \int (1_{(0,1/2]}(u) \log(1/u) + 1_{(1/2,\infty)}(u) u^{-2}) R_\xi(du) < \infty. \tag{3.10}$$

Proof of (2.14). The inclusion $T(\mathbb{R}^d) \subset L(\mathbb{R}^d)$ is evident from Proposition 2.2 and Definition 2.3. If $k_\xi(r)$ is completely monotone, then so is $k_\xi(r) r^{-1}$, since the product of completely monotone functions is completely monotone. Hence $T(\mathbb{R}^d) \subset B(\mathbb{R}^d)$.

For $d = 1$, let us construct $\mu \in B(\mathbb{R}) \cap L(\mathbb{R})$ such that $\mu \notin T(\mathbb{R})$. Let

$$k(r) = e^{-a_1 r} - e^{-b_1 r} + e^{-a_2 r}, \qquad r > 0$$

with $0 < a_1 < b_1 < a_2$ and let $l(r) = k(r) r^{-1}$. Then $k(r)$ is not completely monotone, since $k(r) = \int_{(0,\infty)} e^{-ru} Q(du)$ with a signed measure Q such that $Q(\{b_1\}) < 0$. But $l(r)$ is completely monotone, since

$$l(r) = \frac{e^{-a_1 r} - e^{-b_1 r}}{r} + \frac{e^{-a_2 r}}{r} = \int_{a_1}^{b_1} e^{-ru} du + \int_{a_2}^\infty e^{-ru} du.$$

Hence the distribution μ given by $\widehat{\mu}(z) = \exp \int_0^\infty (e^{izr} - 1) l(r) dr$ is in $B(\mathbb{R}) \setminus T(\mathbb{R})$ (μ is in fact a mixture of exponential distributions with parameters a_1 and a_2 by Steutel's theorem; see Sato (1999), Lemma 51.14, or Steutel and van Harn (2004), Chapter VI, Proposition 3.4). We claim that for some choice of a_1, b_1, and a_2, $k(r)$ is decreasing so that $\mu \in L(\mathbb{R})$. Indeed, let $a_1 = 1 - \varepsilon$, $b_1 = 1$, and $a_2 = 1 + \varepsilon$ with $0 < \varepsilon < 1$. Then $k'(r) = e^{-r}(1 - f(r))$ with $f(r) = (1 - \varepsilon) e^{\varepsilon r} + (1 + \varepsilon) e^{-\varepsilon r}$. We have $f(r_0) = \min_{r>0} f(r)$ when $e^{2\varepsilon r_0} = (1 + \varepsilon)/(1 - \varepsilon)$. Hence $f(r_0) = 2(1 - \varepsilon^2)^{1/2} \to 2$ as $\varepsilon \downarrow 0$. It follows that $k'(r) < 0$ for all $r > 0$ if ε is small enough. A d-dimensional example is given by taking this $k(r)$ for the radial component of a Lévy measure. □

Proof of Proposition 2.5. Let $\mu = \mu_{(A,\nu,\gamma)}$. We use a general result (an analogue of Theorem 2.7 of Rajput and Rosinski (1989)) for integrability of functions with respect to an $\mathbb{R}^d$-valued independently scattered random measure. In order to show that a function $f(t)$ is $\{X_t^{(\mu)}\}$-integrable, it suffices to show that, for any $0 < t_0 < \infty$,

$$\int_0^{t_0} \langle z, Az\rangle f(t)^2 dt < \infty,$$

$$\int_0^{t_0} dt \int_{\mathbb{R}^d} (|xf(t)|^2 \wedge 1)\nu(dx) < \infty,$$

$$\int_0^{t_0} \left| \langle \gamma, zf(t)\rangle + \int_{\mathbb{R}^d} (g(zf(t), x) - g(z, xf(t)))\nu(dx) \right| dt < \infty,$$

where

$$g(z, x) = e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle/(1 + |x|^2). \tag{3.11}$$

(The first condition is equivalent to $\int_0^{t_0} f(t)^2 dt < \infty$ if $A \neq 0$.) Hence, in the case of $f(t)$ of (2.15), it suffices to show

$$\int_0^\infty \langle z, Az\rangle s^2 e^{-s} ds < \infty, \tag{3.12}$$

$$\int_0^\infty e^{-s} ds \int_{\mathbb{R}^d} (|sx|^2 \wedge 1)\nu(dx) < \infty, \tag{3.13}$$

$$\int_0^\infty e^{-s} \left| \langle \gamma, sz\rangle + \int_{\mathbb{R}^d} (g(sz, x) - g(z, sx))\nu(dx) \right| ds < \infty. \tag{3.14}$$

Among these (3.12) is evident; (3.13) follows from

$$\begin{aligned} &\int_0^\infty e^{-s} ds \int_{\mathbb{R}^d} (|sx|^2 \wedge 1)\nu(dx) \\ &\qquad = \int_{\mathbb{R}^d} |x|^2 \nu(dx) \int_0^{1/|x|} s^2 e^{-s} ds + \int_{\mathbb{R}^d} \nu(dx) \int_{1/|x|}^\infty e^{-s} ds\,; \end{aligned}$$

(3.14) follows from

$$\int_0^\infty |\langle \gamma, z\rangle| s e^{-s} ds < \infty \tag{3.15}$$

and

$$\int_0^\infty e^{-s} \left| \int_{\mathbb{R}^d} (g(sz, x) - g(z, sx))\nu(dx) \right| ds < \infty; \tag{3.16}$$

(3.15) is evident and (3.16) follows from

$$\begin{aligned} &\int_0^\infty s e^{-s} ds \int_{\mathbb{R}^d} \left| \langle z, x\rangle \left(\frac{1}{1 + s^2|x|^2} - \frac{1}{1 + |x|^2} \right) \right| \nu(dx) \\ &\qquad \leqslant |z| \int_{\mathbb{R}^d} \frac{|x|^3}{1 + |x|^2} \nu(dx) \int_0^\infty \frac{e^{-s} s|1 - s^2|}{1 + s^2|x|^2} ds \\ &\qquad \leqslant |z| I_1 \int_{|x|\leqslant 1} |x|^3 \nu(dx) + |z| \int_{|x|>1} |x| I_2(x)\, \nu(dx), \end{aligned}$$

where

$$I_1 = \int_0^\infty e^{-s} s(1+s^2)ds,$$

$$I_2(x) = \int_0^1 \frac{sds}{1+s^2|x|^2} + \int_1^\infty \frac{e^{-s}s^3ds}{1+|x|^2} = \frac{\log(1+|x|^2)}{2|x|^2} + \frac{\int_1^\infty e^{-s}s^3ds}{1+|x|^2}.$$

No restriction on μ is needed. $\square$

Proof of Proposition 2.7. We begin with a proof of (ii). The assertion (iv) will be proved after (v).

(ii) The assertion $\Upsilon\mu \in ID(\mathbb{R}^d)$ and (2.22)–(2.23) are consequences of general results for $\{X_t^{(\mu)}\}$-integrable functions in Proposition 4.3 of Sato (2004). A direct check of (2.23) is also possible because we have

$$|\mathrm{Re}\ C_\mu(z)| + |\mathrm{Im}\ C_\mu(z)| \leqslant c_0 + c_2|z|^2$$

from (2.5) with positive constants c_0, c_2 depending on μ.

(i) It follows from (2.22)–(2.23) that

$$C_{\Upsilon\mu}(z) = \int_0^\infty C_\mu(sz)e^{-s}ds \tag{3.17}$$

and hence, for $u > 0$,

$$C_{\Upsilon\mu}(u^{-1}z) = u\int_0^\infty C_\mu(vz)e^{-uv}dv.$$

That is, for each $z \in \mathbb{R}^d$, $u^{-1}C_{\Upsilon\mu}(u^{-1}z)$, $u > 0$, is the Laplace transform of $C_\mu(vz)$, $v > 0$. Therefore $C_\mu(vz)$ is determined by $C_{\Upsilon\mu}$ for almost every $v > 0$. Since $C_\mu(vz)$ is continuous in v, it is determined for all $v > 0$. Now let $v = 1$ to get our assertion.

(iii) Obvious from $\{X_t^{(\mu_1 * \mu_2)}\} \stackrel{\mathrm{d}}{=} \{X_t^{(\mu_1)} + X_t^{(\mu_2)}\}$, where $\{X_t^{(\mu_1)}\}$ and $\{X_t^{(\mu_2)}\}$ are independent.

(v) By a general result (see Lemma 2.7 and Corollary 4.4 of Sato (2004)),

$$\begin{aligned}
\widetilde{A} &= \int_0^1 (\log(1/t))^2 dt A, \\
\widetilde{\nu}(B) &= \int_0^1 dt \int_{\mathbb{R}^d} 1_B(x\log(1/t))\nu(dx), \qquad B \in \mathcal{B}(\mathbb{R}^d), \\
\widetilde{\gamma} &= \int_0^1 \left(\gamma \log\frac{1}{t} - \int_{\mathbb{R}^d} x\left(\log\frac{1}{t}\right)\left(\frac{1}{1+|x|^2} - \frac{1}{1+|(\log(1/t))x|^2}\right)\nu(dx)\right) dt.
\end{aligned}$$

It follows that

$$\widetilde{A} = \int_0^\infty s^2 e^{-s} ds A,$$

$$\widetilde{\nu}(B) = \int_0^\infty ds \int_{\mathbb{R}^d} 1_B(xs)\nu(dx),$$

$$\widetilde{\gamma} = \int_0^\infty e^{-s}\left(\gamma s - \int_{\mathbb{R}^d} xs\left(\frac{1}{1+|x|^2} - \frac{1}{1+|sx|^2}\right)\nu(dx)\right) ds.$$

That is, (2.24)–(2.26) hold.

(iv) Assume that $\mu_n = \mu_{(A_n,\nu_n,\gamma_n)} \to \mu = \mu_{(A,\mu,\nu)}$ as $n \to \infty$. Then $C_{\mu_n}(z) \to C_\mu(z)$, and $\operatorname{tr} A_n$, $\int(|x|^2 \wedge 1)\nu_n(dx)$, and $|\gamma_n|$ are bounded. Since $\Upsilon\mu_n$ and $\Upsilon\mu$ have cumulant functions expressed as in (2.22) or (3.17) and since we have already proved (v), we can use the dominated convergence theorem to get $C_{\Upsilon\mu_n}(z) \to C_{\Upsilon\mu}(z)$, that is, $\Upsilon\mu_n \to \Upsilon\mu$.

Conversely, assume that $\widetilde{\mu}_n = \Upsilon\mu_n \to \widetilde{\mu}$. Let $(\widetilde{A}_n, \widetilde{\nu}_n, \widetilde{\gamma}_n)$ and (A_n, ν_n, γ_n) be the triplets of $\widetilde{\mu}_n$ and μ_n. We claim that $\{\mu_n\}$ is precompact. The following conditions[2] are necessary and sufficient for precompactness of $\{\mu_n\}$:

$$\sup_n \operatorname{tr} A_n < \infty, \tag{3.18}$$

$$\sup_n \int_{\mathbb{R}^d} (|x|^2 \wedge 1)\nu_n(dx) < \infty, \tag{3.19}$$

$$\lim_{l\to\infty} \sup_n \int_{|x|>l} \nu_n(dx) = 0, \tag{3.20}$$

$$\sup_n |\gamma_n| < \infty. \tag{3.21}$$

Since $\{\widetilde{\mu}_n\}$ is precompact, these four relations already hold with (A_n, ν_n, γ_n) replaced by $(\widetilde{A}_n, \widetilde{\nu}_n, \widetilde{\gamma}_n)$. We denote them by (3.18)˜–(3.21)˜. Then (3.18) follows from (2.24) and (3.18)˜; (3.19) follows from (3.19)˜ since, by (2.25),

$$\begin{aligned}
\int_{\mathbb{R}^d} (|x|^2 \wedge 1)\widetilde{\nu}_n(dx) &= \int_0^\infty e^{-s} ds \int_{\mathbb{R}^d} (|sx|^2 \wedge 1)\nu_n(dx) \\
&= \int_{\mathbb{R}^d} |x|^2 \nu_n(dx) \int_0^{1/|x|} s^2 e^{-s} ds + \int_{\mathbb{R}^d} \nu_n(dx) \int_{1/|x|}^\infty e^{-s} ds \\
&\geqslant \int_{|x|\leqslant 1} |x|^2 \nu_n(dx) \int_0^1 s^2 e^{-s} ds + \int_{|x|>1} \nu_n(dx) \int_1^\infty e^{-s} ds;
\end{aligned}$$

(3.20) is obtained from (3.20)˜ because

$$\int_{|x|>l} \widetilde{\nu}_n(dx) = \int_0^\infty e^{-s} ds \int_{|x|>l/s} \nu_n(dx) \geqslant \int_1^\infty e^{-s} ds \int_{|x|>l} \nu_n(dx).$$

[2]There is an error in E 12.5 of Sato (1999); a condition corresponding to (3.20) should be added.

To see (3.21), use (3.21)˜ and the estimate

$$\sup_n \left| \int_{\mathbb{R}^d} \frac{x|x|^2}{1+|x|^2} \nu_n(dx) \int_0^\infty \frac{e^{-s} s(1-s^2)}{1+s^2|x|^2} ds \right| < \infty,$$

which is a consequence of (3.19) as in the proof of (3.16). This finishes the proof of precompactness of $\{\mu_n\}$. Now we can choose a convergent subsequence $\{\mu_{n'}\}$ of $\{\mu_n\}$. Thus there is $\mu \in ID(\mathbb{R}^d)$ such that $\mu_{n'} \to \mu$. Hence $\Upsilon \mu_{n'} \to \Upsilon \mu$ and $\Upsilon \mu = \widetilde{\mu}$. It follows from (i) that μ does not depend on the choice of the subsequence. Hence $\mu_n \to \mu$.

(vi) Let $X_t = X_t^{(\mu)}$. Let $X'_t = X_1 - X_{(1-t)-}$ for $0 \leqslant t < 1$. Then $\{X'_t \colon 0 \leqslant t < 1\}$ is a process identical in law with $\{X_t \colon 0 \leqslant t < 1\}$ (Proposition 41.8 of Sato (1999)). We have

$$\int_s^1 \log \frac{1}{t} dX'_t = \int_0^{1-s} \log \frac{1}{1-t} dX_t,$$

and the function

$$\widetilde{f}(t) = \begin{cases} \log(1/(1-t)) & \text{for } 0 \leqslant t < 1 \\ 0 & \text{for } t \geqslant 1 \end{cases}$$

is $\{X_t\}$-integrable similarly to $f(t)$. Hence (2.27).

In order to show (2.28), first note that $\int_s^1 \log \frac{1}{t} dX_t$ tends to $\int_0^1 \log \frac{1}{t} dX_t$ a. s. as $s \downarrow 0$, since $\int_B f(t) dX_t$, $B \in \mathcal{B}^0_{[0,\infty)}$, is an independently scattered random measure. By Theorem 4.7 of Sato (2004),

$$\int_s^1 \log \frac{1}{t} dX_t = \int_s^1 dX_t \int_t^1 \frac{1}{u} du = \int_s^1 \frac{du}{u} \int_s^u dX_t = \int_s^1 \frac{X_u}{u} du - X_s \log \frac{1}{s}.$$

It is known that $X_s \log(1/s) \to 0$ a. s. as $s \downarrow 0$ (apply Proposition 47.11 of Sato (1999) to the components of $\{X_t\}$). Therefore $\lim_{s\downarrow 0} \int_s^1 (X_u/u) du$ exists a. s. and (2.28) holds. $\square$

Proof of Theorem A. Let $\mu \in ID(\mathbb{R}^d)$ and $\widetilde{\mu} = \Upsilon \mu$. Let ν and $\widetilde{\nu}$ be the Lévy measures of μ and $\widetilde{\mu}$, respectively. Then (2.18) holds by Proposition 2.7 (v). Thus, if $\nu = 0$, then $\widetilde{\nu} = 0$ and $\widetilde{\mu} \in B(\mathbb{R}^d)$. Assume that $\nu \neq 0$ and ν has polar decomposition (λ, ν_ξ). Then, for any nonnegative measurable function f,

$$\int_{\mathbb{R}^d} f(x) \widetilde{\nu}(dx) = \int_0^\infty e^{-s} ds \int_{\mathbb{R}^d} f(sx) \nu(dx) = \int_0^\infty e^{-s} ds \int_S \lambda(d\xi) \int_0^\infty f(sr\xi) \nu_\xi(dr)$$
$$= \int_S \lambda(d\xi) \int_0^\infty \nu_\xi(dr) r^{-1} \int_0^\infty e^{-s/r} f(s\xi) ds = \int_S \lambda(d\xi) \int_0^\infty f(s\xi) \widetilde{l}_\xi(s) ds,$$

where $\widetilde{l}_\xi(s)$ is defined by (2.19). Define a measure $\widetilde{Q}_\xi$ by

$$\widetilde{Q}_\xi(B) = \int_0^\infty 1_B(r^{-1})r^{-1}\nu_\xi(dr), \qquad B \in \mathcal{B}((0,\infty)).$$

Then $\widetilde{Q}_\xi(B)$ is measurable in ξ and

$$\widetilde{l}_\xi(s) = \int_{(0,\infty)} e^{-su}\widetilde{Q}_\xi(du) \quad \text{for } s > 0. \tag{3.22}$$

Hence $\widetilde{l}_\xi$ is completely monotone. Letting $\widetilde{\lambda} = \lambda$ and $\widetilde{\nu}_\xi(dr) = \widetilde{l}_\xi(r)dr$, we see that $(\widetilde{\lambda}, \widetilde{\nu}_\xi)$ is a polar decomposition of $\widetilde{\nu}$ and that $\widetilde{\mu} \in B(\mathbb{R}^d)$.

Conversely, suppose that $\widetilde{\mu} \in B(\mathbb{R}^d)$ with triplet $(\widetilde{A}, \widetilde{\nu}, \widetilde{\gamma})$. If $\widetilde{\nu} = 0$, then $\widetilde{\mu} = \Upsilon\mu$ with $\mu = \mu_{(\widetilde{A}/2,0,\widetilde{\gamma})}$ by Proposition 2.7 (v). Suppose that $\widetilde{\nu} \neq 0$. Then, in a decomposition $(\widetilde{\lambda}, \widetilde{\nu}_\xi)$ of $\widetilde{\nu}$, we have $\widetilde{\nu}_\xi(dr) = \widetilde{l}_\xi(r)dr$, where $\widetilde{l}_\xi(r)$ is completely monotone in r and measurable in ξ. Thus there are measures $\widetilde{Q}_\xi$ on $(0,\infty)$ satisfying (3.22) such that $\widetilde{Q}_\xi(B)$ is measurable in ξ for each $B \in \mathcal{B}((0,\infty))$. Now define

$$\nu_\xi(B) = \int_{(0,\infty)} 1_B(u^{-1})u^{-1}\widetilde{Q}_\xi(du).$$

Then ν_ξ is a measure on $(0,\infty)$ for each ξ and

$$\int_0^\infty f(r)\nu_\xi(dr) = \int_{(0,\infty)} f(u^{-1})u^{-1}\widetilde{Q}_\xi(du)$$

for all nonnegative measurable functions f on $(0,\infty)$. Notice that it follows that

$$\int_{(0,\infty)} f(r)\widetilde{Q}_\xi(dr) = \int_0^\infty f(u^{-1})u^{-1}\nu_\xi(du)$$

for all nonnegative measurable functions f on $(0,\infty)$. Hence we have (2.19). Let $\lambda = \widetilde{\lambda}$. Then

$$\begin{aligned}\int_S \lambda(d\xi)\int_0^\infty (r^2 \wedge 1)\nu_\xi(dr) &= \int_S \widetilde{\lambda}(d\xi)\int_{(0,\infty)} (u^{-2} \wedge 1)u^{-1}\widetilde{Q}_\xi(du) \\ &= \int_S \widetilde{\lambda}(d\xi)\left(\int_{(0,1]} u^{-1}\widetilde{Q}_\xi(du) + \int_{(1,\infty)} u^{-3}\widetilde{Q}_\xi(du)\right) < \infty\end{aligned}$$

by (3.5) for $\widetilde{Q}_\xi$ in place of Q_ξ. Define ν by (2.8). Then ν is the Lévy measure of an infinitely divisible distribution and we can check

$$\int_0^\infty e^{-s}ds\int_{\mathbb{R}^d} f(sx)\nu(dx) = \int_{\mathbb{R}^d} f(x)\widetilde{\nu}(dx)$$

for all nonnegative measurable functions f on $\mathbb{R}^d$. Define A and γ by (2.24) and (2.26) and let $\mu = \mu_{(A,\nu,\gamma)}$. Then $\Upsilon\mu = \widetilde{\mu}$. Thus $\widetilde{\mu} \in \Upsilon(ID(\mathbb{R}^d))$. This finishes the proof of Theorem A. $\square$

Proof of Theorem B. Let $\mu \in L(\mathbb{R}^d)$ and $\widetilde{\mu} = \Upsilon\mu$. Let ν and $\widetilde{\nu}$ be the Lévy measures of μ and $\widetilde{\mu}$, respectively. If $\nu = 0$, then $\widetilde{\nu} = 0$ and $\widetilde{\mu} \in T(\mathbb{R}^d)$. Assume that $\nu \neq 0$ and let λ, $\nu_\xi(dr) = k_\xi(r)r^{-1}dr$ be a polar decomposition of ν with k-function $k_\xi(r)$. We claim that $\widetilde{\mu} \in T(\mathbb{R}^d)$. For any nonnegative measurable function f on $\mathbb{R}^d$,

$$\begin{aligned}\int f(x)\widetilde{\nu}(dx) &= \int_0^\infty e^{-s}ds \int f(sx)\nu(dx) = \int_0^\infty e^{-s}ds \int_S \lambda(d\xi) \int_0^\infty f(sr\xi)k_\xi(r)r^{-1}dr \\ &= \int_0^\infty e^{-s}ds \int_S \lambda(d\xi) \int_0^\infty f(r\xi)k_\xi(rs^{-1})r^{-1}dr \\ &= \int_S \lambda(d\xi) \int_0^\infty f(r\xi)r^{-1}dr \int_0^\infty k_\xi(rs^{-1})e^{-s}ds.\end{aligned}$$

Define $\widetilde{k}_\xi(r)$ by the first equality in (2.21). Let $k^\sharp_\xi(u) = \lim_{u'\downarrow u} k_\xi(1/u')$. Notice that $\lim_{r\to\infty} k_\xi(r) = 0$ for λ-a. e. ξ. Then

$$\begin{aligned}-\int 1_{[a,\infty)}(v)dk_\xi(v) &= \lim_{a'\uparrow a} k_\xi(a') = k^\sharp_\xi(a^{-1}) = \int 1_{(0,a^{-1}]}(u)dk^\sharp_\xi(u) \\ &= \int 1_{[a,\infty)}(u^{-1})dk^\sharp_\xi(u)\end{aligned}$$

for all $a > 0$. More generally,

$$-\int_{(0,\infty)} g(v)dk_\xi(v) = \int_{(0,\infty)} g(u^{-1})dk^\sharp_\xi(u)$$

for any nonnegative measurable function g on $(0,\infty)$. Then

$$\begin{aligned}\widetilde{k}_\xi(r) &= -\int_0^\infty e^{-s}ds \int_{(r/s,\infty)} dk_\xi(v) = -\int_{(0,\infty)} dk_\xi(v) \int_{r/v}^\infty e^{-s}ds \\ &= -\int_{(0,\infty)} e^{-r/v}dk_\xi(v) = \int_{(0,\infty)} e^{-ru}dk^\sharp_\xi(u).\end{aligned} \tag{3.23}$$

Since $k^\sharp_\xi(u)$ is increasing, it follows that $\widetilde{k}_\xi(r)$ is completely monotone. Hence $\widetilde{\mu} \in T(\mathbb{R}^d)$.

Conversely, suppose $\widetilde{\mu} \in T(\mathbb{R}^d)$ with triplet $(\widetilde{A}, \widetilde{\nu}, \widetilde{\gamma})$. If $\widetilde{\nu} = 0$, then $\widetilde{\mu} = \Upsilon\mu$ with μ Gaussian and hence $\widetilde{\mu} \in \Upsilon(L(\mathbb{R}^d))$. Suppose $\widetilde{\nu} \neq 0$. Then, we have a decomposition $(\widetilde{\lambda}, \widetilde{\nu}_\xi)$ of $\widetilde{\nu}$ with $\widetilde{\nu}_\xi(dr) = \widetilde{k}_\xi(r)r^{-1}dr$, where $\widetilde{k}_\xi(r)$ is completely monotone in r and measurable in ξ. We have $\widetilde{k}_\xi(r) = \int_{(0,\infty)} e^{-ru}\widetilde{R}_\xi(du)$ with $\widetilde{R}_\xi(du)$ described in Remark 3.2. Define $k^\sharp_\xi(u) = \widetilde{R}_\xi((0,u])$, $u > 0$, and $k_\xi(v) = \lim_{v'\downarrow v} k^\sharp_\xi(1/v')$, $v > 0$. Then $k_\xi(v)$ is right-continuous and decreasing in v. The calculation in (3.23) shows the first equality in (2.21). Hence we have

$$\int_{\mathbb{R}^d} f(x)\widetilde{\nu}(dx) = \int_0^\infty e^{-s}ds \int_S \widetilde{\lambda}(d\xi) \int_0^\infty f(sr\xi)k_\xi(r)r^{-1}dr$$

for all nonnegative measurable functions $f(x)$. Define $\lambda = \widetilde{\lambda}$ and

$$\nu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi) k_\xi(r) r^{-1} dr.$$

Then we have (2.25) and

$$\begin{aligned}
\int_{\mathbb{R}^d} (|x|^2 \wedge 1)\nu(dx) &= \int_S \lambda(d\xi) \int_0^\infty (r^2 \wedge 1) k_\xi(r) r^{-1} dr \\
&= \int_S \widetilde{\lambda}(d\xi) \int_0^\infty (r^2 \wedge 1) k_\xi^\sharp(r^{-1}) r^{-1} dr = \int_S \widetilde{\lambda}(d\xi) \int_0^\infty (r \wedge r^{-1}) dr \int_{(0,r^{-1}]} \widetilde{R}_\xi(du) \\
&= \int_S \widetilde{\lambda}(d\xi) \left(\int_0^1 r dr \int_{(0,r^{-1}]} \widetilde{R}_\xi(du) + \int_1^\infty r^{-1} dr \int_{(0,r^{-1}]} \widetilde{R}_\xi(du) \right) \\
&= \int_S \widetilde{\lambda}(d\xi) \left(\int_{(0,\infty)} \widetilde{R}_\xi(du) \int_0^{1\wedge(1/u)} r dr + \int_{(0,1]} \widetilde{R}_\xi(du) \int_1^{1/u} r^{-1} dr \right) \\
&= \int_S \widetilde{\lambda}(d\xi) \left(\frac{1}{2} \int_{(0,1]} \widetilde{R}_\xi(du) + \frac{1}{2} \int_{(1,\infty)} u^{-2} \widetilde{R}_\xi(du) + \int_{(0,1]} \log \frac{1}{u} \widetilde{R}_\xi(du) \right) < \infty
\end{aligned}$$

by using (3.10) for $\widetilde{R}_\xi$ in place of R_ξ. Hence, ν is the Lévy measure of a distribution. Letting $A = \frac{1}{2}\widetilde{A}$ and choosing γ to satisfy (2.26), we have $\widetilde{\mu} = \Upsilon\mu$ for $\mu = \mu_{(A,\nu,\gamma)} \in L(\mathbb{R}^d)$. $\square$

4. Proof of Theorems C and D

We give the proofs of Theorems C and D together with some general results on complete closedness in the strong sense.

Proof of Theorem C. (i) Let $\mu \in ID(\mathbb{R}^d)$ and $\widetilde{\mu} = \Upsilon\mu$. Let ν and $\widetilde{\nu}$ be the Lévy measures of μ and $\widetilde{\mu}$. We have

$$\begin{aligned}
\int_{|x|>2} \log|x| \widetilde{\nu}(dx) &= \int_0^\infty e^{-s} ds \int_{|x|>2/s} \log(s|x|)\nu(dx) \\
&= \int_{\mathbb{R}^d} \nu(dx) \int_{2/|x|}^\infty e^{-s} \log(s|x|) ds = \int_{\mathbb{R}^d} h(x)\nu(dx),
\end{aligned}$$

where

$$h(x) = \int_{2/|x|}^\infty e^{-s} \log s \, ds + e^{-2/|x|} \log|x|.$$

Note that $h(x) = o(|x|^2)$ as $|x| \downarrow 0$ and $h(x) \sim \log|x|$ as $|x| \to \infty$. Thus, $\int_{|x|>2} \log|x|\widetilde{\nu}(dx) < \infty$ if and only if $\int_{|x|>2} \log|x|\nu(dx) < \infty$.

(ii) If $\mu \in ID_{\log}(\mathbb{R}^d)$, then

$$\int_0^\infty |C_\mu(e^{-t}z)| dt < \infty \quad \text{and} \quad C_{\Phi\mu}(z) = \int_0^\infty C_\mu(e^{-t}z) dt$$

(see the references given for (2.2) and (2.3)). If $\mu \in ID(\mathbb{R}^d)$, then

$$\int_0^\infty e^{-s}|C_\mu(sz)|ds < \infty \quad \text{and} \quad C_{\Upsilon\mu}(z) = \int_0^\infty e^{-s}C_\mu(sz)ds$$

by (2.23) and (2.22). Let $\mu \in ID_{\log}(\mathbb{R}^d)$. Using $\Upsilon\mu \in ID_{\log}(\mathbb{R}^d)$ in (i), we have

$$C_{\Phi\Upsilon\mu}(z) = \int_0^\infty dt \int_0^\infty e^{-s}C_\mu(e^{-t}sz)ds, \tag{4.1}$$

$$C_{\Upsilon\Phi\mu}(z) = \int_0^\infty e^{-s}ds \int_0^\infty C_\mu(e^{-t}sz)dt. \tag{4.2}$$

We claim that

$$\int_0^\infty e^{-s}ds \int_0^\infty |C_\mu(e^{-t}sz)|dt < \infty \quad \text{for } z \in \mathbb{R}^d. \tag{4.3}$$

If this is proved, then we can interchange the order of the integrations in (4.1) and (4.2) and get $\Phi\Upsilon\mu = \Upsilon\Phi\mu$.

The proof of (4.3) is as follows. Let $\mu = \mu_{(A,\nu,\gamma)}$. Then

$$|C_\mu(z)| \leqslant \frac{1}{2}(\operatorname{tr} A)|z|^2 + |\gamma||z| + \int |g(z,x)|\nu(dx),$$

where $g(z,x)$ is given by (3.11). Hence

$$|C_\mu(e^{-t}sz)| \leqslant I_1 + I_2 + I_3 + I_4$$

with

$$I_1 = \frac{1}{2}(\operatorname{tr} A)e^{-2t}s^2|z|^2, \qquad I_2 = |\gamma|e^{-t}s|z|,$$

$$I_3 = \int |g(z, e^{-t}sx)|\nu(dx), \qquad I_4 = \int |g(e^{-t}sz, x) - g(z, e^{-t}sx)|\nu(dx).$$

Finiteness of $\int_0^\infty e^{-s}ds \int_0^\infty (I_1 + I_2)dt$ is straightforward. To deal with the similar integrals of I_3 and I_4, note that

$$|g(z,x)| \leqslant c_z|x|^2/(1+|x|^2), \tag{4.4}$$

with a constant c_z depending on z, and that, for any $a \in \mathbb{R}$,

$$\begin{aligned} |g(az,x) - g(z,ax)| &= |\langle az, x\rangle| \left| \frac{|x|^2}{1+|x|^2} - \frac{|ax|^2}{1+|ax|^2} \right| \\ &= |\langle az, x\rangle| \frac{|x|^2|1-a^2|}{(1+|x|^2)(1+|ax|^2)} \leqslant |z| \frac{|x|^3(|a|+|a|^3)}{(1+|x|^2)(1+|ax|^2)}. \end{aligned}$$

Then

$$\int_0^\infty e^{-s}ds\int_0^\infty I_3 dt \leqslant c_z\int_{\mathbb{R}^d}\nu(dx)\int_0^\infty e^{-s}ds\int_0^\infty \frac{(e^{-t}s|x|)^2}{1+(e^{-t}s|x|)^2}dt$$
$$= c_z\int \nu(dx)\int_0^\infty e^{-s}ds\int_0^{s|x|}\frac{u}{1+u^2}du$$
$$= \frac{c_z}{2}\int \nu(dx)\int_0^\infty e^{-s}\log(1+s^2|x|^2)ds = J_3, \quad \text{say.}$$

Since $\log(1+v) \leqslant c(v1_{(0,2]}(v) + (\log v)1_{(2,\infty)}(v))$ for $v > 0$ with an absolute constant c, we have

$$J_3 \leqslant \frac{cc_z}{2}\int_{\mathbb{R}^d}|x|^2\nu(dx)\int_0^{\sqrt{2}/|x|}e^{-s}s^2 ds + cc_z\int_{\mathbb{R}^d}\nu(dx)\int_{\sqrt{2}/|x|}^\infty e^{-s}(\log s + \log|x|)ds,$$

which is finite since $\int_{|x|\leqslant 2}|x|^2\nu(dx) < \infty$ and $\int_{|x|>2}\log|x|\nu(dx) < \infty$. Concerning I_4, we have

$$\int_0^\infty I_4 dt \leqslant |z|\int_{\mathbb{R}^d}\frac{|x|^3}{1+|x|^2}\nu(dx)\int_0^\infty \frac{e^{-t}s+e^{-3t}s^3}{1+e^{-2t}s^2|x|^2}dt$$
$$= |z|\int_{\mathbb{R}^d}\frac{|x|^3}{1+|x|^2}\nu(dx)\int_0^{s|x|}\frac{u|x|^{-1}+u^3|x|^{-3}}{(1+u^2)u}du$$
$$\leqslant \frac{\pi}{2}|z|\int_{\mathbb{R}^d}\frac{|x|^2}{1+|x|^2}\nu(dx) + |z|\int_{\mathbb{R}^d}\frac{1}{1+|x|^2}\nu(dx)\int_0^{s|x|}\frac{u^2 du}{1+u^2}$$
$$= J_{4,1} + J_{4,2}, \quad \text{say.}$$

Then $\int_0^\infty e^{-s}J_{4,1}ds < \infty$ is evident and

$$\int_0^\infty e^{-s}J_{4,2}ds = |z|\int_{\mathbb{R}^d}\frac{\nu(dx)}{1+|x|^2}\int_0^\infty \frac{u^2 du}{1+u^2}\int_{u/|x|}^\infty e^{-s}ds$$
$$\leqslant |z|\int_{\mathbb{R}^d}\frac{\nu(dx)}{1+|x|^2}\left(\int_0^1 u^2 e^{-u/|x|}du + \int_1^\infty e^{-u/|x|}du\right)$$
$$= |z|\int_{\mathbb{R}^d}\frac{|x|^3}{1+|x|^2}\nu(dx)\int_0^{1/|x|}u^2 e^{-u}du + |z|\int_{\mathbb{R}^d}\frac{|x|e^{-1/|x|}}{1+|x|^2}\nu(dx) < \infty.$$

This finishes the proof of (4.3).

It follows from (4.1) and (4.3) that

$$C_{\Phi\Upsilon\mu}(z) = \int_0^\infty dt\int_0^\infty C_\mu(uz)e^{t-ue^t}du = \int_0^\infty C_\mu(uz)e^{-u}u^{-1}du$$
$$= -\int_0^\infty C_\mu(uz)de_1(u) = \int_0^\infty C_\mu(e_1^*(t)z)dt, \tag{4.5}$$

$$\int_0^\infty |C_\mu(e_1^*(t)z)|dt < \infty \tag{4.6}$$

for $\mu \in ID_{\log}(\mathbb{R}^d)$.

The function $e_1^*(t)$ is $\{X_t^{(\mu)}\}$-integrable for every $\mu = \mu_{(A,\nu,\gamma)} \in ID(\mathbb{R}^d)$. This is because, for each $t_0 \in (0,\infty)$, the integrals

$$\int_0^{t_0} \langle z, Az\rangle e_1^*(t)^2 dt, \qquad \int_0^{t_0} dt \int_{R^d} (|xe_1^*(t)|^2 \wedge 1)\nu(dx),$$

$$\int_0^{t_0} |\langle \gamma, z\rangle| e_1^*(t) dt, \qquad \int_0^{t_0} dt \int_{R^d} |g(e_1^*(t)z, x) - g(z, e_1^*(t)x)|\nu(dx)$$

are finite. Indeed,

$$\int_0^{t_0} dt \int_{R^d} (|xe_1^*(t)|^2 \wedge 1)\nu(dx) = \int_{e_1^*(t_0)}^{\infty} e^{-s}s^{-1} ds \int_{R^d} (|xs|^2 \wedge 1)\nu(dx) < \infty$$

like (3.13), and finiteness of the other integrals is shown similarly. It follows from (4.5) and (4.6) that, if $\mu \in ID_{\log}(\mathbb{R}^d)$, then $\int_0^\infty e_1^*(t) dX_t^{(\mu)}$ exists and equals $\Phi\Upsilon\mu$ in distribution.

It remains to show that $\int_0^\infty e_1^*(t) dX_t^{(\mu)}$ does not exists if $\mu \in ID(\mathbb{R}^d) \setminus ID_{\log}(\mathbb{R}^d)$. It is easy to see that

$$e_1(s) \sim e^{-s}s^{-1} \text{ as } s \to \infty, \qquad e_1(s) \sim \log(1/s) \text{ as } s \downarrow 0.$$

Then we have

$$e_1^*(t) \sim ce^{-t} \text{ as } t \to \infty, \qquad e_1^*(t) \sim \log(1/t) \text{ as } t \downarrow 0 \tag{4.7}$$

with some positive constant c, for we have

$$\lim_{t\to\infty} \frac{e_1^*(t)}{e^{-t}} = \lim_{s\downarrow 0} \frac{s}{e^{-e_1(s)}} = \lim_{s\downarrow 0} e^{e_1(s)+\log s} = \exp\left(\int_1^\infty e^{-u}u^{-1}du - \int_0^1 (1-e^{-u})u^{-1}du\right),$$

$$\lim_{t\downarrow 0} \frac{e_1^*(t)}{\log(1/t)} = \lim_{s\to\infty} \frac{s}{-\log e_1(s)} = \lim_{s\to\infty} \frac{1}{-e_1'(s)/e_1(s)} = \lim_{s\to\infty} \frac{e_1(s)}{e^{-s}s^{-1}} = 1.$$

Let $\mu \in ID(\mathbb{R}^d)$ and suppose that $\int_0^\infty e_1^*(t) dX_t^{(\mu)}$ exists and has distribution $\widetilde{\mu}$. Let $t_n \to \infty$ and denote $\widetilde{\mu}_n = \mathcal{L}\left(\int_0^{t_n} e_1^*(t) dX_t^{(\mu)}\right)$. Then $\widetilde{\mu}_n \to \widetilde{\mu}$. Let $\widetilde{\nu}_n$ and $\widetilde{\nu}$ be the Lévy measures of $\widetilde{\mu}_n$ and $\widetilde{\mu}$. Then $\int f(x)\widetilde{\nu}_n(dx) \to \int f(x)\widetilde{\nu}(dx)$ for all bounded continuous functions f vanishing on a neighborhood of 0 (Sato (1999) Theorem 8.7). Choose $t_0 > 0$ such that $e_1^*(t) \geqslant ce^{-t}/2$ for $t > t_0$. Since

$$\widetilde{\nu}_n(B) = \int_0^{t_n} dt \int 1_B(e_1^*(t)x)\nu(dx), \qquad B \in \mathcal{B}(\mathbb{R}^d),$$

where ν is the Lévy measure of μ, we get

$$\int_{|x|>1} \widetilde{\nu}_n(dx) = \int_0^{t_n} dt \int 1_{\{|x|>1/e_1^*(t)\}}(x)\nu(dx) \geqslant \int_0^{t_n} dt \int 1_{\{|x|>2e^t/c\}}(x)\nu(dx)$$

$$= \int_{\mathbb{R}^d} \nu(dx) \int_{(t_0,t_n]\cap(0,\log(c|x|/2))} dt \to \int_{\{\log(c|x|/2)>t_0\}} (\log(c|x|/2) - t_0)\nu(dx).$$

Hence $\int_{|x|>a} \log |x| \nu(dx) < \infty$ for some a, that is, $\mu \in ID_{\log}(\mathbb{R}^d)$.

(iii) is a consequence of (i) and (ii) combined with Theorems A and B. $\square$

As in Maejima, Sato and Watanabe (1999), a class M of distributions on $\mathbb{R}^d$ is said to be *completely closed in the strong sense* if it satisfies the following five conditions:

(ccs1) M is a subclass of $ID(\mathbb{R}^d)$,

(ccs2) $\mu_1, \mu_2 \in M$ implies $\mu_1 * \mu_2 \in M$,

(ccs3) $\mu_n \in M$ $(n = 1, 2, \ldots)$ and $\mu_n \to \mu$ imply $\mu \in M$,

(ccs4) if X is an $\mathbb{R}^d$-valued random variable with $\mathcal{L}(X) \in M$, then $\mathcal{L}(aX + b) \in M$ for any $a > 0$ and $b \in \mathbb{R}^d$,

(ccs5) $\mu \in M$ implies $\mu^{s*} \in M$ for any $s > 0$.

In the following we sometimes omit $\mathbb{R}^d$ in writing $ID_{\log}(\mathbb{R}^d)$, $L_m(\mathbb{R}^d)$, or $T_m(\mathbb{R}^d)$.

Lemma 4.1. *Let M be a class of distributions on $\mathbb{R}^d$, completely closed in the strong sense. Then the following statements are true.*

(i) *The classes $\Upsilon(M)$ and $\Phi(M \cap ID_{\log}(\mathbb{R}^d))$ are subclasses of M.*

(ii) *The classes $\Upsilon(M)$ and $\Phi(M \cap ID_{\log}(\mathbb{R}^d))$ are completely closed in the strong sense.*

(iii) $\Phi(M \cap ID_{\log}(\mathbb{R}^d)) = \{\sigma \in L(\mathbb{R}^d) \colon \rho_b^{(\sigma)} \in M$ *for all* $b > 1\}$*, where $\rho_b^{(\sigma)}$ is defined by* (2.32) *with σ in place of μ.*

Proof. (i) Let $\mu \in M$ and $X_t = X_t^{(\mu)}$. Let $\sigma = \Upsilon\mu = \mathcal{L}(I)$ where $I = \int_0^1 \log(1/t) dX_t$. For any $s_n \downarrow 0$ let $\sigma_n = \mathcal{L}(I_n)$ where $I_n = \int_{s_n}^1 \log(1/t) dX_t$. By Proposition 4.5 of Sato (2004), I_n is the limit in probability of a sequence $\int_{s_n}^1 f_m(t) dX_t$ as $m \to \infty$, where $f_m(t)$ is a nonnegative step function for each m. We see that $\mathcal{L}\left(\int_{s_n}^1 f_m(t) dX_t\right) \in M$ from (ccs2), (ccs4), and (ccs5). Thus $\sigma_n \in M$ by (ccs3). As $n \to \infty$, I_n tends to I in probability and thus $\sigma_n \to \sigma$. Hence $\sigma \in M$. Proof that $\Phi\mu \in M$ for $\mu \in M \cap ID_{\log}$ is similar, using (2.3).

(ii) The properties (ccs1)–(ccs3) for $\Upsilon(M)$ follows from Proposition 2.7. To see (ccs4), note that $\mathcal{L}\left(a \int_0^1 \log(1/t) dX_t^{(\mu)} + b\right) = \mathcal{L}\left(\int_0^1 \log(1/t) dX_t'\right)$, where $\{X_t'\}$ is a Lévy process with $\mathcal{L}(X_1') = \mathcal{L}(aX_1^{(\mu)} + b)$. Here we have used $\int_0^1 \log(1/t) dt = 1$. To see (ccs5), note that

$$sC_{\Upsilon\mu}(z) = s \int_0^1 C_\mu(z \log(1/t)) dt = \int_0^1 C_{\mu^{s*}}(z \log(1/t)) dt.$$

Similarly we can prove (ccs1)–(ccs5) for $\Phi(M \cap ID_{\log})$ except (ccs3). Proof of (ccs3) for $\Phi(M \cap ID_{\log})$ will be given after we show (iii).

(iii) Suppose that $\mu \in M \cap ID_{\log}$ and $\sigma = \Phi\mu$. Use (2.2) and (2.3). Then $\sigma \in L(\mathbb{R}^d)$. Notice that, for $X_t = X_t^{(\mu)}$ and $b > 1$,

$$b^{-1} \int_0^\infty e^{-t} dX_t = \int_0^\infty e^{-(t+\log b)} dX_t \overset{\mathrm{d}}{=} \int_{\log b}^\infty e^{-t} dX_t,$$

$$\int_0^\infty e^{-t} dX_t = \int_{\log b}^\infty e^{-t} dX_t + \int_0^{\log b} e^{-t} dX_t,$$

and thus $\rho_b^{(\sigma)} = \mathcal{L}\left(\int_0^{\log b} e^{-t} dX_t\right)$. Hence $\rho_b^{(\sigma)} \in M$ as in the proof of (i).

Conversely, suppose that $\sigma \in L$ and $\rho_b^{(\sigma)} \in M$ for all $b > 1$. Choosing $\mu \in ID_{\log}$ with $\Phi\mu = \sigma$, we see that $C_{\rho_b^{(\sigma)}}(z) = \int_0^{\log b} C_\mu(e^{-t}z)dt$. Let $g_b(z)$ be the cumulant function of $(\rho_b^{(\sigma)})^{(1/\log b)*} \in M$. Then $g_b(z) = (1/\log b) \int_0^{\log b} C_\mu(e^{-t}z)dt$, which tends to $C_\mu(z)$ as $b \downarrow 1$. Hence $(\rho_b^{(\sigma)})^{(1/\log b)*} \to \mu$ and $\mu \in M$.

Now we give the proof that $\widetilde{M} = \Phi(M \cap ID_{\log})$ has property (ccs3). Let $\sigma_n \in \widetilde{M}$ and $\sigma_n \to \sigma$ as $n \to \infty$. Then, by (iii), $\rho_b^{(\sigma_n)} \in M$. Since the characteristic function of $\rho_b^{(\sigma_n)}$ equals $\widehat{\sigma}_n(z)/\widehat{\sigma}_n(b^{-1}z)$, which tends to a continuous function $\widehat{\sigma}(z)/\widehat{\sigma}(b^{-1}z)$ as $n \to \infty$, $\rho_b^{(\sigma_n)}$ tends to some $\rho \in M$. We have $\widehat{\sigma}(z) = \widehat{\sigma}(b^{-1}z)\widehat{\rho}(z)$. Hence $\sigma \in \widetilde{M}$ again by (iii). $\square$

Proof of Corollary to Theorem C. By Lemma 4.1 (ii) it follows from Theorem A (i) that $B(\mathbb{R}^d)$ is completely closed in the strong sense. Hence, by Lemma 4.1 (iii), we get (2.35) from (2.30) of Theorem C. $\square$

Proof of Theorem D. Let us prove (2.38). Although (2.43) and the complete closedness in the strong sense of $L_m(\mathbb{R}^d)$ are known facts, it is more natural to reprove them and to prove the complete closedness in the strong sense of $T_m(\mathbb{R}^d)$, together with the proof of (2.38). For $m = 0$ (2.38) is already proved in Theorem B. To prove it for $m = 1$, first note that L_0 is completely closed in the strong sense by Lemma 4.1 (ii) and (2.2). Hence so is T_0 by (2.38) for $m = 0$. Lemma 4.1 (iii) says that $L_1 = \Phi(L_0 \cap ID_{\log})$. Now we have

$$\begin{aligned} T_1 &= \Phi(T_0 \cap ID_{\log}) = \Phi(\Upsilon(L_0) \cap ID_{\log}) = \Phi(\Upsilon(L_0 \cap ID_{\log})) \\ &= \Upsilon\Phi(L_0 \cap ID_{\log}) = \Upsilon(L_1), \end{aligned}$$

using definition (2.36) of T_1, Lemma 4.1 (iii), Theorem C (i), and Theorem C (ii), consecutively. This is (2.38) for $m = 1$. Continuing this procedure, we get (2.38), (2.42), (2.43), and the complete closedness in the strong sense of $L_m(\mathbb{R}^d)$ and $T_m(\mathbb{R}^d)$

for all finite m. It follows that (2.38) holds also for $m = \infty$. Moreover (2.42) and (2.43) also hold for $m = \infty$, since we get from (2.36)

$$T_\infty = \{\mu \in L \colon \rho_b^{(\mu)} \in T_\infty \text{ for every } b > 1\},$$

and similarly for L_∞.

Let us show (2.39). Denote by $\mathfrak{S}(\alpha, \mathbb{R}^d)$ the class of α-stable distributions on $\mathbb{R}^d$. It is enough to show that

$$\Upsilon(\mathfrak{S}(\alpha, \mathbb{R}^d)) = \mathfrak{S}(\alpha, \mathbb{R}^d). \tag{4.8}$$

This is evident in the case $\alpha = 2$ (Gaussian). Let $\mu \in \mathfrak{S}(\alpha, \mathbb{R}^d)$ with $0 < \alpha < 2$. Then it has k-function $k_\xi(r) = r^{-\alpha}$. Thus by (2.21) $\Upsilon\mu$ has k-function $\int_0^\infty r^{-\alpha} s^\alpha e^{-s} ds = \Gamma(\alpha+1)\, r^{-\alpha}$. Thus $\Upsilon\mu \in \mathfrak{S}(\alpha, \mathbb{R}^d)$. On the other hand, this shows that, for any $\widetilde{\mu} \in \mathfrak{S}(\alpha, \mathbb{R}^d)$, there is a $\mu \in \mathfrak{S}(\alpha, \mathbb{R}^d)$ such that $\widetilde{\mu} = \Upsilon\mu$.

The assertion $T_m \subset L_m$ for all finite m is a consequence of (2.38) and Lemma 4.1 (i). But we have to show the inclusion is strict. Define

$$\begin{aligned} ID_{\log^n}(\mathbb{R}^d) &= \left\{\mu \in ID(\mathbb{R}^d) \colon \int_{|x|>2} (\log|x|)^n \mu(dx) < \infty\right\} \\ &= \left\{\mu \in ID(\mathbb{R}^d) \colon \int_{|x|>2} (\log|x|)^n \nu^{(\mu)}(dx) < \infty\right\}, \end{aligned}$$

for $n = 1, 2, \ldots$. Let $ID_{\log^0}(\mathbb{R}^d) = ID(\mathbb{R}^d)$. It is known that

$$\Phi(ID_{\log^{n+1}}(\mathbb{R}^d)) = L(\mathbb{R}^d) \cap ID_{\log^n}(\mathbb{R}^d) \quad \text{for } n = 0, 1, \ldots, \tag{4.9}$$

$$L_m(\mathbb{R}^d) = \Phi^{m+1}(ID_{\log^{m+1}}(\mathbb{R}^d)) \quad \text{for } m = 0, 1, \ldots \tag{4.10}$$

(see the references given after (2.43)). The proof of (2.14) actually showed that

$$B \cap L_0 \cap ID_{\log^n} \supsetneqq T_0 \cap ID_{\log^n} \quad \text{for } n = 0, 1, \ldots. \tag{4.11}$$

Hence $L_0 \cap ID_{\log^n} \supsetneqq T_0 \cap ID_{\log^n}$ for $n = 0, 1, \ldots$. Applying Φ and using (2.42) and (2.43), we get $L_1 \cap ID_{\log^n} \supsetneqq T_1 \cap ID_{\log^n}$ for $n = 0, 1, \ldots$. Repeating this, we have $L_m \cap ID_{\log^n} \supsetneqq T_m \cap ID_{\log^n}$ for $m = 0, 1, \ldots$ and $n = 0, 1, \ldots$. For $n = 0$ this is (2.40).

The proof of (2.41) is as follows. It follows from $T_m \subset L_m$ for finite m that $T_\infty \subset L_\infty$. On the other hand we know that $\mathfrak{S} \subset T_\infty$ and that T_∞ is completely closed in the strong sense. Since L_∞ is the smallest class containing $\mathfrak{S}$ and closed under convolution and convergence, we have $T_\infty \supset L_\infty$. $\square$

5. Proof of Theorem E

For $a > 0$, let Δ_a be the difference operator, $\Delta_a f(u) = f(u+a) - f(u)$, $u \in \mathbb{R}$, and let Δ_a^n be its nth iteration. Clearly

$$\Delta_a^n f(u) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} f(u+ja)$$

for $n = 0, 1, \ldots$. We say that a function $f(u)$ is *monotone of order* n if

$$\Delta_a^j f(u) \geqslant 0 \quad \text{for any } a > 0,\ u \in \mathbb{R},\ j = 0, 1, \ldots, n.$$

When f is monotone of order n for all $n = 0, 1, \ldots$, f is *absolutely monotone.* Then a characterization of distributions in the class $L_m(\mathbb{R}^d)$ in terms of Lévy measures is given as follows (Sato (1980)).

Proposition 5.1. *Let $\mu \in L_0(\mathbb{R}^d)$ with Lévy measure ν such that $\nu = 0$ or $\nu \neq 0$ with spherical component λ and h-function $h_\xi(u)$.*

(i) *Let $m \in \{1, 2, \ldots\}$. Then $\mu \in L_m(\mathbb{R}^d)$ if and only if either $\nu = 0$ or $\nu \neq 0$ with $h_\xi(u)$ being monotone of order $m+1$ in u for λ-a. e. ξ.*

(ii) *We have $\mu \in L_\infty(\mathbb{R}^d)$ if and only if either $\nu = 0$ or $\nu \neq 0$ with $h_\xi(u)$ being absolutely monotone in u for λ-a. e. ξ.*

Proof of Theorem E. Let us denote by T'_m the class of $\mu \in L_0$ such that either $\nu = 0$ or $\nu \neq 0$ with h-function satisfying (2.44). First, notice that condition (2.44) is equivalent to the condition that

$$h_\xi^{(j)}(-\log r) \text{ is completely monotone in } r > 0 \text{ for } j = 0, 1, \ldots, m,\ \lambda\text{-a. e. } \xi. \tag{5.1}$$

Indeed, this clearly implies (2.44). On the other hand, if (2.44) holds, then $-(d/dr)(h^{(m-1)}(-\log r)) = h^{(m)}(-\log r) r^{-1}$ is completely monotone as the product of two completely monotone functions, and thus $h^{(m-1)}(-\log r)$ is itself completely monotone since $h^{(m-1)} \geqslant 0$, and so on. Since $h_\xi(-\log r) = k_\xi(r)$, we have $T'_0 = T_0$ by Definition 2.3. Let us prove $T'_m = T_m$ for all finite m.

Part 1. (Proof that $T_m \subset T'_m$.) Assume that $1 \leqslant m < \infty$ and let $\widetilde{\mu} \in T_m$. By virtue of (2.38) of Theorem D, there is $\mu \in L_m$ such that $\widetilde{\mu} = \Upsilon\mu$. Let ν and $\widetilde{\nu}$ be the Lévy measures of μ and $\widetilde{\mu}$. If $\nu = 0$, then $\widetilde{\nu} = 0$ and $\widetilde{\mu} \in T'_m$. Assume that $\nu \neq 0$ and let k_ξ and h_ξ be the k-function and h-function of ν. For notational simplicity, we omit ξ in $k_\xi(r)$ and $h_\xi(u)$. By Proposition 5.1, h is monotone of order $m+1$, and by Lemma 3.2 of Sato (1980), h is $m-1$ times continuously differentiable, $h^{(j)}$ is nonnegative for $j = 0, 1, \ldots, m-1$, and $h^{(m-1)}$ is increasing and convex. Thus there exists the Radon–Nikodým derivative $h^{(m)}$ of $h^{(m-1)}$ such that $h^{(m)}$ is nonnegative and increasing. We take $h^{(m)}$ as a right-continuous function having left limits. We see that, for $j = 1, \dots, m$, $h^{(j)}$ is nonnegative, increasing, and satisfies $h^{(j)}(-\infty) = 0$ and $h^{(j-1)}(u) = \int_{-\infty}^{u} h^{(j)}(v)dv$. Let us use (2.21) of Theorem B (ii). Thus

$$\widetilde{k}(r) = \int_{(0,\infty)} e^{-ru} dh(\log u). \tag{5.2}$$

Define, for $j = 1, 2, \dots$,

$$e_j(t) = \int_t^{\infty} e_{j-1}(s)s^{-1}ds \tag{5.3}$$

with $e_0(t) = e^{-t}$. This is definable, because we can inductively prove that $e_{j-1}(t) \sim e^{-t}t^{-(j-1)}$ as $t \to \infty$. This definition is consistent with the definition of $e_1(t)$ in Section 2. We are now going to show that

$$\widetilde{k}(r) = \int_{(0,\infty)} e_j(ru)dh^{(j)}(\log u) \tag{5.4}$$

for $j = 0, 1, \dots, m$.

By (5.2), (5.4) is true for $j = 0$. Suppose that (5.4) is true for some $j < m$. Then

$$\begin{aligned}
\widetilde{k}(r) &= \int_0^{\infty} e_j(ru)h^{(j+1)}(\log u)u^{-1}du = \int_0^{\infty} e_j(ru)u^{-1}du \int_{(-\infty,\log u]} dh^{(j+1)}(v) \\
&= \int_{(-\infty,\infty)} dh^{(j+1)}(v) \int_{e^v}^{\infty} e_j(ru)u^{-1}du = \int_{(-\infty,\infty)} dh^{(j+1)}(v) \int_{re^v}^{\infty} e_j(u)u^{-1}du \\
&= \int_{(-\infty,\infty)} e_{j+1}(re^v)dh^{(j+1)}(v) = \int_{(0,\infty)} e_{j+1}(ru)dh^{(j+1)}(\log u),
\end{aligned}$$

which is (5.4) for $j+1$ in place of j. Hence (5.4) is true for $j = 0, 1, \dots, m$. Thus the h-function, $\widetilde{h}(u) = \widetilde{k}(e^{-u})$, of $\widetilde{\nu}$ satisfies

$$\widetilde{h}(-\log r) = \int_{(0,\infty)} e_j(ru)dh^{(j)}(\log u)$$

for $j = 0, 1, \dots, m$. Thus for any $r_1 < r_2$,

$$\begin{aligned}
\int_{r_1}^{r_2} \widetilde{h}'(-\log r)r^{-1}dr &= -(\widetilde{h}(-\log r_2) - \widetilde{h}(-\log r_1)) \\
&= -\int_{(0,\infty)} (e_j(r_2u) - e_j(r_1u))dh^{(j)}(\log u) \\
&= \int_{(0,\infty)} dh^{(j)}(\log u) \int_{r_1}^{r_2} e_{j-1}(ru)r^{-1}dr \\
&= \int_{r_1}^{r_2} r^{-1}dr \int_{(0,\infty)} e_{j-1}(ru)dh^{(j)}(\log u),
\end{aligned}$$

and hence

$$\widetilde{h}'(-\log r)=\int_{(0,\infty)} e_{j-1}(ru)dh^{(j)}(\log u),\qquad \text{a. e. } r>0 \tag{5.5}$$

for $j=1,\ldots,m$. Recall that $\widetilde{h}$ is of C^∞, because $\widetilde{\mu}\in T_m\subset T_0$. On the other hand, the right-hand side of (5.5) is continuous and decreasing in r. Hence (5.5) holds for all $r>0$. Repeating this argument, we have, for $0\leqslant l\leqslant m$,

$$\widetilde{h}^{(l)}(-\log r)=\int_{(0,\infty)} e_{j-l}(ru)dh^{(j)}(\log u)\quad \text{for } j=l,\ldots,m.$$

Hence

$$\widetilde{h}^{(l)}(-\log r)=\int_{(0,\infty)} e^{-ru}dh^{(l)}(\log u)\quad \text{for } l=0,\ldots,m, \tag{5.6}$$

which implies the complete monotonicity of $\widetilde{h}^{(l)}(-\log r)$, and thus $\widetilde{\mu}\in T'_m$.

Part 2. (Proof that $T'_m\subset T_m$.) We use induction in m. We already know that $T'_0=T_0$. Given $1\leqslant m<\infty$, assume that $T'_{m-1}\subset T_{m-1}$. Let $\widetilde{\mu}\in T'_m$. Then $\widetilde{\mu}\in T_0$ and we can find $\mu\in L_0$ such that $\widetilde{\mu}=\Upsilon\mu$. In order to show $\widetilde{\mu}\in T_m$, it is enough to show $\mu\in L_m$, again by (2.38) of Theorem D. Let $\widetilde{\nu}$ and ν be the Lévy measures of $\widetilde{\mu}$ and μ. If $\widetilde{\nu}=0$, then $\widetilde{\mu}$ and μ are Gaussian and $\mu\in L_m$. Suppose $\widetilde{\nu}\neq 0$. Recalling the converse part in the proof of Theorem B, we can give the k-function of ν as $k_\xi(v)=\lim_{v'\downarrow v}k^\sharp_\xi(1/v')$ where $k^\sharp_\xi(u)=\widetilde{R}_\xi((0,u])$ and $\widetilde{k}_\xi(r)=\int_{(0,\infty)}e^{-ru}\widetilde{R}_\xi(du)$. The h-function h_ξ of ν is given by $h_\xi(\log u)=\lim_{u'\downarrow u}k_\xi(1/u')=\widetilde{R}_\xi((0,u])$. Hence

$$\widetilde{h}_\xi(-\log r)=\lim_{r'\uparrow r}\widetilde{k}_\xi(r')=\int_{(0,\infty)} e^{-ru}dh_\xi(\log u).$$

Since $\widetilde{\mu}\in T'_m\subset T'_{m-1}$, we have $\mu\in L_{m-1}$ by the induction hypothesis. Thus

$$\widetilde{h}^{(m-1)}(-\log r)=\int_{(0,\infty)} e^{-ru}dh^{(m-1)}(\log u) \tag{5.7}$$

by (5.6) of Part 1 (we are omitting ξ in the subscript). It follows from $\widetilde{\mu}\in T'_m$ that, for $j=0,\ldots,m$, $\widetilde{h}^{(j)}(-\log r)$ is not only completely monotone but also $\widetilde{h}^{(j)}(-\infty)=0$. Indeed, $\widetilde{h}(-\infty)=0$ since $\widetilde{k}(\infty)=0$, $\widetilde{h}'(-\infty)=0$ since $\widetilde{h}(u_2)-\widetilde{h}(u_1)=\int_{u_1}^{u_2}\widetilde{h}'(u)du$, and so on. Therefore, there is a measure σ on $(0,\infty)$ such that

$$\widetilde{h}^{(m)}(\log r)=\int_{(0,\infty)} e^{-ru}\sigma(du).$$

Now,

$$\begin{aligned}\widetilde{h}^{(m-1)}(-\log r)&=\int_r^\infty \widetilde{h}^{(m)}(-\log u)u^{-1}du=\int_r^\infty u^{-1}du\int_{(0,\infty)}e^{-uv}\sigma(dv)\\&=\int_{(0,\infty)}\sigma(dv)\int_r^\infty e^{-uv}u^{-1}du=\int_{(0,\infty)}\sigma(dv)\int_v^\infty e^{-ru}u^{-1}du\end{aligned}$$

$$= \int_0^\infty e^{-ru} u^{-1} \sigma((0,u])du.$$

This together with (5.7) implies that $dh^{(m-1)}(\log u) = u^{-1}\sigma((0,u])du$. It follows that the Radon–Nikodým derivative $h^{(m)}(u)$ of $h^{(m-1)}(u)$ exists and we have $h^{(m)}(\log u) = \sigma((0,u])$. Hence $h^{(m)}$ is nonnegative and increasing, meaning that h is monotone of order $m+1$. Thus $\mu \in L_m$, completing the proof. $\square$

Remark 5.2. It follows from Theorem E and (5.1) that $\mu \in T_\infty(\mathbb{R}^d)$ if and only if $\mu \in L(\mathbb{R}^d)$ and the Lévy measure ν of μ is either $\nu = 0$ or $\nu \neq 0$ having h-function $h_\xi(u)$ such that

$$h_\xi^{(j)}(-\log r) \text{ is completely monotone in } r > 0 \text{ for all } j = 0, 1, \ldots, \ \lambda\text{-a.e. } \xi \qquad (5.8)$$

where λ is the spherical component of ν. The property (5.8) is equivalent to the absolute monotonicity of $h_\xi(u)$ in u, λ-a.e. ξ. We can prove this directly, but this is also a consequence of $T_\infty = L_\infty$ in (2.41) and of Proposition 5.1 (ii).

6. Proof of Theorem F

We prove the characterization of $B(\mathbb{R}^d)$ and $T(\mathbb{R}^d)$ by elementary mixed-exponential variables and elementary Γ-variables in $\mathbb{R}^d$.

Proof of Theorem F. Part 1. (Characterization of $B(\mathbb{R}^d)$.) Let B^0 be the smallest class of distributions on $\mathbb{R}^d$ closed under convolution and convergence and containing the distributions of all elementary mixed-exponential variables in $\mathbb{R}^d$. In order to prove $B^0 = B(\mathbb{R}^d)$, it is enough to check the following facts:

$$B(\mathbb{R}^d) \text{ is closed under convolution and convergence,} \qquad (6.1)$$

$$\mathcal{L}(Ux) \in B(\mathbb{R}^d) \text{ for all elementary mixed-exponential variables } Ux \text{ in } \mathbb{R}^d, \qquad (6.2)$$

$$\delta_x \in B^0 \text{ for all } x \in \mathbb{R}^d, \qquad (6.3)$$

$$\text{if } \mu = \mu_{(0,\nu,0)} \in B(\mathbb{R}^d), \text{ then } \mu \in B^0, \qquad (6.4)$$

$$\text{if } \mu = \mu_{(A,0,0)}, \text{ then } \mu \in B^0. \qquad (6.5)$$

Indeed, (6.1) and (6.2) imply $B(\mathbb{R}^d) \supset B^0$; (6.3)–(6.5) imply $B(\mathbb{R}^d) \subset B^0$.

Proof of (6.1). Closedness under convolution is evident. Since $B(\mathbb{R}^d) = \Upsilon(ID(\mathbb{R}^d))$, closedness under convergence is proved in Proposition 2.7 (iv). Moreover, by Lemma 4.1, $B(\mathbb{R}^d)$ is completely closed in the strong sense.

Proof of (6.2). Let

$$P(U \in B) = \sum_{j=1}^{n} c_j \int_{B\cap(0,\infty)} a_j e^{-a_j s} ds, \qquad B \in \mathcal{B}(\mathbb{R})$$

with $c_j > 0$, $\sum_{j=1}^{n} c_j = 1$, and $0 < a_1 < \cdots < a_n < \infty$. Then, by Lemma 51.14 of Sato (1999),

$$Ee^{ivU} = \exp \int_0^\infty (e^{ivr} - 1) l(r) dr, \qquad v \in \mathbb{R},$$
$$l(r) = \int_0^\infty e^{-ru} \sum_{j=1}^{n} 1_{(a_j, a_j')}(u) du$$

with $a_1 < a_1' < a_2 < a_2' < a_3 < \cdots < a_n < a_n' = \infty$. Hence, for $x \in \mathbb{R}^d \setminus \{0\}$,

$$\begin{aligned} C_{Ux}(z) &= \int_0^\infty (e^{i\langle z,x\rangle r} - 1) l(r) dr = \int_S \delta_{x/|x|}(d\xi) \int_0^\infty (e^{i\langle z,\xi\rangle |x| r} - 1) l(r) dr \\ &= \int_S \delta_{x/|x|}(d\xi) \int_0^\infty (e^{i\langle z,\xi\rangle r} - 1) l(r/|x|) dr/|x|, \qquad z \in \mathbb{R}^d. \end{aligned}$$

Therefore $\mathcal{L}(Ux) \in B(\mathbb{R}^d)$.

Proof of (6.3) *and* (6.4). Let $B^0(\mathbb{R}_+)$ be the smallest class closed under convolution and convergence and containing all finite mixtures of exponential distributions. Then $\mu^0 \in B^0(\mathbb{R}_+)$ if and only if

$$C_{\mu^0}(v) = \int_0^\infty (e^{ivr} - 1) l(r) dr + ir^0 v, \qquad v \in \mathbb{R},$$

with $r^0 \geqslant 0$ and with $l(r)$ being completely monotone and satisfying $\int_0^\infty (r \wedge 1) l(r) dr < \infty$ (Theorem 51.10 of Sato (1999)). Therefore, if $l(r)$ is such a function and if $\mu \in ID(\mathbb{R}^d)$ satisfying

$$C_\mu(z) = \int_0^\infty (e^{i\langle z,\xi^0\rangle r} - 1) l(r) dr + i\langle z, \xi^0\rangle r^0, \qquad z \in \mathbb{R}^d$$

with some $\xi^0 \in S$ and $r^0 \geqslant 0$, then $\mu \in B^0$. Choosing $l(r) = 0$, we get (6.3).

Consider $\mu \in ID(\mathbb{R}^d)$ such that

$$C_\mu(z) = \int_S \lambda(d\xi) \int_0^\infty (e^{i\langle z,\xi\rangle r} - 1) l_\xi(r) dr + i\langle \gamma^0, z\rangle, \qquad z \in \mathbb{R}^d \tag{6.6}$$

with $\gamma^0 \in \mathbb{R}^d$, $l_\xi(r)$ completely monotone, $\int_S \lambda(d\xi) \int_0^\infty (r \wedge 1) l_\xi(r) dr < \infty$, and $\mathrm{Supp}\,(\lambda)$ being a finite set. Then $\mu \in B^0$ by the discussion above.

Next consider $\mu \in ID(\mathbb{R}^d)$ such that

$$C_\mu(z) = \int_S \lambda(d\xi) \int_0^\infty g(z, r\xi) l_\xi(r) dr$$

with $l_\xi(r)$ completely monotone and $\int_S \lambda(d\xi)\int_0^\infty (r^2\wedge 1)l_\xi(r)dr < \infty$. Here g is the function of (3.11). This is a general form of $\mu = \mu_{(0,\nu,0)} \in B(\mathbb{R}^d)$. Using Remark 3.2, write

$$C_\mu(z) = \int_S \lambda(d\xi)\int_{(0,\infty)} Q_\xi(du)\int_0^\infty g(z, r\xi)e^{-ru}dr,$$

where we have (3.3) with $a(u)$ of (3.4). We can choose finite measures λ_n and $Q_{n,\xi}$ ($n = 1, 2, \ldots$) such that $\mathrm{Supp}\,(\lambda_n)$ is a finite set for each n, $\mathrm{Supp}\,(Q_{n,\xi})$ is a finite set for each n and ξ, and

$$\int_S \lambda_n(d\xi)\int_{(0,\infty)} a(u)f(u,\xi)Q_{n,\xi}(du) \to \int_S \lambda(d\xi)\int_{(0,\infty)} a(u)f(u,\xi)Q_\xi(du)$$

for any bounded continuous function $f(u,\xi)$ on $(0,\infty)\times S$. Let

$$\nu_n(B) = \int_S \lambda_n(d\xi)\int_{(0,\infty)} Q_{n,\xi}(du)\int_0^\infty 1_B(r\xi)e^{-ru}dr,$$

and let μ_n be such that $C_{\mu_n}(z) = \int g(z,x)\nu_n(dx)$. Then, noticing that $\int_S \lambda(d\xi)\int_0^\infty (r\wedge 1)l_\xi(r)dr < \infty$ is equivalent to $\int_S \lambda(d\xi)\int_{(0,\infty)} a_0(u)Q_\xi(du) < \infty$ with $a_0(u) = u^{-2}\int_0^u ve^{-v}dv + u^{-1}e^{-u}$ (thus $a_0(u) \sim u^{-1}$ as $u \downarrow 0$ and $a_0(u) \sim u^{-2}$ as $u \to \infty$), we see that $C_{\mu_n}(z)$ is of the form (6.6). Hence $\mu_n \in B^0$. Denote

$$f_z(u,\xi) = a(u)^{-1}\int_0^\infty g(z, r\xi)e^{-ru}dr. \tag{6.7}$$

Then $f_z(u,\xi)$ is bounded and continuous in $(u,\xi) \in (0,\infty)\times S$, since

$$\int_0^\infty |g(z,r\xi)|e^{-ru}dr \leqslant c_z\int_0^\infty r^2(1+r^2)^{-1}e^{-ru}dr \leqslant c_z\int_0^\infty (r^2\wedge 1)e^{-ru}dr = c_z a(u)$$

with c_z of (4.4). Thus we have $\int g(z,x)\nu_n(dx) \to \int g(z,x)\nu(dx)$, that is, $\mu_n \to \mu$. Hence $\mu \in B^0$.

Proof of (6.5). Let $\mu = \mu_{(A,0,0)}$, the Gaussian distribution with mean 0 and covariance matrix A. We claim that $\mu \in B^0$. We use the function $f_z(u,\xi)$ in (6.7). Let us show that

$$\lim_{u\to\infty} f_z(u,\xi) = -\frac{1}{2}\langle z,\xi\rangle^2. \tag{6.8}$$

Indeed,

$$\begin{aligned} f_z(u,\xi) &= \frac{1}{a(u)}\int_0^\infty (e^{i\langle z,\xi\rangle r} - 1 - i\langle z,\xi\rangle r)e^{-ru}dr + \frac{i\langle z,\xi\rangle}{a(u)}\int_0^\infty \frac{r^3}{1+r^2}e^{-ru}dr \\ &= \frac{1}{ua(u)}\int_0^\infty (e^{i\langle z,\xi\rangle r/u} - 1 - i\langle z,\xi\rangle r/u)e^{-r}dr + \frac{i\langle z,\xi\rangle}{ua(u)}\int_0^\infty \frac{(r/u)^3}{1+(r/u)^2}e^{-r}dr, \end{aligned}$$

and, since $a(u) \sim 2u^{-3}$ as $u \to \infty$, the second term in the last expression tends to 0 and the first term tends to $-(1/2)\langle z, \xi\rangle^2$ because

$$e^{i\langle z,\xi\rangle r/u} - 1 - i\langle z, \xi\rangle r/u \sim -(1/2)\langle z, \xi\rangle^2 r^2/u^2$$

and

$$(ua(u))^{-1}|e^{i\langle z,\xi\rangle r/u} - 1 - i\langle z, \xi\rangle r/u| \leqslant (1/2)|\langle z, \xi\rangle|^2 r^2$$

uniformly for large u. In addition to (6.8),

$$\begin{aligned}|f_z(u, \xi)| &\leqslant \frac{c_z}{a(u)} \int_0^\infty \frac{r^2}{1 + r^2} e^{-ru} dr \leqslant \frac{c_z}{ua(u)} \int_0^\infty \frac{(r/u)^2}{1 + (r/u)^2} e^{-r} dr \\ &\leqslant c_z u^2 \int_0^\infty \frac{r^2}{u^2 + r^2} e^{-r} dr \leqslant 2c_z\end{aligned}$$

for u so large that $a(u) \geqslant u^{-3}$. Let X be a Gaussian random variable on $\mathbb{R}^d$ with $\mathcal{L}(X) = \mu$ and let

$$\lambda(B) = E(1_B(X/|X|)\,|X|^2) \qquad \text{for } B \in \mathcal{B}(S).$$

Define μ_n as

$$C_{\mu_n}(z) = \int_S \lambda(d\xi) \int_{(0,\infty)} \delta_n(du) f_z(u, \xi),$$

where δ_n is the δ-distribution located at n. Then $\mu_n \in B^0$ by (6.4) and

$$C_{\mu_n}(z) = \int_S \lambda(d\xi) f_z(n, \xi) \to -\frac{1}{2} \int_S \langle z, \xi\rangle^2 \lambda(d\xi).$$

This means $\mu_n \to \mu$, since

$$\int_S \langle z, \xi\rangle^2 \lambda(d\xi) = E(\langle z, X/|X|\rangle^2 |X|^2) = E(\langle z, X\rangle^2) = \sum_{j,l=1}^d E(z_j z_l X_j X_l) = \langle z, Az\rangle.$$

Now we have $\mu \in B^0$.

Part 2. (Characterization of $T(\mathbb{R}^d)$.) We can give a proof similar to that for $B(\mathbb{R}^d)$. Let T^0 be the smallest class of distributions on $\mathbb{R}^d$ closed under convolution and convergence and containing the distributions of all elementary Γ-variables in $\mathbb{R}^d$. This time it is enough to check the following:

$$T(\mathbb{R}^d) \text{ is closed under convolution and convergence,} \tag{6.9}$$

$$\mathcal{L}(Ux) \in T(\mathbb{R}^d) \text{ for all elementary } \Gamma\text{-variables } Ux \text{ in } \mathbb{R}^d, \tag{6.10}$$

$$\delta_x \in T^0 \text{ for all } x \in \mathbb{R}^d, \tag{6.11}$$

$$\text{if } \mu = \mu_{(0,\nu,0)} \in T(\mathbb{R}^d), \text{ then } \mu \in T^0, \tag{6.12}$$

$$\text{if } \mu = \mu_{(A,0,0)}, \text{ then } \mu \in T^0. \tag{6.13}$$

The proof of (6.9) is similar to that of (6.1). If U is a real Γ-distributed variable, then

$$Ee^{ivU} = \exp \int_0^\infty (e^{ivr} - 1) a e^{-br} r^{-1} dr, \qquad v \in \mathbb{R}$$

with some $a > 0$ and $b > 0$ and, for any $x \in \mathbb{R}^d \setminus \{0\}$, the elementary Γ-variable Ux satisfies

$$C_{Ux}(z) = \int_0^\infty (e^{i\langle z,x\rangle r} - 1) a e^{-br} r^{-1} dr = \int_S \delta_{x/|x|}(d\xi) \int_0^\infty (e^{i\langle z,\xi\rangle r} - 1) a e^{-br/|x|} r^{-1} dr.$$

Hence (6.10).

To see $\delta_x \in T^0$ for $x \neq 0$, note that

$$n|x| \int_0^\infty (e^{i\langle z,x\rangle r} - 1) e^{-n|x|r} r^{-1} dr = n|x| \int_0^\infty (e^{i\langle z,x\rangle r/(n|x|)} - 1) e^{-r} r^{-1} dr \to i\langle z, x\rangle$$

as $n \to \infty$, since $n|x|r^{-1}(e^{i\langle z,x\rangle r/(n|x|)} - 1)$ tends to $i\langle z,x\rangle$ boundedly by $|\langle z,x\rangle|$. That is, δ_x is approximated by distributions of elementary Γ-variables if $x \neq 0$. Evidently $\delta_0 \in T^0$, since $Ux_n \to 0$ as $x_n \to 0$. Hence (6.11).

The proof of (6.12) is similar to that of (6.4). In this case a general $\mu = \mu_{(0,\nu,0)}$ in $T(\mathbb{R}^d)$ satisfies

$$C_\mu(z) = \int_S \lambda(d\xi) \int_{(0,\infty)} R_\xi(du) \int_0^\infty g(z, r\xi) e^{-ru} r^{-1} dr,$$

where R_ξ satisfies (3.8) with $b(u)$ of (3.9). Instead of $f_z(u,\xi)$ we use

$$h_z(u,\xi) = b(u)^{-1} \int_0^\infty g(z, r\xi) e^{-ur} r^{-1} dr,$$

which is bounded and continuous in $(u,\xi) \in (0,\infty) \times S$.

Finally, (6.13) is proved like (6.5), by using

$$\lim_{u\to\infty} h_z(u,\xi) = -\frac{1}{2}\langle z,\xi\rangle^2$$

for the function $h_z(u,\xi)$ above. $\square$

7. Examples

Example 7.1. *Tempered stable distributions of Rosiński.* Rosiński (2002) introduced tempered stable distributions on $\mathbb{R}^d$. His definition is as follows. Let $0 < \alpha < 2$. A distribution μ on $\mathbb{R}^d$ is tempered stable if $\mu \in ID(\mathbb{R}^d)$ with triplet (A, ν, γ) such that $A = 0$ and

$$\nu(B) = \int_{\mathbb{R}^d} \rho(dx) \int_0^\infty 1_B(sx) s^{-\alpha-1} e^{-s} ds, \qquad B \in \mathcal{B}(\mathbb{R}^d), \tag{7.1}$$

where ρ is a measure on $\mathbb{R}^d$ such that

$$\rho(\{0\}) = 0 \text{ and } \int_{\mathbb{R}^d} |x|^\alpha \rho(dx) < \infty. \tag{7.2}$$

This is a generalization of tilted stable laws where ρ is concentrated on a sphere centered at the origin. We can prove that a measure ν of the form (7.1) is the Lévy measure of some infinitely divisible distribution if and only if ρ satisfies

$$\rho(\{0\}) = 0 \text{ and } \int_{\mathbb{R}^d} (|x|^2 \wedge |x|^\alpha)\rho(dx) < \infty. \tag{7.3}$$

The condition (7.2) is stronger than (7.3) regarding the singularity of ρ around the origin. Rosiński finds that ν, not identically zero, satisfies (7.1) together with (7.2) if and only if ν has polar decomposition

$$\nu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi) r^{-\alpha-1} q_\xi(r) dr, \tag{7.4}$$

where $q_\xi(r)$ is completely monotone in r, measurable in ξ and satisfies

$$q_\xi(+\infty) = 0 \text{ and } \int_S q_\xi(0+)\lambda(d\xi) < \infty. \tag{7.5}$$

We can prove that, under the assumption that ν, not identically zero, satisfies (7.4) with $q_\xi(r)$ completely monotone in r, measurable in ξ, and $q_\xi(+\infty) = 0$, then ν is the Lévy measure of some infinitely divisible distribution if and only if

$$\int_S \lambda(d\xi) \left(\int_{(0,1]} Q_\xi(du) + \int_{(1,\infty)} u^{\alpha-2} Q_\xi(du) \right) < \infty, \tag{7.6}$$

where $Q_\xi(du)$ satisfies $q_\xi(r) = \int_{(0,\infty)} e^{-ru} Q_\xi(du)$. This condition is clearly weaker than $\int_S q_\xi(0+)\lambda(d\xi) < \infty$. Let us call α in (7.1)–(7.2) the index of the corresponding tempered stable distribution μ. Following Rosiński, we denote by $TS(\alpha) = TS(\alpha, \mathbb{R}^d)$ the class of tempered stable distributions on $\mathbb{R}^d$ with index α. Notice that, by the representation (7.4)–(7.5), $TS(\alpha) \cap TS(\alpha')$ consists only of δ-distributions if $\alpha \neq \alpha'$.

Rosiński studied Lévy processes $\{X_t\}$ with $\mathcal{L}(X_1) \in TS(\alpha)$ and showed their functional limit theorems for small t and for large t, their absolute continuity on path spaces with respect to some α-stable Lévy processes, and their series representations.

Fix the dimension d arbitrarily. Omitting $\mathbb{R}^d$ in $T(\mathbb{R}^d)$, $T_1(\mathbb{R}^d)$, $L_1(\mathbb{R}^d)$ and so on, we make the following statements.

(i) For every $0 < \alpha < 2$, $TS(\alpha) \subset T$. This is obvious since $r^{-\alpha} q_\xi(r)$ is completely monotone whenever $q_\xi(r)$ is.

(ii) If $1 \leqslant \alpha < 2$, then $TS(\alpha) \subset T_1$.

(iii) If $2/3 \leqslant \alpha < 2$, then $TS(\alpha) \subset L_2$.

(iv) If $1/4 \leqslant \alpha < 2$, then $TS(\alpha) \subset L_1$.

(v) Let $0 < \alpha < 1/4$. If μ is in $TS(\alpha)$ with $q_\xi(r) = c(\xi)e^{-b(\xi)r}$ for all ξ in a set of positive λ-measure, where $c(\xi)$ and $b(\xi)$ are positive measurable functions of ξ, then $\mu \notin L_1$ and consequently $\mu \notin T_1$.

The proof is as follows. Let $\mu \in TS(\alpha)$. The k-function of μ is $k_\xi(r) = r^{-\alpha}q_\xi(r)$. We suppress the subscript ξ in $k_\xi(r)$, $h_\xi(u)$, $q_\xi(r)$, and $Q_\xi(dv)$. Then

$$\begin{aligned}
h(u) &= e^{\alpha u}q(e^{-u}),\\
h'(u) &= \alpha e^{\alpha u}q(e^{-u}) - e^{(\alpha-1)u}q'(e^{-u}),\\
h''(u) &= \alpha^2 e^{\alpha u}q(e^{-u}) - (2\alpha-1)e^{(\alpha-1)u}q'(e^{-u}) + e^{(\alpha-2)u}q''(e^{-u}),\\
h'''(u) &= \alpha^3 e^{\alpha u}q(e^{-u}) - (3\alpha^2-3\alpha+1)e^{(\alpha-1)u}q'(e^{-u})\\
&\qquad + 3(\alpha-1)e^{(\alpha-2)u}q''(e^{-u}) - e^{(\alpha-3)u}q'''(e^{-u}).
\end{aligned}$$

Recall that $q(r)$ is completely monotone. If $1 \leqslant \alpha < 2$, then $h'(-\log r) = \alpha r^{-\alpha}q(r) - r^{1-\alpha}q'(r)$ is completely monotone and hence $\mu \in T_1$ by Theorem E. We have $h'(u) \geqslant 0$ for all $0 < \alpha < 2$ and $h''(u) \geqslant 0$ for $1/4 \leqslant \alpha < 2$ since

$$\begin{aligned}
h''(-\log r) &= r^{-\alpha}[\alpha^2 q(r) - (2\alpha-1)rq'(r) + r^2q''(r)]\\
&= r^{-\alpha}\int_{(0,\infty)} (\alpha^2 + (2\alpha-1)rv + r^2v^2)e^{-rv}Q(dv)\\
&= r^{-\alpha}\int_{(0,\infty)} ((rv+\alpha-1/2)^2 + \alpha - 1/4)e^{-rv}Q(dv).
\end{aligned}$$

Thus $\mu \in L_1$ if $1/4 \leqslant \alpha < 2$. If $0 < \alpha < 1/4$ and if $q(r)$ is as is assumed in (v), then, for ξ in a set of positive λ-measure,

$$h''(-\log r) = cr^{-\alpha}((rb+\alpha-1/2)^2 + \alpha - 1/4)e^{-rb} < 0$$

for $r = (1/2-\alpha)/b$ and hence $\mu \notin L_1$. If $2/3 \leqslant \alpha < 2$, then

$$h'''(-\log r) = r^{-\alpha}\int_{(0,\infty)} [\alpha^3 + (3\alpha^2-3\alpha+1)rv + 3(\alpha-1)r^2v^2 + r^3v^3)e^{-rv}Q(dv) \geqslant 0,$$

as we can check $g(w) = \alpha^3 + (3\alpha^2-3\alpha+1)w + 3(\alpha-1)w^2 + w^3$ is nonnegative for $w \geqslant 0$, because $g'(w) \geqslant 0$ for $w \geqslant 0$ and $g(0) \geqslant 0$.

The simplest case of tempered stable distributions is given by μ on $\mathbb{R}$ with

$$C_\mu(z) = c\int_0^\infty (e^{izx}-1)x^{-\alpha-1}e^{-bx}dx$$

with $0 < \alpha < 1$ and positive constants c and b. This μ is the distribution of a tilted stable subordinator at time 1. The relation with $L_1(\mathbb{R})$ of this was discussed in Maejima, Sato and Watanabe (2000) p. 397. When $\alpha = 1/2$ this gives an inverse Gaussian distribution. Thus $\mu \in L_1(\mathbb{R})$ and $\Upsilon\mu \in T_1(\mathbb{R})$ for an inverse Gaussian μ.

Example 7.2. As mentioned near the end of Section 2, many examples of distributions in $T(\mathbb{R})$ supported on $\mathbb{R}_+$ are given in Bondesson (1992) and Steutel and van Harn (2004). Distributions in $T(\mathbb{R})$ with support $\mathbb{R}$ can be constructed by the transformation Υ if we have selfdecomposable distributions with support $\mathbb{R}$. For such selfdecomposable distributions as well as other examples, see Jurek (1997). We have discussed T_m. Since we have several examples of distributions in $L_m(\mathbb{R})$, $m = 1, 2$, with explicit densities, we can construct concrete examples of distributions in $T_m(\mathbb{R})$, $m = 1, 2$.

Let $\{\Gamma_t^{(a)}\}$ be a Γ-process with scale parameter $a > 0$, $\{Y_t\}$ a strictly α-stable subordinator $(0 < \alpha < 1)$, and $\{Z_t\}$ a symmetric α'-stable Lévy process $(0 < \alpha' \leqslant 2)$. We have, for $t > 0$,

$$\begin{aligned} P(\Gamma_t^{(a)} \in B) &= \frac{a^t}{\Gamma(t)} \int_{B \cap (0,\infty)} x^{t-1} e^{-ax} dx, \quad B \in \mathcal{B}(\mathbb{R}), \\ E(e^{-uY_t}) &= \exp(-btu^{\alpha}), \quad u \geqslant 0, \\ E(e^{izZ_t}) &= \exp(-ct|z|^{\alpha'}), \quad z \in \mathbb{R}, \end{aligned}$$

where b and c are positive constants. The distribution of $\log \Gamma_t^{(a)}$ has density

$$(a^t/\Gamma(t)) \exp(tx - ae^x), \quad x \in \mathbb{R}$$

for $t > 0$. Linnik and Ostrovskii (1977) (Chap. 2, Sect. 6, Example 3) shows that this distribution is infinitely divisible with triplet $(0, \nu, \gamma)$ with

$$\nu(dx) = 1_{(-\infty,0)}(x)|x|^{-1}(1 - e^x)^{-1} e^{tx} dx$$

and some γ (see also Jurek (1997) and Sato (1999) E 18.19). Thus $\mathcal{L}(\log \Gamma_t^{(a)}) \in L(\mathbb{R})$ for all $t > 0$ and $a > 0$. Akita and Maejima (2002) showed the following.

(i) $\mathcal{L}(\log \Gamma_t^{(a)}) \in L_1(\mathbb{R})$ for $t \geqslant 1/2$.

(ii) $\mathcal{L}(\log \Gamma_t^{(a)}) \in L_2(\mathbb{R})$ for $t \geqslant 1$.

(iii) $\mathcal{L}(\log Y_t) \in L_1(\mathbb{R})$ for $t > 0$.

(iv) $\mathcal{L}(\log |Z_t|) \in L_1(\mathbb{R})$ for $t > 0$.

Applying the mapping Υ to these distributions, we get examples of $T_1(\mathbb{R})$ and $T_2(\mathbb{R})$. In particular $\Upsilon(\mathcal{L}(\log \Gamma_t^{(a)}))$ has Lévy measure

$$1_{(-\infty,0)}(x) \left(\int_0^{\infty} \frac{e^{tx/s - s}}{1 - e^{x/s}} ds \right) \frac{dx}{|x|}$$

and belongs to $T_1(\mathbb{R})$ for $t \geqslant 1/2$ and to $T_2(\mathbb{R})$ for $t \geqslant 1$. The generating triplets of $\mathcal{L}(\log Y_t)$ and $\mathcal{L}(\log |Z_t|)$ can be obtained by the method of the proofs of (iii) and (iv) in Akita and Maejima (2002). They are purely non-Gaussian. The Lévy measure of $\mathcal{L}(\log Y_t)$ is

$$1_{(0,\infty)}(x)\frac{(e^{-\alpha x}-e^{-x})dx}{(1-e^{-\alpha x})(1-e^{-x})x}$$

for any $t>0$ if $b=1$ and that of $\mathcal{L}(\log |Z_t|)$ is

$$\left(1_{(-\infty,0)}(x)\frac{e^x}{1-e^{2x}}+1_{(0,\infty)}(x)\frac{e^{-\alpha' x}-e^{-2x}}{(1-e^{-2x})(1-e^{-\alpha' x})}\right)\frac{dx}{|x|}$$

for any $t>0$ if $c=1$. The explicit distributions for $\alpha=1/2$ and $\alpha'=1$ are

$$P(\log Y_t \in B)=\frac{t}{2\pi^{1/2}}\int_B \exp\left(-\frac{1}{2}x-\frac{t^2}{4}e^{-x}\right)dx \quad \text{for } b=1,$$

$$P(\log |Z_t| \in B)=\frac{2t}{\pi}\int_B \frac{e^x}{e^{2x}+t^2}dx \quad \text{for } c=1.$$

Recall that $\mathcal{L}(Y_t)=\mathcal{L}(1/\Gamma_{1/2}^{(t^2/4)})$ for this Y_t with $\alpha=1/2$ and $b=1$.

Example 7.3. Let $\{X_t\}$ be Brownian motion on $\mathbb{R}^d$ with drift $\gamma\in\mathbb{R}^d$, that is, $\{X_t\}$ is the Lévy process with $\mathcal{L}(X_t)=\mu_{(tI,0,t\gamma)}$, where I is the $d\times d$ unit matrix. Let $\{Z_t\}$ be a subordinator such that $\mathcal{L}(Z_t)$ is a generalized Γ-convolution (equivalently, $\mathcal{L}(Z_t)$ is in $T(\mathbb{R})$ and has support in $\mathbb{R}_+$). Subordination of $\{X_t\}$ by $\{Z_t\}$ gives a Lévy process $\{Y_t\}$ on $\mathbb{R}^d$. That is, $Y_t=X_{Z_t}$, where $\{X_t\}$ and $\{Z_t\}$ are independent. Assume that $\mathcal{L}(Z_t)$ is not a δ-distribution. Let $\mu^t=\mathcal{L}(Y_t)$. In the case $d=1$, Halgreen (1979) showed that $\mu^t\in L(\mathbb{R})$ for any γ. Then Takano (1989, 1990) showed that in the case $d\geqslant 2$ one had a different phenomenon; under some additional assumption on the so-called U-measure of the generalized Γ-convolution $\mathcal{L}(Z_1)$, he proved that, for any fixed $t>0$, $\mu^t\in L(\mathbb{R}^d)$ if and only if $\gamma=0$.

Generalized inverse Gaussian distributions are in the class of generalized Γ-convolutions (Halgreen (1979)). If $\mathcal{L}(Z_t)$ is a generalized inverse Gaussian, then the explicit expression of the density of $\mathcal{L}(Y_t)$ using modified Bessel functions is obtained by Barndorff-Nielsen (1977, 1978); the process $\{Y_t\}$ is referred to as a generalized hyperbolic motion, the finite dimensional laws of $\{Y_t\}$ being of the generalized hyperbolic type.

Let us assume that $\{Z_t\}$ is the Γ-process with scale parameter 1. This is a special case of the generalized inverse Gaussian. We have

$$\widehat{\mu^t}(z)=(1+2^{-1}|z|^2-i\langle\gamma,z\rangle)^{-t}.$$

The expression of the density of μ^t, $t>0$, mentioned above is in this case

$$c(t,\gamma)|x|^{t-(d/2)}K_{t-(d/2)}((2+|\gamma|^2)^{1/2}|x|)\,e^{\langle\gamma,x\rangle}$$

with $c(t,\gamma)=2(2\pi)^{-d/2}\Gamma(t)^{-1}(2+|\gamma|^2)^{-(t-(d/2))/2}$. Here $K_{t-(d/2)}$ is the modified Bessel function of the third kind with index $t-(d/2)$. In particular, $\mu^{(d+1)/2}$ has density

$$c\exp(-\sqrt{2+|\gamma|^2}\,|x|+\langle\gamma,x\rangle)$$

with a normalizing constant c. We can prove the following for every $t>0$.

(i) Let $d=1$. Then $\mu^t\in T(\mathbb{R})$ and $\mu^t\notin L_1(\mathbb{R})$ (hence $\mu^t\notin T_1(\mathbb{R})$), irrespective of whether $\gamma=0$ or $\gamma\neq 0$.

(ii) Let $d\geqslant 2$. If $\gamma=0$, then $\mu^t\in L(\mathbb{R}^d)$, $\mu^t\notin T(\mathbb{R}^d)$, and $\mu^t\notin L_1(\mathbb{R}^d)$. If $\gamma\neq 0$, then $\mu^t\notin L(\mathbb{R}^d)$ (hence $\mu^t\notin T(\mathbb{R}^d)$).

Proof of (i). Choose $\lambda=\delta_{+1}+\delta_{-1}$. It is known that $\mu^t\in L$ with k-function

$$k_\xi(r)=\begin{cases} t\exp[-(\sqrt{2+\gamma^2}-\gamma)\,r] & \text{for } \xi=+1\\ t\exp[-(\sqrt{2+\gamma^2}+\gamma)\,r] & \text{for } \xi=-1.\end{cases}$$

Hence $k_\xi(r)$ is completely monotone and $\mu^t\in T$. The fact that $\mu^t\notin L_1(\mathbb{R})$ is observed by Maejima, Sato and Watanabe (2000) p. 397.

Proof of (ii). As is shown by Takano (1989), the Lévy measure of μ^t has polar decomposition $\lambda(d\xi)$, $\nu_\xi(dr)$ where λ is the Lebesgue measure on the $(d-1)$-dimensional unit sphere S and

$$\nu_\xi(dr)=2t\,e^{\langle\gamma,\xi\rangle r}L_{d/2}(\sqrt{2+|\gamma|^2}\,r)r^{-1}dr \tag{7.7}$$

with $L_{d/2}(u)=(2\pi)^{-d/2}u^{d/2}K_{d/2}(u)$.

If $\gamma\neq 0$, then $\mu^t\notin L(\mathbb{R}^d)$, which is a special case of Takano (1990).

Now assume that $\gamma=0$. Write $p=d/2$ and $k(r)=r^pK_p(r)$. Since

$$k'(r)=-r^pK_{p-1}(r)<0,$$

we have $\mu^t\in L(\mathbb{R}^d)$ (this is also a consequence of the general result that subordination of a strictly stable Lévy process by a selfdecomposable subordinator gives a selfdecomposable Lévy process). Furthermore,

$$k''(r)=r^pK_{p-2}(r)-r^{p-1}K_{p-1}(r)=2^{-p}r^{2p-2}\int_0^\infty e^{-s-r^2/(4s)}s^{-p}(2s-1)ds$$

by the well-known integral representation of the modified Bessel function ((30.28) of Sato (1999)). Since (here we use that $d\geqslant 2$)

$$\int_0^{1/2}e^{-s-r^2/(4s)}s^{-p}(2s-1)ds\to-\infty\qquad\text{as } r\downarrow 0,$$

$k''(r) < 0$ when r is small enough. Hence $k(r)$ is not completely monotone and $\mu^t \notin T(\mathbb{R}^d)$. For the function $h(u) = k(e^{-u})$ we have

$$h''(u) = k''(e^{-u})e^{-2u} + k'(e^{-u})e^{-u} < 0 \quad \text{for some } u,$$

and hence $\mu^t \notin L_1(\mathbb{R}^d)$.

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