CHARACTERIZATIONS OF SUBCLASSES OF TYPE G DISTRIBUTIONS ON $\mathbb{R}^d$ BY STOCHASTIC INTEGRAL REPRESENTATIONS

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Summary:

The class of type G distributions on $\mathbb{R}^d$ and its nested subclasses are studied. An analytic characterization in terms of Lévy measures for the class of type G distributions is known. In this paper, probabilistic characterizations by stochastic integral representations for all classes are shown as well as analytic characterizations for the nested subclasses are given in terms of Lévy measures.

Keywords: infinitely divisible distribution on $\mathbb{R}^d$; type G distribution; stochastic integral representation; Lévy process

1. Introduction

Throughout the paper, $I(\mathbb{R}^d)$ (resp. $I_{sym}(\mathbb{R}^d)$) stands for the class of all infinitely divisible (resp. all symmetric infinitely divisible) distributions on $\mathbb{R}^d$. The characteristic function $\widehat{\mu}(z), z \in \mathbb{R}^d$, of an infinitely divisible distribution $\mu \in I(\mathbb{R}^d)$ has the so-called Lévy-Khintchine representation as follows:

$$\widehat{\mu}(z) = \exp\left[-\frac{1}{2}\langle z, Az\rangle + \mathrm{i}\langle \gamma, z\rangle + \int_{\mathbb{R}^d}\left(e^{\mathrm{i}\langle z,x\rangle} - 1 - \frac{\mathrm{i}\langle z, x\rangle}{1+|x|^2}\right)\nu(dx)\right],\ z \in \mathbb{R}^d,$$

where A is a symmetric nonnegative-definite $d\times d$ matrix, $\gamma \in \mathbb{R}^d$ and ν is a measure (called the Lévy measure) on $\mathbb{R}^d$ satisfying

$$\nu(\{0\}) = 0 \text{ and } \int_{\mathbb{R}^d}(|x|^2 \wedge 1)\nu(dx) < \infty.$$

The triplet (A, ν, γ) is called the generating triplet of $\mu \in I(\mathbb{R}^d)$. Let $C_\mu(z) = \log \widehat{\mu}(z)$ be the cumulant of $\mu \in I(\mathbb{R}^d)$. Summarizing the discussions in Rosinski (1991) and Maejima and Rosiński (2001, 2002), we use the following definition of type G distributions on $\mathbb{R}^d$.

Definition 1.1. A probability measure $\mu_0 \in I_{sym}(\mathbb{R}^d)$ is said to be of type G if its Lévy measure ν_0 is given by

$$\nu_0(B) = E\left[\nu(Z^{-1}B)\right], \quad B \in \mathcal{B}_0(\mathbb{R}^d), \tag{1.1}$$

where ν is another Lévy measure on $\mathbb{R}^d$ and Z is the real valued standard normal random variable. Here $\mathcal{B}_0(\mathbb{R}^d)$ is the class of all Borel sets B in $\mathbb{R}^d$ such that $B \subset \{|x| > \varepsilon\}$ for some $\varepsilon > 0$.

Remark 1.2. ν in (1.1) is not necessarily unique. However, if we let $\bar{\nu}$ be the symmetrization of ν defined by $\bar{\nu}(B) = \frac{1}{2}(\nu(B) + \nu(-B))$, then

$$\nu_0(B) = E\left[\bar{\nu}(Z^{-1}B)\right] = E\left[\bar{\nu}(|Z|^{-1}B)\right]$$

also holds and $\bar{\nu}$ is uniquely determined, (see Maejima and Rosiński (2002)).

Definition 1.1 is a multidimensional extension of the well-known notion of type G distributions on $\mathbb{R}$. (Another type of multidimensional extension is discussed in Barndorff-Nielsen and Pérez-Abreu (2002).) In one dimensional case, a type G random variable X can be expressed as $X \stackrel{\mathrm{d}}{=} V^{1/2}Z$, where $\stackrel{\mathrm{d}}{=}$ means equality in law, V is a nonnegative infinite divisible random variable, independent of Z. Among others, some examples of $\mathbb{R}$-valued type G distributions are symmetric stable distributions, convolution of symmetric stable distributions of different stability indices, symmetric gamma distributions (a special case of which is Laplace distribution), student t-distributions and normal inverse Gaussian distributions. The first two have multidimensional extensions.

In Maejima and Rosiński (2001), they introduced an operator $K : I_{sym}(\mathbb{R}^d) \to I_{sym}(\mathbb{R}^d)$, where $K(\mu)$ is a symmetric infinitely divisible distribution having the same Gaussian component as μ and the Lévy measure ν_0 in (1.1), where ν is the Lévy measure of $\mu \in I_{sym}(\mathbb{R}^d)$. Let $G_0(\mathbb{R}^d)$ be the class of all type G distributions on $\mathbb{R}^d$ and define, for $m \in \mathbb{N}$,

$$\begin{aligned} G_m(\mathbb{R}^d) = \{\mu_0 \in G_0(\mathbb{R}^d) : &\ \nu \text{ in (1.1) is the Lévy measure of} \\ &\text{some symmetric infinitely divisible distribution in } G_{m-1}(\mathbb{R}^d)\}. \end{aligned}$$

Also, define $G_\infty(\mathbb{R}^d) = \cap_{m\ge 0} G_m(\mathbb{R}^d)$. The classes $G_m(\mathbb{R}^d), 1 \le m \le \infty$, were introduced in Maejima and Rosiński (2001), and if we use the operator K,

$$G_0(\mathbb{R}^d) = K(I_{sym}(\mathbb{R}^d)) \tag{1.2}$$

and $G_m(\mathbb{R}^d) = K(G_{m-1}(\mathbb{R}^d))$. It was also shown in the paper that

$$I_{sym}(\mathbb{R}^d) \supset G_0(\mathbb{R}^d) \supset G_1(\mathbb{R}^d) \supset \cdots \supset G_m(\mathbb{R}^d) \supset \cdots \supset G_\infty(\mathbb{R}^d) \supset S_{sym}(\mathbb{R}^d),$$

where $S_{sym}(\mathbb{R}^d)$ is the class of all symmetric stable distributions on $\mathbb{R}^d$, and $G_\infty(\mathbb{R}^d)$ is the largest subclass of $I_{sym}(\mathbb{R}^d)$ which is invariant under the operation K.

2. The case $m = 0$

We start with the case $m = 0$. The following is a known characterization of the Lévy measures of type G distributions.

Proposition 2.1. (Maejima and Rosiński (2002).) A probability distribution $\mu_0 \in I_{sym}(\mathbb{R}^d)$ is of type G if and only if its Lévy measure ν_0 is either zero or it can be represented as

$$\nu_0(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi) g_\xi(r^2) dr, \quad B \in \mathcal{B}_0(\mathbb{R}^d),$$

where λ is a symmetric probability measure on the unit sphere S in $\mathbb{R}^d$ and $g_\xi(r)$ is a jointly measurable function such that $g_\xi = g_{-\xi}, \lambda - a.e.$ for any fixed $\xi \in S$, $g_\xi(\cdot)$ is completely monotone on $(0, \infty)$ and satisfies

$$\int_0^\infty (1 \wedge r^2) g_\xi(r^2) dr = c \in (0, \infty)$$

with c independent of ξ.

One of our purposes of this paper is to give a characterization of type G distributions by stochastic integrals with respect to Lévy processes. This is a probabilistic characterization, while Proposition 2.1 is an analytic characterization in terms of Fourier transform of the probability distribution. As to the definition of stochastic integrals of nonrandom functions with respect to Lévy processes $\{X_t\}$ on $\mathbb{R}^d$, we follow the definition in Sato (2004, 2005), whose idea is to define the integrals with respect to $\mathbb{R}^d$-valued independently scattered random measure induced by a Lévy process on $\mathbb{R}^d$. This idea was used in Urbanik and Woyczyński (1967) and Rajput and Rosinski (1989) for the case $d = 1$. See also Barndorff-Nielsen et al. (2006).

We call $\mu \in I(\mathbb{R}^d)$ to be selfdecomposable if for every $b \in (0, 1)$, there exists a distribution ρ_b on $\mathbb{R}^d$ such that $\widehat{\mu}(z) = \widehat{\mu}(bz)\widehat{\rho}_b(z)$. We know that the class of all selfdecomposable distributions can be characterized by stochastic integrals with respect to Lévy processes, namely μ is selfdecomposable if and only if there exists a Lévy process $\{X_t\}$ such that $E[\log^+ |X_1|] < \infty$ and $\mu = \mathcal{L}\left(\int_0^\infty e^{-t} dX_t\right)$, where $\mathcal{L}(Y)$ stands for the law of Y. In Jurek (1985), he defined s-selfdecomposable distributions. $\mu \in I(\mathbb{R}^d)$ is s-selfdecomposable if for every $b \in (0,1)$, there exists $\rho_b \in I(\mathbb{R}^d)$ such that $\widehat{\mu}(z) = \widehat{\mu}(bz)^b \widehat{\rho}_b(z)$, and he gave a stochastic integral characterization that μ is s-selfdecomposable if and only if $\mu = \mathcal{L}\left(\int_0^1 t dX_t\right)$ for some Lévy process $\{X_t\}$. However, only a few classes of infinitely divisible distributions were characterised in this way. Recently, Barndorff-Nielsen et al. (2006) found such characterizations for the Goldie-Steutel-Bondesson class (by their naming) and the Thorin class. (For the details, see Barndorff-Nielsen et al. (2006).) Our study is on the line of this history.

The following result for the integrability of stochastic integrals is due to Sato (2005), who studied more general stochastic integrals of matrix valued integrands with respect to additive processes. We state parts of Propositions 2.7 and 3.4 of Sato (2005) as a lemma below for our use.

Lemma 2.2. (Sato (2005).) Let $\mu \in I(\mathbb{R}^d)$ and $\{X_t^{(\mu)}\}$ the Lévy process with $\mathcal{L}(X_1^{(\mu)}) = \mu$ on $\mathbb{R}^d$ and let $f(t)$ be a real-valued measurable function on $[0,1]$. If

$$\int_0^1 f(t)^2 dt < \infty, \tag{2.1}$$

then $Y := \int_0^1 f(t) dX_t^{(\mu)}$ is integrable, $\int_0^1 |C_\mu(f(t)z)| dt < \infty$ and $C_{\mathcal{L}(Y)}(z) = \int_0^1 C_\mu(f(t)z) dt$. Furthermore, if we let (A, ν, γ) and (A_Y, ν_Y, γ_Y) be the generating triplets of μ and $\mathcal{L}(Y)$, respectively, then

$$A_Y = A \int_0^1 f(t)^2 dt, \tag{2.2}$$

$$\nu_Y(B) = \int_0^1 dt \int_{\mathbb{R}^d} 1_B(f(t)x) \nu(dx) \tag{2.3}$$

and

$$\gamma_Y = \int_0^1 f(t)\gamma + f(t) \int_{\mathbb{R}^d} x \left(\frac{1}{1+|f(t)x|^2} - \frac{1}{1+|x|^2} \right) \nu(dx) dt. \tag{2.4}$$

Let

$$\phi(u) = (\sqrt{2\pi})^{-1} e^{-u^2/2}$$

and

$$h(x) = \int_x^\infty \phi(u) du, \quad x \in \mathbb{R}.$$

Define the inverse function of h by h^*, namely, $x = h^*(t)$ if and only if $h(x) = t$. The stochastic integrals we need can be shown to be integrable as follows.

Theorem 2.3. *The stochastic integral*

$$\int_0^1 h^*(t)dX_t^{(\mu)}$$

is integrable for every $\mu \in I(\mathbb{R}^d)$.

Proof of Theorem 2.3. It is enough to show that $f(t) = h^*(t)$ satisfies the conditions in Lemma 2.2 for every $\mu \in I(\mathbb{R}^d)$. Since

$$\int_0^1 h^*(t)^2 dt = \int_{-\infty}^{\infty} r^2 \phi(r)dr = 1,$$

we have (2.1). This completes the proof. $\square$

Definition 2.4. For any $\mu \in I(\mathbb{R}^d)$, define a mapping $\mathcal{G} : I(\mathbb{R}^d) \to I(\mathbb{R}^d)$ by

$$\mathcal{G}(\mu) = \mathcal{L}\left(\int_0^1 h^*(t)dX_t^{(\mu)}\right).$$

Proposition 2.5. (*i*) *For any* $\mu \in I(\mathbb{R}^d)$,

$$\int_0^1 |C_\mu(zh^*(t))|dt < \infty \tag{2.5}$$

and

$$C_{\mathcal{G}(\mu)}(z) = \int_0^1 C_\mu(zh^*(t))dt, \quad z \in \mathbb{R}^d. \tag{2.6}$$

(*ii*) *The mapping* $\mathcal{G}$ *is many-to-one from* $I(\mathbb{R}^d)$ *into* $I_{sym}(\mathbb{R}^d)$, *and one-to-one from* $I_{sym}(\mathbb{R}^d)$ *into* $I_{sym}(\mathbb{R}^d)$.
(*iii*) *For any* $\mu_1, \mu_2 \in I(\mathbb{R}^d)$, $\mathcal{G}(\mu_1 * \mu_2) = \mathcal{G}(\mu_1) * \mathcal{G}(\mu_2)$.
(*iv*) *Let* $\mu_n \in I(\mathbb{R}^d), n = 1, 2, \cdots$. *If* $\mu_n \to \mu$, *then* $\mathcal{G}(\mu_n) \to \mathcal{G}(\mu)$.
(*v*) *Let* (A, ν, γ) *be the triplet of* μ *and* $(\widetilde{A}, \widetilde{\nu}, \widetilde{\gamma})$ *the triplet of* $\widetilde{\mu} = \mathcal{G}(\mu)$. *Then*

$$\widetilde{A} = A,$$

$$\widetilde{\nu}(B) = \int_0^1 dt \int_{\mathbb{R}^d} 1_B(h^*(t)x)\nu(dx) = E\left[\nu(Z^{-1}B)\right],$$

$$\widetilde{\gamma} = 0.$$

Proof. (i) (2.5) and (2.6) follow from Lemma 2.2. (ii) Since $\widehat{\mathcal{G}(\mu)}(z) = \exp\{C_{\mathcal{G}(\mu)}(z)\}$, in order to show $\mathcal{G}(\mu) \in I_{sym}(\mathbb{R}^d)$, it is enough to show that $C_{\mathcal{G}(\mu)}(z)$ is symmetric in z. Actually, we have

$$\begin{aligned} C_{\mathcal{G}(\mu)}(-z) &= \int_0^1 C_\mu(-zh^*(t))dt = -\int_{-\infty}^{\infty} C_\mu(-zr)dh(r) \\ &= \int_{-\infty}^{\infty} C_\mu(-zr)\phi(r)dr = \int_{-\infty}^{\infty} C_\mu(zs)\phi(s)ds \\ &= -\int_{-\infty}^{\infty} C_\mu(zr)dh(r) = \int_0^1 C_\mu(zh^*(t))dt \\ &= C_{\mathcal{G}(\mu)}(z), \end{aligned}$$

and thus $C_{\mathcal{G}(\mu)}(z)$ is symmetric. This shows that the mapping $\mathcal{G}$ is from $I(\mathbb{R}^d)$ into $I_{sym}(\mathbb{R}^d)$. The fact that $\mathcal{G}$ is one-to-one from $I_{sym}(\mathbb{R}^d)$ into $I_{sym}(\mathbb{R}^d)$ can be shown by Remark 1.2. (iii) and (iv) can be proved by the same idea of Proposition 2.7 (iii) and (iv) of Barndorff-Nielsen et al. (2006). (v) follows from (2.2)-(2.4) if we notice that $\int_0^1 h^*(t)dt = 0$ and $\int_0^1 h^*(t)^2dt = 1$. □

The following theorem shows that each type G distribution admits the stochastic integral representation defined in Definition 2.4.

Theorem 2.6.

$$G_0(\mathbb{R}^d) = \mathcal{G}(I(\mathbb{R}^d)).$$

Proof. Let $\mu \in I(\mathbb{R}^d)$ and $\widetilde{\mu} = \mathcal{G}(\mu)$. Then by Proposition 2.5 (v), we have (1.1), and thus $\widetilde{\mu} \in G_0(\mathbb{R}^d)$, concluding $\mathcal{G}(I(\mathbb{R}^d)) \subset G_0(\mathbb{R}^d)$.

Conversely, suppose that $\widetilde{\mu} \in G_0(\mathbb{R}^d)$. Then by Definition 1.1 and Proposition 2.5 (v) again, we wee that $\widetilde{\mu} = \mathcal{L}\left(\int_0^1 h^*(t)dX_t^{(\mu)}\right)$ for some $\mu \in I(\mathbb{R}^d)$. This means that $\widetilde{\mu} \in \mathcal{G}(I(\mathbb{R}^d)$ and $G_0(\mathbb{R}^d) \subset \mathcal{G}(I(\mathbb{R}^d))$, completing the proof. □

Corollary 2.7. *Let H be a subclass of $I(\mathbb{R}^d)$ and let*

$$G_H(\mathbb{R}^d) = \{\mu_0 \in I_{sym}(\mathbb{R}^d) \ : \ \nu_{\mu_0}(B) = E[\nu(Z^{-1}B)], \ B \in \mathcal{B}_0(\mathbb{R}^d), \ \textit{for some } \mu_\nu \in H\},$$

where ν_μ s the Lévy measure of $\mu \in I(\mathbb{R}^d)$ and μ_ν is the infinitely divisible distribution with Lévy measure ν. Then we have

$$G_H(\mathbb{R}^d) = \mathcal{G}(H).$$

Remark 2.8. If $H = I(\mathbb{R}^d)$, then the corollary above is nothing but Theorem 2.6. The proof of the corollary can be carried out in the same way as for Theorem 2.6.

Also, we see from the discussions above that as mappings from $I_{sym}(\mathbb{R}^d)$ into $I_{sym}(\mathbb{R}^d)$, two mappings K and $\mathcal{G}$ are the same.

3. Lévy measures of distributions in $G_m(\mathbb{R}^d), m \in \mathbb{N}$.

In this section, we characterize Lévy measures of distributions in $G_m, m \in \mathbb{N}$. Write $\phi_0(x) = \phi(x), h_0(x) = h(x)$ and $h_0^*(t) = h^*(t)$.

For $m \in \mathbb{N}$, let $\phi_m(x)$ be the probability density function of the product of $(m+1)$ independent standard normal random variables. Then we have the following.

Lemma 3.1. *For each $m \in \mathbb{N}$,*
(i)

$$\phi_m(x) = \phi_m(-x),$$

(ii)

$$\int_{-\infty}^{\infty} \phi_m(x)dx = 1,$$

(iii)

$$\int_{-\infty}^{\infty} |x|\phi_m(x)dx < \infty \quad \textit{and} \int_{-\infty}^{\infty} x\phi_m(x)dx = 0,$$

(iv)

$$\int_{-\infty}^{\infty} x^2\phi_m(x)dx = 1.$$

(v)

$$\phi_m(x) = \int_{-\infty}^{\infty} \phi_0(u)\phi_{m-1}(x|u|^{-1})|u|^{-1}du. \tag{3.1}$$

Proof. (i)-(iv) are trivial. (v) is a consequence of a standard calculation. □

For $m \in \mathbb{N}$, let

$$h_m(x) = \int_x^{\infty} \phi_m(u)du, \quad x \in \mathbb{R}$$

and define its inverse $x = h_m^*(t)$ by $t = h_m(x)$. We note that for each $m \in \mathbb{N} \cup \{0\}$,

$$h_m(+\infty) = 0, \quad h_m(-\infty) = 1,$$
$$\int_0^1 h_m^*(t)dt = 0 \quad \text{and} \quad \int_0^1 h_m^*(t)^2dt = 1,$$

where the last two integrals are given by Proposition 3.2 (iii) and (iv).

Theorem 3.2. *For each $m \in \mathbb{N}$, let $\mu_m \in I_{sym}(\mathbb{R}^d)$ and denote its Lévy measure by ν_m. Then $\mu_m \in G_m(\mathbb{R}^d)$ if and only if*

$$\nu_m(B) = \int_{-\infty}^{\infty} \nu_0(u^{-1}B)\phi_{m-1}(u)du, \tag{3.2}$$

where ν_0 is the Lévy measure of some $\mu_0 \in G_0(\mathbb{R}^d)$.

Proof. ("Only if" part.) Let $m = 1$. Then, by the definition

$$\nu_1(B) = E\left[\nu_0(Z^{-1}B)\right] = \int_{-\infty}^{\infty} \nu_0(u^{-1}B)\phi_0(u)du$$

for some Lévy measure ν_0 whose distribution is in G_0. Suppose the statement is true for some $m \in \mathbb{N}$. The Lévy measure ν_{m+1} of $\mu_{m+1} \in G_{m+1}(\mathbb{R}^d)$ is given by

$$\nu_{m+1}(B) = E\left[\nu_m(Z^{-1}B)\right]$$

for some Lévy measure ν_m of a distribution $\mu_m \in G_m(\mathbb{R}^d)$. Then by the induction hypothesis

$$\begin{aligned}
\nu_{m+1}(B) &= \int_{-\infty}^{\infty} \phi_0(u)\nu_m(u^{-1}B)du \\
&= \int_{-\infty}^{\infty} \phi_0(u)du \int_{-\infty}^{\infty} \nu_0(u^{-1}v^{-1}B)\phi_{m-1}(v)dv \\
&= \int_{-\infty}^{\infty} \phi_0(u)du \int_{-\infty}^{\infty} \nu_0(y^{-1}B)\phi_{m-1}(y|u|^{-1})|u|^{-1}dy \\
&= \int_{-\infty}^{\infty} \nu_0(y^{-1}B)\phi_m(y)dy
\end{aligned}$$

by (3.1).

("If" part.) Let $m = 1$. Then, by the definition, if a Lévy measure ν_1 is represented as

$$\nu_1(B) = \int_{-\infty}^{\infty} \nu_0(u^{-1}B)\phi_0(u)du$$

for some ν_0, the Lévy measure of some $\mu_0 \in G_0(\mathbb{R}^d)$, then $\mu_1 \in G_1(\mathbb{R}^d)$. Suppose that the statement ("if" part) is true for some $m \in \mathbb{N}$. By the same calculation as above (from the bottom to the top), we have

$$\begin{aligned}
\nu_{m+1}(B) &= \int_{-\infty}^{\infty} \nu_0(u^{-1}B)\phi_m(u)du \\
&= \int_{-\infty}^{\infty} \phi_0(u)\nu_m(u^{-1}B)du \\
&= E[\nu_m(Z^{-1}B)]
\end{aligned}$$

for some Lévy measure ν_m having the representation (3.2). Then, by the induction hypothesis, μ_m with the Lévy measure ν_m belong to $G_m(\mathbb{R}^d)$. Thus, $\mu_{m+1} \in G_{m+1}(\mathbb{R}^d)$. This completes the proof. $\square$

The following is a G_m–version of Proposition 2.1, and it characterizes Lévy measures of distributions in $G_m(\mathbb{R}^d)$.

Theorem 3.3. *Let $m \in \mathbb{N}$. A $\mu_m \in I_{sym}(\mathbb{R}^d)$ belongs to $G_m(\mathbb{R}^d)$ if and only if its Lévy measure ν_m is either zero or it can be represented as*

$$\nu_m(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi) g_{m,\xi}(r^2) dr, \quad B \in \mathcal{B}_0(\mathbb{R}^d),$$

where λ is a symmetric measure on the unit sphere S on $\mathbb{R}^d$ and $g_{m,\xi}(r)$ is represented as

$$g_{m,\xi}(s) = \int_{-\infty}^\infty \phi_{m-1}(\sqrt{s}|r|^{-1})|r|^{-1} g_\xi(r^2) dr,$$

for some function g_ξ on $(0, \infty)$ which has the same properties as in Proposition 2.1.

Proof. We see by Propositions 3.3 and 2.1, $\mu_m \in G_m(\mathbb{R}^d)$ if and only if ν_m is represented as

$$\begin{aligned}\nu_m(B) &= \int_{-\infty}^\infty \nu_0(u^{-1}B)\phi_{m-1}(u)du \\ &= \int_{-\infty}^\infty \phi_{m-1}(u)du \int_S \lambda(d\xi) \int_0^\infty 1_{u^{-1}B}(r\xi) g_\xi(r^2) dr.\end{aligned}$$

If we use here the facts that $\lambda(d\xi) = \lambda(-d\xi), g_\xi = g_{-\xi}$ and $\phi_{m-1}(u) = \phi_{m-1}(-u)$, then we have

$$\begin{aligned}\nu_m(B) &= \int_{-\infty}^\infty \phi_{m-1}(y|r|^{-1})|r|^{-1}dy \int_S \lambda(d\xi) \int_0^\infty 1_B(y\xi) g_\xi(r^2) dr \\ &= \int_S \lambda(d\xi) \int_{-\infty}^\infty 1_B(y\xi) g_{m,\xi}(y^2) dy\end{aligned}$$

where

$$g_{m,\xi}(s) = \int_{-\infty}^\infty \phi_{m-1}(\sqrt{s}|r|^{-1})|r|^{-1} g_\xi(r^2) dr.$$

This completes the proof. $\square$

4. Stochastic integral characterizations of $G_m(\mathbb{R}^d), m \in \mathbb{N}$.

In this section, we characterize distributions in $G_m(\mathbb{R}^d)$ by stochastic integral representations.

Theorem 4.1. *For each $m \in \mathbb{N}$, the stochastic integral*

$$Y_m := \int_0^1 h_m^*(t) dX_t^{(\mu)}$$

is integrable for every $\mu \in I(\mathbb{R}^d)$,

$$\int_0^1 |C_\mu(h_m^*(t)z)| dt < \infty$$

and

$$C_{\mathcal{L}(Y_m)}(z) = \int_0^1 C_\mu(h_m^*(t)z) dt.$$

Proof. Since

$$\int_0^1 h_m^*(t)^2 dt = \int_{-\infty}^\infty |x|^2 \phi_m(x) dx < \infty,$$

we have the assertion by Lemma 2.2. □

Let $\mathcal{G}_1 = \mathcal{G}^1 = \mathcal{G}$.

Definition 4.2. Let $m \in \mathbb{N}$. Define a mapping $\mathcal{G}_{m+1}$ by

$$\mathcal{G}_{m+1}(\mu) = \mathcal{L}\left(\int_0^1 h_m^*(t) dX_t^{(\mu)}\right), \quad \mu \in I(\mathbb{R}^d)$$

and

$$\mathcal{G}^{m+1}(\mu) = \mathcal{G}(\mathcal{G}^m((\mu)), \quad \mu \in I(\mathbb{R}^d).$$

Proposition 4.3. *For $m \in \mathbb{N}$,*

$$G_m(\mathbb{R}^d) = \mathcal{G}(G_{m-1}(\mathbb{R}^d)).$$

Proof. The proof is almost the same as that of Theorem 2.6. Let $\mu_{m-1} \in G_{m-1}(\mathbb{R}^d)$ and $\mu_m = \mathcal{G}(\mu_{m-1})$. Also let ν_{m-1} and ν_m be the Lévy measures of μ_{m-1} and μ_m, respectively. Then by Proposition 2.5 (v), we have $\nu_m(B) = E[\nu_{m-1}(Z^{-1}B)]$. Thus $\mu_m \in G_m(\mathbb{R}^d)$, and $\mathcal{G}(G_{m-1}(\mathbb{R}^d)) \subset G_m(\mathbb{R}^d)$.

Conversely, suppose that $\mu_m \in G_m(\mathbb{R}^d)$. Then by the definition of $G_m(\mathbb{R}^d)$ and Proposition 2.5 (v) again, we see that $\mu_m = \mathcal{L}\left(\int_0^1 h^*(t) dX_t^{(\mu)}\right)$ for some $\mu \in G_{m-1}(\mathbb{R}^d)$. This means that $\mu_m \in \mathcal{G}(G_{m-1}(\mathbb{R}^d))$, and $G_m(\mathbb{R}^d) \subset \mathcal{G}(G_{m-1}(\mathbb{R}^d))$, completing the proof. □

Corollary 4.4. *For $m \in \mathbb{N}$,*

$$G_m(\mathbb{R}^d) = \mathcal{G}^{m+1}(I(\mathbb{R}^d)).$$

We next show

Theorem 4.5. *For $m \in \mathbb{N}$*

$$\mathcal{G}_{m+1}(I(\mathbb{R}^d)) = \mathcal{G}^{m+1}(I(\mathbb{R}^d)).$$

Proof. We note that

$$\widetilde{\mu} \in \mathcal{G}_{m+1}(I(\mathbb{R}^d)) \text{ if and only if } \widetilde{\mu} = \mathcal{L}\left(\int_0^1 h_m^*(t)dX_t^{(\mu)}\right), \quad \mu \in I(\mathbb{R}^d)$$

and that

$$\widetilde{\mu} \in \mathcal{G}^{m+1}(I(\mathbb{R}^d)) \text{ if and only if } \widetilde{\mu} = \mathcal{L}\left(\int_0^1 h_0^*(t)dX_t^{(\mu)}\right), \quad \mu \in \mathcal{G}^m(I(\mathbb{R}^d)).$$

We next claim that

$$\int_{-\infty}^{\infty} \phi_0(u)du \int_{-\infty}^{\infty} |C_\mu(uvz)|\phi_{m-1}(v)dv < \infty, \ z \in \mathbb{R}^d. \tag{4.1}$$

If it would be proved, we can exchange the order of the integrals in the calculation of cumulants below.

The proof of (4.1) is as follows. The idea is from Barndorff–Nielsen et al. (2006). If the generating triplet of μ is (A, ν, γ), then

$$|C_\mu(z)| \le 2^{-1}(\mathrm{tr}A)|z|^2 + |\gamma||z| + \int_{\mathbb{R}^d} |g(z,x)|\nu(dx),$$

where

$$g(z,x) = e^{i\langle z,x\rangle} - 1 - i\langle z,x\rangle(1+|x|^2)^{-1}.$$

Hence

$$\begin{aligned}|C_\mu(uvz)| &\le 2^{-1}(\mathrm{tr}A)u^2v^2|z|^2 + |\gamma||u||v||z| + \int_{\mathbb{R}^d} |g(z,uvx)|\nu(dx) \\ &\quad + \int_{\mathbb{R}^d} |g(uvz,x) - g(z,uvx)|\nu(dx) =: I_1 + I_2 + I_3 + I_4,\end{aligned}$$

say. The finiteness of $\int_{-\infty}^{\infty} \phi_0(u)du \int_{-\infty}^{\infty} (I_1 + I_2)\phi_{m-1}(v)dv$ follows from Lemma 3.1. Noting that $|g(z,x)| \le c_z|x|^2(1+|x|^2)^{-1}$ with a positive constant c_z depending on z, we have

$$\begin{aligned}&\int_{-\infty}^{\infty} \phi_0(u)du \int_{-\infty}^{\infty} I_3\phi_{m-1}(v)dv \\ &\quad \le c_z \int_{\mathbb{R}^d} \nu(dx) \int_{-\infty}^{\infty} \phi_0(u)du \int_{-\infty}^{\infty} \frac{(uv|x|)^2}{1+(uv|x|)^2}\phi_{m-1}(v)dv \\ &\quad = c_z \left(\int_{|x|\le 1} \nu(dx) + \int_{|x|>1} \nu(dx)\right) \int_{-\infty}^{\infty} \phi_0(u)du \int_{-\infty}^{\infty} \frac{(uv|x|)^2}{1+(uv|x|)^2}\phi_{m-1}(v)dv \\ &\quad =: I_{31} + I_{32},\end{aligned}$$

say, and

$$I_{31} \le c_z \int_{|x|\le 1} |x|^2 \nu(dx) \int_{-\infty}^{\infty} u^2 \phi_0(u) du \int_{-\infty}^{\infty} v^2 \phi_{m-1}(v) dv < \infty,$$
$$I_{32} \le c_z \int_{|x|>1} \nu(dx) \int_{-\infty}^{\infty} \phi_0(u) du \int_{-\infty}^{\infty} \phi_{m-1}(v) dv < \infty.$$

As to I_4, note that for $a \in \mathbb{R}$,

$$\begin{aligned} |g(az,x)-g(z,ax)| &= \frac{|\langle az, x\rangle||x|^2|1-a^2|}{(1+|x|^2)(1+|ax|^2)} \\ &\le \frac{|z||x|^3(|a|+|a|^3)}{(1+|x|^2)(1+|ax|^2)} \\ &\le \frac{|z||x|^2(1+|a|^2)}{2(1+|x|^2)}, \end{aligned}$$

since $|b|(1+b^2)^{-1} \le 2^{-1}$. Then

$$\begin{aligned} &\int_{-\infty}^{\infty} \phi_0(u) du \int_{-\infty}^{\infty} I_4 \phi_{m-1}(v) dv \\ &\quad \le |z| \int_{\mathbb{R}^d} \frac{|x|^2}{1+|x|^2} \nu(dx) \int_{-\infty}^{\infty} \phi_0(u) du \int_{-\infty}^{\infty} (1+u^2v^2) \phi_{m-1}(v) dv < \infty. \end{aligned}$$

This completes the proof of (4.1).

If we calculate the necessary cumulants, we have

$$\begin{aligned}
C_{\mathcal{G}_{m+1}(\mu)}(z) &= \int_0^1 C_\mu(h_m^*(t)z)dt \\
&= -\int_{-\infty}^{\infty} C_\mu(uz)dh_m(u) \\
&= \int_{-\infty}^{\infty} C_\mu(uz)\phi_m(u)du \\
C_{\mathcal{G}^{m+1}(\mu)}(z) &= \int_0^1 C_{\mathcal{G}^m(\mu)}(h_0^*(t)z)dt \\
&= \int_0^1 dt \int_0^1 C_\mu(h_0^*(t)h_{m-1}^*(s)z)ds \\
&= \int_{-\infty}^{\infty} dh_0(u) \int_{-\infty}^{\infty} C_\mu(uvz)dh_{m-1}(v) \\
&= \int_{-\infty}^{\infty} \phi_0(u)du \int_{-\infty}^{\infty} C_\mu(uvz)\phi_{m-1}(v)dv \\
&= \int_{-\infty}^{\infty} C_\mu(yz)dy \int_{-\infty}^{\infty} \phi_0(u)\phi_{m-1}(y|u|^{-1})|u|^{-1}du \\
&= \int_{-\infty}^{\infty} C_\mu(yz)\phi_m(y)dy \\
&= C_{\mathcal{G}^{m+1}(\mu)}(z).
\end{aligned}$$

This completes the proof of Theorem 4.5. □

The following is a goal of this section and a G_m–version of Theorem 2.6. Namely, any $\mu \in G_m(\mathbb{R}^d)$ has the stochastic integral representation defined in Definition 4.2.

Theorem 4.6.

$$G_m(\mathbb{R}^d) = \mathcal{G}_{m+1}(I(\mathbb{R}^d)).$$

Proof. The statement is an immediate consequence of Corollary 4.4 and Theorem 4.5. □

5. The case $m = \infty$

We conclude this paper with two statements for $G_\infty(\mathbb{R}^d)$.

Proposition 5.1.

$$\mathcal{G}(G_\infty(\mathbb{R}^d)) = G_\infty(\mathbb{R}^d)$$

Proposition 5.2. *$S_{sym}(\mathbb{R}^d)$ is invariant under $\mathcal{G}$–mapping and $G_\infty(\mathbb{R}^d)$ is the largest class which is invariant under $\mathcal{G}$–mapping.*

These two propositions are given by Remark 2.8 in section 2 and Theorem 2.3 of Maejima and Rosiński (2001).

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