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SEVERAL FORMS OF STOCHASTIC INTEGRAL REPRESENTATIONS OF GAMMA RANDOM VARIABLES AND RELATED TOPICS

BY

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Abstract. Gamma distributions can be characterized as the laws of stochastic integrals with respect to many different Lévy processes with different nonrandom integrands. A Lévy process corresponds to an infinitely divisible distribution. Therefore, many infinitely divisible distributions can birth a gamma distribution through stochastic integral mappings with different integrands. In this paper, we pick up several integrands which have appeared in characterizing well-studied classes of infinitely divisible distributions, and find inverse images of a gamma distribution through each stochastic integral mapping. As a byproduct of our approach to stochastic integral representations of gamma random variables, we find a remarkable new general characterization of classes of infinitely divisible distributions, which were already considered by James, Roynette and Yor (2008) and Aoyama, Lindner and Maejima (2010) in some special cases.

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1. Introduction and preliminaries

Let $I(\mathbb{R}^d)$ be the class of all infinitely divisible distributions on $\mathbb{R}^d$ and $I_{\log}(\mathbb{R}^d) = \{\mu \in I(\mathbb{R}^d)\colon \int_{\mathbb{R}^d} \log^+|x|\mu(dx) < \infty\}$, where $\log^+|x| = (\log|x|) \vee 0$. Let $\{X_t^{(\mu)}\}$ be a Lévy process on $\mathbb{R}^d$ with $\mathcal{L}(X_1^{(\mu)}) = \mu \in I(\mathbb{R}^d)$, where and in what follows $\mathcal{L}$ denotes "the law of." Recently, many stochastic integral mappings have been studied. Namely, for a nonrandom measurable function f, we define

$$\widetilde{\mu} = \Phi_f(\mu) = \mathcal{L}\left(\int_0^\infty f(t)dX_t^{(\mu)}\right), \quad \mu \in \mathfrak{D}(\Phi_f) \subset I(\mathbb{R}^d),$$

where $\mathfrak{D}(\Phi_f)$ is the domain of the mapping Φ_f that is the class of $\mu \in I(\mathbb{R}^d)$ for which $\int_0^\infty f(t)dX_t^{(\mu)}$ is definable. (See, e.g., [18].) The well-studied mappings are the following.

(i) $\mathcal{U}$-mapping ([8]). For $\mu \in \mathfrak{D}(\mathcal{U}) = I(\mathbb{R}^d)$, $\mathcal{U}(\mu) = \mathcal{L}\left(\int_0^1 t dX_t^{(\mu)}\right)$.
(ii) Υ-mapping ([4]). For $\mu \in \mathfrak{D}(\Upsilon) = I(\mathbb{R}^d)$, $\Upsilon(\mu) = \mathcal{L}\left(\int_0^1 \log(t^{-1})dX_t^{(\mu)}\right)$.
(iii) Φ-mapping ([10, 20, 21]). For $\mu \in \mathfrak{D}(\Phi) = I_{\log}(\mathbb{R}^d)$, $\Phi(\mu) = \mathcal{L}\left(\int_0^\infty e^{-t}dX_t^{(\mu)}\right)$.
(iv) Ψ-mapping ([4]). Let $p(x) = \int_x^\infty e^{-u}u^{-1}du, x > 0$, and denote its inverse function by $p^*(t)$. For $\mu \in \mathfrak{D}(\Psi) = I_{\log}(\mathbb{R}^d)$, $\Psi(\mu) = \mathcal{L}\left(\int_0^\infty p^*(t)dX_t^{(\mu)}\right)$.
(v) $\mathcal{G}$-mapping ([15]). Let $g(x) = \int_x^\infty e^{-u^2}du$, $x > 0$, and denote its inverse function by $g^*(t)$. For $\mu \in \mathfrak{D}(\mathcal{G}) = I(\mathbb{R}^d)$, $\mathcal{G}(\mu) = \mathcal{L}\left(\int_0^{\pi^{1/2}/2} g^*(t)dX_t^{(\mu)}\right)$.
(vi) $\mathcal{M}$-mapping ([13]). Let $m(x) = \int_x^\infty e^{-u^2}u^{-1}du$, $x > 0$, and denote its inverse function by $m^*(t)$. For $\mu \in \mathfrak{D}(\mathcal{M}) = I_{\log}(\mathbb{R}^d)$, $\mathcal{M}(\mu) = \mathcal{L}\left(\int_0^\infty m^*(t)dX_t^{(\mu)}\right)$.

We also use the notation Φ_f^L, $\mathcal{U}^L$, Υ^L, Φ^L, Ψ^L, $\mathcal{G}^L$ and $\mathcal{M}^L$ as the transformations of Lévy measures in each mapping. For instance, if ν is the Lévy measure of μ, then $\Phi_f^L(\nu)$ is the Lévy measure of $\Phi_f(\mu)$, provided that $\Phi_f(\mu)$ is definable.

These mappings are related to the following subclasses of $I(\mathbb{R}^d)$, which are defined in terms of Lévy measures. To explain it, we need the polar decomposition of Lévy measures, (see, e.g., [4]).

Let ν be the Lévy measure of the characteristic function of some $\mu \in I(\mathbb{R}^d)$

with $0 < \nu(\mathbb{R}^d) \le \infty$. Then there exist a measure λ on $S = \{\xi \in \mathbb{R}^d : |\xi| = 1\}$ with $0 < \lambda(S) \le \infty$ and a family $\{\nu_\xi, \xi \in S\}$ of measures on $(0,\infty)$ such that $\nu_\xi(B)$ is measurable in ξ for each $B \in \mathcal{B}((0,\infty))$, $0 < \nu_\xi((0,\infty)) \le \infty$ for each $\xi \in S$, and

$$\nu(B) = \int_S \lambda(d\xi) \int_0^\infty \mathbf{1}_B(r\xi)\nu_\xi(dr), \quad B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}). \tag{1.1}$$

Here λ and $\{\nu_\xi\}$ are uniquely determined by ν up to multiplication of a measurable function $c(\xi)$ and $c(\xi)^{-1}$ with $0 < c(\xi) < \infty$, and ν_ξ is called the radial component of ν. If ν fulfills (1.1), then we say that (λ, ν_ξ) is a polar decomposition of ν.

Classes in $I(\mathbb{R}^d)$ we are going to be concerned with in this paper are defined in the following way in terms of ν_ξ.

(i) The class $U(\mathbb{R}^d)$ (the Jurek class): $\nu_\xi(dr) = \ell_\xi(r)dr$, where $\ell_\xi(r)$ is measurable in $\xi \in S$ and nonincreasing and right-continuous in $r \in (0,\infty)$.

(ii) The class $B(\mathbb{R}^d)$ (the Goldie-Steutel-Bondesson class): $\nu_\xi(dr) = \ell_\xi(r)dr$, where $\ell_\xi(r)$ is measurable in $\xi \in S$ and completely monotone in $r \in (0,\infty)$.

(iii) The class $L(\mathbb{R}^d)$ (the class of selfdecomposable distributions): $\nu_\xi(dr) = r^{-1} k_\xi(r)dr$, where $k_\xi(r)$ is measurable in $\xi \in S$ and nonincreasing and right-continuous in $r \in (0,\infty)$.

(iv) The class $T(\mathbb{R}^d)$ (the Thorin class): $\nu_\xi(dr) = r^{-1}k_\xi(r)dr$, where $k_\xi(r)$ is measurable in $\xi \in S$ and completely monotone in $r \in (0,\infty)$.

(v) The class $G(\mathbb{R}^d)$ (the class of generalized type G distributions): $\nu_\xi(dr) = g_\xi(r^2)dr$, where $g_\xi(r)$ is measurable in $\xi \in S$ and completely monotone in $r \in (0,\infty)$.

(vi) The class $M(\mathbb{R}^d)$ (the class M): $\nu_\xi(dr) = g_\xi(r^2)r^{-1}dr$, where $g_\xi(r)$ is measurable in $\xi \in S$ and completely monotone in $r \in (0,\infty)$.

Then we know the following characterizations of these classes by the mappings above.

PROPOSITION 1.1. (i) $U(\mathbb{R}^d) = \mathcal{U}(I(\mathbb{R}^d))$. *([8])*

(ii) $B(\mathbb{R}^d) = \Upsilon(I(\mathbb{R}^d))$. *([4])*

(iii) $L(\mathbb{R}^d) = \Phi(I_{\log}(\mathbb{R}^d))$. *([10, 20, 21])*

(iv) $T(\mathbb{R}^d) = \Psi(I_{\log}(\mathbb{R}^d))$. *([4])*

(v) $G(\mathbb{R}^d) = \mathcal{G}(I(\mathbb{R}^d))$. *([2] for symmetric case and [15] for general case.)*

(vi) $M(\mathbb{R}^d) = \mathcal{M}(I_{\log}(\mathbb{R}^d))$. *([3] for symmetric case and [13] for general case.)*

The relations among the classes are the following.

$$U(\mathbb{R}^d) \supset B(\mathbb{R}^d) \cup L(\mathbb{R}^d), \quad B(\mathbb{R}^d) \cap L(\mathbb{R}^d) \supset T(\mathbb{R}^d), \tag{1.2}$$

$$G(\mathbb{R}^d) \supset B(\mathbb{R}^d) \tag{1.3}$$

and

$$M(\mathbb{R}^d) \supset T(\mathbb{R}^d). \tag{1.4}$$

(1.2) can be seen from their definitions, (1.3) is shown in [15] and (1.4) is proved in [3] for symmetric case, but it is also true for general case. Therefore, $T(\mathbb{R}^d)$ is the smallest class among these six classes. Many concrete one-dimensional infinitely divisible distributions belonging to these classes are known. (See [11].)

Since one of the main topics in this paper is a gamma distribution, we henceforth consider only distributions in $I(\mathbb{R})$.

Let $\gamma_{c,\lambda}$ be a gamma random variable with parameters $c > 0$ and $\lambda > 0$. Namely,

$$P(\gamma_{c,\lambda} \in B) = \lambda^c \Gamma(c)^{-1} \int_{B \cap (0,\infty)} x^{c-1} e^{-\lambda x} dx, \quad B \in \mathcal{B}(\mathbb{R}),$$

and

$$E\left[e^{iz\gamma_{c,\lambda}}\right] = \exp\left\{c \int_0^\infty \left(e^{izx} - 1\right) x^{-1} e^{-\lambda x} dx\right\}.$$

Therefore $\mathcal{L}(\gamma_{c,\lambda})$ belongs to all six classes mentioned above. Since the six mappings above are injective (see, e.g., [4, 8, 10, 13]), the inverse image of $\mathcal{L}(\gamma_{c,\lambda})$

through each mapping is unique. Our first interest is what they are. Namely, since

$$\gamma_{c,\lambda} \stackrel{\mathrm{d}}{=} \begin{cases} \int_0^1 t\,dX_t^{(\mu_{\mathcal{U}})} & \text{for a unique } \mu_{\mathcal{U}} \in I(\mathbb{R}), \\ \int_0^1 \log(t^{-1})dX_t^{(\mu_\Upsilon)} & \text{for a unique } \mu_\Upsilon \in I(\mathbb{R}), \\ \int_0^\infty e^{-t}dX_t^{(\mu_\Phi)} & \text{for a unique } \mu_\Phi \in I_{\log}(\mathbb{R}), \\ \int_0^\infty p^*(t)dX_t^{(\mu_\Psi)} & \text{for a unique } \mu_\Psi \in I_{\log}(\mathbb{R}), \\ \int_0^\infty g^*(t)dX_t^{(\mu_{\mathcal{G}})} & \text{for a unique } \mu_{\mathcal{G}} \in I(\mathbb{R}), \\ \int_0^\infty m^*(t)dX_t^{(\mu_{\mathcal{M}})} & \text{for a unique } \mu_{\mathcal{M}} \in I_{\log}(\mathbb{R}), \end{cases} \tag{1.5}$$

where $\stackrel{\mathrm{d}}{=}$ means equality in distribution, we want to find $\mu_{\mathcal{U}}, \mu_\Upsilon, \mu_\Phi, \mu_\Psi, \mu_{\mathcal{G}}$ and $\mu_{\mathcal{M}}$.

This paper is organized as follows. Section 2 answers this question. The proofs are given in Section 3 in a more general setting. This general setting allows us to find a new general characterization of classes of infinitely divisible distributions, which is discussed in Section 4.

2. Results

Let $\widehat{\mu}$ be the characteristic function of $\mu \in I(\mathbb{R})$. The answer to the question stated after (1.5) is the following.

PROPOSITION 2.1.

$$\widehat{\mu}_{\mathcal{U}}(z) = \exp\left\{c\int_0^\infty \left(e^{izx}-1\right)e^{-\lambda x}\left(\lambda + x^{-1}\right)dx\right\}, \tag{2.1}$$

$$\widehat{\mu}_\Upsilon(z) = \exp\left\{c\int_0^{1/\lambda} \left(e^{izx}-1\right)x^{-1}dx\right\}, \tag{2.2}$$

$$\widehat{\mu}_\Phi(z) = \exp\left\{c\lambda\int_0^\infty \left(e^{izx}-1\right)e^{-\lambda x}dx\right\}, \tag{2.3}$$

$$\widehat{\mu}_\Psi(z) = \exp\left\{c\left(e^{iz\lambda^{-1}}-1\right)\right\}, \tag{2.4}$$

$$\widehat{\mu}_{\mathcal{G}}(z) = \exp\left\{2\pi^{-1/2}c\int_0^\infty \left(e^{izx}-1\right)x^{-1}e^{-\lambda^2x^2/4}dx\right\}, \tag{2.5}$$

$$\widehat{\mu}_{\mathcal{M}}(z) = \exp\left\{\pi^{-1/2}c\lambda\int_0^\infty \left(e^{izx}-1\right)e^{-\lambda^2x^2/4}dx\right\}. \tag{2.6}$$

We can prove Proposition 2.1 by the direct calculation of the cumulant functions of the stochastic integrals, but we give another proof in a more general setting in the next section.

The explicit form of $\mu_{\mathcal{U}}$, μ_Φ, μ_Ψ and $\mu_{\mathcal{M}}$ are given as follows.

COROLLARY 2.1. *Let* $\{N(t)\}$ *be a Poisson process with parameter* 1*. Let us consider* $\{\gamma_{t,\lambda}, t \geq 0\}$*, with* $\gamma_{0,\lambda} = 0$*, a gamma process with parameter* $\lambda > 0$ *and let* $\{\widetilde{\gamma}_{t,\lambda}\}$ *be an independent copy of* $\{\gamma_{t,\lambda}\}$*. Let* $W_1, W_2, \ldots$ *be i.i.d. standard normal random variables. Assume that the processes and the random variables above are independent. Then we have the following.*

$$\gamma_{c,\lambda} \stackrel{\mathrm{d}}{=} \begin{cases} \displaystyle\int_0^1 t d(\gamma_{N(ct),\lambda} + \widetilde{\gamma}_{ct,\lambda}), & \left(\mu_{\mathcal{U}} = \mathcal{L}\left(\gamma_{N(c),\lambda} + \widetilde{\gamma}_{c,\lambda}\right)\right), \\ \displaystyle\int_0^\infty e^{-t} d\gamma_{N(ct),\lambda}, & \left(\mu_\Phi = \mathcal{L}\left(\gamma_{N(c)\lambda}\right)\right), \\ \displaystyle\lambda^{-1}\int_0^\infty p^*(t) dN(ct), & \left(\mu_\Psi = \mathcal{L}\left(\lambda^{-1}N(c)\right)\right), \\ \displaystyle 2^{1/2}\lambda^{-1}\int_0^\infty m^*(t) d\left(\sum_{k=1}^{N(ct)} |W_k|\right), & \left(\mu_{\mathcal{M}} = \mathcal{L}\left(2^{1/2}\lambda^{-1}\sum_{k=1}^{N(c)} |W_k|\right)\right). \end{cases}$$

Here and in what follows, $\sum_{k=1}^0$ *is regarded as* 0*. The last expression suggests us the following result about symmetrized gamma distributions.*

$$\gamma_{c,\lambda} - \widetilde{\gamma}_{c,\lambda} \stackrel{\mathrm{d}}{=} 2^{1/2}\lambda^{-1}\int_0^\infty m^*(t) dB(N(2ct)),$$

where $\{B(t)\}$ *is a Brownian motion independent of* $\{N(t)\}$*.*

Let $I(\mathbb{R}_+)$ be the totality of $\mu \in I(\mathbb{R})$ whose support is included in $\mathbb{R}_+ = [0, \infty)$. We use the symbol $U(\mathbb{R}_+) := U(\mathbb{R}) \cap I(\mathbb{R}_+)$. We also use $B(\mathbb{R}_+)$, $L(\mathbb{R}_+)$, $T(\mathbb{R}_+)$, $G(\mathbb{R}_+)$, $M(\mathbb{R}_+)$ and $I_{\log}(\mathbb{R}_+)$ in the same way.

The following is a comment on μ_Υ.

REMARK 2.1. *What is* μ_Υ*? We do not know an answer. However, this distribution is important in* $L(\mathbb{R}_+)$ *in the sense that some property holds (for example, the unimodality of distributions on* $\mathbb{R}_+$*) for this special distribution, then the same property holds for all distributions in* $L(\mathbb{R}_+)$*. (See, e.g., [22, 17].) We call this dis-*

tribution a building-block of $L(\mathbb{R}_+)$. *(This remark is based on a private discussion with Ken-iti Sato.)*

We do not know a probabilistic meaning of $\mu_{\mathcal{G}}$.

3. More general facts including the proof of Proposition 2.1

Let $\gamma = \gamma_{1,1}$. Since $E[e^{iz\gamma_{c,\lambda}}] = \left(E[e^{iz\lambda^{-1}\gamma}]\right)^c$, we may assume $c = \lambda = 1$ without loss of generality. Indeed, if $\gamma \overset{\mathrm{d}}{=} \int_0^\infty f(t)dX_t$ with a nonrandom measurable function f and a Lévy process $\{X_t\}$, then $\gamma_{c,\lambda} \overset{\mathrm{d}}{=} \lambda^{-1}\int_0^\infty f(t)dX_{ct} = \lambda^{-1}\int_0^\infty f(t/c)dX_t$.

Let $\nu_\gamma(dx) := x^{-1}e^{-x}dx$, which is the Lévy measure of γ. Let f be a nonnegative nonrandom measurable function and $X = \{X_t\}$ a subordinator without drift, i.e. $E[e^{izX_t}] = \exp\left\{t\int_{\mathbb{R}_+}\left(e^{izx}-1\right)\nu_X(dx)\right\}$, where $\nu_X(\{0\}) = 0$ and $\int_{\mathbb{R}_+}(x \wedge 1)\nu_X(dx) < \infty$. In this case, in order to prove that $\int_0^\infty f(t)dX_t$ is definable and $\gamma \overset{\mathrm{d}}{=} \int_0^\infty f(t)dX_t$, it is only enough to check that

$$\nu_\gamma(B) = \left(\Phi_f^L(\nu_X)\right)(B) \qquad (3.1)$$
$$\left(= \int_0^\infty ds \int_{(0,\infty)} \mathbf{1}_B(f(s)x)\nu_X(dx)\right), \quad B \in \mathcal{B}((0,\infty)),$$

due to Theorems 2.6 and 3.5 of [19]. Indeed, since γ and X_1 are nonnegative infinitely divisible random variables, γ and X_1 have no Gaussian part. Also, ν_X is transformed by Φ_f^L to the Lévy measure ν_γ due to (3.1). Furthermore, by (3.1),

$$\begin{aligned}(3.2)\qquad &\int_0^\infty \left| f(s)\left(\int_{(0,\infty)} x(1+x^2)^{-1}\nu_X(dx)\right.\right.\\ &\qquad \left.\left. + \int_{(0,\infty)} x\left((1+f(s)^2x^2)^{-1} - (1+x^2)^{-1}\right)\nu_X(dx)\right)\right| ds\\ &= \int_0^\infty ds \int_{(0,\infty)} f(s)x(1+f(s)^2x^2)^{-1}\nu_X(dx)\\ &= \int_{(0,\infty)} x(1+x^2)^{-1}\nu_\gamma(dx) < \infty,\end{aligned}$$

$$\begin{aligned}(3.3)\qquad &\lim_{p\downarrow 0, q\uparrow\infty}\int_p^q f(s)ds\left(\int_{(0,\infty)} x(1+x^2)^{-1}\nu_X(dx)\right.\\ &\qquad \left. + \int_{(0,\infty)} x\left((1+f(s)^2x^2)^{-1} - (1+x^2)^{-1}\right)\nu_X(dx)\right)\end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty ds \int_{(0,\infty)} f(s)x(1+f(s)^2x^2)^{-1}\nu_X(dx) \\
&= \int_{(0,\infty)} x(1+x^2)^{-1}\nu_\gamma(dx) < \infty.
\end{aligned}
$$

These yield that $\int_0^\infty f(s)dX_s$ is definable and has no drift since X_1 has no drift. Thus it is only enough to check the condition (3.1).

Now, for two σ-finite measures ρ and η on $(0,\infty)$, we define

$$(\rho \circledast \eta)(B) = \int_{(0,\infty)^2} \mathbf{1}_B(xy)\rho(dx)\eta(dy).$$

Trivially, $\rho \circledast \eta = \eta \circledast \rho$ and $\rho \circledast \delta_1 = \rho$, where δ_a is the Dirac measure at a. If two $(0,\infty)$-valued random variables X and Y are independent, then $\mathcal{L}(X) \circledast \mathcal{L}(Y) = \mathcal{L}(XY)$. For a σ-finite measure η on $(0,\infty)$, define a transformation Υ_η of a σ-finite measure ρ on $\mathbb{R}$ to a measure $\Upsilon_\eta(\rho)$ on $\mathbb{R}$ by

$$\left(\Upsilon_\eta(\rho)\right)(B) := \int_{(0,\infty)} \rho(x^{-1}B)\eta(dx), \quad B \in \mathcal{B}(\mathbb{R}),$$

which is called the Upsilon transformation with dilation measure η, (cf. [5]). For σ-finite measures ρ and η on $(0,\infty)$, it is easy to see that

$$\Upsilon_\eta(\rho) = \rho \circledast \eta = \eta \circledast \rho = \Upsilon_\rho(\eta).$$

Let η be a measure on $(0,\infty)$ satisfying

$$\varepsilon_\eta(\xi) := \eta((\xi,\infty)) < \infty \tag{3.4}$$

for all $\xi > 0$. (It is permitted that $\varepsilon_\eta(0) = \eta((0,\infty)) = \infty$.) Let

$$\varepsilon_\eta^*(t) := \inf\{\xi > 0\colon \varepsilon_\eta(\xi) \le t\}, \quad t > 0.$$

Then, ε_η and ε_η^* are nonincreasing càdlàg functions. We have that $\varepsilon_\eta^*(t) \le \xi$ if and only if $\varepsilon_\eta(\xi) \le t$, so that

$$\mathrm{Leb}(\varepsilon_\eta^*)^{-1}((\xi,\infty)) = \mathrm{Leb}(\{t > 0\colon \varepsilon_\eta^*(t) > \xi\}) \tag{3.5}$$

$$= \mathrm{Leb}(\{t>0\colon \varepsilon_\eta(\xi)>t\})$$
$$= \varepsilon_\eta(\xi) = \eta((\xi,\infty)), \quad \xi>0,$$

where Leb denotes the Lebesgue measure and $\mathrm{Leb}(\varepsilon_\eta^*)^{-1}$ means the image measure of Leb under ε_η^*. Hence we have that for any σ-finite measure ρ on $(0,\infty)$,

$$\begin{aligned}
(\Upsilon_\eta(\rho))(B) &= \int_{(0,\infty)} \eta(dx) \int_{(0,\infty)} \mathbf{1}_B(xy)\rho(dy) \\
&= \int_{(0,\infty)} \mathrm{Leb}(\varepsilon_\eta^*)^{-1}(dx) \int_{(0,\infty)} \mathbf{1}_B(xy)\rho(dy) \\
&= \int_0^{\varepsilon_\eta(0)} ds \int_{(0,\infty)} \mathbf{1}_B(\varepsilon_\eta^*(s)y)\rho(dy) \\
&= \left(\Phi^L_{\varepsilon_\eta^*}(\rho)\right)(B), \quad B\in\mathcal{B}((0,\infty)).
\end{aligned}$$

Therefore, if $\nu_\gamma = \Upsilon_\eta(\nu_X)(=\eta\circledast\nu_X)$, then

$$\gamma \overset{\mathrm{d}}{=} \int_0^{\varepsilon_\eta(0)} \varepsilon_\eta^*(t)dX_t,$$

namely,

$$\mathcal{L}(\gamma) = \Phi_{\varepsilon_\eta^*}(\mu) \quad \text{with} \quad \widehat{\mu}(z) = \exp\left\{\int_{(0,\infty)} \left(e^{izx}-1\right)\nu_X(dx)\right\}, \tag{3.6}$$

where γ, ν_γ, $\{X_t\}$ and ν_X are the ones in the previous paragraph.

Also, the fact that $\nu_\gamma = \rho\circledast\eta$ for some σ-finite measures ρ and η has another meaning. In what follows, $\Upsilon_\eta\Upsilon_\rho$, $\Phi^L_f\Phi^L_g$ and $\Phi_f\Phi_g$ mean the composites of two transformations (or mappings). Proposition 4.1 of [5] yields that $\Upsilon_{\nu_\gamma} = \Upsilon_\rho\Upsilon_\eta = \Upsilon_\eta\Upsilon_\rho$. If η satisfies (3.4) and ρ satisfies (3.4) with the replacement of η by ρ, then, by the argument above, we have $\Upsilon_\eta = \Phi^L_{\varepsilon_\eta^*}$ and $\Upsilon_\rho = \Phi^L_{\varepsilon_\rho^*}$. Since $\Upsilon_{\nu_\gamma} = \Psi^L$, we have $\Psi^L = \Phi^L_{\varepsilon_\rho^*}\Phi^L_{\varepsilon_\eta^*} = \Phi^L_{\varepsilon_\eta^*}\Phi^L_{\varepsilon_\rho^*}$, which suggests in many cases that

$$\Psi = \Phi_{\varepsilon_\rho^*}\Phi_{\varepsilon_\eta^*} = \Phi_{\varepsilon_\eta^*}\Phi_{\varepsilon_\rho^*}.$$

On the base of the arguments above, we have the following. Let $\nu_\gamma = \nu_1\circledast\nu_2$ for some Lévy measures $\nu_j, j=1,2$, on $(0,\infty)$ satisfying $\int_{(0,\infty)}(x\wedge 1)\nu_j(dx)<\infty$.

Denote by $\{X_j(t)\}$ a subordinator without drift whose Lévy measure at $t=1$ is ν_j. Then, the following are true.

$$\gamma \stackrel{\mathrm{d}}{=} \int_0^{\varepsilon_{\nu_1(0)}} \varepsilon^*_{\nu_1}(t)dX_2(t) \stackrel{\mathrm{d}}{=} \int_0^{\varepsilon_{\nu_2(0)}} \varepsilon^*_{\nu_2}(t)dX_1(t), \tag{3.7}$$

$$\Psi^L = \Phi^L_{\varepsilon^*_{\nu_1}} \Phi^L_{\varepsilon^*_{\nu_2}} = \Phi^L_{\varepsilon^*_{\nu_2}} \Phi^L_{\varepsilon^*_{\nu_1}}. \tag{3.8}$$

EXAMPLE 3.1. $\nu_\gamma = \nu_\gamma \circledast \delta_1$.

EXAMPLE 3.2. Let $-\infty < \alpha < \beta < \infty$, $\eta_\alpha(dx) = x^{-\alpha-1}e^{-x}dx$ and $\eta_{\beta,\alpha}(dx) = (\Gamma(\alpha-\beta))^{-1}(1-x)^{\alpha-\beta-1}x^{-\alpha-1}\mathbf{1}_{(0,1)}(x)dx$. Write $\Psi_\alpha := \Phi_{\varepsilon^*_{\eta_\alpha}}$ and $\Phi_{\beta,\alpha} := \Phi_{\varepsilon^*_{\eta_{\beta,\alpha}}}$. These stochastic integral mappings were introduced by Sato [18] and he proved the following formula about composition of these mappings:

$$\Psi_\alpha = \Psi_\beta \Phi_{\beta,\alpha} = \Phi_{\beta,\alpha}\Psi_\beta, \quad \text{for } -\infty < \beta < \alpha < \infty,$$

which entails that $\Upsilon_{\eta_\alpha}(\nu) = \left(\Upsilon_{\eta_\beta}\Upsilon_{\eta_{\beta,\alpha}}\right)(\nu)$ for Lévy measures ν of infinitely divisible distributions in the domain of the mapping above. Noting that $\eta_0 = \nu_\gamma$, we have

$$\nu_\gamma = \Upsilon_{\eta_0}(\delta_1) = \left(\Upsilon_{\eta_\beta}\Upsilon_{\eta_{\beta,0}}\right)(\delta_1) = \eta_\beta \circledast \eta_{\beta,0}, \quad \text{for } \beta < 0.$$

It follows that for $\beta < 0$,

$$\begin{aligned} &\mathcal{L}(\gamma) = \Psi_\beta(\mu_{\beta,0}) \\ &\quad \text{with} \quad \widehat{\mu}_{\beta,0}(z) = \exp\left\{(\Gamma(-\beta))^{-1}\int_0^1 \left(e^{izx}-1\right)(1-x)^{-\beta-1}x^{-1}dx\right\}, \end{aligned} \tag{3.9}$$

and

$$\mathcal{L}(\gamma) = \Phi_{\beta,0}(\mu_\beta) \quad \text{with} \quad \widehat{\mu}_\beta(z) = \exp\left\{\int_0^\infty \left(e^{izx}-1\right)x^{-\beta-1}e^{-x}dx\right\}. \tag{3.10}$$

The latter expression has the meaning that

$$\gamma \stackrel{\mathrm{d}}{=} \int_0^\infty \varepsilon_{\eta^*_{\beta,\alpha}}(t)d\gamma_{-\beta N(\Gamma(-\beta)t),1},$$

where $\{\gamma_{t,1}\}$ and $\{N(t)\}$ are the processes defined in Corollary 2.1. Also, it holds that

$$\nu_\gamma = \eta_\beta \circledast \eta_{\beta,0} = \Upsilon_{\eta_\beta}(\delta_1) \circledast \eta_{\beta,0} = \left(\Upsilon_{\eta_{\beta'}}\Upsilon_{\eta_{\beta',\beta}}\right)(\delta_1) \circledast \eta_{\beta,0} = \eta_{\beta'} \circledast \eta_{\beta',\beta} \circledast \eta_{\beta,0}$$

for $\beta' < \beta < 0$. Note that $\rho_{\beta',\beta} := \eta_{\beta'} \circledast \eta_{\beta,0}$ is the Lévy measure of a subordinator by Theorem 3.4 (ii) of [5] and its density is

$$\begin{aligned}
\rho_{\beta',\beta}(dt) &= dt \int_0^1 (ts^{-1})^{-\beta'-1} e^{-ts^{-1}} s^{-1} (\Gamma(-\beta))^{-1} (1-s)^{-\beta-1} s^{-1} ds \\
&= (\Gamma(-\beta))^{-1} t^{-\beta'-1} e^{-t} dt \int_0^\infty x^{-\beta-1} (x+1)^{\beta-\beta'} e^{-tx} dx \\
&= t^{-(\beta'+3)/2} e^{-t/2} W_{(2\beta-\beta'+1)/2,-\beta'/2}(t) dt,
\end{aligned}$$

where $W_{a,b}$ is the Whittaker function. Since $\nu_\gamma = \eta_{\beta',\beta} \circledast \rho_{\beta',\beta}$, we have

$$\mathcal{L}(\gamma) = \Phi_{\beta',\beta}(\mu^{(1)}_{\beta',\beta}) \quad \text{with} \quad \widehat{\mu}^{(1)}_{\beta',\beta}(z) = \exp\left\{\int_0^\infty \left(e^{izx} - 1\right) \rho_{\beta',\beta}(dx)\right\},$$

and

$$\mathcal{L}(\gamma) = \Phi_{\varepsilon^*_{\rho_{\beta',\beta}}}\left(\mu^{(2)}_{\beta',\beta}\right)$$

with

$$\widehat{\mu}^{(2)}_{\beta',\beta}(z) = \exp\left\{(\Gamma(\beta-\beta'))^{-1} \int_0^1 \left(e^{izx} - 1\right) (1-x)^{\beta-\beta'-1} x^{-\beta-1} dx\right\}.$$

The latter expression has the meaning that

$$\gamma \stackrel{\mathrm{d}}{=} \int_0^\infty \varepsilon_{\eta^*_{\beta,\alpha}}(t) d\left(\sum_{k=1}^{N(\Gamma(-\beta)\Gamma(-\beta')^{-1}t)} X_k^{(-\beta,\beta-\beta')}\right),$$

where $\{N(t)\}$ is a Poisson process with parameter 1 and $X_k^{(-\beta,\beta-\beta')}, k \in \mathbb{N}$, are i.i.d. beta random variables with parameters $-\beta$ and $\beta - \beta'$, independent of $\{N(t)\}$.

EXAMPLE 3.3. Let $\rho(dx) = e^{-x^2}dx$ and $\eta(dx) = x^{-1}e^{-x^2}dx$. Then

$$(\rho \circledast \eta)(dt) = dt \int_0^\infty e^{-(t/s)^2} s^{-1} s^{-1} e^{-s^2} ds$$

$$= 2^{-1}dt\int_0^\infty e^{-t^2u}u^{-1/2}e^{-1/u}du$$

$$= \pi^{1/2}(2te^{2t})^{-1}dt = \pi^{1/2}2^{-1}\nu_\gamma(2dt).$$

Hence

$$\nu_\gamma = 2\pi^{-1/2}\rho(2^{-1}\cdot)\circledast\eta = \rho\circledast 2\pi^{-1/2}\eta(2^{-1}\cdot). \tag{3.11}$$

This also entails that $\nu_\gamma = 2\pi^{-1/2}\delta_2\circledast\rho\circledast\eta$, that is, $\Upsilon_{\nu_\gamma} = \Upsilon_{2\pi^{-1/2}\delta_2}\Upsilon_\rho\Upsilon_\eta$, (cf. Proposition 4.1 of [5]). Hence

$$\Psi^L = \Upsilon_{2\pi^{-1/2}\delta_2}\mathcal{G}^L\mathcal{M}^L = \Phi^L_{\varepsilon^*_{2\pi^{-1/2}\delta_2}}\mathcal{G}^L\mathcal{M}^L, \tag{3.12}$$

where $\Phi_{\varepsilon^*_{2\pi^{-1/2}\delta_2}}(\mu) = \mathcal{L}(2X_{2\pi^{-1/2}}) = \mathcal{L}\left(2\int_0^{2\pi^{-1/2}}dX_t^{(\mu)}\right)$.

EXAMPLE 3.4. Let $\alpha < 0$. Let $\eta_\alpha(dx) = x^{-\alpha-1}\mathbf{1}_{(0,1)}(x)dx$ and let $\rho_\alpha(dx) = (1-\alpha x^{-1})e^{-x}dx$. Note that

$$\varepsilon^*_{\eta_\alpha}(t) = (1+\alpha t)^{-1/\alpha}\mathbf{1}_{(0,-1/\alpha)}(t). \tag{3.13}$$

Define $\Phi_\alpha := \Phi_{\varepsilon^*_{\eta_\alpha}}$, which was studied in [9, 18, 12]. It holds that

$$\eta_\alpha\circledast\rho_\alpha(dt) = dt\int_t^\infty (t/s)^{-\alpha-1}s^{-1}(1-\alpha s^{-1})e^{-s}ds$$

$$= t^{-\alpha-1}dt\int_t^\infty(1-\alpha s^{-1})s^\alpha e^{-s}ds.$$

Since $d(t^\alpha e^{-t})/dt = (\alpha t^{-1}-1)t^\alpha e^{-t}$, we have

$$\eta_\alpha\circledast\rho_\alpha(dt) = t^{-1}e^{-t}dt = \nu_\gamma(dt). \tag{3.14}$$

This also yields that

$$\gamma \stackrel{\mathrm{d}}{=} \begin{cases} \displaystyle\int_0^{-1/\alpha}(1+\alpha t)^{-1/\alpha}d(\gamma_{N(t),1}+\widetilde{\gamma}_{-\alpha t,1}), \\ \displaystyle\int_0^\infty \varepsilon^*_{\rho_\alpha}(t)d\left(\sum_{k=1}^{N(-\alpha^{-1}t)}X_k^{(-\alpha,1)}\right), \end{cases}$$

where $\{\gamma_{t,1}\}$, $\{\widetilde{\gamma}_{t,1}\}$ and $\{N(t)\}$ are the processes defined in Corollary 2.1, and $X_k^{(-\alpha,1)}, k\in\mathbb{N}$, are i.i.d. beta random variables with parameters $-\alpha$ and 1, independent of $\{N(t)\}$.

We are now ready to prove Proposition 2.1.

Proof of Proposition 2.1. By (3.13), $\Phi_{\varepsilon^*_{\eta_{-1}}} = \mathcal{U}$. Hence (3.14) with $\alpha = -1$ and (3.6) yield (2.1).

Letting $\beta = -1$ in (3.9) and (3.10), we have (2.2) and (2.3), respectively.

Note that $\varepsilon_{\nu_\gamma}(\xi) = p(\xi)$, which entails $\Phi_{\varepsilon^*_{\nu_\gamma}} = \Psi$. Hence Example 3.1 and (3.6) yield (2.4).

The measures ρ and η in Example 3.3 satisfy $\Phi_{\varepsilon^*_\rho} = \mathcal{G}$ and $\Phi_{\varepsilon^*_\eta} = \mathcal{M}$. Therefore (3.11) and (3.6) yield (2.5) and (2.6), respectively. ■

In the following, we give more expressions of γ and ν_γ.

EXAMPLE 3.5. Let $\rho_{\alpha,\beta}(dx) = x^{-\alpha-1}e^{-x^\beta}dx$ and $\eta(dx) = 2\pi^{-1}(1-x^2)^{-1/2}$ $\mathbf{1}_{(0,1)}(x)dx$. Maejima et al. [14] proved that $\mathcal{G}^L = \Upsilon_{2\delta_1}\Upsilon_{\rho_{-2,2}}\Upsilon_\eta$. It follows from (3.12) that

$$\Psi^L = \Upsilon_{2\pi^{-1/2}\delta_2}\mathcal{M}^L\Upsilon_{2\delta_1}\Upsilon_{\rho_{-2,2}}\Upsilon_\eta = \Upsilon_{4\pi^{-1/2}\delta_2}\mathcal{M}^L\Upsilon_{\rho_{-2,2}}\Upsilon_\eta.$$

Hence

$$\nu_\gamma = 4\pi^{-1/2}\delta_2 \circledast \rho_{0,2} \circledast \rho_{-2,2} \circledast \eta.$$

Noting that $\rho_{0,2} \circledast \rho_{-2,2} = \Upsilon_{\rho_{-2,2}}(\rho_{0,2})$ and $\rho_{0,2} \circledast \eta = \Upsilon_\eta(\rho_{0,2})$, by Theorem 3.4(ii) of [5], we have that $\rho_{0,2} \circledast \rho_{-2,2}$ and $\rho_{0,2} \circledast \eta$ are Lévy measures on $(0,\infty)$ satisfying $\int_{(0,\infty)}(x \wedge 1)(\rho_{0,2} \circledast \rho_{-2,2})(dx) + \int_{(0,\infty)}(x \wedge 1)(\rho_{0,2} \circledast \eta)(dx) < \infty$. Moreover, we get

$$\begin{aligned}
(\rho_{0,2} \circledast \rho_{-2,2})(dt) &= dt\int_0^\infty (t/s)^{-1}e^{-(t/s)^2}s^{-1}se^{-s^2}ds \\
&= (2t)^{-1}dt\int_0^\infty u^{-2}e^{-1/u}e^{-t^2u}du = K_1(2t)dt, \\
(\rho_{0,2} \circledast \eta)(dt) &= dt\int_0^1 (t/s)^{-1}e^{-(t/s)^2}s^{-1}2\pi^{-1}(1-s^2)^{-1/2}ds \\
&= (\pi t)^{-1}dt\int_1^\infty e^{-t^2u}(u-1)^{-1/2}u^{-1}du \\
&= (\pi^{1/2}t)^{-1}dt\int_{t^2}^\infty u^{-1/2}e^{-u}du \\
&= (\pi^{1/2}t)^{-1}2dt\int_t^\infty e^{-v^2}dv,
\end{aligned}$$

where K_1 is the modified Bessel function of order 1. It also holds that

$$\gamma \overset{\mathrm{d}}{=} \begin{cases} 2\int_0^\infty \varepsilon^*_{\rho_{0,2}\circledast\rho_{-2,2}}(t)d\left(\sum_{k=1}^{N(4\pi^{-1/2}t)} X_k\right), \\ 2\int_0^\infty \varepsilon^*_{\rho_{0,2}\circledast\eta}(t)d\left(\sum_{k=1}^{N(2\pi^{-1/2}t)} Y_k\right), \end{cases}$$

where we have used $\varepsilon_{\rho_{0,2}\circledast\rho_{-2,2}}(0) = \int_0^\infty K_1(2t)dt = \infty$, due to the fact that $K_1(x) \sim x^{-1}$ as $x \downarrow 0$, and where $\{N(t)\}$ is a Poisson process with parameter 1, $X_k, k \in \mathbb{N}$, are i.i.d. random variables with arcsine law independent of $\{N(t)\}$, and $Y_k, k \in \mathbb{N}$, are i.i.d. Weibull random variables with parameter 2 independent of $\{N(t)\}$. We also have that

$$\gamma \overset{\mathrm{d}}{=} 2\int_0^1 \cos(2^{-1}\pi t)dX_t^{(\mu)}$$

with

$$\widehat{\mu}(z) = \exp\left\{4\pi^{-1/2}\int_0^\infty \left(e^{izx}-1\right)K_1(2x)dx\right\}.$$

EXAMPLE 3.6. Let $\alpha \in (-\infty,1)\cup(1,2)$ and $\beta > 0$. Let $\rho_{\alpha,\beta}$ and η_α be the ones in Examples 3.5 and 3.4, respectively. Maejima and Ueda [16] proved that $\rho_{\alpha,\beta} = \beta\eta_\alpha \circledast \rho_{\alpha-\beta,\beta}$. On the other hand, by (3.12), $\nu_\gamma = 2\pi^{-1/2}\delta_2 \circledast \rho_{-1,2} \circledast \rho_{0,2}$. Hence

$$\nu_\gamma = 8\pi^{-1/2}\delta_2 \circledast \eta_{-1} \circledast \eta_0 \circledast \rho_{-3,2} \circledast \rho_{-2,2}.$$

This yields several results as in the examples above.

4. A remarkable characterization of classes of infinitely divisible distributions

Let $I^\sharp(\mathbb{R}_+)$ denote the totality of $\mu \in I(\mathbb{R}_+)$ without drift. Let $I^{\mathrm{sym},0}(\mathbb{R})$ be the totality of symmetric $\mu \in I(\mathbb{R})$ without Gaussian part. We use the symbols $U^\sharp(\mathbb{R}_+) := U(\mathbb{R}) \cap I^\sharp(\mathbb{R}_+)$ and $U^{\mathrm{sym},0}(\mathbb{R}) := U(\mathbb{R}) \cap I^{\mathrm{sym},0}(\mathbb{R})$. Also, we define $B^\sharp(\mathbb{R}_+)$, $B^{\mathrm{sym},0}(\mathbb{R})$, $L^\sharp(\mathbb{R}_+)$, $L^{\mathrm{sym},0}(\mathbb{R})$, $T^\sharp(\mathbb{R}_+)$, $T^{\mathrm{sym},0}(\mathbb{R})$, $G^\sharp(\mathbb{R}_+)$, $G^{\mathrm{sym},0}(\mathbb{R})$, $M^\sharp(\mathbb{R}_+)$, $M^{\mathrm{sym},0}(\mathbb{R})$, $I^\sharp_{\log}(\mathbb{R}_+)$ and $I^{\mathrm{sym},0}_{\log}(\mathbb{R})$ in the same way.

The class $T(\mathbb{R}_+)$ is well known to be the class of generalized gamma convolutions (GGC) (cf. [6]). $T^\sharp(\mathbb{R}_+)$ is the totality of generalized gamma convolutions without drift. James et al. [7] characterized the class $T^\sharp(\mathbb{R}_+)$ in the following way.

PROPOSITION 4.1 ([7]). *Let $\{\gamma_t\}$ be a gamma process, which is a Lévy process with $\mathcal{L}(\gamma_1) = \mathcal{L}(\gamma)$. Then*

$$T^\sharp(\mathbb{R}_+) = \left\{ \mathcal{L}\left(\int_0^\infty h(t) d\gamma_t \right) : h \geq 0,\ \int_0^\infty \log(1 + h(t)) dt < \infty \right\}.$$

Also, Aoyama et al. [1] showed a similar proposition about the class $B^\sharp(\mathbb{R}_+)$ as follows. For a Lévy process $Y = \{Y_t\}$, denote by $\mathbf{L}_{(0,\infty)}(Y)$ the totality of locally Y-integrable measurable functions on $(0,\infty)$ (cf. [19]), and let

$$\begin{aligned} \mathrm{Dom}(Y) &:= \left\{ h \in \mathbf{L}_{(0,\infty)}(Y) \colon \int_0^\infty h(t) dY_t \text{ is definable} \right\}, \\ \mathrm{Dom}_+(Y) &:= \{ h \in \mathrm{Dom}(Y) \colon h \geq 0 \}. \end{aligned}$$

PROPOSITION 4.2 ([1]). *Let $\{N(t)\}$ be a Poisson process with parameter* 1. *Then*

$$B^\sharp(\mathbb{R}_+) = \left\{ \mathcal{L}\left(\int_0^\infty h(t) d\gamma_{N(t)} \right) : h \in \mathrm{Dom}_+\left(\gamma_{N(\cdot)}\right) \right\}.$$

Recall Proposition 1.1. Then we have the following.

$$\begin{aligned} &\left\{ \mathcal{L}\left(\int_0^\infty p^*(t) dX_t^{(\mu)} \right) : \mu \in I_{\log}^\sharp(\mathbb{R}_+) \right\} \qquad (4.1) \\ &= \left\{ \mathcal{L}\left(\int_0^\infty h(t) d\gamma_t \right) : h \geq 0,\ \int_0^\infty \log(1 + h(t)) dt < \infty \right\}, \\ &\left\{ \mathcal{L}\left(\int_0^1 \log(t^{-1}) dX_t^{(\mu)} \right) : \mu \in I^\sharp(\mathbb{R}_+) \right\} \qquad (4.2) \\ &= \left\{ \mathcal{L}\left(\int_0^\infty h(t) d\gamma_{N(t)} \right) : h \in \mathrm{Dom}_+\left(\gamma_{N(\cdot)}\right) \right\}. \end{aligned}$$

Let ν_γ and $\nu_{\gamma(N(\cdot))}$ be the Lévy measures of $\{\gamma_t\}$ and $\{\gamma_{N(t)}\}$, respectively. Noting that $p(\xi) = \int_\xi^\infty e^{-u} u^{-1} du = \nu_\gamma((\xi,\infty))$ and that the inverse function of $t \mapsto \log t^{-1}$ is $\xi \mapsto e^{-\xi} = \nu_{\gamma(N(\cdot))}((\xi,\infty))$, we can understand that (4.1) and (4.2) come from a kind of the commutativity (3.7) of integrands and driving processes of stochastic integrals which we have considered in the previous section. Using this method, we can characterize some classes of infinitely divisible distributions in two ways. One is by fixing an integrand and by taking some possible driving Lévy processes, and another is by fixing a driving Lévy process and by taking some possible integrands.

Fix a Lévy measure ν on $(0,\infty)$ satisfying $\int_{(0,1]} x\nu(dx) < \infty$. We denote by $\{X_\nu(t)\}$ a subordinator without drift whose Lévy measure is ν. Also $\{X_\nu^{\mathrm{sym}}(t)\}$ denotes a symmetric Lévy process without Gaussian part with Lévy measure $\mathbf{1}_{(0,\infty)}(x)\nu(dx) + \mathbf{1}_{(-\infty,0)}(x)\nu(-dx)$. For $j = 1,2$, let

$$\mathbf{L}^{\downarrow j} := \Big\{ h\colon h \text{ is a nonnegative nonincreasing right-continuous function satisfying } \int_0^\infty (h(t)^j \wedge 1)dt < \infty \Big\}.$$

Then we have the following.

THEOREM 4.1. (i) *We have*

$$\Phi_{\varepsilon_\nu^*}\left(\mathfrak{D}(\Phi_{\varepsilon_\nu^*}) \cap I^\sharp(\mathbb{R}_+)\right) = \left\{ \mathcal{L}\left(\int_0^\infty h(t)dX_\nu(t)\right) : h \in \mathrm{Dom}_+(X_\nu) \cap \mathbf{L}^{\downarrow 1} \right\}.$$

(ii) *If the class* $\Phi_{\varepsilon_\nu^*}\left(\mathfrak{D}(\Phi_{\varepsilon_\nu^*}) \cap I^\sharp(\mathbb{R}_+)\right)$ *is closed under weak convergence, then*

$$\Phi_{\varepsilon_\nu^*}\left(\mathfrak{D}(\Phi_{\varepsilon_\nu^*}) \cap I^\sharp(\mathbb{R}_+)\right) = \left\{ \mathcal{L}\left(\int_0^\infty h(t)dX_\nu(t)\right) : h \in \mathrm{Dom}_+(X_\nu) \right\}.$$

Proof. (i) ($\subset$) Suppose $\mu \in \mathfrak{D}(\Phi_{\varepsilon_\nu^*}) \cap I^\sharp(\mathbb{R}_+)$ with Lévy measure ν_μ. Let $h := \varepsilon_{\nu_\mu}^*$. Then h is a nonnegative nonincreasing càdlàg function satisfying

$$\begin{aligned}\int_0^\infty (h(t)\wedge 1)dt &= \int_0^{\varepsilon_{\nu_\mu}(0)} (\varepsilon_{\nu_\mu}^*(t)\wedge 1)dt \\ &= \int_{(0,\infty)} (x\wedge 1)\mathrm{Leb}(\varepsilon_{\nu_\mu}^*)^{-1}(dx) \\ &= \int_{(0,\infty)} (x\wedge 1)\nu_\mu(dx) < \infty,\end{aligned}$$

where we have used the relation (3.5) with the replacement of η by ν_μ. Hence $h \in \mathbf{L}^{\downarrow 1}$. Since $\mu \in \mathfrak{D}(\Phi_{\varepsilon_\nu^*})$, $\int_0^{\varepsilon_\nu(0)} \varepsilon_\nu^*(t)dX_t^{(\mu)}$ is definable and its Lévy measure is

$$\int_0^{\varepsilon_\nu(0)} ds \int_{(0,\infty)} \mathbf{1}_B(\varepsilon_\nu^*(s)x)\nu_\mu(dx) \tag{4.3}$$

$$
\begin{aligned}
&= \int_{(0,\infty)} \mathrm{Leb}(\varepsilon_\nu^*)^{-1}(dy) \int_{(0,\infty)} \mathbf{1}_B(xy)\nu_\mu(dx) \\
&= \int_{(0,\infty)} \nu(dy) \int_{(0,\infty)} \mathbf{1}_B(xy)\nu_\mu(dx) = (\nu \circledast \nu_\mu)(B),
\end{aligned}
$$

which satisfies $\int_{(0,\infty)}(x \wedge 1)(\nu \circledast \nu_\mu)(dx) < \infty$ since $\int_0^{\varepsilon_\nu(0)} \varepsilon_\nu^*(t)dX_t^{(\mu)} \geq 0$. Since $\{X_t^{(\mu)}\}$ has no drift, the location parameter of $\int_0^{\varepsilon_\nu(0)} \varepsilon_\nu^*(t)dX_t^{(\mu)}$ is $\int_{(0,\infty)} x(1+x^2)^{-1}$ $(\nu \circledast \nu_\mu)(dx)$. Noting that

$$
\begin{aligned}
&\int_0^\infty ds \int_{(0,\infty)} \mathbf{1}_B(h(s)x)\nu(dx) \\
&= \int_0^{\varepsilon_{\nu_\mu}(0)} ds \int_{(0,\infty)} \mathbf{1}_B(\varepsilon_{\nu_\mu}^*(s)x)\nu(dx) = (\nu_\mu \circledast \nu)(B),
\end{aligned}
\tag{4.4}
$$

we have, by similar calculations to (3.2) and (3.3), that $h \in \mathrm{Dom}_+(X_\nu)$ and that $\int_0^\infty h(t)dX_\nu(t) \stackrel{\mathrm{d}}{=} \int_0^{\varepsilon_\nu(0)} \varepsilon_\nu^*(t)dX_t^{(\mu)}$.
($\supset$) Suppose $h \in \mathrm{Dom}_+(X_\nu) \cap \mathbf{L}^{\downarrow 1}$. Let

$$
h^*(x) := \inf\{t > 0\colon h(t) \leq x\}, \quad x > 0. \tag{4.5}
$$

Then, h is a nonincreasing càdlàg function satisfying $h^*(\infty) = 0$. We have that $h^*(x) \leq t$ if and only if $h(t) \leq x$, and hence

$$
\begin{aligned}
\mathrm{Leb}\, h^{-1}((x,\infty)) &= \mathrm{Leb}(\{t > 0\colon h(t) > x\}) \\
&= \mathrm{Leb}(\{t > 0\colon h^*(x) > t\}) \\
&= h^*(x) = -\int_x^\infty dh^*(y), \quad x > 0.
\end{aligned}
$$

Hence

$$
\begin{aligned}
-\int_0^\infty (x \wedge 1)dh^*(x) &= \int_{(0,\infty)} (x \wedge 1)\mathrm{Leb}\, h^{-1}(dx) \\
&= \int_0^{h^*(0)} (h(t) \wedge 1)dt = \int_0^\infty (h(t) \wedge 1)dt < \infty.
\end{aligned}
$$

Let $\mu \in I^\sharp(\mathbb{R}_+)$ with Lévy measure $\nu_\mu(dx) = -dh^*(x)$. Then $\varepsilon_{\nu_\mu} = h^*$, and hence $\varepsilon_{\nu_\mu}^* = h$. Since $h \in \mathrm{Dom}_+(X_\nu)$, $\int_0^\infty h(t)dX_\nu(t)$ is definable and its Lévy measure is (4.4), which satisfies $\int_{(0,\infty)}(x \wedge 1)(\nu \circledast \nu_\mu)(dx) < \infty$ since $\int_0^\infty h(t)dX_\nu(t) \geq 0$. Since $\{X_\nu(t)\}$ has no drift, the location parameter of $\int_0^\infty h(t)dX_\nu(t)$ is $\int_{(0,\infty)} x(1+x^2)^{-1}(\nu \circledast \nu_\mu)(dx)$. Due to (4.3) and similar calculations to (3.2) and (3.3), we have that $\mu \in \mathfrak{D}(\Phi_{\varepsilon_\nu^*})$ and that $\int_0^\infty h(t)dX_\nu(t) \stackrel{\mathrm{d}}{=} \int_0^{\varepsilon_\nu(0)} \varepsilon_\nu^*(t)dX_t^{(\mu)}$.

(ii) ($\subset$) This follows from (i).
($\supset$) It follows from (i) that

$$\mathcal{L}(X_\nu(1)) = \mathcal{L}\left(\int_0^\infty \mathbf{1}_{(0,1)}(t)dX_\nu(t)\right) \in \Phi_{\varepsilon_\nu^*}(\mathfrak{D}(\Phi_{\varepsilon_\nu^*}) \cap I^\sharp(\mathbb{R}_+))$$

since $\mathbf{1}_{(0,1)} \in \mathrm{Dom}_+(X_\nu) \cap \mathbf{L}^{\downarrow 1}$. Then, by the definition of stochastic integrals, $\int_0^\infty h(t)dX_\nu(t) \in \Phi_{\varepsilon_\nu^*}\big(\mathfrak{D}(\Phi_{\varepsilon_\nu^*}) \cap I^\sharp(\mathbb{R}_+)\big)$ for any $h \in \mathrm{Dom}_+(X_\nu)$ if $\Phi_{\varepsilon_\nu^*}\big(\mathfrak{D}(\Phi_{\varepsilon_\nu^*}) \cap I^\sharp(\mathbb{R}_+)\big)$ is closed under weak convergence. ■

About the symmetric cases, we have the following.

THEOREM 4.2. (i) *We have*

$$\begin{aligned}&\Phi_{\varepsilon_\nu^*}\left(\mathfrak{D}(\Phi_{\varepsilon_\nu^*}) \cap I^{\mathrm{sym},0}(\mathbb{R})\right)\\ &= \left\{\mathcal{L}\left(\int_0^\infty h(t)dX_\nu^{\mathrm{sym}}(t)\right) : h \in \mathrm{Dom}_+(X_\nu^{\mathrm{sym}}) \cap \mathbf{L}^{\downarrow 2}\right\}.\end{aligned}$$

(ii) *If the class* $\Phi_{\varepsilon_\nu^*}\left(\mathfrak{D}(\Phi_{\varepsilon_\nu^*}) \cap I^{\mathrm{sym},0}(\mathbb{R})\right)$ *is closed under weak convergence, then*

$$\begin{aligned}&\Phi_{\varepsilon_\nu^*}\left(\mathfrak{D}(\Phi_{\varepsilon_\nu^*}) \cap I^{\mathrm{sym},0}(\mathbb{R})\right)\\ &= \left\{\mathcal{L}\left(\int_0^\infty h(t)dX_\nu^{\mathrm{sym}}(t)\right) : h \in \mathrm{Dom}_+(X_\nu^{\mathrm{sym}})\right\}.\end{aligned}$$

P r o o f. (i) ($\subset$) Suppose $\mu \in \mathfrak{D}(\Phi_{\varepsilon_\nu^*}) \cap I^{\mathrm{sym},0}(\mathbb{R})$ with Lévy measure ν_μ. Note that $\nu_\mu(B) = \nu_\mu(-B)$ and that the location parameter of μ is 0. Let $\nu_\mu^+ := \nu_\mu|_{\mathcal{B}((0,\infty))}$, which is the restriction of ν_μ on $\mathcal{B}((0,\infty))$. and let $h := \varepsilon_{\nu_\mu^+}^*$. Then h is a nonnegative nonincreasing càdlàg function satisfying

$$\begin{aligned}\int_0^\infty (h(t)^2 \wedge 1)dt &= \int_0^{\varepsilon_{\nu_\mu^+}(0)} (\varepsilon_{\nu_\mu^+}^*(t)^2 \wedge 1)dt\\ &= \int_{(0,\infty)} (x^2 \wedge 1)\mathrm{Leb}(\varepsilon_{\nu_\mu^+}^*)^{-1}(dx)\\ &= \int_{(0,\infty)} (x^2 \wedge 1)\nu_\mu(dx) < \infty,\end{aligned}$$

where we have used the relation (3.5) with the replacement of η by ν_μ^+. Hence $h \in \mathbf{L}^{\downarrow 2}$. Since $\mu \in \mathfrak{D}(\Phi_{\varepsilon_\nu^*})$, $\int_0^{\varepsilon_\nu(0)} \varepsilon_\nu^*(t)dX_t^{(\mu)}$ is definable and its Lévy measure is

$$\int_0^{\varepsilon_\nu(0)} ds \int_{\mathbb{R}} \mathbf{1}_B(\varepsilon_\nu^*(s)x)\nu_\mu(dx) \tag{4.6}$$

$$
\begin{aligned}
&= \int_{(0,\infty)} \mathrm{Leb}(\varepsilon_\nu^*)^{-1}(dy) \int_{\mathbb{R}} \mathbf{1}_B(xy)\nu_\mu(dx) \\
&= \int_{(0,\infty)} \nu(dy) \int_{\mathbb{R}} \mathbf{1}_B(xy)\nu_\mu(dx) \\
&= \int_{(0,\infty)} \nu(dy) \left(\int_{(0,\infty)} \mathbf{1}_B(xy)\nu_\mu(dx) + \int_{(0,\infty)} \mathbf{1}_B(-xy)\nu_\mu(dx) \right) \\
&= \left(\nu \circledast \nu_\mu^+\right)(B \cap (0,\infty)) + \left(\nu \circledast \nu_\mu^+\right)((-B)\cap(0,\infty)), \quad B \in \mathcal{B}(\mathbb{R}\setminus\{0\}).
\end{aligned}
$$

Since $\{X_t^{(\mu)}\}$ is symmetric without Gaussian part, so is $\int_0^{\varepsilon_\nu(0)} \varepsilon_\nu^*(t)dX_t^{(\mu)}$. Note that for any symmetric Lévy process $Y = \{Y_t\}$ without Gaussian part having Lévy measure ν_Y and for any measurable function $f\colon (0,\infty) \to \mathbb{R}$, a necessary and sufficient condition for that $\mathcal{L}(Y_1) \in \mathfrak{D}(\Phi_f)$ is that $\int_0^\infty dt \int_{\mathbb{R}} \mathbf{1}_B(f(t)x)\nu_Y(dx), B \in \mathcal{B}(\mathbb{R}\setminus\{0\})$, is a Lévy measure (cf. Theorems 2.6 and 3.5 of [19]). Note that

$$
\begin{aligned}
(4.7)\quad &\int_0^\infty ds \int_{\mathbb{R}} \mathbf{1}_B(h(s)x) \left(\mathbf{1}_{(0,\infty)}(x)\nu(dx) + \mathbf{1}_{(-\infty,0)}(x)\nu(-dx)\right) \\
&= \int_0^{\varepsilon_{\nu_\mu^+}(0)} ds \int_{\mathbb{R}} \mathbf{1}_B(\varepsilon^*_{\nu_\mu^+}(s)x) \left(\mathbf{1}_{(0,\infty)}(x)\nu(dx) + \mathbf{1}_{(-\infty,0)}(x)\nu(-dx)\right) \\
&= \int_{(0,\infty)} \nu_\mu^+(dy) \int_{\mathbb{R}} \mathbf{1}_B(xy) \left(\mathbf{1}_{(0,\infty)}(x)\nu(dx) + \mathbf{1}_{(-\infty,0)}(x)\nu(-dx)\right) \\
&= \int_{(0,\infty)} \nu_\mu^+(dy) \int_{(0,\infty)} \mathbf{1}_B(xy)\nu(dx) + \int_{(0,\infty)} \nu_\mu^+(dy) \int_{(0,\infty)} \mathbf{1}_B(-xy)\nu(dx) \\
&= \left(\nu \circledast \nu_\mu^+\right)(B \cap (0,\infty)) + \left(\nu \circledast \nu_\mu^+\right)((-B)\cap(0,\infty)), \quad B \in \mathcal{B}(\mathbb{R}\setminus\{0\}),
\end{aligned}
$$

which is equal to the Lévy measure (4.6) of $\int_0^{\varepsilon_\nu(0)} \varepsilon_\nu^*(t)dX_t^{(\mu)}$. Thus $h \in \mathrm{Dom}_+(X_\nu^{\mathrm{sym}})$ and $\int_0^\infty h(t)dX_\nu^{\mathrm{sym}}(t) \overset{\mathrm{d}}{=} \int_0^{\varepsilon_\nu(0)} \varepsilon_\nu^*(t)dX_t^{(\mu)}$.
($\supset$) Suppose $h \in \mathrm{Dom}_+(X_\nu^{\mathrm{sym}}) \cap \mathbf{L}^{\downarrow 2}$. Define h^* by (4.5). Then

$$
\begin{aligned}
-\int_0^\infty (x^2 \wedge 1)dh^*(x) &= \int_{(0,\infty)} (x^2 \wedge 1)\mathrm{Leb}\, h^{-1}(dx) \\
&= \int_0^{h^*(0)} (h(t)^2 \wedge 1)dt = \int_0^\infty (h(t)^2 \wedge 1)dt < \infty.
\end{aligned}
$$

Let $\mu \in I^{\mathrm{sym},0}(\mathbb{R})$ with Lévy measure

$$
\nu_\mu(B) = -\int_{B\cap(0,\infty)} dh^*(x) - \int_{(-B)\cap(0,\infty)} dh^*(x), \quad B \in \mathcal{B}(\mathbb{R}\setminus\{0\}).
$$

Then $\varepsilon_{\nu_\mu^+} = h^*$ and thus $\varepsilon^*_{\nu_\mu^+} = h$. Since $h \in \mathrm{Dom}_+(X_\nu^{\mathrm{sym}})$, $\int_0^\infty h(t)dX_\nu^{\mathrm{sym}}(t)$ is definable and its Lévy measure is (4.7). Due to (4.6), which is equal to (4.7), we have that $\mu \in \mathfrak{D}(\Phi_{\varepsilon_\nu^*})$ and $\int_0^\infty h(t)dX_\nu^{\mathrm{sym}}(t) \overset{\mathrm{d}}{=} \int_0^{\varepsilon_\nu(0)} \varepsilon_\nu^*(t)dX_t^{(\mu)}$.

(ii) ($\subset$) This follows from (i).
($\supset$) It follows from (i) that

$$\mathcal{L}(X_\nu^{\mathrm{sym}}(1)) = \mathcal{L}\left(\int_0^\infty \mathbf{1}_{(0,1)}(t)dX_\nu^{\mathrm{sym}}(t)\right) \in \Phi_{\varepsilon_\nu^*}\left(\mathfrak{D}(\Phi_{\varepsilon_\nu^*}) \cap I^{\mathrm{sym},0}(\mathbb{R})\right)$$

since $\mathbf{1}_{(0,1)} \in \mathrm{Dom}_+(X_\nu^{\mathrm{sym}}) \cap \mathbf{L}^{\downarrow 2}$. Then, by the definition of stochastic integrals, $\int_0^\infty h(t)dX_\nu^{\mathrm{sym}}(t) \in \Phi_{\varepsilon_\nu^*}\left(\mathfrak{D}(\Phi_{\varepsilon_\nu^*}) \cap I^{\mathrm{sym},0}(\mathbb{R})\right)$ for any $h \in \mathrm{Dom}_+(X_\nu^{\mathrm{sym}})$ if $\Phi_{\varepsilon_\nu^*}\big(\mathfrak{D}(\Phi_{\varepsilon_\nu^*}) \cap I^{\mathrm{sym},0}(\mathbb{R})\big)$ is closed under weak convergence. ■

In the following, we give some new stochastic integral characterizations of well-known classes of infinitely divisible distributions.

EXAMPLE 4.1. Let $\nu = \nu_\gamma$. Then Proposition 1.1 and Theorem 4.1 (ii) yield that

$$T^\sharp(\mathbb{R}_+) = \Psi\left(I_{\log}^\sharp(\mathbb{R}_+)\right) = \left\{\mathcal{L}\left(\int_0^\infty h(t)d\gamma_t\right) : h \in \mathrm{Dom}_+(\gamma)\right\}.$$

Also, $h \in \mathrm{Dom}_+(\gamma)$ if and only if $h \geq 0$ and $\int_0^\infty dt \int_0^\infty h(t)x(h(t)x+1)^{-1}\nu_\gamma(dx) < \infty$. The integrability condition $\int_0^\infty dt \int_0^\infty h(t)x(h(t)x+1)^{-1}\nu_\gamma(dx) < \infty$ is equivalent to $\int_0^\infty \log(1+h(t))dt < \infty$. Since, for h with $h \geq 0$ and $\int_0^\infty \log(1+h(t))dt < \infty$, we have

$$\begin{aligned}
&\int_0^\infty dt \int_0^\infty h(t)x(h(t)x+1)^{-1}\nu_\gamma(dx) \\
&= \int_0^\infty dt \int_0^\infty h(t)(h(t)x+1)^{-1}e^{-x}dx \\
&\leq \int_0^\infty dt \left(\int_0^1 h(t)(h(t)x+1)^{-1}dx + \int_1^\infty h(t)(h(t)+1)^{-1}e^{-x}dx\right) \\
&= \int_0^\infty \left\{\log(1+h(t)) + h(t)(h(t)+1)^{-1}e^{-1}\right\} dt \\
&\leq (1+e^{-1})\int_0^\infty \log(1+h(t))dt < \infty.
\end{aligned}$$

Conversely, for h with $h \geq 0$ and $\int_0^\infty dt \int_0^\infty h(t)x(h(t)x+1)^{-1}\nu_\gamma(dx) < \infty$, we have

$$\begin{aligned}
&\int_0^\infty \log(1+h(t))dt \\
&= \int_0^\infty dt \int_0^1 h(t)(h(t)x+1)^{-1}dx
\end{aligned}$$

$$\le \int_0^\infty dt \int_0^1 h(t)x(h(t)x+1)^{-1}x^{-1}e^{1-x}dx$$
$$\le e\int_0^\infty dt \int_0^\infty h(t)x(h(t)x+1)^{-1}\nu_\gamma(dx) < \infty.$$

Hence

$$T^\sharp(\mathbb{R}_+) = \left\{ \mathcal{L}\left(\int_0^\infty h(t)d\gamma_t\right) : h \ge 0,\ \int_0^\infty \log(1+h(t))dt < \infty \right\}.$$

This is another proof of Proposition 4.1.

EXAMPLE 4.2. Let $\nu = \nu_{\gamma(N(\cdot))}$. Then Proposition 1.1 and Theorem 4.1 (ii) yield that

$$B^\sharp(\mathbb{R}_+) = \Upsilon\left(I^\sharp(\mathbb{R}_+)\right) = \left\{ \mathcal{L}\left(\int_0^\infty h(t)d\gamma_{N(t)}\right) : h \in \mathrm{Dom}_+(\gamma_{N(\cdot)}) \right\}$$
$$= \left\{ \mathcal{L}\left(\int_0^\infty h(t)d\gamma_{N(t)}\right) : h \ge 0,\ \int_0^\infty (h(t)\wedge 1)dt < \infty \right\}.$$

EXAMPLE 4.3. Let $\nu(dx) = \mathbf{1}_{(0,1)}(x)dx$, that is, $X_\nu(t) = \sum_{k=1}^{N(t)} Y_k$, where $\{N(t)\}$ is a Poisson process with parameter 1 and $Y_k, k \in \mathbb{N}$, are i.i.d. uniform random variables on $(0,1)$ independent of $\{N(t)\}$. Then Proposition 1.1 and Theorem 4.1 (ii) yield that

$$U^\sharp(\mathbb{R}_+) = \mathcal{U}\left(I^\sharp(\mathbb{R}_+)\right) = \left\{ \mathcal{L}\left(\int_0^\infty h(t)dX_\nu(t)\right) : h \in \mathrm{Dom}_+(X_\nu) \right\}$$
$$= \left\{ \mathcal{L}\left(\int_0^\infty h(t)dX_\nu(t)\right) : h \ge 0,\ \int_0^\infty \mathbf{1}_{\{h(t)>0\}} \left\{1-(2h(t))^{-1}\right\} dt < \infty \right\}.$$

EXAMPLE 4.4. Let $\nu(dx) = x^{-1}\mathbf{1}_{(0,1)}(x)dx$, which is the Lévy measure of a building-block of $L(\mathbb{R}_+)$. Then Proposition 1.1 and Theorem 4.1 (ii) yield that

$$L^\sharp(\mathbb{R}_+) = \Phi\left(I^\sharp_{\log}(\mathbb{R}_+)\right) = \left\{ \mathcal{L}\left(\int_0^\infty h(t)dX_\nu(t)\right) : h \in \mathrm{Dom}_+(X_\nu) \right\}$$
$$= \left\{ \mathcal{L}\left(\int_0^\infty h(t)dX_\nu(t)\right) : h \ge 0,\ \int_0^\infty \log(1+h(t))dt < \infty \right\}.$$

EXAMPLE 4.5. Let $\nu(dx) = e^{-x^2}dx$, that is, $X_\nu^{\mathrm{sym}}(t) = 2^{-1/2}B(N(\pi^{1/2}t))$, where $\{B(t)\}$ is a Brownian motion independent of a Poisson process $\{N(t)\}$ with parameter 1. Then Proposition 1.1 and Theorem 4.2 (ii) yield that

$$G^{\mathrm{sym},0}(\mathbb{R}) = \mathcal{G}\left(I^{\mathrm{sym},0}(\mathbb{R})\right) = \left\{\mathcal{L}\left(\int_0^\infty h(t)dX_\nu^{\mathrm{sym}}(t)\right) : h \in \mathrm{Dom}_+(X_\nu^{\mathrm{sym}})\right\}$$
$$= \left\{\mathcal{L}\left(\int_0^\infty h(t)dX_\nu^{\mathrm{sym}}(t)\right) : h \geq 0,\ \int_0^\infty dt \int_0^\infty (|h(t)x|^2 \wedge 1)e^{-x^2}dx < \infty\right\}.$$

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