A SUBCLASS OF TYPE G SELFDECOMPOSABLE DISTRIBUTIONS ON $\mathbb{R}^d$

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(Running head : Type G selfdecomposable distributions on $\mathbb{R}^d$)

Abstract. A new class of type G selfdecomposable distributions on $\mathbb{R}^d$ is introduced and characterized in terms of stochastic integrals with respect to Lévy processes. This class is a strict subclass of the class of type G and selfdecomposable distributions, and in dimension one, it is strictly bigger than the class of variance mixtures of normal distributions by selfdecomposable distributions. The relation to several other known classes of infinitely divisible distributions is established.

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1. Introduction

Let $I(\mathbb{R}^d)$ denote the set of all infinitely divisible distributions on $\mathbb{R}^d$. The characteristic function $\widehat{\mu}(z), z \in \mathbb{R}^d$, of an infinitely divisible distribution $\mu \in I(\mathbb{R}^d)$ has the Lévy-Khintchine representation as follows:

$$\widehat{\mu}(z) = \exp\left\{-\frac{1}{2}\langle z, Az\rangle + \mathrm{i}\langle \gamma, z\rangle + \int_{\mathbb{R}^d}\left(e^{\mathrm{i}\langle z,x\rangle} - 1 - \frac{\mathrm{i}\langle z, x\rangle}{1+|x|^2}\right)\nu(dx)\right\},\ z \in \mathbb{R}^d,$$

where A is a symmetric nonnegative-definite $d \times d$ matrix, $\gamma \in \mathbb{R}^d$, and ν is a measure on $\mathbb{R}^d$ satisfying

$$\nu(\{0\}) = 0 \text{ and } \int_{\mathbb{R}^d}(|x|^2 \wedge 1)\nu(dx) < \infty.$$

(A, ν, γ) is uniquely determined by μ and is called the triplet of μ. $\nu = \nu_\mu$ is called the Lévy measure of μ. $I_{sym}(\mathbb{R}^d)$ will denote the subset of $I(\mathbb{R}^d)$ consisting of symmetric distributions.

The polar decomposition of Lévy measures on $\mathbb{R}^d$ is the following: Let ν be the Lévy measure of some $\mu \in I(\mathbb{R}^d)$ with $0 < \nu(\mathbb{R}^d) \le \infty$. Then there exist a measure

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λ on $S = \{\xi \in \mathbb{R}^d : |\xi| = 1\}$ with $0 < \lambda(S) \le \infty$ and a family $\{\nu_\xi \colon \xi \in S\}$ of measures on $(0,\infty)$ such that $\nu_\xi(B)$ is measurable in ξ for each $B \in \mathcal{B}((0,\infty))$, $0 < \nu_\xi((0,\infty)) \le \infty$ for each $\xi \in S$ and that

$$\nu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi)\nu_\xi(dr),\ \ B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}).$$

Here λ and $\{\nu_\xi\}$ are uniquely determined by ν up to multiplication of a measurable function $c(\xi)$ and $c(\xi)^{-1}$ with $0 < c(\xi) < \infty$. We say that ν has the polar decomposition (λ, ν_ξ) and ν_ξ is called the radial component of ν. (See, e.g. [2] Lemma 2.1.)

We also define the cumulant function $C_\mu(z)$ of $\mu \in I(\mathbb{R}^d)$ as follows: $C_\mu(z)$ is the unique complex-valued continuous function on $\mathbb{R}^d$ satisfying $C_\mu(0) = 0$ and $\widehat{\mu}(z) = e^{C_\mu(z)}$.

We can characterize five classes of infinitely divisible distributions in terms of ν_ξ:

(i) Class $U(\mathbb{R}^d)$ (*Jurek class*, see [4].)

$\nu_\xi(dr) = l_\xi(r)dr$ and $l_\xi(r)$ is nonincreasing.

(ii) Class $B(\mathbb{R}^d)$ (*Goldie–Steutel–Bondesson class*, see, e.g. [2].)

$\nu_\xi(dr) = l_\xi(r)dr$ and $l_\xi(r)$ is completely monotone.

(iii) Class $L(\mathbb{R}^d)$ (*Class of selfdecomposable distributions*, see, e.g. [8].)

$\nu_\xi(dr) = k_\xi(r)r^{-1}dr$ and $k_\xi(r)$ is nonincreasing.

(iv) Class $T(\mathbb{R}^d)$ (*Thorin class*, see, e.g. [2].)

$\nu_\xi(dr) = k_\xi(r)r^{-1}dr$ and $k_\xi(r)$ is completely monotone.

(v) Class $G(\mathbb{R}^d)$ (*Class of type G distributions*, see [3].)

$\nu_\xi(dr) = g_\xi(r^2)dr$ and $g_\xi(r)$ is completely monotone; in this case we also assume that μ is symmetric.

Let $I_{\log}(\mathbb{R}^d) = \{\mu \in I(\mathbb{R}^d) : \int_{|x|>2} \log|x|\mu(dx) < \infty\}$ and let $\phi(x) = (2\pi)^{-1/2}e^{-x^2/2}$ denote the standard normal density on $\mathbb{R}$.

Being motivated by the relations among classes (i)–(v), it is natural to introduce and consider the following new class.

Definition 1.1 (*Class $M(\mathbb{R}^d)$*)**.** $\mu \in M(\mathbb{R}^d)$ if and only if $\mu \in I_{sym}(\mathbb{R}^d)$ with

$$\nu_\xi(dr) = g_\xi(r^2)r^{-1}dr \text{ and } g_\xi(r) \text{ is completely monotone.} \tag{1.1}$$

It is easy to see that $M(\mathbb{R}^d) \subset L(\mathbb{R}^d) \cap G(\mathbb{R}^d)$, i.e., the elements of $M(\mathbb{R}^d)$ are type G selfdecomposable distributions. In Theorem 3.1 below we will prove that this inclusion is strict. The purpose of this paper is to characterize the class $M(\mathbb{R}^d)$ by stochastic integrals with respect to Lévy processes, and compare it with other known classes.

2. Characterization of the class $M(\mathbb{R}^d)$ by stochastic integrals with respect to Lévy processes

Throughout this paper, $\mathcal{L}(X)$ denotes the law of a random variable X on $\mathbb{R}^d$. Let $m(x) = \int_x^\infty \phi(s)s^{-1}ds, x > 0$, and denote its inverse by $m^*(t)$, that is, $t = m(x)$ if and only if $x = m^*(t)$.

Theorem 2.1. *(i) Let $\mu \in I(\mathbb{R}^d)$. Then the stochastic integral*

$$\int_0^\infty m^*(t)dX_t^{(\mu)}$$

exists if and only if $\mu \in I_{\log}(\mathbb{R}^d)$, where $\{X_t^{(\mu)}\}$ is a Lévy process on $\mathbb{R}^d$ with $\mathcal{L}(X_1^{(\mu)}) = \mu$.

Proof of "if" part. For the proof, we need the following lemma, which is a special case of Proposition 5.5 of [9].

Lemma 2.2. *Let $\{X_t^{(\mu)}\}$ be a Lévy process on $\mathbb{R}^d$ and $f(t)$ a real-valued measurable function on $[0,\infty)$. Let (A,ν,γ) be the triplet of μ. Then $\int_0^\infty f(t)dX_t^{(\mu)}$ exists if the following conditions are satisfied:*

$$\int_0^\infty f(t)^2dt < \infty, \tag{2.1}$$

$$\int_0^\infty dt \int_{\mathbb{R}^d}(|f(t)x|^2 \wedge 1)\nu(dx) < \infty, \tag{2.2}$$

$$\int_0^\infty \left| f(t)\gamma + f(t)\int_{\mathbb{R}^d} x\left(\frac{1}{1+|f(t)x|^2} - \frac{1}{1+|x|^2}\right)\nu(dx)\right| dt < \infty. \tag{2.3}$$

For the proof of "if" part, it is enough to show that $f(t) = m^*(t)$ satisfies (2.1) – (2.3) in Lemma 2.2 for every $\mu \in I_{\log}(\mathbb{R}^d)$. Note that $m(+0) = \infty$ and $m(\infty) = 0$. Since

$$\int_0^\infty m^*(t)^2dt = \int_0^\infty s\phi(s)ds < \infty,$$

we have (2.1).

As to (2.2), we have

$$\begin{aligned}
\int_0^\infty dt & \int_{\mathbb{R}^d} (|m^*(t)x|^2 \wedge 1)\nu(dx) \\
&= -\int_0^\infty dm(s) \int_{\mathbb{R}^d} (|sx|^2 \wedge 1)\nu(dx) \\
&= \int_0^\infty \phi(s)s^{-1}ds \left(\int_{|x|\le 1/s} |sx|^2\nu(dx) + \int_{|x|>1/s} \nu(dx) \right) \\
&=: (I_1 + I_2),
\end{aligned}$$

say. Here

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}^d} |x|^2\nu(dx) \int_0^{1/|x|} s\phi(s)ds \\
&= \left(\int_{|x|\le 1} + \int_{|x|>1} \right) |x|^2\nu(dx) \int_0^{1/|x|} s\phi(s)ds \\
&=: I_{11} + I_{12},
\end{aligned}$$

say, and

$$\begin{aligned}
I_{11} &\le \int_{|x|\le 1} |x|^2\nu(dx) \int_0^\infty s\phi(s)ds < \infty, \\
I_{12} &\le \int_{|x|>1} |x|^2\nu(dx) \int_0^{1/|x|} sds \le 2^{-1} \int_{|x|>1} \nu(dx) < \infty.
\end{aligned}$$

Also,

$$\begin{aligned}
I_2 &= \int_{\mathbb{R}^d} \nu(dx) \int_{1/|x|}^\infty \phi(s)s^{-1}ds \\
&= \left(\int_{|x|\le 1} + \int_{|x|>1} \right) \nu(dx) \int_{1/|x|}^\infty \phi(s)s^{-1}ds \\
&=: I_{21} + I_{22},
\end{aligned}$$

say, and

$$\begin{aligned}
I_{21} &\le C_1 \int_{|x|\le 1} x^2\nu(dx) < \infty, \\
I_{22} &\le \int_{|x|>1} \nu(dx) \left\{ \int_{1/|x|}^1 s^{-1}ds + \int_1^\infty \phi(s)s^{-1}ds \right\} \\
&= \int_{|x|>1} (\log|x| + C_2)\nu(dx) < \infty,
\end{aligned}$$

since $\mu \in I_{\log}(\mathbb{R}^d)$, where $C_1, C_2 > 0$. This shows (2.2).

For (2.3), we have

$$\begin{aligned}
&\int_0^\infty \left| m^*(t)\gamma + m^*(t)\int_{\mathbb{R}^d} x\left(\frac{1}{1+|m^*(t)x|^2} - \frac{1}{1+|x|^2}\right)\nu(dx)\right| dt \\
&\quad \le -|\gamma|\int_0^\infty s dm(s) - \int_0^\infty \left| s\int_{\mathbb{R}^d} x\left(\frac{1}{1+|sx|^2} - \frac{1}{1+|x|^2}\right)\nu(dx)\right| dm(s) \\
&\quad =: I_3 + I_4,
\end{aligned}$$

say, where

$$\begin{aligned}
I_3 &\le |\gamma|\int_0^\infty \phi(s)ds < \infty, \\
I_4 &\le \int_0^\infty \phi(s)ds \left|\int_{\mathbb{R}^d}\left(\frac{x|x|^2|s^2-1|}{(1+|sx|^2)(1+|x|^2)}\right)\nu(dx)\right| \\
&\le \int_0^\infty |s^2-1|\phi(s)ds \int_{\mathbb{R}^d}\frac{|x|^3}{(1+|sx|^2)(1+|x|^2)}\nu(dx) \\
&= \int_0^\infty |s^2-1|\phi(s)ds \left(\int_{|x|\le 1} + \int_{|x|>1}\right)\frac{|x|^3}{(1+|sx|^2)(1+|x|^2)}\nu(dx) \\
&=: I_{41} + I_{42},
\end{aligned}$$

say. Here

$$I_{41} \le \int_0^\infty |s^2-1|\phi(s)ds \int_{|x|\le 1}\frac{|x|^3}{1+|x|^2}\nu(dx) < \infty,$$

and

$$\begin{aligned}
I_{42} &\le \int_{|x|>1}\frac{|x|^3}{1+|x|^2}\nu(dx)\int_0^\infty \frac{s^2+1}{1+|sx|^2}\phi(s)ds \\
&= \int_{|x|>1}\frac{|x|^3}{1+|x|^2}\nu(dx)\left(\int_0^1 + \int_1^\infty\right)\frac{s^2+1}{1+|sx|^2}\phi(s)ds \\
&=: I_{421} + I_{422},
\end{aligned}$$

say. Furthermore,

$$\begin{aligned}
I_{421} &\le \int_{|x|>1}\frac{|x|^3}{1+|x|^2}\nu(dx)\int_0^1 \frac{1}{1+|sx|^2}ds \\
&\le \int_{|x|>1}\frac{|x|^2}{1+|x|^2}\nu(dx)\int_0^\infty \frac{1}{1+t^2}dt < \infty,
\end{aligned}$$

and

$$I_{422} \le \int_{|x|>1}\frac{|x|^3}{(1+|x|^2)^2}\nu(dx)\int_1^\infty (s^2+1)\phi(s)ds < \infty.$$

Thus we have (2.3). This completes the proof of “if” part. □

Proof of "only if" part. Suppose $\int_0^\infty m^*(t)dX_t^{(\mu)}$ exists and let $\widetilde{\nu}$ be its Lévy measure. We have

$$\begin{aligned}\int_{|x|>1}\widetilde{\nu}(dx) &= \int_0^\infty dt\int 1_{\{|m^*(t)x|>1\}}(x)\nu(dx)\\ &= -\int_0^\infty dm(s)\int 1_{\{|x|>1/s\}}(x)\nu(dx)\\ &= -\int_{\mathbb{R}^d}\nu(dx)\int_{1/|x|}^\infty dm(s)\\ &\geq \int_{|x|>1}\nu(dx)\int_{1/|x|}^\infty \phi(s)s^{-1}ds\\ &\geq \int_{|x|>1}\nu(dx)(C_1\log|x|+C_2),\end{aligned}$$

for some $C_1, C_2 > 0$. Thus, $\mu \in I_{\log}(\mathbb{R}^d)$. This competes the proof of "only if" part. ☐

Definition 2.3. For any $\mu \in I_{\log}(\mathbb{R}^d)$, define the mapping $\mathcal{M}$ by

$$\mathcal{M}\mu = \mathcal{L}\left(\int_0^\infty m^*(t)dX_t^{(\mu)}\right).$$

The statement (i) below is one of the main results in this paper.

Theorem 2.4. *(i)*

$$M(\mathbb{R}^d) = \mathcal{M}(I_{\log}(\mathbb{R}^d)) \cap I_{sym}(\mathbb{R}^d).$$

(ii) Let ν and $\widetilde{\nu}$ be the Lévy measures of $\mu \in I_{\log}(\mathbb{R}^d)$ and $\mathcal{M}\mu$, respectively. Then

$$\widetilde{\nu}(B) = \int_0^\infty \nu(s^{-1}B)\phi(s)s^{-1}ds,\ B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}).$$

Proof.

We will first prove (ii). By a general result on stochastic integral with respect to Lévy process, we have

$$\begin{aligned}\widetilde{\nu}(B) &= \int_0^\infty dt\int_{\mathbb{R}^d} 1_B(xm^*(t))\nu(dx)\\ &= -\int_0^\infty dm(s)\int_{\mathbb{R}^d} 1_B(xs)\nu(dx)\\ &= \int_0^\infty \nu(s^{-1}B)\phi(s)s^{-1}ds.\end{aligned}$$

Now we consider part (i). Let $\mu \in I_{\log}(\mathbb{R}^d)$ and $\widetilde{\mu} = \mathcal{M}\mu$. Let ν and $\widetilde{\nu}$ be the Lévy measures of μ and $\widetilde{\mu}$, respectively. Then (ii) holds. Thus, if $\nu = 0$, then $\widetilde{\nu} = 0$

and $\widetilde{\mu} \in M(\mathbb{R}^d)$. Assume that $\nu \neq 0$ and ν has the polar decomposition (λ, ν_ξ). Then, for any nonnegative measurable function f,

$$\begin{aligned}\int_{\mathbb{R}^d} f(x)\widetilde{\nu}(dx) &= \int_0^\infty \phi(s)s^{-1}ds \int_{\mathbb{R}^d} f(sx)\nu(dx)\\ &= \int_0^\infty \phi(s)s^{-1}ds \int_S \lambda(d\xi) \int_0^\infty f(sr\xi)\nu_\xi(dr)\\ &= \int_S \lambda(d\xi) \int_0^\infty \nu_\xi(dr) \int_0^\infty \phi(s/r)f(s\xi)s^{-1}ds\\ &= \int_S \lambda(d\xi) \int_0^\infty f(s\xi)\widetilde{g}_\xi(s^2)s^{-1}ds,\end{aligned}$$

where

$$\widetilde{g}_\xi(x) = \int_0^\infty \phi(x^{1/2}/r)\nu_\xi(dr) = (2\pi)^{-1/2}\int_0^\infty e^{-x/(2r^2)}\nu_\xi(dr).$$

Define a measure $\widetilde{Q}_\xi$ by

$$\widetilde{Q}_\xi(B) = (2\pi)^{-1/2}\int_0^\infty 1_B(1/(2r^2))\nu_\xi(dr), \qquad B \in \mathcal{B}((0,\infty)).$$

Then $\widetilde{Q}_\xi(B)$ is measurable in ξ and

$$\widetilde{g}_\xi(x) = \int_0^\infty e^{-xu}\widetilde{Q}_\xi(du) \quad \text{for } x > 0.$$

Hence $\widetilde{g}_\xi$ is completely monotone. Letting $\widetilde{\lambda} = \lambda$ and $\widetilde{\nu}_\xi(dr) = \widetilde{g}_\xi(r^2)r^{-1}dr$, we see that $(\widetilde{\lambda}, \widetilde{\nu}_\xi)$ is a polar decomposition of $\widetilde{\nu}$ and that $\widetilde{\mu} \in M(\mathbb{R}^d)$. Thus, $\mathcal{M}(I_{\log}(\mathbb{R}^d)) \cap I_{sym}(\mathbb{R}^d) \subset M(\mathbb{R}^d)$.

Conversely, suppose that $\widetilde{\mu} \in M(\mathbb{R}^d)$ with triplet $(\widetilde{A}, \widetilde{\nu}, \widetilde{\gamma})$. If $\widetilde{\nu} = 0$, then $\widetilde{\mu} = \mathcal{M}\mu$ with some $\widetilde{A}$ and $\widetilde{\gamma}$. Suppose that $\widetilde{\nu} \neq 0$. Then, in a polar decomposition $(\widetilde{\lambda}, \widetilde{\nu}_\xi)$ of $\widetilde{\nu}$, we have $\widetilde{\nu}_\xi(dr) = \widetilde{g}_\xi(r^2)r^{-1}dr$, where $\widetilde{g}_\xi(x)$ is completely monotone in x and measurable in ξ. Thus there are measures $\widetilde{Q}_\xi$ on $(0,\infty)$ satisfying

$$\widetilde{g}_\xi(x) = \int_0^\infty e^{-xu}\widetilde{Q}_\xi(du)$$

such that $\widetilde{Q}_\xi(B)$ is measurable in ξ for each $B \in \mathcal{B}((0,\infty))$. Now define

$$\nu_\xi(B) = (2\pi)^{1/2}\int_0^\infty 1_B((2u)^{-1/2})\widetilde{Q}_\xi(du).$$

Then ν_ξ is a measure on $(0,\infty)$ for each ξ and

$$\int_0^\infty f(r)\nu_\xi(dr) = (2\pi)^{1/2}\int_0^\infty f((2u)^{-1/2})\widetilde{Q}_\xi(du)$$

for all nonnegative measurable functions f on $(0,\infty)$.

Let $\lambda = \widetilde{\lambda}$. Then

$$\int_S \lambda(d\xi) \int_0^\infty (r^2 \wedge 1)\nu_\xi(dr) = (2\pi)^{1/2} \int_S \widetilde{\lambda}(d\xi) \int_{(0,\infty)} ((2u)^{-1} \wedge 1)\widetilde{Q}_\xi(du)$$

$$= (2\pi)^{1/2} \int_S \widetilde{\lambda}(d\xi) \left(\int_0^{1/2} \widetilde{Q}_\xi(du) + \int_{1/2}^\infty (2u)^{-1}\widetilde{Q}_\xi(du) \right) < \infty,$$

where the finiteness of the integral is assured by

$$\int_0^\infty (r^2 \wedge 1)\widetilde{g}_\xi(r^2)r^{-1}dr < \infty,$$

which can be shown by a standard calculation based on the fact that $\widetilde{g}_\xi$ is the Laplace transform of $\widetilde{Q}_\xi$. Define ν by

$$\nu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi)\nu_\xi(dr) \quad \text{for } B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}).$$

Then ν is the Lévy measure of an infinitely divisible distribution and we can check

$$\int_0^\infty \phi(s)s^{-1}ds \int_{\mathbb{R}^d} f(sx)\nu(dx) = \int_{\mathbb{R}^d} f(x)\widetilde{\nu}(dx)$$

for all nonnegative measurable functions f on $\mathbb{R}^d$. This relation can be checked as follows:

$$\begin{aligned}
\int_{\mathbb{R}^d} f(x)\widetilde{\nu}(dx) &= \int_S \widetilde{\lambda}(d\xi) \int_0^\infty f(r\xi)\widetilde{\nu}_\xi(dr) \\
&= \int_S \widetilde{\lambda}(d\xi) \int_0^\infty f(r\xi)\widetilde{g}_\xi(r^2)r^{-1}dr \\
&= \int_S \widetilde{\lambda}(d\xi) \int_0^\infty f(r\xi)r^{-1}dr \int_0^\infty e^{-r^2u}\widetilde{Q}_\xi(du) \\
&= (2\pi)^{-1/2} \int_S \widetilde{\lambda}(d\xi) \int_0^\infty f(r\xi)r^{-1}dr \int_0^\infty e^{-r^2/(2u^2)}\nu_\xi(du) \\
&= (2\pi)^{-1/2} \int_S \widetilde{\lambda}(d\xi) \int_0^\infty e^{-r^2/(2u^2)}r^{-1}dr \int_0^\infty f(r\xi)\nu_\xi(du) \\
&= (2\pi)^{-1/2} \int_0^\infty e^{-y^2/2}y^{-1}dy \int_S \widetilde{\lambda}(d\xi) \int_0^\infty f(yu\xi)\nu_\xi(du) \\
&= \int_0^\infty \phi(s)s^{-1}ds \int_{\mathbb{R}^d} f(sx)\nu(dx).
\end{aligned}$$

Define A and γ suitably and let μ be a distribution with the triplet (A, ν, γ). Then $\mathcal{M}\mu = \widetilde{\mu}$, namely $\mathcal{L}\left(\int_0^\infty m^*(t)dX_t^{(\mu)}\right) = \widetilde{\mu}$. Thus by Theorem 2.1, we see that $\mu \in I_{\log}(\mathbb{R}^d)$ and that $\widetilde{\mu} \in \mathcal{M}(I_{\log}(\mathbb{R}^d))$. Since $\widetilde{\mu} \in I_{sym}(\mathbb{R}^d)$, $\widetilde{\mu} \in \mathcal{M}(I_{\log}(\mathbb{R}^d)) \cap I_{sym}(\mathbb{R}^d)$. This completes the proof of Theorem 2.4. □

3. Relations of $M(\mathbb{R}^d)$ with other classes (I)

We have the following relations of $M(\mathbb{R}^d)$ with other classes.

Theorem 3.1. *We have*

$$T(\mathbb{R}^d) \cap I_{sym}(\mathbb{R}^d) \subsetneqq M(\mathbb{R}^d) \subsetneqq L(\mathbb{R}^d) \cap G(\mathbb{R}^d).$$

Proof.

(i) We first show that $M(\mathbb{R}^d) \subsetneqq L(\mathbb{R}^d) \cap G(\mathbb{R}^d)$. Note that $r^{-1/2}$ is completely monotone and the product of two completely monotone functions is also completely monotone. Thus by the definition of $M(\mathbb{R}^d)$, it is clear that $M(\mathbb{R}^d) \subset L(\mathbb{R}^d) \cap G(\mathbb{R}^d)$. To show that $M(\mathbb{R}^d) \neq L(\mathbb{R}^d) \cap G(\mathbb{R}^d)$, it is enough to construct $\mu \in I(\mathbb{R}^d)$ such that $\mu \in L(\mathbb{R}^d) \cap G(\mathbb{R}^d)$ but $\mu \notin M(\mathbb{R}^d)$.

First consider the case $d = 1$. Let

$$\nu(dr) = g(r^2) r^{-1} dr, \; r > 0.$$

For our purpose, it is enough to construct a function $g : (0, \infty) \to (0, \infty)$ such that
(1) $r^{-1/2} g(r)$ is completely monotone on $(0, \infty)$, (meaning that the corresponding μ belongs to $G(\mathbb{R})$),
(2) $g(r^2)$ or, equivalently, $g(r)$ is nonincreasing on $(0, \infty)$, (meaning that the corresponding μ belongs to $L(\mathbb{R})$), and
(3) $g(r)$ is not completely monotone on $(0, \infty)$, (meaning that the corresponding μ does not belong to $M(\mathbb{R})$). Put

$$g(r) = r^{-1/2} \left(e^{-0.9r} - e^{-r} + 0.1 e^{-1.1r} \right), \quad r > 0.$$

(1) We have

$$r^{-1/2} g(r) = r^{-1} \left(e^{-0.9r} - e^{-r} + 0.1 e^{-1.1r} \right) = \int_{0.9}^{1} e^{-ru} du + 0.1 \int_{1.1}^{\infty} e^{-ru} du,$$

which is a sum of two completely monotone functions, and thus $r^{-1/2} g(r)$ is completely monotone.
(2) Put

$$k(r) = e^{-0.9r} - e^{-r} + 0.1 e^{-1.1r}, \; r > 0.$$

If $k(r)$ is nonincreasing, then so is $g(r) = r^{-1/2}k(r)$. To show it, we have

$$k'(r) = -0.9e^{-0.9r} + e^{-r} - 0.11e^{-1.1r} = -0.9e^{-1.1r}\left[\left(e^{0.1r} - \frac{1}{1.8}\right)^2 - \frac{0.604}{3.24}\right]$$
$$\leq -0.9e^{-1.1r}\left[\left(1 - \frac{1}{1.8}\right)^2 - \frac{0.604}{3.24}\right] = -0.01e^{-1.1r} < 0, \quad r > 0.$$

(3) To show (3), we see that

$$k(r) = \int_0^\infty e^{-ru}Q(du),$$

where Q is a signed measure such that $Q = Q_1 + Q_2 + Q_3$ and

$$Q_1(\{0.9\}) = 1,\ Q_2(\{1\}) = -1,\ Q_3(\{1.1\}) = 0.1.$$

On the other hand

$$r^{-1/2} = \pi^{-1/2}\int_0^\infty e^{-ru}u^{-1/2}du =: \int_0^\infty e^{-ru}R(du),$$

where

$$R(du) = (\pi u)^{-1/2}du.$$

Thus

$$g(r) = \int_0^\infty e^{-ru}R(du)\int_0^\infty e^{-rv}Q(dv) = \int_0^\infty e^{-rw}U(dw),$$

where

$$U(B) = \int_0^\infty Q(B-y)R(dy).$$

We are going to show that U is a signed measure, namely, for some interval $(a,b), U\left((a,b)\right) < 0$. If so, g is not completely monotone. We have

$$U\left((a,b)\right) = \pi^{-1/2}\int_0^\infty Q\left((a-y, b-y)\right)y^{-1/2}dy$$
$$= \pi^{-1/2}\sum_{i=1}^{3}\int_0^\infty Q_i\left((a-y,b-y)\right)y^{-1/2}dy$$
$$= \pi^{-1/2}\left[\int_{a-0.9}^{b-0.9} y^{-1/2}dy - \int_{a-1}^{b-1} y^{-1/2}dy + 0.1\int_{a-1.1}^{b-1.1} y^{-1/2}dy\right]$$
$$= 2\pi^{-1/2}\left[\left(\sqrt{b-0.9} - \sqrt{a-0.9}\right) - \left(\sqrt{b-1} - \sqrt{a-1}\right) + 0.1\left(\sqrt{b-1.1} - \sqrt{a-1.1}\right)\right].$$

Take $(a,b) = (1.15, 1.35)$. Then

$$U\left((1.15, 1.35)\right)$$
$$= 2\pi^{-1/2}\left[(\sqrt{0.45} - \sqrt{0.25}) - (\sqrt{0.35} - \sqrt{0.15}) + 0.1(\sqrt{0.25} - \sqrt{0.05})\right]$$
$$< -0.01\pi^{-1/2} < 0.$$

This concludes that g is not completely monotone.

A d–dimensional example of $\mu \in I(\mathbb{R}^d)$ such that $\mu \in L(\mathbb{R}^d) \cap G(\mathbb{R}^d)$ but $\mu \notin M(\mathbb{R}^d)$ is given by taking $\nu(dr)$ for the radial component of a Lévy measure. This completes the proof of $M(\mathbb{R}^d) \subsetneqq L(\mathbb{R}^d) \cap G(\mathbb{R}^d)$.

(ii) We next show that $T(\mathbb{R}^d) \cap I_{sym}(\mathbb{R}^d) \subsetneqq M(\mathbb{R}^d)$. Since $M(\mathbb{R}^d) \subset I_{sym}(\mathbb{R}^d)$, we consider only $\mu \in I_{sym}(\mathbb{R}^d)$. We need the following lemma.

Lemma 3.2. (See Feller [3], p.441, Corollary 2.) *Let ϕ be a completely monotone function on $(0, \infty)$ and let ψ be a nonnegative function on $(0, \infty)$ whose derivative is completely monotone. Then $\phi(\psi)$ is completely monotone.*

If $\mu \in T(\mathbb{R}^d) \cap I_{sym}(\mathbb{R}^d)$, then the radial component of the Lévy measure of μ has the form $\nu_\xi(dr) = k_\xi(r)r^{-1}dr$, where k_ξ is completely monotone. By the lemma above and the fact that $\psi(r) = r^{1/2}$ has a completely monotone derivative, then $g_\xi(r) := k_\xi(r^{1/2})$ is completely monotone. Thus $\nu_\xi(dr)$ can be read as $g_\xi(r^2)r^{-1}dr$, where g_ξ is completely monotone, concluding that $\mu \in M(\mathbb{R}^d)$.

To show that $T(\mathbb{R}^d) \cap I_{sym}(\mathbb{R}^d) \neq M(\mathbb{R}^d)$, it is enough to find a completely monotone function g_ξ such that $k_\xi(r) = g_\xi(r^2)$ is *not* completely monotone. However, the function $g_\xi(r) = e^{-r}$ has such a property. Although e^{-r} is completely monotone, $(-1)^2 \frac{d^2}{dr^2} e^{-r^2} < 0$ for small $r > 0$. This completes the proof of that $T(\mathbb{R}^d) \cap I_{sym}(\mathbb{R}^d) \subsetneqq M(\mathbb{R}^d)$. ☐

Additional remark. The argument above also gives us the following result between classes $B(\mathbb{R}^d)$ and $G(\mathbb{R}^d)$, namely,

$$B(\mathbb{R}^d) \cap I_{sym}(\mathbb{R}^d) \subsetneqq G(\mathbb{R}^d).$$

4. Relations of $M(\mathbb{R}^d)$ with other classes (II)

To give more relation of $M(\mathbb{R}^d)$ with other classes, we introduce two mappings.

Definition 4.1.

$$\Phi : I_{\log}(\mathbb{R}^d) \to I(\mathbb{R}^d), \quad \Phi\mu = \mathcal{L}\left(\int_0^\infty e^{-t} dX_t^{(\mu)}\right),$$

$$\mathcal{G} : I(\mathbb{R}^d) \to I_{sym}(\mathbb{R}^d), \quad \mathcal{G}\mu = \mathcal{L}\left(\int_0^1 h^*(t) dX_t^{(\mu)}\right),$$

$h^*(t)$ is the inverse function of $h(x) = \int_x^\infty \phi(u)du, x \in \mathbb{R}$.

Remark 4.2 (known)**.** (i) $\Phi\mu$ is a selfdecomposable distribution and $\mathcal{G}\mu$ is a type G distribution. $L(\mathbb{R}^d) = \Phi(I_{\log}(\mathbb{R}^d))$ and $G(\mathbb{R}^d) = \mathcal{G}(I(\mathbb{R}^d)) = \mathcal{G}(I_{sym}(\mathbb{R}^d))$. (See, e.g. [4] and [6].)
(ii) $\Phi(B(\mathbb{R}^d) \cap I_{\log}(\mathbb{R}^d)) = T(\mathbb{R}^d)$. (See [1].)
(iii) $\nu_{\mathcal{G}\mu}(B) = E[\nu_\mu(Z^{-1}B)],\ B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$. (See [1] and [7].)

Theorem 4.3. *(i) Let $\mu \in I(\mathbb{R}^d)$. Then $\mathcal{G}\mu \in I_{\log}(\mathbb{R}^d)$ if and only if $\mu \in I_{\log}(\mathbb{R}^d)$.*
(ii) Let

$$a(s) = 2\int_s^\infty u^{-1}du \int_u^\infty \phi(v)dv,\ s > 0,$$

and define the inverse function $s = a^*(t)$ by $t = a(s)$. Then the stochastic integral

$$\int_0^\infty a^*(t)dX_t^{(\mu)}$$

exists if and only if $\mu \in I_{\log}(\mathbb{R}^d)$.
(iii) If $\mu \in I_{\log}(\mathbb{R}^d) \cap I_{sym}(\mathbb{R}^d)$, then

$$\Phi\mathcal{G}\mu = \mathcal{G}\Phi\mu = \mathcal{L}\left(\int_0^\infty a^*(t)dX_t^{(\mu)}\right)$$

and the Lévy measure $\widetilde{\nu}$ of $\mathcal{L}\left(\int_0^\infty a^*(t)dX_t^{(\mu)}\right)$ is

$$\widetilde{\nu}(B) = \int_0^\infty \nu(s^{-1}B)\phi(s)s^{-1}ds,$$

where ν is the Lévy measure of μ.
(iv)

$$M(\mathbb{R}^d) \supsetneqq \mathcal{G}\Phi(I_{\log}(\mathbb{R}^d)) = \mathcal{G}(L(\mathbb{R}^d)) = \Phi(G(\mathbb{R}^d) \cap I_{\log}(\mathbb{R}^d)).$$

Proof of (i). The proof of Theorem C (i) in [2] also works here. □

Proof of (ii). It is almost the same as that of Theorem 2.1, if we replace $\phi(s)s^{-1}$ by $s^{-1}\int_s^\infty \phi(v)dv$. So, we omit it. □

Proof of (iii). Recall that for $\mu \in I_{\log}(\mathbb{R}^d)$

$$C_{\Phi\mu}(z) = \int_0^\infty C_\mu(ze^{-t})dt.$$

and for $\mu \in I(\mathbb{R}^d)$.

$$C_{\mathcal{G}\mu}(z) = \int_0^1 C_\mu(zh^*(s))ds.$$

Let $\mu \in I_{\log}(\mathbb{R}^d)$. We have, for $z \in \mathbb{R}^d$,

$$C_{\Phi\mathcal{G}\mu}(z) = \int_0^\infty C_{\mathcal{G}\mu}(e^{-t}z)dt = \int_0^\infty dt \int_0^1 C_\mu(h^*(s)e^{-t}z)ds$$

$$C_{\mathcal{G}\Phi\mu}(z) = \int_0^1 C_{\Phi\mu}(h^*(s)z)ds = \int_0^1 ds \int_0^\infty C_\mu(e^{-t}h^*(s)z)dt.$$

We claim that

$$\int_0^\infty dt \int_0^1 |C_\mu(h^*(s)e^{-t}z)|ds = \int_0^\infty dt \int_{-\infty}^\infty |C_\mu(ue^{-t}z)|\phi(u)du < \infty. \tag{4.1}$$

Note that $\mathcal{G}\mu$ is symmetric and it is unchanged even if we replace μ by $\bar{\mu}(B) = 2^{-1}(\mu(B) + \mu(-B))$. (See [6].) Hence, without loss of generality, we assume μ is symmetric. Thus to show (4.1), it is enough to show that

$$\int_0^\infty dt \int_0^\infty |C_\mu(ue^{-t}z)|e^{-u^2/2}du < \infty. \tag{4.2}$$

However, in [2] (Equation (4.5)), it was shown that

$$\int_0^\infty e^{-u}du \int_0^\infty |C_\mu(ue^{-t}z)|dt < \infty.$$

A similar calculation works also for getting (4.2). Thus

$$\begin{aligned} C_{\Phi\mathcal{G}\mu}(z) &= \int_0^\infty dt \int_0^1 C_\mu(ze^{-t}h^*(s))ds \\ &= -\int_0^\infty dt \int_{-\infty}^\infty C_\mu(ze^{-t}v)dh(v) \\ &= \int_0^\infty dt \int_0^\infty C_\mu(ze^{-t}v)2\phi(v)dv \\ &= 2\int_0^\infty \phi(v)dv \int_0^\infty C_\mu(ze^{-t}v)dt \\ &= 2\int_0^\infty \phi(v)dv \int_0^v C_\mu(zs)s^{-1}ds \\ &= 2\int_0^\infty C_\mu(zs)s^{-1}ds \int_s^\infty \phi(v)dv, \end{aligned}$$

where the change of the order of integrals is assured by (4.1) and (4.2). Thus we have

$$C_{\Phi\mathcal{G}\mu}(z) = -\int_0^\infty C_\mu(zs)da(s),$$

and hence

$$C_{\Phi\mathcal{G}\mu}(z) = \int_0^\infty C_\mu(za^*(t))dt.$$

The form of $\widetilde{\nu}$ is a direct consequence of a general result on the Lévy measure of stochastic integral with respect to Lévy process. This concludes the proof of (iii). □

Proof of (iv). We first show that the radial component of the Lévy measure of $\Phi\mathcal{G}\mu$ satisfies (1.1). We have

$$\begin{aligned}\widetilde{\nu}(B) = \nu_{\Phi\mathcal{G}\mu}(B) &= \int_0^\infty \nu_{\mathcal{G}\mu}(e^t B)dt \\ &= \int_0^\infty dt \int_S \lambda(d\xi) \int_0^\infty 1_{e^t B}(r\xi) g_\xi(r^2)dr,\end{aligned}$$

where λ is a probability measure appearing in the polar decomposition of ν and g_ξ is the radial component of ν_μ. Then

$$\begin{aligned}\widetilde{\nu}(B) &= \int_S \lambda(d\xi) \int_0^\infty g_\xi(r^2)dr \int_0^\infty 1_B(e^{-t} r\xi)dt \\ &= \int_S \lambda(d\xi) \int_0^\infty g_\xi(r^2)dr \int_0^r 1_B(y\xi)y^{-1}dy \\ &= \int_S \lambda(d\xi) \int_0^\infty 1_B(y\xi)y^{-1}dy \int_y^\infty g_\xi(r^2)dr \\ &=: \int_S \lambda(d\xi) \int_0^\infty 1_B(y\xi)\widetilde{\nu}_\xi(dy),\end{aligned}$$

where

$$\widetilde{\nu}_\xi(dy) = \left(y^{-1} \int_y^\infty g_\xi(r^2)dr \right) dy.$$

This $\widetilde{\nu}_\xi$ satisfies $\int_0^\infty (1 \wedge y^2)\widetilde{\nu}_\xi(dy) < \infty$. For

$$\begin{aligned}&\int_0^\infty (1 \wedge y^2)\widetilde{\nu}_\xi(dy) \\ &\quad = \int_0^1 ydy \int_y^\infty g_\xi(r^2)dr + \int_1^\infty y^{-1}dy \int_y^\infty g_\xi(r^2)dr \\ &\quad = \int_0^1 g_\xi(r^2)dr \int_0^r ydy + \int_1^\infty g_\xi(r^2)dr \int_0^1 ydr + \int_1^\infty g_\xi(r^2)dr \int_1^r y^{-1}dy < \infty,\end{aligned}$$

where the last integral is finite because ν is the Lévy measure of a $\mu \in I_{\log}(\mathbb{R}^d)$. Put

$$\widetilde{g}_\xi(x) = \int_{x^{1/2}}^\infty g_\xi(r^2)dr$$

We then have

$$\frac{d}{dx}\widetilde{g}_\xi(x) = -2^{-1}x^{-1/2}g_\xi(x)$$

Since g_ξ and $x^{-1/2}$ are completely monotone, $x^{-1/2}g_\xi(x)$ is completely monotone. Thus $\widetilde{g}_\xi$ is completely monotone. Hence

$$\widetilde{\nu}_\xi(dy) = \widetilde{g}_\xi(y^2)y^{-1}dy,$$

where $\widetilde{g}_\xi$ is completely monotone. Thus the Lévy measure of $\widetilde{\mu}$ is that of $\Phi\mathcal{G}\mu$ and thus $\widetilde{\mu}$ belongs to the class $M(\mathbb{R}^d)$. Thus $M(\mathbb{R}^d) \supset \mathcal{G}(L(\mathbb{R}^d))$.

The last equality is a consequence of (i) and (iii). Namely, by (i),

$$\mathcal{G}(I_{\log}(\mathbb{R}^d)) = G(\mathbb{R}^d) \cap I_{\log}(\mathbb{R}^d).$$

Thus by (iii).

$$\Phi\mathcal{G}(I_{\log}(\mathbb{R}^d)) = \Phi(G(\mathbb{R}^d) \cap I_{\log}(\mathbb{R}^d)) = \left\{ \mathcal{L}\left(\int_0^\infty a^*(t) dX_t^{(\mu)} \right),\ \mu \in I_{\log}(\mathbb{R}^d) \cap I_{sym}(\mathbb{R}^d) \right\}.$$

It remains to show $M(\mathbb{R}^d) \neq \mathcal{G}(L(\mathbb{R}^d))$. It is enough to show it for $d = 1$.

Consider a Lévy measure $\nu(dr) = \phi(r)|r|^{-1}dr$. Then the corresponding infinitely divisible distribution μ belongs to $M(\mathbb{R})$. Suppose $\mu \in \mathcal{G}(L(\mathbb{R}))$. Then, by (iii), ν also satisfies

$$\nu(B) = \int_0^\infty \nu_0(s^{-1}B)h(s)s^{-1}ds,$$

where $h(s) = \int_s^\infty \phi(x)dx$ and ν_0 is a symmetric Lévy measure. Consider $B \in \mathcal{B}((0,\infty))$. Then we have

$$\begin{aligned}\nu(B) &= \int_0^\infty \int_{\mathbb{R}} 1_B(sx)\nu_0(dx)h(s)s^{-1}\, ds \\ &= \int_0^\infty \int_0^\infty 1_B(r)h(rx^{-1})r^{-1}\, dr\nu_0(dx).\end{aligned}$$

Thus

$$\nu(dr) = \left(\int_0^\infty h(rx^{-1})\nu_0(dx) \right) r^{-1}dr, \quad r > 0.$$

By our assumption, for any $r > 0$,

$$\phi(r) = \int_0^\infty h(rx^{-1})\, \nu_0(dx).$$

Let $h > 0$ and consider

$$\frac{1}{h}(\phi(r+h) - \phi(r)) = \int_0^\infty \frac{1}{h} \left(h((r+h)x^{-1}) - h(rx^{-1}) \right) \nu_0(dx). \tag{4.3}$$

We have

$$|h((r+h)x^{-1}) - h(rx^{-1})| = \phi((r+\theta h)x^{-1})hx^{-1} \le \phi(rx^{-1})hx^{-1},$$

where $0 < \theta < 1$. Thus we can interchange the limit as $h \to 0$ and the integral in (4.3), and we get

$$-r\phi(r) = -\int_0^\infty \phi(rx^{-1})x^{-1}\, \nu_0(dx), \quad \text{for any } r > 0.$$

Changing variable from r to $r^{1/2}$, we get

$$r^{1/2}\phi(r^{1/2}) = \int_0^\infty \phi(r^{1/2}x^{-1})x^{-1}\, \nu_0(dx).$$

The right hand side is completely monotone, but the left had side is not. This contradicts our assumption that $\mu \in \mathcal{G}(L(\mathbb{R}))$. The proof of (iv) is now completed. $\square$

5. More about the classes $M(\mathbb{R})$ and $\mathcal{G}(L(\mathbb{R}))$ when $d = 1$

We first note that

$$\mathcal{G}(L(\mathbb{R}^d)) = \{\mu \in I_{sym}(\mathbb{R}^d) \ : \nu_\mu(B) = E[\nu_0(Z^{-1}B)], \ B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}), \quad \text{for the Lévy measure } \nu_0 \text{ of } \mu_0 \in L(\mathbb{R}^d)\}. \tag{5.1}$$

This follows from Remark 4.2 (iii). When $d = 1$, we also know that μ is of type G if and only if $\mu = \mathcal{L}(V^{1/2}Z)$ for some infinitely divisible nonnegative random variable V independent of the standard normal random variable Z. That is, μ is a variance mixture of normal distributions. The goal here is to characterize the distribution of the random variance V in the case of $\mu \in M(\mathbb{R})$. We begin with the following.

Proposition 5.1. *$\mu \in \mathcal{G}(L(\mathbb{R}))$ if and only if $\mu = \mathcal{L}(V^{1/2}Z)$ with $\mathcal{L}(V) \in L(\mathbb{R}_+)$.*

Proof. The "only if" part: Suppose $\mu \in \mathcal{G}(L(\mathbb{R}))$. Since $\mu \in G(\mathbb{R})$, there exists V such that $\mu = \mathcal{L}(V^{1/2}Z)$ and $\mathcal{L}(V) \in I(\mathbb{R}_+)$. Also from (5.1), there exists a Lévy measure ν_0 of an element in $L(\mathbb{R})$ such that $\nu_\mu(B) = E[\nu_0(Z^{-1}B)]$. It is known ([5]) that for every $x > 0$,

$$\nu_0([x, \infty)) = 2^{-1}\nu_V([x^2, \infty)). \tag{5.2}$$

Since ν_0 is the Lévy measure of some $\mu_0 \in L(\mathbb{R})$,

$$\nu_0(dx) = k_0(x)x^{-1}dx, \ x > 0, \tag{5.3}$$

for some nonincreasing function k_0. It follows from (5.2) and (5.3) that

$$\int_x^\infty k_0(y)y^{-1}dy = 2^{-1}\int_{x^2}^\infty \nu_V(dy), \ x > 0.$$

By the change of variables $u = y^2$ on the left hand side above, we have

$$2^{-1}\int_{x^2}^\infty k_0(u^{1/2})u^{-1}du = 2^{-1}\int_{x^2}^\infty \nu_V(dy), \ x > 0.$$

Thus, we have

$$\nu_V(dy) = k_1(y)y^{-1}dy,$$

where $k_1(y) = k_0(y^{1/2})$ is nonincreasing. Hence $\mathcal{L}(V) \in L(\mathbb{R}_+)$.

The "if" part: Suppose $\mu = \mathcal{L}(V^{1/2}Z)$ and $\mathcal{L}(V) \in L(\mathbb{R}_+)$. Then there exits a nonincreasing function $k_1(y)$ such that

$$\nu_V(dy) = k_1(y)y^{-1}dy.$$

Then by (5.2),

$$\int_x^\infty \nu_0(dy) = 2^{-1}\int_{x^2}^\infty k_1(y)y^{-1}dy$$
$$= \int_x^\infty k_1(u^2)u^{-1}du,\ x>0.$$

Thus, $\nu_0(dy) = k_0(y)y^{-1}dy$, where $k_0(y) = k_1(y^2)$ is nonincreasing. Hence ν_0 is the Lévy measure of some $\mu_0 \in L(\mathbb{R})$. Since $\nu_\mu(B) = E[\nu_0(Z^{-1}B)]$, where ν_0 is defined by (5.2) from ν_V, we have $\mu \in \mathcal{G}(L(\mathbb{R}))$. This completes the proof. □

We have the following.

Theorem 5.2. *$\mu \in M(\mathbb{R})$ if and only if $\mu = \mathcal{L}(V^{1/2}Z)$, where $\mathcal{L}(V) \in I(\mathbb{R}_+)$ has an absolutely continuous Lévy measure ν_V of the form*

$$\nu_V(dr) = \ell(r)r^{-1}\,dr, \quad r>0. \tag{5.4}$$

The function ℓ is given by

$$\ell(r) = \int_r^\infty (x-r)^{-1/2}\,\rho(dx), \tag{5.5}$$

where ρ is a measure on $(0,\infty)$ satisfying the integrability condition

$$\int_0^1 x^{1/2}\,\rho(dx) + \int_1^\infty (1+\log x)x^{-1/2}\,\rho(dx) < \infty. \tag{5.6}$$

Proof. (i) The "only if" part: Suppose $\mu \in M(\mathbb{R})$. Since $M(\mathbb{R}) \subset G(\mathbb{R})$, we have $\mu = \mathcal{L}(V^{1/2}Z)$ for some $V \in I(\mathbb{R}_+)$. Thus, we get for $z \in \mathbb{R}$,

$$\begin{aligned} E\left[e^{izV^{1/2}Z}\right] &= E\left[e^{-Vz^2/2}\right] \\ &= \exp\left\{-2^{-1}Az^2 + \int_{0+}^\infty (e^{-vz^2/2}-1)\,\nu_V(dv)\right\} \\ &= \exp\left\{-2^{-1}Az^2 + \int_{0+}^\infty \nu_V(dv)\int_{-\infty}^\infty (e^{izv^{1/2}u}-1)\phi(u)\,du\right\} \\ &= \exp\left\{-2^{-1}Az^2 + \int_{-\infty}^\infty (e^{izx}-1)dx\int_{0+}^\infty \phi(v^{-1/2}x)v^{-1/2}\,\nu_V(dv)\right\}, \end{aligned}$$

where $A \geq 0$. Therefore, the Lévy measure ν_μ of μ is of the form

$$\nu_\mu(dx) = \left(\int_{0+}^\infty \phi(v^{-1/2}x)v^{-1/2}\,\nu_V(dv)\right)\,dx. \tag{5.7}$$

By the definition, $\mu \in M(\mathbb{R})$ if and only if $\nu_\mu(dx) = |x|^{-1}g(x^2)dx$, where g is completely monotone. Thus, g can be written as

$$g(r) = \int_0^\infty e^{-ry/2}\,Q(dy), \quad r>0,$$

for some measure Q on $(0,\infty)$. By (5.7), we get

$$\int_{0+}^{\infty} \phi(v^{-1/2}x)v^{-1/2}\,\nu_V(dv) = |x|^{-1}g(x^2). \tag{5.8}$$

Since

$$r^{-1/2} = (2\pi)^{-1/2}\int_0^{\infty} e^{-rw/2}w^{-1/2}\,dw, \quad r>0,$$

we obtain

$$\begin{aligned} r^{-1/2}g(r) &= (2\pi)^{-1/2}\int_0^{\infty}\int_0^{\infty} e^{-r(w+y)/2}w^{-1/2}\,dwQ(dy) \\ &= (2\pi)^{-1/2}\int_0^{\infty} Q(dy)\int_y^{\infty} e^{-ru/2}(u-y)^{-1/2}\,du \\ &= (2\pi)^{-1/2}\int_0^{\infty} e^{-ru/2}du\int_0^{u}(u-y)^{-1/2}\,Q(dy). \end{aligned}$$

Taking $x = r^{1/2} > 0$ in (5.8), we get

$$(2\pi)^{-1/2}\int_{0+}^{\infty} e^{-r/2v}v^{-1/2}\,\nu_V(dv) = (2\pi)^{-1/2}\int_0^{\infty} e^{-ru/2}du\int_0^{u}(u-y)^{-1/2}\,Q(dy). \tag{5.9}$$

Let

$$\rho(dx) = -x^{1/2}Q(d(x^{-1})).$$

Then $\ell(r)$ in (5.5) becomes

$$\begin{aligned} \ell(r) &= -\int_r^{\infty}(x-r)^{-1/2}x^{1/2}Q(d(x^{-1})) \\ &= \int_0^{r^{-1}}(y^{-1}-r)^{-1/2}y^{-1/2}Q(dy) \\ &= \int_0^{r^{-1}}(1-yr)^{-1/2}Q(dy) \\ &= r^{-1/2}\int_0^{r^{-1}}(r^{-1}-y)^{-1/2}Q(dy). \end{aligned}$$

Thus by (5.9),

$$\int_{0+}^{\infty} e^{-r/2v}v^{-1/2}\,\nu_V(dv) = \int_0^{\infty} e^{-ru/2}u^{-1/2}\ell(u^{-1})\,du$$

or

$$\int_{0+}^{\infty} e^{-r/2v}v^{-1/2}\,\nu_V(dv) = \int_0^{\infty} e^{-r/2v}v^{-3/2}\ell(v)\,dv, \quad r>0.$$

Therefore

$$v^{-1/2}\,\nu_V(dv) = v^{-3/2}\ell(v)\,dv, \quad v>0,$$

which yields (5.4).

The integrability condition (5.6) for ρ is obtained from the fact that

$$\infty > \int_{\mathbb{R}} (x^2 \wedge 1)\, \nu_\mu(dx) = \int_{\mathbb{R}} (|x| \wedge |x|^{-1}) g(x^2) dx.$$

For, this yields that

$$\int_0^1 x dx \int_0^\infty e^{-x^2 y/2}\, Q(dy) < \infty \quad \text{and} \quad \int_1^\infty x^{-1} dx \int_0^\infty e^{-x^2 y/2}\, Q(dy) < \infty,$$

and hence

$$\int_0^\infty \left[y^{-1}(1 - e^{-y/2}) + 2^{-1} \int_y^\infty u^{-1} e^{-u/2}\, du \right] Q(dy) < \infty.$$

It is obvious that the above condition is equivalent to

$$\int_0^1 (1 + \log y^{-1}) Q(dy) + \int_1^\infty y^{-1} Q(dy) < \infty. \tag{5.10}$$

On the other hand,

$$\int_0^1 x^{1/2} \rho(dx) = -\int_0^1 x Q(d(x^{-1})) = \int_1^\infty y^{-1} Q(dy)$$

and

$$\int_1^\infty (1 + \log x) x^{-1/2} \rho(dx) = -\int_1^\infty (1 + \log x) Q(d(x^{-1})) = \int_0^1 (1 + \log y^{-1}) Q(dy).$$

Thus, we get (5.6) from (5.10). The “only if” part is thus proved.

(ii) The “if” part. Suppose $\mu = \mathcal{L}(V^{1/2} Z)$ and the Lévy measure ν_V of V satisfies (5.4)–(5.6).

We first claim that the integrability condition (5.6) implies that ν_V is really a Lévy measure on $(0, \infty)$ of a positive infinitely divisible random variable, namely it satisfies

$$\int_0^\infty (r \wedge 1) \nu_V(dr) < \infty. \tag{5.11}$$

We have

$$\int_0^\infty (r \wedge 1) \nu_V(dr) = \int_0^1 r \nu_V(dr) + \int_1^\infty \nu_V(dr).$$

As to the first integral, we have

$$\begin{aligned}\int_0^1 r\nu_V(dr) &= \int_0^1 \ell(r)dr = \int_0^1 dr \int_r^\infty (x-r)^{-1/2}\rho(dx) \\ &= \int_0^1 \rho(dx)\int_0^x (x-r)^{-1/2}dr + \int_1^\infty \rho(dx)\int_0^1 (x-r)^{-1/2}dr \\ &= 2\int_0^1 x^{1/2}\rho(dx) + 2\int_1^\infty \left(x^{1/2}-(x-1)^{1/2}\right)\rho(dx) \\ &\le 2\int_0^1 x^{1/2}\rho(dx) + C\int_1^\infty x^{-1/2}\rho(dx),\end{aligned}$$

where $C > 0$ is a constant. Next, as to the second integral,

$$\begin{aligned}\int_1^\infty \nu_V(dr) &= \int_1^\infty r^{-1}\ell(r)dr \\ &= \int_1^\infty r^{-1}dr\int_r^\infty (x-r)^{-1/2}\rho(dx) \\ &= \int_1^\infty \rho(dx)\int_1^x r^{-1}(x-r)^{-1/2}dr \\ &= \int_1^\infty 2x^{-1/2}\log(x^{1/2}+(x-1)^{1/2})\rho(dx).\end{aligned}$$

Therefore, (5.6) implies (5.11). Furthermore, as we have already seen, ν_μ is expressed as in (5.7). So, to complete the proof, it is enough to show that when we put

$$g(x^2) = |x|\int_0^\infty \phi(v^{-1/2}x)v^{-1/2}\nu_V(dv),$$

then $g(r)$ is completely monotone on $(0,\infty)$. However, for that, it is enough to follow the proof of the "only if" part from the bottom to the top. This concludes the proof. $\square$

Example 5.3. Suppose that the measure ρ in Theorem 5.2 has the density and for some $0 < \alpha < 1$,

$$\rho(dx) = x^{-\alpha-1/2}dx.$$

This ρ satisfies the integrability condition (5.6). Then $\ell(r)$ in (5.5) turns out to be

$$\ell(r) = Kr^{-\alpha}, \quad \text{where} \quad K = \int_1^\infty (u-1)^{-1/2}u^{-\alpha-1/2}du < \infty.$$

Thus, ν_V in (5.4) is the Lévy measure of a positive α-stable distribution, and thus $\mu \in \mathcal{G}(L(\mathbb{R})) \subsetneqq M(\mathbb{R})$.

Example 5.4. (Another example of μ such that $\mu \in M(\mathbb{R})$ but $\mu \notin \mathcal{G}(L(\mathbb{R}))$.) Let ρ in (5.5) satisfy (5.6) and that

$$\rho([r_1, r_2]) = 0 \quad \text{for some } 0 < r_1 < r_2 < \infty$$

and $\rho((r_2, \infty)) > 0$. Then the resulting μ belongs to $M(\mathbb{R})$. However,

$$\begin{aligned} \ell(r_1) &= \int_{r_1}^{\infty} (x - r_1)^{-1/2} \rho(dx) = \int_{r_2}^{\infty} (x - r_1)^{-1/2} \rho(dx) \\ &< \int_{r_2}^{\infty} (x - r_2)^{-1/2} \rho(dx) = \ell(r_2). \end{aligned}$$

Thus $\ell(r)$ is not a nonincreasing function so that $\mathcal{L}(V) \notin L((0, \infty))$. It follows from Proposition 5.1 that $\mu = \mathcal{L}(V^{1/2} Z) \notin \mathcal{G}(L(\mathbb{R}))$.

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