STOCHASTIC INTEGRAL CHARACTERIZATIONS OF SEMI-SELFDECOMPOSABLE DISTRIBUTIONS AND RELATED ORNSTEIN-UHLENBECK TYPE PROCESSES

MAKOTO MAEJIMA AND YOHEI UEDA

Abstract. In this paper, three topics on semi-selfdecomposable distributions are studied. The first one is to characterize semi-selfdecomposable distributions by stochastic integrals with respect to Lévy processes. This characterization defines a mapping from an infinitely divisible distribution with finite log-moment to a semi-selfdecomposable distribution. The second one is to introduce and study a Langevin type equation and the corresponding Ornstein-Uhlenbeck type process whose limiting distribution is semi-selfdecomposable. Also, semi-stationary Ornstein-Uhlenbeck type processes with semi-selfdecomposable distributions are constructed. The third one is to study the iteration of the mapping above. The iterated mapping is expressed as a single mapping with a different integrand. Also, nested subclasses of the class of semi-selfdecomposable distributions are considered, and it is shown that the limit of these nested subclasses is the closure of the class of semi-stable distributions.

1. Introduction

Let $I(\mathbb{R}^d)$ be the class of all infinitely divisible distributions on $\mathbb{R}^d$ and let $\{X_t^{(\mu)}, t \geq 0\}$ be an $\mathbb{R}^d$-valued Lévy process with $\mu \in I(\mathbb{R}^d)$ as its distribution at time 1. Many subclasses of $I(\mathbb{R}^d)$ have recently been investigated in many aspects. Among those, there are characterizations of those classes in terms of stochastic integrals with respect to Lévy processes. In such cases, we define mappings

$$\Phi_f(\mu) = \mathcal{L}\left(\int_0^\infty f(t)dX_t^{(\mu)}\right), \quad \mu \in \mathfrak{D}(\Phi_f) \subset I(\mathbb{R}^d)$$

for nonrandom measurable functions $f\colon [0,\infty) \to \mathbb{R}$, where $\mathcal{L}(X)$ is the law of a random variable X and $\mathfrak{D}(\Phi_f)$ is the domain of a mapping Φ_f that is the class of $\mu \in I(\mathbb{R}^d)$ for which $\int_0^\infty f(t)dX_t^{(\mu)}$ is definable. For the definition of stochastic integrals with respect to Lévy processes of nonrandom measurable functions, see the next section. When we consider the composition of two mappings Φ_f and Φ_g, denoted by $\Phi_g \circ \Phi_f$, the domain of $\Phi_g \circ \Phi_f$ is $\mathfrak{D}(\Phi_g \circ \Phi_f) = \{\mu \in I(\mathbb{R}^d) : \mu \in \mathfrak{D}(\Phi_f)$ and $\Phi_f(\mu) \in \mathfrak{D}(\Phi_g)\}$. Once we define such a mapping, we can characterize a subclass of $I(\mathbb{R}^d)$ as the range of Φ_f, $\mathfrak{R}(\Phi_f)$, say. Among such classes, there

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are the Jurek class, the class of selfdecomposable distributions, the Goldie-Steutel-Bondesson class, the Thorin class, the class of generalized type G distributions and so on. (For details on these, see, e.g., Maejima and Sato [7].) Also, by iterating a mapping Φ_f, we can define a sequence of nested subclasses $\mathfrak{R}(\Phi_f^m), m \in \mathbb{N}$, where Φ_f^m is the m times composition of the mapping Φ_f itself.

The class of selfdecomposable distributions, denoted by $L(\mathbb{R}^d)$, has the longest history in the study of subclasses of $I(\mathbb{R}^d)$. Let $\widehat{\mu}(z), z \in \mathbb{R}^d$, be the characteristic function of μ. $\mu \in I(\mathbb{R}^d)$ is said to be selfdecomposable if for any $b > 1$, there exists a distribution ρ_b such that $\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)\widehat{\rho}_b(z)$. This ρ_b automatically belongs to $I(\mathbb{R}^d)$. $\mu \in L(\mathbb{R}^d)$ is also a limiting distribution of normalized partial sums of independent random variables under infinitesimal condition, and has the stochastic integral representation with respect to a Lévy process, which is $\mathcal{L}\left(\int_0^\infty e^{-t}dX_t\right)$ with $\mathcal{L}(X_1) \in I_{\log}(\mathbb{R}^d)$, where $I_{\log}(\mathbb{R}^d) = \{\mu \in I(\mathbb{R}^d)\colon \int_{\mathbb{R}^d} \log^+|x|\mu(dx) < \infty\}$, $\log^+|x| = (\log|x|) \vee 0$, and $|x|$ is the Euclidean norm of $x \in \mathbb{R}^d$. Furthermore, $\mu \in L(\mathbb{R}^d)$ is the limiting distribution of the solution of a Langevin equation with Lévy noise. More precisely, let $\{X_t, t \geq 0\}$ be a Lévy process on $\mathbb{R}^d$, $c \in \mathbb{R}$, and let M be an $\mathbb{R}^d$-valued random variable. The Langevin equation is

$$Z_t = M + X_t - c\int_0^t Z_s ds, \quad t \geq 0, \tag{1.1}$$

and the following is known, (see, e.g., Rocha-Arteaga and Sato [10]).

$$Z_t = e^{-ct}M + e^{-ct}\int_0^t e^{cs}dX_s, \quad t \geq 0,$$

is an almost surely unique solution of (1.1), and if $c > 0$, $E\left[\log^+|X_1|\right] < \infty$, and M is independent of $\{X_t\}$, then $\mathcal{L}(Z_t) \to \mu \in L(\mathbb{R}^d)$ as $t \to \infty$. We also know that, for a fixed $c > 0$, the equation

$$Z_t - Z_s = X_t - X_s - c\int_s^t Z_u du, \quad -\infty < s \leq t < \infty, \tag{1.2}$$

has an almost surely unique stationary solution

$$Z_t = \int_{-\infty}^t e^{-c(t-u)}dX_u, \quad t \in \mathbb{R},$$

where $\{X_t, t \in \mathbb{R}\}$ is a h.-i.s.r.m.-process (whose precise definition is given in Section 2) satisfying $E\left[\log^+|X_1|\right] < \infty$. (See, e.g., Rocha-Arteaga and Sato [10] or Maejima and Sato [6].) This stationary solution fulfills that $\mathcal{L}(Z_t) = \mathcal{L}\left(\int_0^\infty e^{-cu}dX_u\right) \in L(\mathbb{R}^d)$ for all $t \in \mathbb{R}$. Also, it is recognized that some selfdecomposable distributions on $\mathbb{R}$ are very important in the area of mathematical finance, (see Carr et al. [2]).

In Maejima and Naito [5], the concept of the selfdecomposability was extended to the semi-selfdecomposability. Here, $\mu \in I(\mathbb{R}^d)$ is called *semi-selfdecomposable* if there exist $b > 1$ and $\rho \in I(\mathbb{R}^d)$ such that $\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)\widehat{\rho}(z)$. We call this b a span of μ, and we denote the class of all semi-selfdecomposable distributions with span b by $L(b, \mathbb{R}^d)$. From the definitions, $L(b, \mathbb{R}^d) \supsetneqq L(\mathbb{R}^d)$ and $L(\mathbb{R}^d) = \bigcap_{b>1} L(b, \mathbb{R}^d)$. $\mu \in L(b, \mathbb{R}^d)$ is also realized as a limiting distribution of normalized partial sums of independent random variables under infinitesimal condition when the limit is taken through a geometric subsequence. A typical example is a semi-stable distribution, where $\mu \in I(\mathbb{R}^d)$ is said to be semi-stable with span b if there exist $a > 1$ and $c \in \mathbb{R}^d$ satisfying $\widehat{\mu}(z)^a = \widehat{\mu}(bz)e^{i\langle c,z\rangle}$. Recently, several natural examples of semi-selfdecomposable distributions have appeared in the literature. We will mention some of them in the next section.

In Maejima and Sato [6], they gave a stochastic integral characterization of $\mu \in L(b, \mathbb{R}^d)$ in terms of, not Lévy process, but natural semi-Lévy process. Here a semi-Lévy process with period $p > 0$ is an additive process with periodically stationary increments with period p and natural additive process was defined in Sato [12] as semimartingale additive process in terms of the Lévy-Khintchine triplet. Namely, they showed that for each $b > 1$, $\mu \in L(b, \mathbb{R}^d)$ if and only if $\mu = \mathcal{L}\left(\int_0^\infty e^{-t} dX_t\right)$, where $\{X_t\}$ is a semi-Lévy process with period $p = \log b$ and $\mathcal{L}(X_p) \in I_{\log}(\mathbb{R}^d)$. Our first topic of this paper is to give a stochastic integral characterization of $\mu \in L(b, \mathbb{R}^d)$ in terms of Lévy process. If all natural semi-Lévy processes can be expressed as stochastic integrals with respect to Lévy processes, this problem is trivial from a result in Maejima and Sato [6] just mentioned now. However, as we will see in Example 3.8 later, it is not the case. Once we could solve this problem, we would define a mapping Φ_b from $\mathfrak{D}(\Phi_b)$ into $I(\mathbb{R}^d)$ and we can enjoy many stories similar to those about $L(\mathbb{R}^d)$. For instance, we can characterize $L(b, \mathbb{R}^d)$ as the range of the mapping Φ_b, that is, $L(b, \mathbb{R}^d) = \Phi_b(I_{\log}(\mathbb{R}^d))$.

Our second topic is to construct and study a Langevin type equation and the corresponding Ornstein-Uhlenbeck type processes related to semi-selfdecomposable distributions, which are analogies of (1.1) and (1.2) in the case of selfdecomposable distributions, not in terms of semi-Lévy processes given in Maejima and Sato [6], but in terms of Lévy processes. Namely, we introduce a Langevin type equation and give its unique solution, which we call an Ornstein-Uhlenbeck type process. We then show that the limit of the Ornstein-Uhlenbeck type process exists in law, when the noise process has finite log-moment, and the limiting distribution is semi-selfdecomposable. We also construct semi-stationary Ornstein-Uhlenbeck type process whose marginal distributions are semi-selfdecomposable.

Our third topic is to look for the ranges of the iterated mappings Φ_b^m and its limit. In Maejima and Naito [5], the nested subclasses of $L(b, \mathbb{R}^d)$, $L_m(b, \mathbb{R}^d), m \in \mathbb{Z}_+$, are defined as follows: $\mu \in L_m(b, \mathbb{R}^d)$ if and only if there exists $\rho \in L_{m-1}(b, \mathbb{R}^d)$ such that $\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)\widehat{\rho}(z)$, where $L_0(b, \mathbb{R}^d) = L(b, \mathbb{R}^d)$. Then we will show

$$L_m(b, \mathbb{R}^d) = \Phi_b^{m+1}\left(I_{\log^{m+1}}(\mathbb{R}^d)\right), \quad m \in \mathbb{Z}_+, \tag{1.3}$$

where $I_{\log^{m+1}}(\mathbb{R}^d) = \{\mu \in I(\mathbb{R}^d) \colon \int_{\mathbb{R}^d} (\log^+ |x|)^{m+1} \mu(dx) < \infty\}$. The relation (1.3) implies that the limit of these nested subclasses is the closure of the class of semi-stable distributions, where the closure is taken under convolution and weak convergence.

Organization of this paper is the following. In Section 2, we explain some notation and give preliminaries and some examples of semi-selfdecomposable distributions. In Section 3, the first topic is considered. In Sections 4–6, we study the second topic. Finally, in Section 7, we treat the third topic.

2. Notation, preliminaries and examples

In this section, we explain necessary notation, and give some preliminaries and examples.

Let J be $\mathbb{R}$ or $[0,\infty)$, and $\mathcal{B}_J^0$ the class of all bounded Borel sets in J. An $\mathbb{R}^d$-valued independently scattered random measure (abbreviated as i.s.r.m.) $X = \{X(B), B \in \mathcal{B}_J^0\}$ is said to be homogeneous if $\mathcal{L}(X(B)) = \mathcal{L}(X(B+a))$ for all $B \in \mathcal{B}_J^0$ and $a \in \mathbb{R}$ satisfying $B + a \in \mathcal{B}_J^0$. See Maejima and Sato [6] and Sato [12, 13], for the definition and deep study of stochastic integrals of nonrandom measurable functions $f\colon J \to \mathbb{R}$ with respect to $\mathbb{R}^d$-valued i.s.r.m.'s X, denoted by $\int_B f(s)X(ds), B \in \mathcal{B}_J^0$. For a fixed $t_0 \in J$, we use the symbol

$$\int_{t_0}^t f(s)X(ds) = \begin{cases} \int_{(t_0,t]} f(s)X(ds), & \text{for } t \in (t_0,\infty), \\ 0, & \text{for } t = t_0, \\ -\int_{(t,t_0]} f(s)X(ds), & \text{for } t \in J \cap (-\infty, t_0), \end{cases}$$

which is understood to be a càdlàg modification, (see Remark 3.16 of Maejima and Sato [6]). If $\{X_t, t \geq 0\}$ is a Lévy process, then there exists a unique $\mathbb{R}^d$-valued homogeneous i.s.r.m. X over $[0,\infty)$ satisfying $X_t = X([0,t])$ a.s. for each $t \geq 0$. Then $\int_0^t f(s)dX_s$ is defined by $\int_0^t f(s)X(ds)$ for $t \in [0,\infty)$. The improper stochastic integral $\int_0^\infty f(s)dX_s$ is defined as the limit in probability of $\int_0^t f(s)dX_s$ as $t \to \infty$ whenever the limit exists. See also Sato [14]. In this paper, we say that a stochastic process $\{X_t, t \in \mathbb{R}\}$ on $\mathbb{R}^d$ is a *h.-i.s.r.m.-process* if there exists an $\mathbb{R}^d$-valued homogeneous i.s.r.m. X over $\mathbb{R}$ such that $X_t = \int_0^t X(du), t \in \mathbb{R}$. We define a stochastic integral $\int_s^t f(u)dX_u, -\infty < s \leq t < \infty$ of a nonrandom measurable function $f\colon \mathbb{R} \to \mathbb{R}$ with respect to this process $\{X_t, t \in \mathbb{R}\}$ by $\int_s^t f(u)X(du)$. See also Rocha-Arteaga and Sato [10] and Maejima and Sato [6]. The improper stochastic integral $\int_{-\infty}^t f(u)dX_u$ is defined as the limit in probability of $\int_s^t f(u)dX_u$ as $s \to -\infty$, provided that this limit exists. If the improper stochastic integral $\int_{-\infty}^t f(u)dX_u$ is definable for $t \in \mathbb{R}$, then we regard it as a càdlàg process, since such a modification always exists.

Throughout this paper, we use the Lévy-Khintchine representation of the characteristic function of $\mu \in I(\mathbb{R}^d)$ in the following way:

$$\widehat{\mu}(z) = \exp\left\{-\frac{1}{2}\langle z, Az\rangle + i\langle \gamma, z\rangle + \int_{\mathbb{R}^d}\left(e^{i\langle z,x\rangle} - 1 - \frac{i\langle z,x\rangle}{1+|x|^2}\right)\nu(dx)\right\}, \quad z \in \mathbb{R}^d,$$

where $\langle\cdot,\cdot\rangle$ denotes Euclidean inner product on $\mathbb{R}^d$, A is a nonnegative-definite symmetric $d\times d$ matrix, $\gamma \in \mathbb{R}^d$, and ν is a measure, called Lévy measure, satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d}(|x|^2 \wedge 1)\nu(dx) < \infty$. We call (A,ν,γ) the Lévy-Khintchine triplet of μ and we write $\mu = \mu_{(A,\nu,\gamma)}$ when we want to emphasize the Lévy-Khintchine triplet. $C_\mu(z), z \in \mathbb{R}^d$, denotes the cumulant function of $\mu \in I(\mathbb{R}^d)$, that is, $C_\mu(z)$ is the unique continuous function satisfying $\widehat{\mu}(z) = e^{C_\mu(z)}$ and $C_\mu(0) = 0$. When a random variable X has the distribution μ, we sometime write $C_X(z)$ for $C_\mu(z)$.

We also use the polar decomposition (2.1) of the Lévy measure ν of $\mu \in I(\mathbb{R}^d)$ with $0 < \nu(\mathbb{R}^d) \leq \infty$. There exist a measure λ on $S := \{x \in \mathbb{R}^d\colon |x| = 1\}$ with

$0 < \lambda(S) \le \infty$ and a family $\{\nu_\xi, \xi \in S\}$ of measures on $(0,\infty)$ such that $\nu_\xi(B)$ is measurable in ξ for each $B \in \mathcal{B}((0,\infty))$, $0 < \nu_\xi((0,\infty)) \le \infty$ for each $\xi \in S$ and

$$\nu(B) = \int_S \lambda(d\xi) \int_0^\infty \mathbf{1}_B(r\xi)\nu_\xi(dr), \quad B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}). \tag{2.1}$$

Here λ and $\{\nu_\xi\}$ are uniquely determined by ν up to multiplication of measurable functions $c(\xi)$ and $c(\xi)^{-1}$, respectively, with $0 < c(\xi) < \infty$. We say that ν has the polar decomposition (λ, ν_ξ), and λ and ν_ξ are called the spherical and the radial components of ν, respectively. (See, e.g., Barndorff-Nielsen et al. [1], Lemma 2.1.)

Recently, several natural examples of semi-selfdecomposable distributions have appeared in the literature. In Watanabe [17], he showed that the distribution of a certain supercritical branching process and the first hitting time of Brownian motion starting at the origin on the unbounded Sierpinski gasket on $\mathbb{R}^2$ are both semi-selfdecomposable. Also, let $\{N_t, t \ge 0\}$ be a Poisson process and $\{X_t\}$ a Lévy process on $\mathbb{R}^d$ independent of $\{N_t\}$. Suppose $E\left[\log^+ |X_1|\right] < \infty$ and $b > 1$. Then $\mathcal{L}\left(\int_0^\infty b^{-N_{t-}} dX_t\right)$ is semi-selfdecomposable with span b. (Theorem 3.2 of Kondo et al. [3].) In a recent paper by Lindner and Sato [4], we can also find several examples of semi-selfdecomposable distributions with the form $\mathcal{L}\left(\int_0^\infty c^{-N_{t-}} dX_t\right)$, where $\{(N_t, X_t)\}$ is a bivariate compound Poisson process with Lévy measure concentrated on the three points $(1,0), (0,1)$ and $(1,1)$. Another recent example is found by Pacheco-González [9] in some financial modeling. These indicate introducing of semi-selfdecomposable distributions allows us more flexibility in stochastic modeling.

3. Stochastic integral characterizations of semi-selfdecomposable distributions

As we mentioned in Introduction, in this section, we introduce a mapping from a subset of $I(\mathbb{R}^d)$ into $I(\mathbb{R}^d)$, by which semi-selfdecomposable distributions can be characterized.

Definition 3.1. Let $b > 1$ and $\mu \in I(\mathbb{R}^d)$. Define a mapping Φ_b by

$$\Phi_b(\mu) := \mathcal{L}\left(\int_0^\infty b^{-[t]} dX_t^{(\mu)}\right), \tag{3.1}$$

provided that this improper stochastic integral is definable, where $[x]$ denotes the largest integer not greater than $x \in \mathbb{R}$.

The domain of the mapping Φ_b, where the improper stochastic integral in (3.1) is definable, is given as follows by Theorem 2.4 of Sato [15].

Proposition 3.2. $\mathfrak{D}(\Phi_b) = I_{\log}(\mathbb{R}^d)$.

We start with the following theorem.

Theorem 3.3. *Fix any $b > 1$. Let μ and ρ be distributions on $\mathbb{R}^d$. Then,*

$$\rho \in I(\mathbb{R}^d) \quad \text{and} \quad \widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)\widehat{\rho}(z) \tag{3.2}$$

if and only if

$$\rho \in I_{\log}(\mathbb{R}^d) \quad \text{and} \quad \mu = \Phi_b(\rho). \tag{3.3}$$

Proof. To show the "if" part, suppose (3.3). Note that

$$\Phi_b(\rho) = \mathcal{L}\left(\int_0^\infty b^{-[t]} dX_t^{(\rho)}\right) = \mathcal{L}\left(\sum_{j=0}^\infty b^{-j}\left(X_{j+1}^{(\rho)} - X_j^{(\rho)}\right)\right). \tag{3.4}$$

Then

$$\widehat{\mu}(z) = \prod_{j=0}^\infty \widehat{\rho}(b^{-j}z) = \prod_{j=1}^\infty \widehat{\rho}(b^{-j}z) \times \widehat{\rho}(z) = \prod_{k=0}^\infty \widehat{\rho}(b^{-k}(b^{-1}z)) \times \widehat{\rho}(z) = \widehat{\mu}(b^{-1}z)\widehat{\rho}(z),$$

which concludes (3.2).

We next show the "only if" part. Assume (3.2). Then as can be seen in Wolfe [18], we have

$$\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)\widehat{\rho}(z) = \widehat{\mu}(b^{-2}z)\widehat{\rho}(b^{-1}z)\widehat{\rho}(z) = \cdots = \widehat{\mu}(b^{-n}z)\prod_{j=0}^{n-1}\widehat{\rho}(b^{-j}z),$$

for all $n \in \mathbb{N}$. Hence it follows that $\prod_{j=0}^\infty \widehat{\rho}(b^{-j}z)$ exists and equals $\widehat{\mu}(z)$, which implies $\rho \in I_{\log}(\mathbb{R}^d)$ by Wolfe [18]. Then $\Phi_b(\rho)$ is definable and satisfies (3.4). Thus we have $\Phi_b(\rho) = \mu$, which yields (3.3). □

Theorem 3.3 yields the following.

Corollary 3.4. *Fix any $b > 1$. Then, the range $\mathfrak{R}(\Phi_b)$ is the class of all semi-selfdecomposable distributions with span b on $\mathbb{R}^d$, namely,*

$$\Phi_b\left(I_{\log}(\mathbb{R}^d)\right) = L(b, \mathbb{R}^d).$$

The injectivity of the mapping Φ_b is shown as follows.

Proposition 3.5. *For each $b > 1$, the mapping Φ_b is injective.*

Proof. Let $\rho_1, \rho_2 \in I_{\log}(\mathbb{R}^d)$ and $\mu = \Phi_b(\rho_1) = \Phi_b(\rho_2)$. Then, Theorem 3.3 yields that

$$\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)\widehat{\rho}_1(z) = \widehat{\mu}(b^{-1}z)\widehat{\rho}_2(z).$$

Since $\widehat{\mu}(b^{-1}z) \neq 0$ for all $z \in \mathbb{R}^d$ by the infinite divisibility of μ, it follows that $\widehat{\rho}_1(z) = \widehat{\rho}_2(z)$. □

Remark 3.6. As mentioned in Introduction, if $\mu \in L(\mathbb{R}^d)$, then there exists $\mu_0 \in I_{\log}(\mathbb{R}^d)$ such that $\mu = \mathcal{L}\left(\int_0^\infty e^{-t} dX_t^{(\mu_0)}\right)$, and it is known that this μ_0 is uniquely determined by μ. We have just shown that if $\mu \in L(b, \mathbb{R}^d)$, then there exists $\mu_b \in I_{\log}(\mathbb{R}^d)$ such that $\mu = \mathcal{L}\left(\int_0^\infty b^{-[t]} dX_t^{(\mu_b)}\right)$, and the uniqueness of μ_b is assured by Proposition 3.5. If $\mu \in L(\mathbb{R}^d)$, then $\mu \in L(b, \mathbb{R}^d)$ for any $b > 1$. Then it is natural to ask what relation there is between μ_0 and μ_b with $b > 1$. We answer this question below. Fix $b > 1$. Let $\mu = \mu_{(A,\nu,\gamma)}$, $\mu_0 = \mu_{0(A_0,\nu_0,\gamma_0)}$ and $\mu_b = \mu_{b(A_b,\nu_b,\gamma_b)}$. Denote the polar decompositions of ν, ν_0 and ν_b by (λ, ν_ξ), $(\lambda_0, \nu_{0,\xi})$ and $(\lambda_b, \nu_{b,\xi})$, respectively. Note that $\mu \in L(\mathbb{R}^d)$ if and only if

$$\nu_\xi(dr) = \frac{k_\xi(r)}{r}dr, \quad r > 0,$$

where $k_\xi(r)$ is a nonnegative function, which is measurable in ξ, and is nonincreasing and right-continuous in r. (See Sato [11], Theorem 15.10.) We have

$$A=\int_0^\infty e^{-2t}A_0dt=2^{-1}A_0, \tag{3.5}$$

$$\begin{aligned}\gamma&=\int_0^\infty e^{-t}dt\left\{\gamma_0+\int_{\mathbb{R}^d}x\left(\frac{1}{1+e^{-2t}|x|^2}-\frac{1}{1+|x|^2}\right)\nu_0(dx)\right\}\\&=\gamma_0+\int_{\mathbb{R}^d\setminus\{0\}}x\left(\frac{\arctan|x|}{|x|}-\frac{1}{1+|x|^2}\right)\nu_0(dx),\end{aligned} \tag{3.6}$$

and it follows from Theorem 41 (ii) of Rocha-Arteaga and Sato [10] that $\lambda_0=\lambda$ and $\nu_{0,\xi}(dr)=-dk_\xi(r)$, up to multiplication of positive finite measurable functions $c(\xi)$ and $c(\xi)^{-1}$. On the other hand, Theorem 3.3 yields that μ_b is an infinitely divisible distribution satisfying $\widehat{\mu}(z)=\widehat{\mu}(b^{-1}z)\widehat{\mu}_b(z)$. Therefore

$$A_b=\left(1-b^{-2}\right)A, \tag{3.7}$$

$$\gamma_b=\left(1-b^{-1}\right)\gamma-\int_{\mathbb{R}^d}x\left(\frac{1}{1+|x|^2}-\frac{1}{1+|bx|^2}\right)\nu(b\,dx), \tag{3.8}$$

and

$$\begin{aligned}\nu_b(B)=\nu(B)-\nu(bB)&=\int_S\lambda(d\xi)\int_0^\infty \mathbf{1}_B(r\xi)\frac{k_\xi(r)-k_\xi(br)}{r}dr\\&=\int_S\lambda(d\xi)\int_0^\infty \mathbf{1}_B(r\xi)\frac{\nu_{0,\xi}\left((r,br]\right)}{r}dr,\qquad B\in\mathcal{B}_0(\mathbb{R}^d).\end{aligned}$$

Then, it follows that

$$\lambda_b=\lambda_0=\lambda\quad\text{and}\quad \nu_{b,\xi}(dr)=\frac{\nu_{0,\xi}\left((r,br]\right)}{r}dr=\frac{k_\xi(r)-k_\xi(br)}{r}dr\ \lambda\text{-a.e. }\xi\in S, \tag{3.9}$$

up to multiplication of positive finite measurable functions $c(\xi)$ and $c(\xi)^{-1}$. One can see the relation between μ_0 and μ_b by (3.5), (3.6), (3.7), (3.8) and (3.9).

As also mentioned in Introduction, Maejima and Sato [6] characterized semi-selfdecomposable distributions by stochastic integrals with respect to natural semi-Lévy processes. The following theorem is another version of Corollary 3.4 in this paper and Corollary 5.4 of Maejima and Sato [6], and connects them. For $b>1$, let G_b denote the totality of bounded periodic measurable functions with period $\log b$.

Theorem 3.7. *Fix any $b>1$. Then we have*

$$L(b,\mathbb{R}^d)=\left\{\mathcal{L}\left(\int_0^\infty e^{-t}g(t)dX_t^{(\mu)}\right):g\in G_b\ \text{and}\ \mu\in I_{\log}(\mathbb{R}^d)\right\}.$$

Proof. Let $\widetilde{\mu}\in L(b,\mathbb{R}^d)$. Then Corollary 3.4 yields that $\widetilde{\mu}=\Phi_b(\mu)$ for some $\mu\in I_{\log}(\mathbb{R}^d)$. If we let

$$g(t):=b^{\frac{t}{\log b}-\left[\frac{t}{\log b}\right]},$$

then $g\in G_b$ and $e^{-t}g(t)=b^{-[t/\log b]}$. It follows that

$$\widetilde{\mu}=\Phi_b(\mu)=\mathcal{L}\left(\int_0^\infty b^{-[t/\log b]}dX_{t/\log b}^{(\mu)}\right)=\mathcal{L}\left(\int_0^\infty e^{-t}g(t)dX_t^{(\mu^{1/\log b})}\right),$$

where for $p > 0$, μ^p is an infinitely divisible distribution with characteristic function $\widehat{\mu}(z)^p$.

Conversely, suppose that $\widetilde{\mu} = \mathcal{L}\left(\int_0^\infty e^{-t}g(t)dX_t^{(\mu)}\right)$ for $g \in G_b$ and $\mu \in I_{\log}(\mathbb{R}^d)$. Putting $Y_t := \int_0^t g(s)dX_s^{(\mu)}$, we have $\widetilde{\mu} = \mathcal{L}\left(\int_0^\infty e^{-t}dY_t\right)$ due to Theorem 4.6 of Sato [12], and we see that $\{Y_t\}$ is a natural semi-Lévy process with period $\log b$. Moreover, we have $E\left[\log^+|Y_{\log b}|\right] < \infty$ since the Lévy measure of $\mathcal{L}(Y_{\log b})$ denoted by $\widetilde{\nu}$ satisfies that

$$\int_{|x|>1} \log|x|\widetilde{\nu}(dx) = \int_0^{\log b} ds \int_{\mathbb{R}^d} \log^+|g(s)x|\nu(dx)$$
$$\leq \log b \int_{\mathbb{R}^d} \log^+\left(\sup_{s\in[0,\log b]}|g(s)||x|\right)\nu(dx) < \infty,$$

where ν is the Lévy measure of μ. Then Corollary 5.4 of Maejima and Sato [6] implies that $\widetilde{\mu} \in L(b,\mathbb{R}^d)$. □

In the proof of Theorem 3.7, $Y_t = \int_0^t g(s)dX_s^{(\mu)}$ with $g \in G_b$ and $\mu \in I_{\log}(\mathbb{R}^d)$ is proved to be a natural semi-Lévy process. However, any natural semi-Lévy process is not necessarily expressed in this way as is shown in the following example.

Example 3.8. We claim that not all natural semi-Lévy processes can be expressed as stochastic integrals with respect to Lévy processes. Fix an arbitrary $p > 0$. Let $\varphi\colon [0,p] \to [0,\infty)$ be a nondecreasing function satisfying $\varphi(0) = 0$ which is continuous but not absolutely continuous, for example, Cantor's function on $[0,p]$. Suppose that $\mu \in I(\mathbb{R}^d) \setminus \{\delta_0\}$. Then there exists a semi-Lévy process $\{Y_t, t \geq 0\}$ with period p such that $C_{Y_t}(z) = \varphi(t)C_\mu(z)$ for $z \in \mathbb{R}^d$ and $t \in [0,p]$, due to Proposition 2.2 of Maejima and Sato [6]. Furthermore, it follows from the monotonicity of φ that $\{Y_t\}$ is natural. However, we cannot express $\{Y_t\}$ in the form that $Y_t = \int_0^t f(s)dX_s$ for any measurable function f and any Lévy process $\{X_t\}$. Indeed, if $\int_0^t f(s)dX_s$ is definable for all $t \in [0,\infty)$, then $C_{\int_0^t f(s)dX_s}(z) = \int_0^t C_{X_1}(f(s)z)ds$ which is absolutely continuous in t, although $C_{Y_t}(z)$ is not absolutely continuous in t by the property of φ.

4. A Langevin type equation

The purpose of this and the following two sections is to find a Langevin type equation like (1.1) or (1.2) related to semi-selfdecomposable distributions. The ideas of proofs of the results below come from Sections 2.2 and 2.4 of Rocha-Arteaga and Sato [10] and Maejima and Sato [6].

For this purpose, we first consider the following *Langevin type equation*:

$$Z_t = M + X_{[ct]/c} - X_{[ct_0]/c} - (b-1)\int_{t_0}^t Z_s d[cs], \quad t \geq t_0, \tag{4.1}$$

where $t_0 \in \mathbb{R}$, $c > 0$, $b > 1$, $\{X_t, t \in \mathbb{R}\}$ is a h.-i.s.r.m.-process on $\mathbb{R}^d$, M is an $\mathbb{R}^d$-valued random variable, and $\int_0^t Z_s d[cs]$ is as follows: $d[cs]$ denotes the Lebesgue-Stieltjes measure associated with $s \mapsto [cs]$, which is equal to $\sum_{k\in\mathbb{Z}} \delta_{k/c}(ds)$, and

$\int_{(\alpha,\beta]} f(s)d[cs]$, written as $\int_\alpha^\beta f(s)d[cs]$, exists for $-\infty < \alpha < \beta < \infty$ and any measurable function f, since it is a finite sum in fact. A stochastic process $\{Z_t\}$ is said to be a *solution* of the Langevin type equation (4.1) or an *Ornstein-Uhlenbeck type process* generated by $\{X_t\}$, b and c starting from $Z_{t_0} = M$ if sample paths of $\{Z_t\}$ are right-continuous with left limits and $\{Z_t\}$ satisfies (4.1) almost surely. We claim that

$$Z_t = b^{-([ct]-[ct_0])}M + b^{-[ct]} \int_{[ct_0]/c}^{[ct]/c} b^{[cs]} dX_s, \quad t \ge t_0, \tag{4.2}$$

is an almost surely unique solution of (4.1).

Theorem 4.1. *Suppose that $\{X_t, t \in \mathbb{R}\}$ is a h.-i.s.r.m.-process on $\mathbb{R}^d$, $t_0 \in \mathbb{R}$, $c > 0$, $b > 1$, and M is an $\mathbb{R}^d$-valued random variable. Then, $\{Z_t\}$ in* (4.2) *is an almost surely unique solution of the equation* (4.1).

Proof. If we define $\{Z_t\}$ by (4.2), then it is a càdlàg process. If $t_0 \le t < t_0 + 1/c$, then $Z_t = M$ and it satisfies (4.1). Let $t \ge t_0 + 1/c$. Then, it follows that

$$\begin{aligned}
&(b-1)\int_{t_0}^t Z_s d[cs] \\
&= (b-1)\sum_{k=[ct_0]+1}^{[ct]} Z_{k/c} = (b-1)\sum_{k=[ct_0]+1}^{[ct]} \left\{ b^{-(k-[ct_0])}M + b^{-k}\int_{[ct_0]/c}^{k/c} b^{[cs]}dX_s \right\} \\
&= \left(1 - b^{-([ct]-[ct_0])}\right) M + (b-1)\sum_{k=[ct_0]+1}^{[ct]} b^{-k} \sum_{\ell=[ct_0]+1}^{k} b^{\ell-1}\left(X_{\ell/c} - X_{(\ell-1)/c}\right) \\
&= \left(1 - b^{-([ct]-[ct_0])}\right) M + (b-1)\sum_{\ell=[ct_0]+1}^{[ct]} b^{\ell-1}\left(X_{\ell/c} - X_{(\ell-1)/c}\right)\sum_{k=\ell}^{[ct]} b^{-k} \\
&= \left(1 - b^{-([ct]-[ct_0])}\right) M + \sum_{\ell=[ct_0]+1}^{[ct]} \left(1 - b^{-[ct]+\ell-1}\right)\left(X_{\ell/c} - X_{(\ell-1)/c}\right) \\
&= \left(1 - b^{-([ct]-[ct_0])}\right) M + X_{[ct]/c} - X_{[ct_0]/c} - b^{-[ct]}\int_{[ct_0]/c}^{[ct]/c} b^{[cs]}dX_s \\
&= M + X_{[ct]/c} - X_{[ct_0]/c} - Z_t.
\end{aligned}$$

This yields (4.1).

It remains to prove the uniqueness of the solution of (4.1). Suppose that two $\{Z_t^{(1)}\}$ and $\{Z_t^{(2)}\}$ are the solutions of (4.1). Setting $Z_t := Z_t^{(1)} - Z_t^{(2)}$, we have

$$Z_t = -(b-1)\int_{t_0}^t Z_s d[cs] \qquad \text{for } t \ge t_0, \quad \text{a.s.} \tag{4.3}$$

Let us show that

$$Z_t = 0, \qquad \text{for } \left(t_0 \vee \frac{[ct_0]+n-1}{c}\right) \le t < \frac{[ct_0]+n}{c}, \quad \text{a.s.} \tag{4.4}$$

for any $n \in \mathbb{N}$ by induction. (4.4) is true for $n = 1$, since the right-hand side of (4.3) is zero for $t_0 \le t < ([ct_0]+1)/c$. Assume that (4.4) holds for $n = 1, 2, \ldots, m$. Then, for $([ct_0]+m)/c \le t < ([ct_0]+m+1)/c$, (4.3) can be reduced to that

$$Z_t = -(b-1)\int_{([ct_0]+m-1)/c}^{([ct_0]+m)/c} Z_s d[cs] = -(b-1)Z_{([ct_0]+m)/c},$$

which implies that $X_{([ct_0]+m)/c} = 0$ by letting $t = ([ct_0]+m)/c$ and thus $Z_t = 0$ for $([ct_0]+m)/c \le t < ([ct_0]+m+1)/c$. Hence (4.4) is true for $n = m+1$. Therefore it holds with probability one that for any $t \ge 0$, $Z_t = Z_t^{(1)} - Z_t^{(2)} = 0$. □

5. Limiting distributions of Ornstein-Uhlenbeck type processes

In this section, we study the Langevin type equation (4.1) with $t_0 = 0$:

$$Z_t = M + X_{[ct]/c} - (b-1)\int_0^t Z_s d[cs], \quad t \ge 0, \tag{5.1}$$

where $\{X_t, t \ge 0\}$ is a Lévy process on $\mathbb{R}^d$, $c > 0$, $b > 1$, and M is an $\mathbb{R}^d$-valued random variable. Theorem 4.1 yields that

$$Z_t = b^{-[ct]}M + b^{-[ct]}\int_0^{[ct]/c} b^{[cs]} dX_s, \quad t \ge 0, \tag{5.2}$$

is an almost surely unique solution of (5.1). We show that if M is independent of $\{X_t\}$ and $\mathcal{L}(X_1) \in I_{\log}(\mathbb{R}^d)$, then the limit of $\mathcal{L}(Z_t)$ exists as $t \to \infty$ and the limiting distribution is semi-selfdecomposable with span b.

Theorem 5.1. *Suppose that $\{X_t, t \ge 0\}$ is a Lévy process on $\mathbb{R}^d$, $c > 0$, $b > 1$, and M is an $\mathbb{R}^d$-valued random variable independent of $\{X_t\}$. Then, $\mathcal{L}(X_1) \in I_{\log}(\mathbb{R}^d)$ if and only if Z_t in (5.2) converges in law to some $\mathbb{R}^d$-valued random variable as $t \to \infty$. This limit $\lim_{t\to\infty} \mathcal{L}(Z_t)$ is equal to $\mathcal{L}\left(\int_0^\infty b^{-[cs]-1} dX_s\right)$ which is semi-selfdecomposable with span b and does not depend on M. Furthermore, if we let $\mathcal{L}(M) := \lim_{t\to\infty} \mathcal{L}(Z_t)$, then $\mathcal{L}(Z_t) = \mathcal{L}(M)$ for all $t \ge 0$.*

Proof. In the rest of the paper, we write $\widehat{\mathcal{L}}(X)(z)$ for the characteristic function of $\mathcal{L}(X)$ for notational simplicity. It follows that

$$\begin{aligned}
\widehat{\mathcal{L}}(Z_t)(z) &= \widehat{\mathcal{L}}(M)\left(b^{-[ct]}z\right) \exp\left\{\int_0^{[ct]/c} C_{X_1}\left(b^{[cs-[ct]]}z\right) ds\right\} \\
&= \widehat{\mathcal{L}}(M)\left(b^{-[ct]}z\right) \exp\left\{\int_0^{[ct]/c} C_{X_1}\left(b^{[-cu]}z\right) du\right\} \\
&= \widehat{\mathcal{L}}(M)\left(b^{-[ct]}z\right) \exp\left\{\int_0^{[ct]/c} C_{X_1}\left(b^{-[cu]-1}z\right) du\right\}.
\end{aligned} \tag{5.3}$$

Note that $\widehat{\mathcal{L}}(M)\left(b^{-[ct]}z\right)$ tends to 1 as $t \to \infty$.

Assume $\mathcal{L}(X_1) \in I_{\log}(\mathbb{R}^d)$. Then, (5.3) tends to the cumulant function of the improper stochastic integral $\int_0^\infty b^{-[cu]-1} dX_u$, whose law is semi-selfdecomposable with span b due to Corollary 3.4. Also, $\lim_{t\to\infty}\mathcal{L}(Z_t)=\mathcal{L}\left(\int_0^\infty b^{-[cu]-1}dX_u\right)$ does not depend on M and if we set $\mathcal{L}(M):=\mathcal{L}\left(\int_0^\infty b^{-[cu]-1}dX_u\right)$, then (5.3) is

$$\begin{aligned}
C_{Z_t}(z) &= \int_0^\infty C_{X_1}\left(b^{-[cu+[ct]]-1}z\right)du+\int_0^{[ct]/c} C_{X_1}\left(b^{-[cu]-1}z\right)du\\
&= \int_{[ct]/c}^\infty C_{X_1}\left(b^{-[cv]-1}z\right)dv+\int_0^{[ct]/c} C_{X_1}\left(b^{-[cu]-1}z\right)du\\
&= \int_0^\infty C_{X_1}\left(b^{-[cu]-1}z\right)du,
\end{aligned}$$

which yields that $\mathcal{L}(Z_t)=\mathcal{L}\left(\int_0^\infty b^{-[cu]-1}dX_u\right)$ for all $t\geq 0$.

Next assume that Z_t converges in law as $t\to\infty$, namely, $\int_0^{n/c} C_{X_1}\left(b^{-[cu]-1}z\right)du$ tends to the cumulant function of some $\mu\in I(\mathbb{R}^d)$ with Lévy measure ν as $n\to\infty$. Denoting the Lévy measure of $\mathcal{L}(X_1)$ by ν_{X_1}, we have

$$\lim_{k\to\infty}\int_0^{n_k/c} du\int_{\mathbb{R}^d}\left(|b^{-[cu]-1}x|^2\wedge 1\right)\nu_{X_1}(dx)=\int_{\mathbb{R}^d}(|x|^2\wedge 1)\nu(dx)$$

for some subsequence, due to the proof of Theorem 8.7 in Sato [11]. It follows from the monotone convergence theorem that $\int_0^\infty du\int_{\mathbb{R}^d}\left(|b^{-[cu]-1}x|^2\wedge 1\right)\nu_{X_1}(dx)<\infty$, which implies $\mathcal{L}(X_1)\in I_{\log}(\mathbb{R}^d)$ by virtue of Lemma 2.7 of Sato [15]. $\square$

Theorem 5.1 yields that if $\mathcal{L}(X_1)\in I_{\log}(\mathbb{R}^d)$, then Z_t in (5.2) converges in law as $t\to\infty$. Then, it might be natural to ask whether or not Z_t converges in probability as $t\to\infty$. The following proposition is the answer.

Proposition 5.2. *Suppose that $\{X_t,t\geq 0\}$ is a Lévy process on $\mathbb{R}^d$, $c>0$, $b>1$, and M is an $\mathbb{R}^d$-valued random variable independent of $\{X_t\}$. If $\mathcal{L}(X_1)$ is not any δ-distribution, then Z_t in* (5.2) *does not converge in probability as $t\to\infty$.*

Proof. Suppose that $\mu:=\mathcal{L}(X_1)$ is not any δ-distribution. Then, Lemma 13.9 of Sato [11] yields that $|\widehat{\mu}(z_0)|<1$ for some $z_0\in\mathbb{R}^d$. As

$$Z_t-Z_{t-1/c}=b^{-[ct]}\int_{[ct-1]/c}^{[ct]/c} b^{[cs]}dX_s+(b^{-[ct]}-b^{-[ct-1]})\left(M+\int_0^{[ct-1]/c} b^{[cs]}dX_s\right)$$

by (5.2), we have

$$\begin{aligned}
\left|\widehat{\mathcal{L}}\left(Z_t-Z_{t-1/c}\right)(z)\right| &\leq \left|\widehat{\mathcal{L}}\left(b^{-[ct]}\int_{[ct-1]/c}^{[ct]/c} b^{[cs]}dX_s\right)(z)\right|\\
&= \left|\widehat{\mathcal{L}}\left(b^{-1}\left(X_{[ct]/c}-X_{[ct-1]/c}\right)\right)(z)\right| = \left|\widehat{\mu}(b^{-1}z)\right|^{1/c}
\end{aligned}$$

for any $t\geq 1/c$ and $z\in\mathbb{R}^d$. This yields that for all $t\geq 1/c$,

$$\left|\widehat{\mathcal{L}}\left(Z_t-Z_{t-1/c}\right)(bz_0)\right|\leq|\widehat{\mu}(z_0)|^{1/c}<1.$$

Then $Z_t-Z_{t-1/c}$ does not tends to zero in probability as $t\to\infty$. Thus Z_t does not converge in probability as $t\to\infty$. $\square$

The following remark is about the relation between the Langevin type equation (5.1) and the mapping Φ_b.

Remark 5.3. Fix $b > 1$ and $c > 0$. Let $\mu \in I_{\log}(\mathbb{R}^d)$. Consider the limiting solution in law $\lim_{t\to\infty} \mathcal{L}(Z_t)$ of the Langevin type equation (5.1) with a Lévy process $\{X_t\}$ satisfying $\mathcal{L}(b^{-1}X_{1/c}) = \mu$ and an $\mathbb{R}^d$-valued random variable M independent of $\{X_t\}$. It follows from Theorem 5.1 that

$$\begin{aligned}\Phi_b(\mu) &= \mathcal{L}\left(\int_0^\infty b^{-[t]} dX_t^{(\mu)}\right) = \mathcal{L}\left(\int_0^\infty b^{-[t]} dX_t^{\left(\mathcal{L}(b^{-1}X_{1/c})\right)}\right) \\ &= \mathcal{L}\left(\int_0^\infty b^{-[t]} d(b^{-1}X_{t/c})\right) = \mathcal{L}\left(\int_0^\infty b^{-[cs]-1} dX_s\right) = \lim_{t\to\infty} \mathcal{L}(Z_t).\end{aligned}$$

Thus the mapping Φ_b can be defined also as the limiting distribution of the solution of the Langevin type equation (5.1).

We conclude this section with the Markov property of our Ornstein-Uhlenbeck type processes.

Proposition 5.4. *Suppose that $\{X_t, t \geq 0\}$ is a Lévy process on $\mathbb{R}^d$, $c > 0$, $b > 1$, and M is an $\mathbb{R}^d$-valued random variable independent of $\{X_t\}$. Then, the process $\{Z_t\}$ in (5.2) is a Markov process satisfying*

$$P\left(Z_t \in B \mid Z_s = x\right) = P\left(b^{-([ct]-[cs])}x + \int_0^{([ct]-[cs])/c} b^{-[cu]-1} dX_u \in B\right) \tag{5.4}$$

for $0 \leq s \leq t$ and $B \in \mathcal{B}(\mathbb{R}^d)$.

Proof. Since

$$Z_t = b^{-([ct]-[cs])} Z_s + b^{-[ct]} \int_{[cs]/c}^{[ct]/c} b^{[cu]} dX_u,$$

we can easily see the Markov property of $\{Z_t\}$ by virtue of the independent increment property of the Lévy process $\{X_t\}$. (5.4) is shown as follows:

$$\begin{aligned}&E\left[e^{i\langle z, Z_t\rangle} \mid Z_s = x\right] \\ &= \exp\left\{ i\langle z, b^{-([ct]-[cs])}x\rangle + \int_{[cs]/c}^{[ct]/c} C_{X_1}\left(b^{[cu-[ct]]}z\right) du \right\} \\ &= \exp\left\{ i\langle z, b^{-([ct]-[cs])}x\rangle + \int_0^{([ct]-[cs])/c} C_{X_1}\left(b^{[-cv]}z\right) du \right\} \\ &= \exp\left\{ i\langle z, b^{-([ct]-[cs])}x\rangle + \int_0^{([ct]-[cs])/c} C_{X_1}\left(b^{-[cv]-1}z\right) du \right\}.\end{aligned}$$

$\square$

6. Semi-stationary Ornstein-Uhlenbeck type processes having semi-selfdecomposable distributions

This section is concerned with the following *Langevin type equation* which has similar properties to (1.2):

$$Z_t - Z_s = X_{[ct]/c} - X_{[cs]/c} - (b-1)\int_s^t Z_u d[cu], \quad -\infty < s \le t < \infty, \tag{6.1}$$

where $c > 0$, $b > 1$, and $\{X_t, t \in \mathbb{R}\}$ is a h.-i.s.r.m.-process on $\mathbb{R}^d$. A stochastic process $\{Z_t\}$ is said to be a *solution* of the Langevin type equation (6.1) or an *Ornstein-Uhlenbeck type process* generated by $\{X_t\}$, b and c if sample paths of $\{Z_t\}$ are right-continuous with left limits and $\{Z_t\}$ satisfies (6.1) almost surely.

We show that if $E[\log^+|X_1|] < \infty$, then $\int_{-\infty}^t b^{[cu]}dX_u$ is definable for each $t \in \mathbb{R}$ and

$$Z_t = b^{-[ct]}\int_{-\infty}^{[ct]/c} b^{[cu]}dX_u, \quad t \in \mathbb{R}, \tag{6.2}$$

is an almost surely unique semi-stationary solution of (6.1), where the *semi-stationarity* of $\{Z_t\}$ means $\{Z_{t+p}\} \overset{\mathrm{d}}{=} \{Z_t\}$ for a fixed $p > 0$. Here $\overset{\mathrm{d}}{=}$ stands for equality in all finite-dimensional distributions. This p is called the *period* of the semi-stationary process $\{Z_t\}$.

To prove this, we prepare two lemmas.

Lemma 6.1. *Let $\{X_t, t \in \mathbb{R}\}$ be a h.-i.s.r.m.-process on $\mathbb{R}^d$. Suppose that $b > 1$ and $c > 0$. Then, the following three statements are equivalent:*

(i) $\mathcal{L}(X_1) \in I_{\log}(\mathbb{R}^d)$,

(ii) $\int_{-\infty}^0 b^{[ct]}dX_t$ *is definable,*

(iii) *there exists an Ornstein-Uhlenbeck type process $\{Z_t\}$ generated by $\{X_t\}$, b and c satisfying* $\operatorname{p-lim}_{t\to-\infty} b^{[ct]}Z_t = 0$, *where* p-lim *stands for limit in probability.*

If (iii) *holds, then $\{Z_t\}$ with the properties in* (iii) *is almost surely unique and expressed as* (6.2).

Proof. We first show that (i) and (ii) are equivalent. Theorem 2.4 of Sato [15] yields that $\mathcal{L}(X_1) = \mathcal{L}\left(-X_{(-1)-}\right) \in I_{\log}(\mathbb{R}^d)$ if and only if $\int_0^\infty b^{[-cu]}d\left(-X_{(-u)-}\right)$ is definable. Lemma 4.8 of Maejima and Sato [6] implies that $\int_0^\infty b^{[-cu]}d\left(-X_{(-u)-}\right)$ is definable if and only if $\int_{-\infty}^0 b^{[cu]}dX_u$ is definable. Thus (i) and (ii) are equivalent.

We next show that (ii) implies (iii). Assume that (ii) holds. Then, $\{Z_t\}$ in (6.2) is definable. It satisfies (6.1) due to Theorem 4.1 by letting $t_0 = s$ and $Z_{t_0} = M = b^{-[cs]}\int_{-\infty}^{[cs]/c} b^{[cu]}dX_u$. It follows from (6.2) and (ii) that

$$\operatorname*{p-lim}_{t\to-\infty} b^{[ct]}Z_t = \operatorname*{p-lim}_{t\to-\infty}\int_{-\infty}^{[ct]/c} b^{[cu]}dX_u = 0.$$

Finally, we show that (iii) implies (i), the uniqueness of $\{Z_t\}$ in (iii), and the expression (6.2). If (iii) holds, then $\{Z_t\}$ in (iii) satisfies (6.1). Theorem 4.1 yields that for each $s \in \mathbb{R}$, with probability one,

$$Z_t = b^{-([ct]-[cs])}Z_s + b^{-[ct]}\int_{[cs]/c}^{[ct]/c} b^{[cu]}dX_u, \qquad \text{for } t \ge s,$$

namely, with probability one,

$$b^{[ct]}Z_t - b^{[cs]}Z_s = \int_{[cs]/c}^{[ct]/c} b^{[cu]}dX_u, \qquad \text{for } t \geq s.$$

By letting $s \to -\infty$, it follows from (iii) that for each $t \in \mathbb{R}$,

$$b^{[ct]}Z_t = \operatorname*{p\text{-}lim}_{k\to\infty} \int_{-k/c}^{[ct]/c} b^{[cu]}dX_u, \qquad \text{a.s.}$$

By a similar argument to that in the proof of Theorem 5.1, we have $\mathcal{L}(X_1) \in I_{\log}(\mathbb{R}^d)$. Thus (i) holds. Then (ii) holds and it follows that for each $t \in \mathbb{R}$,

$$b^{[ct]}Z_t = \int_{-\infty}^{[ct]/c} b^{[cu]}dX_u, \qquad \text{a.s.}$$

However, since the both sides of the equation above have càdlàg paths, we have, with probability one,

$$b^{[ct]}Z_t = \int_{-\infty}^{[ct]/c} b^{[cu]}dX_u, \qquad \text{for all } t \in \mathbb{R}.$$

This yields the almost sure uniqueness of $\{Z_t\}$ in (iii), and the expression (6.2). □

Lemma 6.2. *Let $\{X_t, t \in \mathbb{R}\}$ be a h.-i.s.r.m.-process on $\mathbb{R}^d$, $b > 1$ and $c > 0$. Suppose that $\{Z_t\}$ is an Ornstein-Uhlenbeck type process generated by $\{X_t\}$, b and c. Then, $\{Z_t\}$ is semi-stationary if and only if* p-lim$_{t\to-\infty} b^{[ct]}Z_t = 0$. *Let these conditions be fulfilled. Then, semi-stationary process $\{Z_t\}$ has a period $1/c$. Moreover, $\{Z_t\}$ with the properties above is almost surely unique and expressed as* (6.2).

Proof. We first show the "only if" part. Suppose that $\{Z_t\}$ is a semi-stationary Ornstein-Uhlenbeck type process generated by $\{X_t\}$, b and c with period $p > 0$. Since $\{Z_t\}$ has càdlàg paths, for any sequence $\{t_n, n \in \mathbb{N}\} \subset [0, p]$, there exists its subsequence $\{t_{n_k}, k \in \mathbb{N}\}$ satisfying $Z_{t_{n_k}}$ converges almost surely to some $\mathbb{R}^d$-valued random variable as $k \to \infty$. This implies the relative compactness of $\{\mathcal{L}(Z_t) \colon t \in [0, p]\}$ which is equal to $\{\mathcal{L}(Z_t) \colon t \in \mathbb{R}\}$ by the semi-stationarity of $\{Z_t\}$. Hence $\{\mathcal{L}(Z_t) \colon t \in \mathbb{R}\}$ is tight by Prohorov's theorem. Then, it follows that for any $\varepsilon > 0$,

$$P\left(|b^{[ct]}Z_t| > \varepsilon\right) \leq \sup_{s\in\mathbb{R}} P\left(|Z_s| > b^{-[ct]}\varepsilon\right) \to 0, \quad \text{as } t \to -\infty.$$

Thus p-lim$_{t\to-\infty} b^{[ct]}Z_t = 0$.

We next show the "if" part. Suppose that $\{Z_t\}$ is an Ornstein-Uhlenbeck type process generated by $\{X_t\}$, b and c satisfying p-lim$_{t\to-\infty} b^{[ct]}Z_t = 0$. Then $\{Z_t\}$ has the form (6.2) due to Lemma 6.1. Let $-\infty < t_1 < t_2 < \cdots < t_n < \infty$. Then, for each $j = 2, 3, \ldots, n$, we have

$$\begin{aligned} b^{[ct_j]}Z_{t_j+1/c} - b^{[ct_{j-1}]}Z_{t_{j-1}+1/c} &= b^{-1}\int_{[ct_{j-1}+1]/c}^{[ct_j+1]/c} b^{[cu]}dX_u \\ &= \int_{[ct_{j-1}]/c}^{[ct_j]/c} b^{[cv]}d(X_{v+1/c} - X_{1/c}), \end{aligned}$$

which is equal in law to

$$\int_{[ct_{j-1}]/c}^{[ct_j]/c} b^{[cv]} dX_v = b^{[ct_j]} Z_{t_j} - b^{[ct_{j-1}]} Z_{t_{j-1}}.$$

Since $\{b^{[ct]} Z_{t+1/c}\}$ and $\{b^{[ct]} Z_t\}$ have independent increment property due to the expression (6.2), it follows that $\{b^{[ct]} Z_{t+1/c}\} \stackrel{\mathrm{d}}{=} \{b^{[ct]} Z_t\}$, which yields $\{Z_{t+1/c}\} \stackrel{\mathrm{d}}{=} \{Z_t\}$. This is the semi-stationarity of $\{Z_t\}$ with period $1/c$.

The almost sure uniqueness of $\{Z_t\}$ follows from Lemma 6.1. $\square$

Now, we prove the following theorem on the relation between semi-stationary Ornstein-Uhlenbeck type processes and semi-selfdecomposable distributions.

Theorem 6.3. *Suppose that $\{X_t, t \in \mathbb{R}\}$ is a h.-i.s.r.m.-process on $\mathbb{R}^d$, $c > 0$, and $b > 1$. Then, $\mathcal{L}(X_1) \in I_{\log}(\mathbb{R}^d)$ if and only if* (6.1) *has a semi-stationary solution. If $\mathcal{L}(X_1) \in I_{\log}(\mathbb{R}^d)$, $\{Z_t\}$ in* (6.2) *is an almost surely unique semi-stationary solution of* (6.1) *and it satisfies $\mathcal{L}(Z_t) = \mathcal{L}\left(\int_0^\infty b^{-[cu]-1} dX_u\right) \in L(b, \mathbb{R}^d)$ for all $t \geq 0$. In this case, the semi-stationary process $\{Z_t\}$ has a period $1/c$.*

Proof. The most parts of this theorem have already been proved in Lemmas 6.1 and 6.2. It remains to show the statement that if $\mathcal{L}(X_1) \in I_{\log}(\mathbb{R}^d)$, then $\mathcal{L}(Z_t) = \mathcal{L}\left(\int_0^\infty b^{-[cu]-1} dX_u\right) \in L(b, \mathbb{R}^d)$ for all $t \geq 0$. However, it follows that for all $t \in \mathbb{R}$,

$$\begin{aligned} C_{Z_t}(z) &= \int_{-\infty}^{[ct]/c} C_{X_1}\left(b^{[cu-[ct]]} z\right) du = \int_0^\infty C_{X_1}\left(b^{[-cv]} z\right) dv \\ &= \int_0^\infty C_{X_1}\left(b^{-[cv]-1} z\right) dv = C_{\int_0^\infty b^{-[cv]-1} dX_v}(z), \end{aligned}$$

which is the cumulant function of some semi-selfdecomposable distribution with span b, due to Corollary 3.4. $\square$

The following remark, which is similar to Remark 5.3, is about the relation between the Langevin type equation (6.1) and the mapping Φ_b.

Remark 6.4. Fix $b > 1$ and $c > 0$. Let $\mu \in I_{\log}(\mathbb{R}^d)$. Consider the almost surely unique semi-stationary Ornstein-Uhlenbeck type process $\{Z_t\}$ generated by a h.-i.s.r.m.-process $\{X_t\}$ satisfying $\mathcal{L}(b^{-1} X_{1/c}) = \mu$, and b and c. Theorem 6.3 and the same calculations as those in Remark 5.3 yield that

$$\Phi_b(\mu) = \mathcal{L}(Z_t), \quad \text{for all } t \in \mathbb{R}.$$

Hence the mapping Φ_b can be defined also as the distribution of the semi-stationary solution of the Langevin type equation (6.1).

7. Nested subclasses of $L(b, \mathbb{R}^d)$ given by iterating the mapping Φ_b

We now go back to the mapping Φ_b itself again. The iterated mapping of Φ_b can be expressed by one stochastic integral as follows.

Theorem 7.1. *Suppose $m \in \mathbb{Z}_+$. The domain of Φ_b^{m+1} is*

$$\mathfrak{D}(\Phi_b^{m+1}) = I_{\log^{m+1}}(\mathbb{R}^d). \tag{7.1}$$

Let

$$f_m(u) := \int_0^u \binom{[v]+m}{m} dv,$$

and let f_m^* *be its inverse function. Then*

$$\Phi_b^{m+1}(\mu) = \mathcal{L}\left(\int_0^\infty b^{-[f_m^*(t)]} dX_t^{(\mu)}\right), \quad \textit{for } \mu \in I_{\log^{m+1}}(\mathbb{R}^d).$$

Remark 7.2. If we let

$$\widetilde{f}_m(u) := \int_0^u \frac{v^m}{m!} dv,$$

then its inverse function is $\widetilde{f}_m^*(t) = \{(m+1)!\, t\}^{\frac{1}{m+1}}$ and $e^{-\widetilde{f}_m^*(t)}$ is the integrand of the stochastic integral of the iteration of the mapping in the case of $L(\mathbb{R}^d)$, (see Remark 58 of Rocha-Arteaga and Sato [10]). One can see the difference between f_m and $\widetilde{f}_m$ by

$$f_m(u) = \int_0^u \binom{[v]+m}{m} dv = \int_0^u \frac{([v]+1)([v]+2)\cdots([v]+m)}{m!} dv.$$

Proof of Theorem 7.1. We prove the statement by induction. If $m = 0$, the assertion is true by the definition of Φ_b and Proposition 3.2. Assume that the assertion is true for $0, 1, \ldots, m-1$ in place of m. Let

$$\widetilde{\Phi}_{b,m+1}(\mu) := \mathcal{L}\left(\int_0^\infty b^{-[f_m^*(t)]} dX_t^{(\mu)}\right).$$

Then $\mathfrak{D}^0(\widetilde{\Phi}_{b,m+1}) = \mathfrak{D}(\widetilde{\Phi}_{b,m+1}) = I_{\log^{m+1}}(\mathbb{R}^d)$ due to Proposition 4.3 of Sato [14], where $\mathfrak{D}^0(\Phi_f)$ denotes the set of all $\mu \in I(\mathbb{R}^d)$ satisfying $\int_0^\infty |C_\mu(f(t)z)| dt < \infty$. If $\mu \in I_{\log^{m+1}}(\mathbb{R}^d) \subset I_{\log^m}(\mathbb{R}^d)$, then $\Phi_b^m(\mu) = \mathcal{L}\left(\int_0^\infty b^{-[f_{m-1}^*(t)]} dX_t^{(\mu)}\right)$ by the assumption of induction, and

$$\begin{aligned}
&\int_0^\infty \left| C_{\Phi_b^m(\mu)}(b^{-[t]}z) \right| dt \\
&\quad \le \int_0^\infty dt \int_0^\infty \left| C_\mu\left(b^{-[f_{m-1}^*(s)]-[t]} z\right) \right| ds \\
&\quad = \int_0^\infty dt \int_0^\infty \left| C_\mu\left(b^{-\left[f_{m-1}^*(s)+[t]\right]} z\right) \right| ds \\
&\quad = \int_0^\infty dt \int_{[t]}^\infty \left| C_\mu(b^{-[u]}z) \right| \binom{[u]-[t]+m-1}{m-1} du \\
&\quad = \int_0^\infty \left| C_\mu(b^{-[u]}z) \right| du \int_0^{[u]+1} \binom{[u]-[t]+m-1}{m-1} dt \\
&\quad = \int_0^\infty \left| C_\mu(b^{-[u]}z) \right| \binom{[u]+m}{m} du \\
&\quad = \int_0^\infty \left| C_\mu(b^{-[f_m^*(t)]}z) \right| dt < \infty, \qquad (7.2)
\end{aligned}$$

since $\mu \in I_{\log^{m+1}}(\mathbb{R}^d) = \mathfrak{D}^0(\widetilde{\Phi}_{b,m+1})$. Note that we have used above the formula

$$\int_0^{n-k+1} \binom{n-[t]}{k} dt = \sum_{j=0}^{n-k} \binom{n-j}{k} = \binom{n+1}{k+1} \quad \text{for } k \le n.$$

Hence $I_{\log^{m+1}}(\mathbb{R}^d) \subset \mathfrak{D}(\Phi_b^{m+1})$. By similar calculations to (7.2), we have

$$\int_0^\infty C_{\Phi_b^m(\mu)}(b^{-[t]}z)dt = \int_0^\infty C_\mu(b^{-[f_m^*(t)]}z)dt, \quad \text{for } \mu \in I_{\log^{m+1}}(\mathbb{R}^d),$$

where the use of Fubini's theorem is permitted by the finiteness of (7.2). Thus

$$\Phi_b^{m+1}(\mu) = \widetilde{\Phi}_{b,m+1}(\mu), \quad \text{for } \mu \in I_{\log^{m+1}}(\mathbb{R}^d).$$

To conclude (7.1), it remains to prove that $\mathfrak{D}(\Phi_b^{m+1}) \subset I_{\log^{m+1}}(\mathbb{R}^d)$. If $\mu \in I(\mathbb{R}^d)$ with Lévy measure ν satisfies $\mu \notin I_{\log^{m+1}}(\mathbb{R}^d)$, there exists $n \in \{0, 1, \ldots, m\}$ such that $\mu \in I_{\log^n}(\mathbb{R}^d) \setminus I_{\log^{n+1}}(\mathbb{R}^d)$ (consider $I_{\log^0}(\mathbb{R}^d)$ to be $I(\mathbb{R}^d)$). If $n = 0$, $\mu \notin I_{\log}(\mathbb{R}^d) = \mathfrak{D}(\Phi_b)$ and thus $\mu \notin \mathfrak{D}(\Phi_b^{m+1})$. Suppose $n \ge 1$. Then $\Phi_b^n(\mu)$ is definable and equal to $\widetilde{\Phi}_{b,n}(\mu)$ by the assumption of induction. Denoting the Lévy measure of $\Phi_b^n(\mu)$ by ν_n, we have

$$\begin{aligned}
\int_{|x|>1} \log_b |x| \nu_n(dx) &= \int_0^\infty dt \int_{\mathbb{R}^d} \log_b^+ \left| b^{-[f_{n-1}^*(t)]} x \right| \nu(dx) \\
&= \int_{|x|>1} \nu(dx) \int_0^{f_{n-1}([\log_b |x|]+1)} \left(\log_b |x| - [f_{n-1}^*(t)]\right) dt \\
&\ge \int_{|x|>1} \nu(dx) \int_0^{(\log_b |x|)^n/n!} \left\{\log_b |x| - (n!\, t)^{\frac{1}{n}}\right\} dt \\
&= \int_{|x|>1} \frac{(\log_b |x|)^{n+1}}{(n+1)!} \nu(dx) = \infty.
\end{aligned}$$

Thus, $\Phi_b^n(\mu) \notin I_{\log}(\mathbb{R}^d) = \mathfrak{D}(\Phi_b)$ and hence $\Phi_b^{m+1}(\mu)$ is not definable. Therefore $\mathfrak{D}(\Phi_b^{m+1}) \subset I_{\log^{m+1}}(\mathbb{R}^d)$. □

Recall that the definition of nested subclasses of $L(b, \mathbb{R}^d)$ in Maejima and Naito [5] mentioned in Introduction. The following theorem shows that $L_m(b, \mathbb{R}^d)$ can be realized as $\mathfrak{R}(\Phi_b^{m+1})$.

Theorem 7.3. *Suppose that $b > 1$ and $m \in \mathbb{Z}_+$. Then,*

$$\Phi_b^{m+1}\left(I_{\log^{m+1}}(\mathbb{R}^d)\right) = L_m(b, \mathbb{R}^d).$$

Proof. Let us show the statement by induction. If $m = 0$, the assertion is Corollary 3.4. Assume that the assertion is true for $m-1$ in place of m.

We first show that $\Phi_b^{m+1}\left(I_{\log^{m+1}}(\mathbb{R}^d)\right) \supset L_m(b, \mathbb{R}^d)$. If $\mu \in L_m(b, \mathbb{R}^d)$, there exists $\rho \in L_{m-1}(b, \mathbb{R}^d)$ satisfying $\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)\widehat{\rho}(z)$. The assumption of induction implies that $\rho = \Phi_b^m(\rho_0)$ for some $\rho_0 \in \mathfrak{D}(\Phi_b^m)$. On the other hand, Theorem 3.3 yields that $\rho \in \mathfrak{D}(\Phi_b)$ and $\mu = \Phi_b(\rho)$. Hence $\rho_0 \in \mathfrak{D}(\Phi_b^{m+1})$ and $\mu = \Phi_b^{m+1}(\rho_0)$. Thus $\mu \in \Phi_b^{m+1}\left(I_{\log^{m+1}}(\mathbb{R}^d)\right)$.

To show the converse inclusion of two sets, suppose $\mu \in \Phi_b^{m+1}\left(I_{\log^{m+1}}(\mathbb{R}^d)\right)$. Then, $\mu = \Phi_b^{m+1}(\rho) = \Phi_b\left(\Phi_b^m(\rho)\right)$ for some $\rho \in \mathfrak{D}(\Phi_b^{m+1})$. The assumption of induction implies that $\Phi_b^m(\rho) \in L_{m-1}(b, \mathbb{R}^d)$, and Theorem 3.3 yields that $\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)\widehat{\Phi_b^m(\rho)}(z)$. Thus $\mu \in L_m(b, \mathbb{R}^d)$. □

Let $L_\infty(b, \mathbb{R}^d) = \bigcap_{m=0}^\infty L_m(b, \mathbb{R}^d)$. In Maejima et al. [8], they studied $L_\infty(b, \mathbb{R}^d)$ in the more general setting of operator semi-selfdecomposable distributions and as a special case, they proved that $L_\infty(b, \mathbb{R}^d) = \overline{SS(b, \mathbb{R}^d)}$, where $SS(b, \mathbb{R}^d)$ is the class of all semi-stable distributions with span b and $\overline{SS(b, \mathbb{R}^d)}$ denotes the closure of $SS(b, \mathbb{R}^d)$ taken under convolution and weak convergence. What we want to emphasize here is that we have characterized $L_m(b, \mathbb{R}^d)$ as the range of the mapping Φ_b^{m+1}, and so we can conclude the following. Note that since $L_m(b, \mathbb{R}^d) \supset L_{m+1}(b, \mathbb{R}^d)$, $\lim_{m\to\infty} L_m(b, \mathbb{R}^d) = L_\infty(b, \mathbb{R}^d)$.

Corollary 7.4.

$$\lim_{m\to\infty} \Phi_b^{m+1}\left(I_{\log^{m+1}}(\mathbb{R}^d)\right) = L_\infty(b, \mathbb{R}^d) = \overline{SS(b, \mathbb{R}^d)}.$$

Remark 7.5. In Maejima and Sato [7], they proved that the limits of nested classes of several classes in $I(\mathbb{R}^d)$ are identical with $L_\infty(\mathbb{R}^d)$, which is known to be the same as the closure of the class of all stable distributions on $\mathbb{R}^d$, $\overline{S(\mathbb{R}^d)}$, say. Then a natural question arose. Can we find mappings by which, as the limit of iteration, we get a larger or a smaller class than $\overline{S(\mathbb{R}^d)}$? It is easy to see that $L_\infty(b, \mathbb{R}^d) \supsetneqq L_\infty(\mathbb{R}^d)$ so that $\overline{SS(b, \mathbb{R}^d)} \supsetneqq \overline{S(\mathbb{R}^d)}$. Sato [16] constructed mappings producing a class smaller than $\overline{S(\mathbb{R}^d)}$. Corollary 7.4 shows that a mapping Φ_b produces a larger class than $\overline{S(\mathbb{R}^d)}$ by iteration as a limit.

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Makoto Maejima: Department of Mathematics, Keio University, 3-14-1, Hiyoshi, Kohoku-ku, Yokohama 223-8522, Japan

E-mail address: `maejima@math.keio.ac.jp`

Yohei Ueda: Department of Mathematics, Keio University, 3-14-1, Hiyoshi, Kohoku-ku, Yokohama 223-8522, Japan

E-mail address: `ueda@2008.jukuin.keio.ac.jp`