

NESTED SUBCLASSES OF THE CLASS OF α -SELFDECOMPOSABLE DISTRIBUTIONS

MAKOTO MAEJIMA¹ AND YOHEI UEDA²

Department of Mathematics, Keio University
3-14-1, Hiyoshi, Kohoku-ku
Yokohama 223-8522, Japan

ABSTRACT. A probability distribution μ on \mathbb{R}^d is selfdecomposable if its characteristic function $\widehat{\mu}(z), z \in \mathbb{R}^d$, satisfies that for any $b > 1$, there exists an infinitely divisible distribution ρ_b satisfying $\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)\widehat{\rho}_b(z)$. This concept has been generalized to the concept of α -selfdecomposability by many authors in the following way. Let $\alpha \in \mathbb{R}$. An infinitely divisible distribution μ on \mathbb{R}^d is α -selfdecomposable, if for any $b > 1$, there exists an infinitely divisible distribution ρ_b satisfying $\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)^{b^\alpha}\widehat{\rho}_b(z)$. By denoting the class of all α -selfdecomposable distributions on \mathbb{R}^d by $L^{(\alpha)}(\mathbb{R}^d)$, we define in this paper a sequence of nested subclasses of $L^{(\alpha)}(\mathbb{R}^d)$, and investigate several properties of them by two ways. One is by using limit theorems and the other is by using mappings of infinitely divisible distributions.

1. INTRODUCTION

Let $\mathcal{P}(\mathbb{R}^d)$ and $I(\mathbb{R}^d)$ be the class of all probability distributions on \mathbb{R}^d and the class of all infinitely divisible distributions on \mathbb{R}^d , respectively, and let $I_{\log^m}(\mathbb{R}^d) = \{\mu \in I(\mathbb{R}^d) : \int_{\mathbb{R}^d} (\log^+ |x|)^m \mu(dx) < \infty\}$ for $m \in \mathbb{N}$ and $I_{\log}(\mathbb{R}^d) := I_{\log^1}(\mathbb{R}^d)$, where $|x|$ is the Euclidean norm of $x \in \mathbb{R}^d$ and $\log^+ |x| = (\log |x|) \vee 0$. The terminology of α -selfdecomposability was introduced in Maejima and Ueda (2009a). This is a generalization of selfdecomposability. Here $\mu \in \mathcal{P}(\mathbb{R}^d)$ is said to be selfdecomposable if for each $b > 1$ there exists $\rho_b \in \mathcal{P}(\mathbb{R}^d)$ satisfying $\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)\widehat{\rho}_b(z), z \in \mathbb{R}^d$, where $\widehat{\mu}(z), z \in \mathbb{R}^d$, stands for the characteristic function of $\mu \in \mathcal{P}(\mathbb{R}^d)$. These ρ_b automatically belong to $I(\mathbb{R}^d)$. We denote the totality of selfdecomposable distributions on \mathbb{R}^d by $L(\mathbb{R}^d)$. It is well known that $L(\mathbb{R}^d) \subset I(\mathbb{R}^d)$. Our generalization of selfdecomposability is as follows.

2000 Mathematics Subject Classification. 60E07, 60G51, 60F05.

Key words and phrases. selfdecomposable distribution, α -selfdecomposable distribution, nested subclass, limit theorem, mapping of infinitely divisible distribution.

¹maejima@math.keio.ac.jp

²ueda@2008.jukuin.keio.ac.jp

Definition 1.1 (Maejima and Ueda (2009a)). Let $\alpha \in \mathbb{R}$. We say that $\mu \in I(\mathbb{R}^d)$ is α -selfdecomposable, if for any $b > 1$, there exists $\rho_b \in I(\mathbb{R}^d)$ satisfying

$$(1.1) \quad \widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)^{b^\alpha} \widehat{\rho}_b(z), \quad z \in \mathbb{R}^d.$$

We denote the totality of α -selfdecomposable distributions on \mathbb{R}^d by $L^{(\alpha)}(\mathbb{R}^d)$.

Note that $L^{(0)}(\mathbb{R}^d) = L(\mathbb{R}^d)$. And $L^{(-1)}(\mathbb{R}^d)$ is the class of all s -selfdecomposable distributions on \mathbb{R}^d , which is sometimes written as $U(\mathbb{R}^d)$ and was studied deeply by Jurek, (see, e.g., Jurek (1981, 1985, 2004) or Iksanov et al. (2004)). Also, the classes $L^{(\alpha)}(\mathbb{R}^d)$, $\alpha \in \mathbb{R}$, and similar ones were already studied by several authors. Jurek (1988, 1989, 1992), and Jurek and Schreiber (1992) studied the classes $\mathcal{U}_\beta(Q)$, $\beta \in \mathbb{R}$, of distributions on a real separable Banach space E , where Q is a linear operator on E with certain properties. These classes are equal to $L^{(\alpha)}(\mathbb{R}^d)$ if $\beta = -\alpha$, $E = \mathbb{R}^d$ and Q is the identity operator. As to these classes, they studied the decomposability and stochastic integral characterizations, although some results are only for the case that Q is the identity operator. However, since, for $0 < \alpha < 2$, $L^{(\alpha)}(\mathbb{R}^d)$ contains all α -stable distributions and any $\mu \in L^{(\alpha)}(\mathbb{R}^d)$ belongs to the normal domain of attraction of some α -stable distribution, we adopt the parametrization in Definition 1.1. For details on this history, see Maejima and Ueda (2009a).

$L(\mathbb{R}^d)$ is characterized by, for example, radial components of Lévy measures, a stochastic integral representation, and the relation to Ornstein-Uhlenbeck type processes, (see, e.g., Rocha-Arteaga and Sato (2003)). By Maejima and Ueda (2009a) and others, these characterizations of $L(\mathbb{R}^d)$ were generalized to $L^{(\alpha)}(\mathbb{R}^d)$.

As to nested subclasses of $L(\mathbb{R}^d)$, the following are known, (see, e.g., Rocha-Arteaga and Sato (2003)). Define nested subclasses $L_m(\mathbb{R}^d)$, $m \in \mathbb{Z}_+$ of $L(\mathbb{R}^d)$ in the following way: $\mu \in L_m(\mathbb{R}^d)$ if and only if for each $b > 1$, there exists $\rho_b \in L_{m-1}(\mathbb{R}^d)$ such that $\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)^b \widehat{\rho}_b(z)$, where $L_0(\mathbb{R}^d) := L(\mathbb{R}^d)$. Since, by definition, $L_m(\mathbb{R}^d) \supset L_{m+1}(\mathbb{R}^d)$, these are called nested subclasses. Besides, we introduce an operation $\mathfrak{Q}(\cdot)$ in the following way: Let $H \subset \mathcal{P}(\mathbb{R}^d)$. We say that $\mu \in \mathcal{P}(\mathbb{R}^d)$ belongs to $\mathfrak{Q}(H)$ if there exist sequences $\{X_n\}$ of \mathbb{R}^d -valued independent random variables, $\{a_n\} \subset (0, \infty)$, and $\{c_n\} \subset \mathbb{R}^d$ such that $\{\mathcal{L}(X_n), n \in \mathbb{N}\} \subset H$, $\{a_n^{-1}X_j, 1 \leq j \leq n; n \in \mathbb{N}\}$ is infinitesimal, and

$$\mathcal{L} \left(a_n^{-1} \sum_{j=1}^n X_j + c_n \right) \rightarrow \mu \quad \text{as } n \rightarrow \infty,$$

where $\mathcal{L}(X)$ means the law of a random variable X . Then it is known that $L_0(\mathbb{R}^d) = \mathfrak{Q}(\mathcal{P}(\mathbb{R}^d)) = \mathfrak{Q}(I(\mathbb{R}^d))$ and $L_m(\mathbb{R}^d) = \mathfrak{Q}(L_{m-1}(\mathbb{R}^d))$ for $m \in \mathbb{N}$ so that $L_m(\mathbb{R}^d) = \mathfrak{Q}^{m+1}(\mathcal{P}(\mathbb{R}^d)) = \mathfrak{Q}^{m+1}(I(\mathbb{R}^d))$ for $m \in \mathbb{Z}_+$, where $\mathfrak{Q}^{m+1}(\cdot)$ denotes the $m+1$ times iteration of the $\mathfrak{Q}(\cdot)$ -operation. On the other hand, if we define a mapping Φ by

$$\Phi(\mu) = \mathcal{L} \left(\int_0^\infty e^{-t} dX_t^{(\mu)} \right), \quad \mu \in I_{\log}(\mathbb{R}^d),$$

where $\{X_t^{(\mu)}, t \geq 0\}$ is a Lévy process on \mathbb{R}^d with $\mu \in I(\mathbb{R}^d)$ as its distribution at time 1, then it is known that for $m \in \mathbb{Z}_+$, $L_m(\mathbb{R}^d)$ is realized as the range of the $m+1$ times composition of Φ , namely, $\mathfrak{R}(\Phi^{m+1}) = L_m(\mathbb{R}^d)$, where the domain of Φ^{m+1} is $I_{\log^{m+1}}(\mathbb{R}^d)$. Furthermore, the limit $L_\infty(\mathbb{R}^d) := \lim_{m \rightarrow \infty} L_m(\mathbb{R}^d) = \bigcap_{m=0}^\infty L_m(\mathbb{R}^d)$ is known to be equal to $\overline{S(\mathbb{R}^d)}$, which is the closure under convolution and weak convergence, of the class of all stable distributions. Namely,

$$\lim_{m \rightarrow \infty} \mathfrak{Q}^{m+1}(\mathcal{P}(\mathbb{R}^d)) = \lim_{m \rightarrow \infty} \mathfrak{Q}^{m+1}(I(\mathbb{R}^d)) = \lim_{m \rightarrow \infty} \mathfrak{R}(\Phi^{m+1}) = L_\infty(\mathbb{R}^d) = \overline{S(\mathbb{R}^d)}.$$

The following was already done as to nested subclasses of $L^{(\alpha)}(\mathbb{R}^d)$, $\alpha \in \mathbb{R}$. Jurek (2004) studied nested subclasses of $L^{(-1)}(\mathbb{R}^d)$, Maejima and Sato (2009) found the limit of the nested subclasses of $L^{(\alpha)}(\mathbb{R}^d)$, $-1 \leq \alpha < 0$, defined by mappings, and Maejima et al. (2010) investigated nested subclasses of $L^{(\alpha)}(\mathbb{R}^d)$, $\alpha < 2$, in terms of mappings. However, the study on nested subclasses of $L^{(\alpha)}(\mathbb{R}^d)$, $\alpha \in \mathbb{R}$, in terms of limit theorems and mappings is not completed yet and the purpose of this paper is to do it.

Maejima and Sato (2009) proved that the limits of several nested classes defined by stochastic integral mappings are identical with $\overline{S(\mathbb{R}^d)}$. Then a natural question arose. Can we find mappings by which, as the limit of iteration, we get a larger or a smaller class than $\overline{S(\mathbb{R}^d)}$? Sato (2007–2009) constructed mappings producing a class smaller than $\overline{S(\mathbb{R}^d)}$ and Maejima and Ueda (2009c) found mappings which produce a larger class than $\overline{S(\mathbb{R}^d)}$. In Theorems 4.6, we will see that stochastic integral mappings associated with classes $L^{(\alpha)}(\mathbb{R}^d)$, $\alpha \in (0, 2)$, make smaller classes than $\overline{S(\mathbb{R}^d)}$ as the limits of the ranges of their iteration, which is the same iterated limit as that of Sato's mappings above. Also, in Corollary 4.2, we see a result about nested classes of $L^{(\alpha)}(\mathbb{R}^d)$ based on $H \subset I(\mathbb{R}^d)$ with certain properties instead of $I(\mathbb{R}^d)$, which enable us to find the iterated limit of some other stochastic integral mappings, (see Remark 4.3 and Maejima and Ueda (2009b)).

Organization of this paper is as follows. In Section 2, we explain necessary notation and give some preliminaries. In Section 3, we study nested subclasses of

$L^{(\alpha)}(\mathbb{R}^d)$ in terms of a limit theorem. In Section 4, we investigate nested subclasses of $L^{(\alpha)}(\mathbb{R}^d)$ in terms of a mapping of infinitely divisible distributions, by using the results in Section 3. In Section 5, a supplementary remark is mentioned.

2. NOTATION AND PRELIMINARIES

In this section, we explain necessary notation and give some preliminaries.

Throughout this paper, we use the Lévy-Khintchine representation of the characteristic function of $\mu \in I(\mathbb{R}^d)$ in the following form:

$$\widehat{\mu}(z) = \exp \left\{ -\frac{1}{2} \langle z, Az \rangle + i \langle \gamma, z \rangle + \int_{\mathbb{R}^d} \left(e^{i \langle z, x \rangle} - 1 - \frac{i \langle z, x \rangle}{1 + |x|^2} \right) \nu(dx) \right\}, \quad z \in \mathbb{R}^d,$$

where $\langle \cdot, \cdot \rangle$ is Euclidean inner product on \mathbb{R}^d respectively, A is a nonnegative-definite symmetric $d \times d$ matrix, $\gamma \in \mathbb{R}^d$, and ν is a measure satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d} (|x|^2 \wedge 1) \nu(dx) < \infty$. ν is called the Lévy measure of $\mu \in I(\mathbb{R}^d)$. We also call (A, ν, γ) the Lévy-Khintchine triplet of μ and we write $\mu = \mu_{(A, \nu, \gamma)}$ when we want to emphasize its Lévy-Khintchine triplet. $C_\mu(z), z \in \mathbb{R}^d$, denotes the cumulant function of $\mu \in I(\mathbb{R}^d)$, that is, $C_\mu(z)$ is the unique continuous function satisfying $\widehat{\mu}(z) = e^{C_\mu(z)}$ and $C_\mu(0) = 0$. For $\mu \in I(\mathbb{R}^d)$ and $t > 0$, we call the distribution with characteristic function $\widehat{\mu}(z)^t := e^{tC_\mu(z)}$ the t -th convolution of μ and denote it by μ^t .

A set $H \subset \mathcal{P}(\mathbb{R}^d)$ is said to be closed under type equivalence if $\mathcal{L}(X) \in H$ implies $\mathcal{L}(aX + c) \in H$ for $a > 0$, and $c \in \mathbb{R}^d$. $H \subset I(\mathbb{R}^d)$ is called completely closed in the strong sense (abbreviated as c.c.s.s.) if H is closed under convolution, weak convergence, type equivalence, and t -th convolution for any $t > 0$. Note that $I(\mathbb{R}^d)$ and $L(\mathbb{R}^d)$ are c.c.s.s., but $S(\mathbb{R}^d)$ is not.

$\mathcal{B}_0(\mathbb{R}^d)$ denotes the totality of $B \in \mathcal{B}(\mathbb{R}^d)$ satisfying $\inf_{x \in B} |x| > 0$. Let $S = \{x \in \mathbb{R}^d : |x| = 1\}$ and we write, for $E \in \mathcal{B}((0, \infty))$ and $C \in \mathcal{B}(S)$, $EC := \{x \in \mathbb{R}^d \setminus \{0\} : |x| \in E \text{ and } x/|x| \in C\}$.

We also use stochastic integrals with respect to Lévy processes. Stochastic integrals with respect to Lévy processes $\{X_t, t \geq 0\}$ of nonrandom measurable functions $f: [0, \infty) \rightarrow \mathbb{R}$, which are $\int_0^t f(s) dX_s, t \in [0, \infty)$, are deeply studied in Sato (2004, 2006a), and his way of defining a stochastic integral with respect to a Lévy process is to define a stochastic integral based on the \mathbb{R}^d -valued independently scattered random measure induced by a Lévy process on \mathbb{R}^d . The improper stochastic integral $\int_0^\infty f(s) dX_s$ is defined as the limit in probability of $\int_0^t f(s) dX_s$ as $t \rightarrow \infty$ whenever the limit exists.

Using stochastic integrals with respect to Lévy processes, we can define a mapping

$$(2.1) \quad \Phi_f(\mu) = \mathcal{L} \left(\int_0^\infty f(t) dX_t^{(\mu)} \right), \quad \mu \in \mathfrak{D}(\Phi_f) \subset I(\mathbb{R}^d),$$

for a nonrandom measurable function $f: [0, \infty) \rightarrow \mathbb{R}$, where $\mathfrak{D}(\Phi_f)$ is the domain of a mapping Φ_f that is the class of $\mu \in I(\mathbb{R}^d)$ for which $\int_0^\infty f(t) dX_t^{(\mu)}$ is definable in the sense above. When we consider the composition of two mappings Φ_f and Φ_g , denoted by $\Phi_g \circ \Phi_f$, the domain of $\Phi_g \circ \Phi_f$ is $\mathfrak{D}(\Phi_g \circ \Phi_f) = \{\mu \in I(\mathbb{R}^d): \mu \in \mathfrak{D}(\Phi_f) \text{ and } \Phi_f(\mu) \in \mathfrak{D}(\Phi_g)\}$. Also, for a mapping Φ_f and $m \in \mathbb{N}$, we denote by Φ_f^m the m times composition of Φ_f itself. Once we define such a mapping, we can characterize a subclass of $I(\mathbb{R}^d)$ as the range of Φ_f , $\mathfrak{R}(\Phi_f) := \Phi_f(\mathfrak{D}(\Phi_f))$. See also Sato (2006b).

3. NESTED SUBCLASSES OF THE CLASS OF α -SELFDECOMPOSABLE DISTRIBUTIONS DEFINED BY LIMIT THEOREMS AND THEIR CHARACTERIZATIONS IN TERMS OF LÉVY MEASURES

We start this section with the following definition, which defines a subclass of $I(\mathbb{R}^d)$ through a limit theorem.

Definition 3.1. Let $\alpha \in \mathbb{R}$ and $H \subset I(\mathbb{R}^d)$. $\mu \in \mathcal{P}(\mathbb{R}^d)$ is said to belong to the class $\mathfrak{Q}_\alpha(H)$ if there exist a sequence $\{\mu_j, j \in \mathbb{N}\} \subset I(\mathbb{R}^d)$ satisfying $\{\mu_j, j \geq j_0\} \subset H$ for some $j_0 \in \mathbb{N}$, $a_n > 0$, $\uparrow \infty$ satisfying $a_{n+1}/a_n \rightarrow 1$, $c_n \in \mathbb{R}^d$, and $p_n > 0$ satisfying $p_n/a_n^\alpha \rightarrow 1$ such that

$$(3.1) \quad \lim_{n \rightarrow \infty} \prod_{j=1}^n \widehat{\mu}_j(a_n^{-1}z)^{p_n} e^{i\langle c_n, z \rangle} = \widehat{\mu}(z), \quad \text{for } z \in \mathbb{R}^d.$$

Remark 3.2. In Definition 3.1, we assume H to be a subclass of $I(\mathbb{R}^d)$ because we need the t -th convolution of its elements for $t > 0$. Due to this assumption, we do not need the infinitesimal condition, as Jurek (2004) remarked. Then, Definition 3.1 is similar to the limit theorem characterizing the class of selfdecomposable distributions $L(\mathbb{R}^d)$.

The following is immediately obtained by definition.

Lemma 3.3. *Let $\alpha \in \mathbb{R}$. If $H_1 \subset H_2 \subset I(\mathbb{R}^d)$, then $\mathfrak{Q}_\alpha(H_1) \subset \mathfrak{Q}_\alpha(H_2)$.*

We can characterize the classes $\mathfrak{Q}_\alpha(H)$ by the decomposability and $L^{(\alpha)}(\mathbb{R}^d)$ by the $\mathfrak{Q}_\alpha(\cdot)$ -operation as follows.

Theorem 3.4. Let $\alpha \in \mathbb{R}$ and let $H \subset I(\mathbb{R}^d)$ be c.c.s.s.

(i) $\mu \in \mathfrak{Q}_\alpha(H)$ if and only if $\mu \in I(\mathbb{R}^d)$ and for each $b > 1$ there exists $\rho_b \in H$ satisfying (1.1).

(ii) $\mathfrak{Q}_\alpha(I(\mathbb{R}^d)) = L^{(\alpha)}(\mathbb{R}^d)$.

Proof. (i) We first show the “if” part. Let $\mu \in I(\mathbb{R}^d)$ and for each $b > 1$ there exists $\rho_b \in H$ satisfying (1.1). Then, it suffices to set $\mu_1 := \mu \in I(\mathbb{R}^d)$, $\widehat{\mu}_j(z) := \widehat{\rho}_{j/(j-1)}(jz)^{j^{-\alpha}}$ for $j \geq 2$, $a_n := n$, $p_n := n^\alpha$, and $c_n := 0$. Indeed, $\{\mu_j, j \geq 2\} \subset H$ since H is c.c.s.s., and for all $n \geq 2$,

$$\begin{aligned} \prod_{j=1}^n \widehat{\mu}_j(a_n^{-1}z)^{p_n} e^{i\langle c_n, z \rangle} &= \widehat{\mu} \left(\frac{1}{n}z \right)^{n^\alpha} \prod_{j=2}^n \widehat{\rho}_{j/(j-1)} \left(\frac{j}{n}z \right)^{\left(\frac{n}{j}\right)^\alpha} \\ &= \widehat{\mu} \left(\frac{1}{n}z \right)^{n^\alpha} \prod_{j=2}^n \frac{\widehat{\mu} \left(\frac{j}{n}z \right)^{\left(\frac{n}{j}\right)^\alpha}}{\widehat{\mu} \left(\frac{j-1}{n}z \right)^{\left(\frac{n}{j-1}\right)^\alpha}} = \widehat{\mu}(z), \end{aligned}$$

implying (3.1).

We next show the “only if” part. For any $b > 1$, we can take $n_l, m_l \in \mathbb{N}$ diverging to ∞ such that $m_l < n_l$ and $a_{n_l} a_{m_l}^{-1} \rightarrow b$ as $l \rightarrow \infty$. This is possible, due to the argument in the proof of Theorem 15.3 (i) of Sato (1999). Then,

$$\begin{aligned} \prod_{j=1}^{n_l} \widehat{\mu}_j(a_{n_l}^{-1}z)^{p_{n_l}} e^{i\langle c_{n_l}, z \rangle} &= \left\{ \prod_{j=1}^{m_l} \widehat{\mu}_j(a_{m_l}^{-1}(a_{m_l} a_{n_l}^{-1}z))^{p_{m_l}} e^{i\langle c_{m_l}, a_{m_l} a_{n_l}^{-1}z \rangle} \right\}^{p_{n_l} p_{m_l}^{-1}} \\ &\quad \times \prod_{j=m_l+1}^{n_l} \widehat{\mu}_j(a_{n_l}^{-1}z)^{p_{n_l}} e^{i\langle c_{n_l} - c_{m_l} a_{m_l} a_{n_l}^{-1} p_{n_l} p_{m_l}^{-1}, z \rangle}, \end{aligned}$$

where the left-hand side and the first term of right-hand side tend to $\widehat{\mu}(z)$ and $\widehat{\mu}(b^{-1}z)^{b^\alpha}$ as $l \rightarrow \infty$, respectively, by virtue of the uniform convergence of the characteristic functions. Since $\widehat{\mu}(z)$ is the limit of the sequence of infinitely divisible distributions, μ is also infinitely divisible and thus $\widehat{\mu}(b^{-1}z)^{b^\alpha} \neq 0$ for all $z \in \mathbb{R}^d$. The second term of the right-hand side converges to $\widehat{\mu}(z)/\widehat{\mu}(b^{-1}z)^{b^\alpha}$ which is continuous at $z = 0$ and therefore the characteristic function of some probability measure ρ_b . Then, (1.1) holds. Furthermore, since $\{\mu_j, j \geq j_0\} \subset H$ and H is c.c.s.s., we have $\rho_b \in H$.

(ii) This is an immediate consequence of the part (i) that we have just shown and the definition of $L^{(\alpha)}(\mathbb{R}^d)$. \square

The following holds from Theorem 3.4.

Corollary 3.5. *Let $H(\neq \emptyset) \subset I(\mathbb{R}^d)$ be c.c.s.s. Then, $\mathfrak{Q}_2(H)$ is the class of all Gaussian distributions on \mathbb{R}^d , and for $\alpha > 2$, $\mathfrak{Q}_\alpha(H)$ is the class of all δ -distributions on \mathbb{R}^d .*

Proof. We first prove that $\mathfrak{Q}_\alpha(H)$ includes the class of all Gaussian distributions if $\alpha = 2$, and all δ -distributions if $\alpha > 2$. Indeed, if $H(\neq \emptyset) \subset I(\mathbb{R}^d)$ is c.c.s.s., then $\mu \in H$ exists and for all $\gamma \in \mathbb{R}^d$, $\delta_\gamma = \lim_{n \rightarrow \infty} \mu^{1/n} * \delta_\gamma \in H$ since H is c.c.s.s. If μ is Gaussian, then for each $b > 1$, there is $c_b \in \mathbb{R}^d$ satisfying $\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)^{b^2} e^{i\langle c_b, z \rangle}$. Also, if $\alpha > 2$ and μ is a δ -distribution, then for each $b > 1$, there is $c_b \in \mathbb{R}^d$ satisfying $\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)^{b^\alpha} e^{i\langle c_b, z \rangle}$. Noting that $\delta_{c_b} \in H$, we have the assertion.

We next show that $\mathfrak{Q}_\alpha(H)$ is included in the class of all Gaussian distributions if $\alpha = 2$, and all δ -distributions if $\alpha > 2$. By Lemma 3.3 and Theorem 3.4 (ii), we have $\mathfrak{Q}_\alpha(H) \subset L^{(\alpha)}(\mathbb{R}^d)$. Note that $L^{(\alpha)}(\mathbb{R}^d)$ is equal to the class of all Gaussian distributions if $\alpha = 2$, and all δ -distributions if $\alpha > 2$, (see Maejima and Ueda (2009a)). \square

For $0 < \beta \leq 2$, $S_\beta(\mathbb{R}^d)$ stands for the totality of β -stable distributions on \mathbb{R}^d . Let $S(\mathbb{R}^d) := \bigcup_{\beta \in (0,2]} S_\beta(\mathbb{R}^d)$.

Corollary 3.6. *Let $0 < \alpha \leq 2$. Then, $\mathfrak{Q}_\alpha(\{\delta_\gamma : \gamma \in \mathbb{R}^d\}) = S_\alpha(\mathbb{R}^d)$.*

Proof. Note that $\mu \in S_\alpha(\mathbb{R}^d)$ if and only if for each $b > 1$ there exists $c_b \in \mathbb{R}^d$ satisfying $\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)^{b^\alpha} e^{i\langle c_b, z \rangle}$. Then, Theorem 3.4 (i) implies the statement. \square

For $\alpha < 2$, let

$$\mathcal{C}_\alpha(\mathbb{R}^d) := \left\{ \mu = \mu_{(A,\nu,\gamma)} \in I(\mathbb{R}^d) : \lim_{r \rightarrow \infty} r^\alpha \int_{|x|>r} \nu(dx) = 0 \right\}.$$

Note that $\mathcal{C}_\alpha(\mathbb{R}^d) = I(\mathbb{R}^d)$ if $\alpha \leq 0$. If $\alpha < 2$, $\mu \in \mathcal{C}_\alpha(\mathbb{R}^d)$ and H is c.c.s.s., then, $\mu \in \mathfrak{Q}_\alpha(H)$ can be characterized by a limit theorem slightly different from Definition 3.1 as follows.

Theorem 3.7. *Let $\alpha < 2$ and let $H \subset I(\mathbb{R}^d)$ be c.c.s.s. Assume $\mu \in \mathcal{C}_\alpha(\mathbb{R}^d)$. Then, $\mu \in \mathfrak{Q}_\alpha(H)$ if and only if there exist a sequence $\{\mu_j, j \in \mathbb{N}\} \subset H$, $a_n > 0, \uparrow \infty$ satisfying $a_{n+1}/a_n \rightarrow 1$, $c_n \in \mathbb{R}^d$, and $p_n > 0$ satisfying $p_n/a_n^\alpha \rightarrow 1$ such that*

$$\lim_{n \rightarrow \infty} \prod_{j=1}^n \widehat{\mu}_j(a_n^{-1}z)^{p_n} e^{i\langle c_n, z \rangle} = \widehat{\mu}(z), \quad \text{for } z \in \mathbb{R}^d.$$

Proof. The “if” part is trivial by Definition 3.1.

Let us prove the “only if” part. If $\mu = \mu_{(A,\nu,\gamma)} \in \mathfrak{Q}_\alpha(H)$, then for each $b > 1$, there exists $\rho_b \in H$ satisfying (1.1) by virtue of Theorem 3.4 (i). Then, it suffices to set $\widehat{\mu}_j(z) := \widehat{\rho}_{(j+1)/j}((j+1)z)^{(j+1)^{-\alpha}}$, $a_n := n$, $p_n := n^\alpha$, $c_n := 0$ if $\alpha \leq 0$, and $c_n := n^{\alpha-1}\gamma + n^\alpha \int_{\mathbb{R}^d} x \{(1+|x|^2)^{-1} - (1+|nx|^2)^{-1}\} \nu(n dx)$ if $0 < \alpha < 2$. Indeed, $\{\mu_j, j \in \mathbb{N}\} \subset H$ since H is c.c.s.s. and

$$\prod_{j=1}^n \widehat{\mu}_j(a_n^{-1}z)^{p_n} = \prod_{j=1}^n \widehat{\rho}_{(j+1)/j} \left(\frac{j+1}{n} z \right)^{\left(\frac{n}{j+1}\right)^\alpha} = \prod_{j=1}^n \frac{\widehat{\mu} \left(\frac{j+1}{n} z \right)^{\left(\frac{n}{j+1}\right)^\alpha}}{\widehat{\mu} \left(\frac{j}{n} z \right)^{\left(\frac{n}{j}\right)^\alpha}} = \frac{\widehat{\mu} \left(\frac{n+1}{n} z \right)^{\left(\frac{n}{n+1}\right)^\alpha}}{\widehat{\mu} \left(\frac{1}{n} z \right)^{n^\alpha}}$$

which tends to $\widehat{\mu}(z)$ as $n \rightarrow \infty$, if $\alpha \leq 0$. If $0 < \alpha < 2$, we have

$$n^\alpha C_\mu(n^{-1}z) - i\langle c_n, z \rangle = -\frac{1}{2}n^{\alpha-2}\langle z, Az \rangle + n^\alpha \int_{\mathbb{R}^d} \left(e^{i\langle z, x \rangle} - 1 - \frac{i\langle z, x \rangle}{1+|x|^2} \right) \nu(n dx).$$

For any bounded continuous function $f: \mathbb{R}^d \rightarrow \mathbb{R}$ vanishing on a neighborhood of 0, it follows that

$$\lim_{n \rightarrow \infty} n^\alpha \int_{\mathbb{R}^d} f(x) \nu(n dx) = 0,$$

since $\mu \in \mathcal{C}_\alpha(\mathbb{R}^d)$. Recalling that $\nu(B) \geq n^\alpha \nu(nB)$ for $B \in \mathcal{B}(\mathbb{R}^d)$ from (1.1), we have

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \overline{\lim}_{n \rightarrow \infty} \left| n^{\alpha-2} \langle z, Az \rangle + n^\alpha \int_{|x| \leq \varepsilon} \langle z, x \rangle^2 \nu(n dx) \right| \\ \leq \lim_{n \rightarrow \infty} n^{\alpha-2} |\langle z, Az \rangle| + \lim_{\varepsilon \downarrow 0} \int_{|x| \leq \varepsilon} \langle z, x \rangle^2 \nu(dx) = 0. \end{aligned}$$

Then, it follows from Theorem 8.7 of Sato (1999) that $\lim_{n \rightarrow \infty} \widehat{\mu}(n^{-1}z)^{n^\alpha} e^{-i\langle c_n, z \rangle} = 1$.

Thus

$$\prod_{j=1}^n \widehat{\mu}_j(a_n^{-1}z)^{p_n} e^{i\langle c_n, z \rangle} = \frac{\widehat{\mu} \left(\frac{n+1}{n} z \right)^{\left(\frac{n}{n+1}\right)^\alpha}}{\widehat{\mu} \left(\frac{1}{n} z \right)^{n^\alpha} e^{-i\langle c_n, z \rangle}} \rightarrow \widehat{\mu}(z),$$

as $n \rightarrow \infty$. □

Corollaries 3.5, 3.6 and Theorem 3.7 yield the following.

Corollary 3.8. (i) Let $\alpha \in (-\infty, 0] \cup (2, \infty)$. Then, for all c.c.s.s. $H \subset I(\mathbb{R}^d)$,

$$\mathfrak{Q}_\alpha(H) \subset H.$$

(ii) Let $\alpha \in (0, 2]$. Then, there exists a c.c.s.s. $H \subset I(\mathbb{R}^d)$ satisfying $\mathfrak{Q}_\alpha(H) \not\subset H$.

(iii) Let $\alpha \in (0, 2)$. Then, for all c.c.s.s. $H \subset I(\mathbb{R}^d)$, $\mathfrak{Q}_\alpha(H) \cap \mathcal{C}_\alpha(\mathbb{R}^d) \subset H$.

Proof. (i) If $\alpha \leq 0$, and $H \subset I(\mathbb{R}^d)$ is c.c.s.s., then Theorem 3.7 implies $\mathfrak{Q}_\alpha(H) \subset H$.

Let $\alpha > 2$ and let $H \subset I(\mathbb{R}^d)$ be c.c.s.s. If $H = \emptyset$, then $\mathfrak{Q}_\alpha(\emptyset) = \emptyset$. Assume $H \neq \emptyset$.

Then Corollary 3.5 implies $\mathfrak{Q}_\alpha(H) = \{\delta_\gamma: \gamma \in \mathbb{R}^d\} \subset H$.

(ii) See Corollary 3.6.

(iii) See Theorem 3.7. □

We are ready to define nested subclasses of $L^{(\alpha)}(\mathbb{R}^d)$ by using the $\mathfrak{Q}_\alpha(\cdot)$ -operation. Let $H \subset I(\mathbb{R}^d)$ and $\alpha \in \mathbb{R}$. For $m = 0, 1, 2, \dots, \infty$, we denote the m times iteration of $\mathfrak{Q}_\alpha(\cdot)$ by $\mathfrak{Q}_\alpha^m(\cdot)$, namely,

$$\mathfrak{Q}_\alpha^m(H) = \underbrace{\mathfrak{Q}_\alpha(\mathfrak{Q}_\alpha(\cdots(\mathfrak{Q}_\alpha(H))\cdots))}_m,$$

where $\mathfrak{Q}_\alpha^0(H) = H$, and $\mathfrak{Q}_\alpha^\infty(H) = \bigcap_{m=1}^\infty \mathfrak{Q}_\alpha^m(H)$. By Corollary 3.8 (ii), it is not always true that $\mathfrak{Q}_\alpha^1(H) \subset \mathfrak{Q}_\alpha^0(H)(= H)$. However, it will be seen in Proposition 3.10 (iii) that if $H \subset I(\mathbb{R}^d)$ is c.c.s.s., then $\mathfrak{Q}_\alpha^{m+1}(H) \subset \mathfrak{Q}_\alpha^m(H), m \in \mathbb{N}$, so that $\lim_{m \rightarrow \infty} \mathfrak{Q}_\alpha^m(H) = \bigcap_{m=1}^\infty \mathfrak{Q}_\alpha^m(H)$, if we regard $\mathfrak{Q}_\alpha^m(H)$ as a sequence with $m \in \mathbb{N}$.

For $0 < \alpha < 2$, let

$$I_\alpha(\mathbb{R}^d) := \left\{ \mu \in I(\mathbb{R}^d) : \int_{\mathbb{R}^d} |x|^\alpha \mu(dx) < \infty \right\}.$$

We first prepare the following lemma.

Lemma 3.9. *Let $0 < \alpha < 2$. Suppose $\mu \in L^{(\alpha)}(\mathbb{R}^d)$. Then, for all $b > 1$, ρ_b in Definition 1.1 satisfies $\rho_b \in I_\alpha(\mathbb{R}^d)$.*

Proof. Let $b > 1$. Denoting the Lévy measures of μ and ρ_b by ν and ν_b , respectively, we have that $\nu_b(B) = \nu(B) - b^\alpha \nu(bB)$ for $B \in \mathcal{B}_0(\mathbb{R}^d)$ by (1.1). Then, it follows that

$$\begin{aligned} \int_{|x|>1} |x|^\alpha \nu_b(dx) &= \sum_{k=0}^{\infty} \int_{|x| \in (b^k, b^{k+1}]} |x|^\alpha \nu_b(dx) \leq \sum_{k=0}^{\infty} b^{\alpha(k+1)} \nu_b((b^k, b^{k+1}]S) \\ &= \sum_{k=0}^{\infty} b^{\alpha(k+1)} \{ \nu((b^k, b^{k+1}]S) - b^\alpha \nu((b^{k+1}, b^{k+2}]S) \} \\ &= \sum_{k=0}^{\infty} \{ b^{\alpha(k+1)} \nu((b^k, b^{k+1}]S) - b^{\alpha(k+2)} \nu((b^{k+1}, b^{k+2}]S) \} \\ &= \lim_{n \rightarrow \infty} \{ b^\alpha \nu((1, b]S) - b^{\alpha(n+2)} \nu((b^{n+1}, b^{n+2}]S) \} \\ &\leq b^\alpha \nu((1, b]S) < \infty. \end{aligned}$$

This implies $\mu \in I_\alpha(\mathbb{R}^d)$, due to Corollary 25.8 of Sato (1999). □

We now prove several properties of $\mathfrak{Q}_\alpha^m(H)$.

Proposition 3.10. *Let $H \subset I(\mathbb{R}^d)$ be c.c.s.s. Then, we have the following.*

(i) For $\alpha \in \mathbb{R}$ and $m \in \{0, 1, 2, \dots, \infty\}$, $\mathfrak{Q}_\alpha^m(H)$ is also c.c.s.s.

(ii) For $\alpha \in \mathbb{R}$ and $m \in \mathbb{Z}_+$, $\mu \in \mathfrak{Q}_\alpha^{m+1}(H)$ if and only if $\mu \in I(\mathbb{R}^d)$ and for each $b > 1$ there exists $\rho_b \in \mathfrak{Q}_\alpha^m(H)$ satisfying (1.1).

(iii) Let $\alpha \in \mathbb{R}$. Then, $\mathfrak{Q}_\alpha^m(H)$ is decreasing in $m \in \mathbb{N}$ with respect to set inclusion, namely,

$$(3.2) \quad \mathfrak{Q}_\alpha^m(H) \supset \mathfrak{Q}_\alpha^{m+1}(H) \quad \text{for } m \in \mathbb{N}.$$

(iv) Let $\alpha \in \mathbb{R}$. Then $\mathfrak{Q}_\alpha^\infty(H)$ is invariant under the $\mathfrak{Q}_\alpha^\infty(\cdot)$ -operation, that is,

$$\mathfrak{Q}_\alpha(\mathfrak{Q}_\alpha^\infty(H)) = \mathfrak{Q}_\alpha^\infty(H),$$

which is equivalent to that $\mu \in \mathfrak{Q}_\alpha^\infty(H)$ if and only if $\mu \in I(\mathbb{R}^d)$ and for each $b > 1$ there exists $\rho_b \in \mathfrak{Q}_\alpha^\infty(H)$ satisfying (1.1).

(v) Let $m \in \{0, 1, 2, \dots, \infty\}$. If $\mathfrak{Q}_\alpha(H) \subset H$ for all $\alpha \in (0, 2]$, then $\mathfrak{Q}_\alpha^m(H)$ is decreasing in $\alpha \in \mathbb{R}$ with respect to set inclusion, namely,

$$(3.3) \quad \mathfrak{Q}_{\alpha_1}^m(H) \supset \mathfrak{Q}_{\alpha_2}^m(H) \quad \text{for } \alpha_1 < \alpha_2.$$

Proof. (i) Let us show the statement for $m \in \mathbb{Z}_+$ by induction. The case for $m = 0$ is obvious. Assume that $\mathfrak{Q}_\alpha^{m-1}(H)$ is c.c.s.s. Then, Theorem 3.4 (i) yields that $\mu \in \mathfrak{Q}_\alpha^m(H)$ if and only if $\mu \in I(\mathbb{R}^d)$ and for each $b > 1$ there exists $\rho_b \in \mathfrak{Q}_\alpha^{m-1}(H)$ satisfying (1.1). By using this decomposability, it is easy to see that $\mathfrak{Q}_\alpha^m(H)$ is c.c.s.s. Thus $\mathfrak{Q}_\alpha^m(H)$ is c.c.s.s. for all $m \in \mathbb{Z}_+$. Recalling that the intersection of c.c.s.s. classes is again c.c.s.s., we have the assertion for $m = \infty$.

(ii) Noting (i), we can apply Theorem 3.4 to the class $\mathfrak{Q}_\alpha^m(H)$ in place of H .

(iii) We first show the case for $\alpha \in (-\infty, 0] \cup (2, \infty)$. It follows from Corollary 3.8 that $\mathfrak{Q}_\alpha(H) \subset H$. Then Lemma 3.3 yields (3.2). We next show the case for $\alpha \in (0, 2)$. Suppose that $m \in \mathbb{N}$ and $\mu \in \mathfrak{Q}_\alpha^{m+1}(H)$. Then it follows from (ii) that for each $b > 1$ there exists $\rho_b \in \mathfrak{Q}_\alpha^m(H)$ satisfying (1.1). Then $\mu \in L^{(\alpha)}(\mathbb{R}^d)$ and hence $\rho_b \in I_\alpha(\mathbb{R}^d) \subset \mathcal{C}_\alpha(\mathbb{R}^d)$ by Lemma 3.9. Therefore $\rho_b \in \mathfrak{Q}_\alpha^m(H) \cap \mathcal{C}_\alpha(\mathbb{R}^d) \subset \mathfrak{Q}_\alpha^{m-1}(H)$ by Corollary 3.8. Then it follows from (ii) that $\mu \in \mathfrak{Q}_\alpha^m(H)$. Thus (3.2) holds. We finally show the case for $\alpha = 2$. If $H = \emptyset$, then $\mathfrak{Q}_\alpha^m(H) = \emptyset$ for $m \in \mathbb{N}$ and thus (3.2) is true. Let $H \neq \emptyset$. It is sufficient to show that

$$(3.4) \quad \text{for all } m \in \mathbb{N}, \mathfrak{Q}_2^m(H) \text{ is the class of all Gaussian distributions.}$$

Let us show this statement by induction. If $m = 1$, the assertion is Corollary 3.5. Assume that the assertion is valid for m . Then $\mathfrak{Q}_2^{m+1}(H) = \mathfrak{Q}_2(\mathfrak{Q}_2^m(H)) =$

$\mathfrak{Q}_2(\{\mu \in \mathcal{P}(\mathbb{R}^d): \mu \text{ is Gaussian}\})$, which is equal to the class of all Gaussian distributions on \mathbb{R}^d by Corollary 3.5. Then the statement is true for $m + 1$. Therefore the statement is true for all $m \in \mathbb{N}$.

(iv) It follows from (iii) that $\mathfrak{Q}_\alpha^m(H) \supset \mathfrak{Q}_\alpha^\infty(H)$ for all $m \in \mathbb{N}$. Then, Lemma 3.3 entails that $\mathfrak{Q}_\alpha^{m+1}(H) \supset \mathfrak{Q}_\alpha(\mathfrak{Q}_\alpha^\infty(H))$ for all $m \in \mathbb{N}$. Therefore $\mathfrak{Q}_\alpha^\infty(H) \supset \mathfrak{Q}_\alpha(\mathfrak{Q}_\alpha^\infty(H))$. To prove the converse inclusion, let $\mu \in \mathfrak{Q}_\alpha^\infty(H)$. Then $\mu \in \mathfrak{Q}_\alpha^{m+1}(H)$ for all $m \in \mathbb{Z}_+$. Therefore it follows from (ii) that for any $b > 1$ there exists $\rho_{m,b} \in \mathfrak{Q}_\alpha^m(H)$ such that $\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)^{b^\alpha} \widehat{\rho}_{m,b}(z)$. Since $\mu \in I(\mathbb{R}^d)$, $\widehat{\mu}(b^{-1}z)^{b^\alpha}$ does not vanish. Therefore $\widehat{\rho}_{m,b}(z) = \widehat{\mu}(z) / \widehat{\mu}(b^{-1}z)^{b^\alpha}$, which is independent of m . Denoting it by $\widehat{\rho}_{\infty,b}(z)$, we have $\rho_{\infty,b} \in \mathfrak{Q}_\alpha^m(H)$ for all $m \in \mathbb{Z}_+$, namely, $\rho_{\infty,b} \in \bigcap_{m=0}^\infty \mathfrak{Q}_\alpha^m(H) \subset \mathfrak{Q}_\alpha^\infty(H)$. Then $\mu \in \mathfrak{Q}_\alpha(\mathfrak{Q}_\alpha^\infty(H))$ by Theorem 3.4. Hence $\mathfrak{Q}_\alpha^\infty(H) \subset \mathfrak{Q}_\alpha(\mathfrak{Q}_\alpha^\infty(H))$.

(v) Note that $\mathfrak{Q}_\alpha(H) \subset H$ for all $\alpha \in \mathbb{R}$ by Corollary 3.8 (i) and the assumption. Let us show the statement for $m \in \mathbb{Z}_+$ by induction. The case for $m = 0$ is trivial. Assume that the assertion is valid for $m - 1$. If $\mu \in \mathfrak{Q}_{\alpha_2}^m(H)$, then, by (ii), for each $b > 1$ there exists $\rho_b \in \mathfrak{Q}_{\alpha_2}^{m-1}(H)$ satisfying (1.1) for α_2 in place of α . Noting that $b^{\alpha_2} - b^{\alpha_1} > 0$, we have

$$\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)^{b^{\alpha_1}} \{ \widehat{\mu}(b^{-1}z)^{b^{\alpha_2} - b^{\alpha_1}} \widehat{\rho}_b(z) \}.$$

By (iii), we have $\mu \in \mathfrak{Q}_{\alpha_2}^m(H) \subset \mathfrak{Q}_{\alpha_2}^{m-1}(H)$. Then, the assumption of induction entails that $\mu, \rho_b \in \mathfrak{Q}_{\alpha_2}^{m-1}(H) \subset \mathfrak{Q}_{\alpha_1}^{m-1}(H)$. Since $\mathfrak{Q}_{\alpha_1}^{m-1}(H)$ is c.c.s.s. from (i), the distribution with characteristic function $\widehat{\mu}(b^{-1}z)^{b^{\alpha_2} - b^{\alpha_1}} \widehat{\rho}_b(z)$ also belongs to $\mathfrak{Q}_{\alpha_1}^{m-1}(H)$. Hence $\mu \in \mathfrak{Q}_{\alpha_1}(H)$ by virtue of (ii). Therefore the statement is true for all $m \in \mathbb{Z}_+$. Taking the intersection under $m \in \mathbb{N}$ of the both sides of (3.3), we have the assertion for $m = \infty$. \square

For $H \subset \mathcal{P}(\mathbb{R}^d)$, we write \overline{H} for the closure of H under weak convergence and convolution. Some facts related to the class of stable distributions are the following.

Proposition 3.11. *Let $H \subset I(\mathbb{R}^d)$ be c.c.s.s. and $m \in \{1, 2, \dots, \infty\}$.*

- (i) *If $\alpha \leq 0$ and $H \supset S(\mathbb{R}^d)$, then $\mathfrak{Q}_\alpha^m(H) \supset \overline{S(\mathbb{R}^d)}$.*
- (ii) *If $0 < \alpha < 2$ and $H \supset \bigcup_{\beta \in [\alpha, 2]} S_\beta(\mathbb{R}^d)$, then $\mathfrak{Q}_\alpha^m(H) \supset \overline{\bigcup_{\beta \in [\alpha, 2]} S_\beta(\mathbb{R}^d)}$.*
- (iii) *If $\alpha = 2$ and $H \neq \emptyset$, then $\mathfrak{Q}_2^m(H)$ is the class of all Gaussian distributions.*
- (iv) *If $\alpha > 2$ and $H \neq \emptyset$, then $\mathfrak{Q}_\alpha^m(H)$ is the class of all δ -distributions.*

Proof. (i) Let $\mu \in S(\mathbb{R}^d)$. Then, there exists $\beta \in (0, 2]$ such that for each $b > 1$ there is $c_b \in \mathbb{R}^d$ satisfying $\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)^{b^\beta} e^{i\langle c_b, z \rangle}$. Noting that $\alpha < \beta$ and letting

$$(3.5) \quad \widehat{\rho}_b(z) := \widehat{\mu}(b^{-1}z)^{b^\beta - b^\alpha} e^{i\langle c_b, z \rangle},$$

we have (1.1). Since $\mu \in S(\mathbb{R}^d) \subset H$ and hence $\rho_b \in H$, it follows that $\mu \in \mathfrak{Q}_\alpha(H)$. Then, looking at (3.5) and taking into account that $\mathfrak{Q}_\alpha(H)$ is c.c.s.s., we have $\rho_b \in \mathfrak{Q}_\alpha(H)$, which implies $\mu \in \mathfrak{Q}_\alpha^2(H)$ by Proposition 3.10 (ii). Iterating this argument, we have $\mu \in \mathfrak{Q}_\alpha^m(H)$ for all $m \in \mathbb{N}$. Therefore $\mathfrak{Q}_\alpha^m(H) \supset S(\mathbb{R}^d)$ for all $m \in \mathbb{N}$. Since $\mathfrak{Q}_\alpha^m(H)$ is c.c.s.s., it follows that $\mathfrak{Q}_\alpha^m(H) \supset \overline{S(\mathbb{R}^d)}$ for all $m \in \mathbb{N}$. Thus $\mathfrak{Q}_\alpha^\infty(H) = \bigcap_{m=1}^\infty \mathfrak{Q}_\alpha^m(H) \supset \overline{S(\mathbb{R}^d)}$.

(ii) It is proved in a similar way to (i).

(iii) For $m \in \mathbb{N}$, what we have to show is (3.4) itself, which is already shown. For $m = \infty$, we have that $\mathfrak{Q}_2^\infty(H) = \bigcap_{m=1}^\infty \mathfrak{Q}_2^m(H) = \{\mu \in \mathcal{P}(\mathbb{R}^d) : \mu \text{ is Gaussian}\}$.

(iv) For $m \in \mathbb{N}$, the statement can be proved in the same way as that for (3.4). For $m = \infty$, it is proved in the same way as (iii). \square

We now define $L_m^{(\alpha)}(\mathbb{R}^d)$, the nested subclasses of $L^{(\alpha)}(\mathbb{R}^d)$. Define $L_m^{(\alpha)}(\mathbb{R}^d)$ by $\mathfrak{Q}_\alpha^{m+1}(I(\mathbb{R}^d))$ for $\alpha \in \mathbb{R}$ and $m \in \{0, 1, 2, \dots, \infty\}$. Take into account that $L_0^{(\alpha)}(\mathbb{R}^d) = L^{(\alpha)}(\mathbb{R}^d)$. Noting that $\mathfrak{Q}_\alpha(I(\mathbb{R}^d)) \subset I(\mathbb{R}^d)$ for all $\alpha \in (0, 2]$ and $I(\mathbb{R}^d) \supset S(\mathbb{R}^d)$, we have the following two propositions immediately from Propositions 3.10 and 3.11.

Proposition 3.12. *The following hold.*

- (i) For $\alpha \in \mathbb{R}$ and $m \in \{0, 1, 2, \dots, \infty\}$, $L_m^{(\alpha)}(\mathbb{R}^d)$ is c.c.s.s.
- (ii) For $\alpha \in \mathbb{R}$ and $m \in \mathbb{Z}_+$, $\mu \in L_{m+1}^{(\alpha)}(\mathbb{R}^d)$ if and only if $\mu \in I(\mathbb{R}^d)$ and for each $b > 1$ there exists $\rho_b \in L_m^{(\alpha)}(\mathbb{R}^d)$ satisfying (1.1).
- (iii) Let $\alpha \in \mathbb{R}$. Then $L_m^{(\alpha)}(\mathbb{R}^d) \supset L_{m+1}^{(\alpha)}(\mathbb{R}^d)$ for $m \in \mathbb{Z}_+$.
- (iv) Let $\alpha \in \mathbb{R}$. Then, $\mathfrak{Q}_\alpha(L_\infty^{(\alpha)}(\mathbb{R}^d)) = L_\infty^{(\alpha)}(\mathbb{R}^d)$, namely, $\mu \in L_\infty^{(\alpha)}(\mathbb{R}^d)$ if and only if $\mu \in I(\mathbb{R}^d)$ and for each $b > 1$ there exists $\rho_b \in L_\infty^{(\alpha)}(\mathbb{R}^d)$ satisfying (1.1).
- (v) Let $m \in \{0, 1, 2, \dots, \infty\}$. Then $L_m^{(\alpha_1)}(\mathbb{R}^d) \supset L_m^{(\alpha_2)}(\mathbb{R}^d)$ for $\alpha_1 < \alpha_2$.

Proposition 3.13. *Let $m \in \{0, 1, 2, \dots, \infty\}$.*

- (i) If $\alpha \leq 0$, then $L_m^{(\alpha)}(\mathbb{R}^d) \supset \overline{S(\mathbb{R}^d)}$.
- (ii) If $0 < \alpha < 2$, then $L_m^{(\alpha)}(\mathbb{R}^d) \supset \overline{\bigcup_{\beta \in [\alpha, 2]} S_\beta(\mathbb{R}^d)}$.
- (iii) If $\alpha = 2$, then $L_m^{(2)}(\mathbb{R}^d)$ is the class of all Gaussian distributions.
- (iv) If $\alpha > 2$, then $L_m^{(\alpha)}(\mathbb{R}^d)$ is the class of all δ -distributions.

We next characterize $L_m^{(\alpha)}(\mathbb{R}^d)$ in terms of Lévy measures. For $m = 0$, Maejima and Ueda (2009a) proved the following.

Theorem 3.14. Let $\alpha < 2$. Then, $\mu \in I(\mathbb{R}^d)$ with Lévy measure ν belongs to $L_0^{(\alpha)}(\mathbb{R}^d)$ if and only if

$$\nu(B) = \int_S \lambda(d\xi) \int_0^\infty \mathbb{1}_B(r\xi) r^{-\alpha-1} k_\xi(r) dr, \quad B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}),$$

where λ is a probability measure on S and $k_\xi(r)$ is right-continuous and nonincreasing in $r \in (0, \infty)$ and measurable in $\xi \in S$, and for all $\xi \in S$,

$$\int_0^\infty (r^2 \wedge 1) r^{-\alpha-1} k_\xi(r) dr = \int_{\mathbb{R}^d} (|x|^2 \wedge 1) \nu(dx),$$

which is independent of ξ . If $\nu \neq 0$, then this λ is uniquely determined by ν , and this $k_\xi(\cdot)$ is uniquely determined by ν up to ξ of λ -measure 0.

For characterizations of $L_m^{(\alpha)}(\mathbb{R}^d)$, we need some preparation.

Definition 3.15. Let $\alpha < 2$. For $\mu \in L_0^{(\alpha)}(\mathbb{R}^d)$ with Lévy measure $\nu \neq 0$, we call $k_\xi(r)$ in Theorem 3.14 the *k-function* of ν (or μ). If $\nu = 0$, then we define the *k-function* of ν (or μ) as the zero-function. And we call the function $h_\xi(u), u \in \mathbb{R}$ defined by $h_\xi(u) := k_\xi(e^{-u})$ the *h-function* of ν (or μ).

For $f: \mathbb{R} \rightarrow \mathbb{R}$, we introduce the *difference operator* as follows:

$$\Delta_\varepsilon^n f(u) := \sum_{j=0}^n (-1)^{n-j} \binom{n}{j} f(u + j\varepsilon), \quad \text{for } u \in \mathbb{R}, \varepsilon > 0 \text{ and } n \in \mathbb{Z}_+.$$

For $m \in \mathbb{Z}_+$, $f: \mathbb{R} \rightarrow \mathbb{R}$ is said to be *monotone of order m* if $\Delta_\varepsilon^n f(u) \geq 0$ for all $u \in \mathbb{R}, \varepsilon > 0$ and $n = 0, 1, 2, \dots, m$. $f: \mathbb{R} \rightarrow \mathbb{R}$ is said to be *absolutely monotone* if f is monotone of order m for all $m \in \mathbb{Z}_+$.

The following four statements are proved by similar arguments to those in Section 1.2 of Rocha-Arteaga and Sato (2003), originally done in Sato (1980), so we omit their proofs.

Theorem 3.16. Suppose $\alpha < 2$.

- (i) Let $m \in \mathbb{Z}_+$. Then $\mu \in L_m^{(\alpha)}(\mathbb{R}^d)$ if and only if $\mu \in L_0^{(\alpha)}(\mathbb{R}^d)$ and the *h-function* $h_\xi(u)$ of μ is monotone of order $m + 1$ in $u \in \mathbb{R}$ for λ -a.e. $\xi \in S$.
- (ii) $\mu \in L_\infty^{(\alpha)}(\mathbb{R}^d)$ if and only if $\mu \in L_0^{(\alpha)}(\mathbb{R}^d)$ and the *h-function* $h_\xi(u)$ of μ is absolutely monotone in $u \in \mathbb{R}$ for λ -a.e. $\xi \in S$.

Lemma 3.17. Let $\alpha < 2$ and $0 < c < \infty$. A function $h_\xi(u)$ is absolutely monotone in $u \in \mathbb{R}$ and measurable in $\xi \in S$ and satisfies

$$\int_{-\infty}^\infty (e^{-2u} \wedge 1) e^{\alpha u} h_\xi(u) du = c$$

for all $\xi \in S$ if and only if

$$e^{\alpha u} h_\xi(u) = \int_{(0,2) \cap [\alpha,2)} e^{\beta u} \Gamma_\xi(d\beta),$$

where Γ_ξ is a measure on $(0,2) \cap [\alpha,2)$ for each $\xi \in S$ satisfying

$$\int_{(0,2) \cap [\alpha,2)} \left(\frac{1}{\beta} + \frac{1}{2-\beta} \right) \Gamma_\xi(d\beta) = c$$

and $\Gamma_\xi(B)$ is measurable in $\xi \in S$ for every $B \in \mathcal{B}((0,2) \cap [\alpha,2))$.

Theorem 3.18. *Let $\alpha < 2$.*

(i) *If $\mu \in L_\infty^{(\alpha)}(\mathbb{R}^d)$ with Lévy measure ν , then*

$$\nu(B) = \int_{(0,2) \cap [\alpha,2)} \Gamma(d\beta) \int_S \lambda_\beta(d\xi) \int_0^\infty \mathbf{1}_B(r\xi) r^{-\beta-1} dr, \quad B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}),$$

where Γ is a measure on $(0,2) \cap [\alpha,2)$ satisfying

$$\int_{(0,2) \cap [\alpha,2)} \left(\frac{1}{\beta} + \frac{1}{2-\beta} \right) \Gamma(d\beta) < \infty,$$

and λ_β is a probability measure on S for each $\beta \in (0,2) \cap [\alpha,2)$, and $\lambda_\beta(C)$ is measurable in $\beta \in (0,2) \cap [\alpha,2)$ for every $C \in \mathcal{B}(S)$. This Γ is uniquely determined by μ and this λ_β is uniquely determined by μ up to β of Γ -measure 0.

(ii) *If $\mu \in I(\mathbb{R}^d)$ with Lévy measure ν is expressible as in (i), then $\mu \in L_\infty^{(\alpha)}(\mathbb{R}^d)$.*

Theorem 3.19. (i) *If $\alpha \leq 0$, then $L_\infty^{(\alpha)}(\mathbb{R}^d) \subset \overline{S(\mathbb{R}^d)}$.*

(ii) *If $0 < \alpha < 2$, then $L_\infty^{(\alpha)}(\mathbb{R}^d) \subset \overline{\bigcup_{\beta \in [\alpha,2)} S_\beta(\mathbb{R}^d)}$.*

Combining this theorem with Proposition 3.13 with $m = \infty$, we conclude

Theorem 3.20. (i) *If $\alpha \leq 0$, then $L_\infty^{(\alpha)}(\mathbb{R}^d) = \overline{S(\mathbb{R}^d)}$.*

(ii) *If $0 < \alpha < 2$, then $L_\infty^{(\alpha)}(\mathbb{R}^d) = \overline{\bigcup_{\beta \in [\alpha,2)} S_\beta(\mathbb{R}^d)}$.*

To conclude this section, we go back once to the case for a general c.c.s.s. $H \subset I(\mathbb{R}^d)$.

Theorem 3.21. *Let $H \subset I(\mathbb{R}^d)$ be c.c.s.s.*

(i) *If $\alpha \leq 0$ and $H \supset S(\mathbb{R}^d)$, then $\mathfrak{Q}_\alpha^\infty(H) = \overline{S(\mathbb{R}^d)}$.*

(ii) *If $0 < \alpha < 2$ and $H \supset \bigcup_{\beta \in [\alpha,2)} S_\beta(\mathbb{R}^d)$, then $\mathfrak{Q}_\alpha^\infty(H) = \overline{\bigcup_{\beta \in [\alpha,2)} S_\beta(\mathbb{R}^d)}$.*

Proof. We only prove (i), since (ii) is similarly proved. Proposition 3.11 yields that $\mathfrak{Q}_\alpha^\infty(H) \supset \overline{S(\mathbb{R}^d)}$. Using Lemma 3.3 repeatedly, we have $\mathfrak{Q}_\alpha^{m+1}(H) \subset \mathfrak{Q}_\alpha^{m+1}(I(\mathbb{R}^d)) = L_m^{(\alpha)}(\mathbb{R}^d)$ for $m \in \mathbb{Z}_+$. Hence $\mathfrak{Q}_\alpha^\infty(H) \subset L_\infty^{(\alpha)}(\mathbb{R}^d) = \overline{S(\mathbb{R}^d)}$ by Theorem 3.20. Thus we have $\mathfrak{Q}_\alpha^\infty(H) = \overline{S(\mathbb{R}^d)}$. \square

4. NESTED SUBCLASSES OF THE CLASS OF α -SELFDECOMPOSABLE DISTRIBUTIONS IN TERMS OF MAPPING

For $\alpha \in \mathbb{R}$, Maejima et al. (2010) defined mappings $\Phi_\alpha: \mathfrak{D}(\Phi_\alpha) \rightarrow I(\mathbb{R}^d)$ by

$$(4.1) \quad \Phi_\alpha(\mu) = \begin{cases} \mathcal{L} \left(\int_0^{-1/\alpha} (1 + \alpha t)^{-1/\alpha} dX_t^{(\mu)} \right), & \text{when } \alpha < 0, \\ \mathcal{L} \left(\int_0^\infty e^{-t} dX_t^{(\mu)} \right), & \text{when } \alpha = 0, \\ \mathcal{L} \left(\int_0^\infty (1 + \alpha t)^{-1/\alpha} dX_t^{(\mu)} \right), & \text{when } \alpha > 0. \end{cases}$$

Due to Theorems 2.4 and 2.8 of Sato (2006b), the domains $\mathfrak{D}(\Phi_\alpha)$ are as follows, (see also p. 49 of Sato (2006b)).

$$\mathfrak{D}(\Phi_\alpha) = \begin{cases} I(\mathbb{R}^d), & \text{when } \alpha < 0, \\ I_{\log}(\mathbb{R}^d), & \text{when } \alpha = 0, \\ I_\alpha(\mathbb{R}^d), & \text{when } 0 < \alpha < 1, \\ I_1^*(\mathbb{R}^d), & \text{when } \alpha = 1, \\ I_\alpha^0(\mathbb{R}^d), & \text{when } 1 < \alpha < 2, \\ \{\delta_0\}, & \text{when } \alpha \geq 2, \end{cases}$$

where

$$I_\alpha^0(\mathbb{R}^d) = \left\{ \mu \in I_\alpha(\mathbb{R}^d) : \int_{\mathbb{R}^d} x \mu(dx) = 0 \right\}, \quad \text{for } 1 \leq \alpha < 2,$$

$$I_1^*(\mathbb{R}^d) = \left\{ \mu = \mu_{(A, \nu, \gamma)} \in I_1^0(\mathbb{R}^d) : \lim_{T \rightarrow \infty} \int_1^T t^{-1} dt \int_{|x| > t} x \nu(dx) \text{ exists in } \mathbb{R}^d \right\}.$$

As to the ranges $\mathfrak{R}(\Phi_\alpha)$, Theorem 4.6 of Maejima et al. (2010) says the following.

$$(4.2) \quad \mathfrak{R}(\Phi_\alpha) = \begin{cases} L^{(\alpha)}(\mathbb{R}^d), & \text{when } \alpha < 0, \\ L^{(0)}(\mathbb{R}^d), & \text{when } \alpha = 0, \\ L^{(\alpha)}(\mathbb{R}^d) \cap \mathcal{C}_\alpha(\mathbb{R}^d), & \text{when } 0 < \alpha < 1, \\ L^{(1)}(\mathbb{R}^d) \cap \mathcal{C}_1^*(\mathbb{R}^d), & \text{when } \alpha = 1, \\ L^{(\alpha)}(\mathbb{R}^d) \cap \mathcal{C}_\alpha^0(\mathbb{R}^d), & \text{when } 1 < \alpha < 2, \\ \{\delta_0\}, & \text{when } \alpha \geq 2, \end{cases}$$

where

$$\mathcal{C}_1^*(\mathbb{R}^d) = \left\{ \tilde{\mu}_{(\tilde{A}, \tilde{\nu}, \tilde{\gamma})} \in L^{(1)}(\mathbb{R}^d) \cap \mathcal{C}_1(\mathbb{R}^d) : \tilde{\nu}(B) = \int_S \tilde{\lambda}(d\xi) \int_0^\infty \mathbf{1}_B(r\xi) r^{-2} \tilde{k}_\xi(r) dr, \right. \\ \left. \lim_{\varepsilon \downarrow 0} \int_\varepsilon^1 t dt \int_S \xi \tilde{\lambda}(d\xi) \int_0^\infty \frac{r^2}{1+t^2 r^2} d\tilde{k}_\xi(r) \text{ exists in } \mathbb{R}^d \text{ and equals } \tilde{\gamma} \right\},$$

$$\mathcal{C}_\alpha^0(\mathbb{R}^d) = \mathcal{C}_\alpha(\mathbb{R}^d) \cap I_1^0(\mathbb{R}^d), \quad \text{for } 1 < \alpha < 2.$$

Now, we characterize $\Phi_\alpha^m(H \cap \mathfrak{D}(\Phi_\alpha^m))$ with a c.c.s.s. $H \subset I(\mathbb{R}^d)$ using the results in the previous section. Note that, for $\alpha < 0$, $\mathfrak{D}(\Phi_\alpha^m) = I(\mathbb{R}^d)$, $m \in \mathbb{N}$, since $\mathfrak{D}(\Phi_\alpha) = I(\mathbb{R}^d)$. However, henceforth we do not treat the case for $\alpha \geq 2$, since it is obvious that $\Phi_\alpha^m(\{\delta_0\}) = \{\delta_0\}$ for all $m \in \mathbb{N}$.

Theorem 4.1. *Let $H \subset I(\mathbb{R}^d)$ be c.c.s.s., and let $m \in \mathbb{N}$.*

- (i) *When $\alpha < 0$, $\Phi_\alpha^m(H) = \mathfrak{Q}_\alpha^m(H)$.*
- (ii) *When $\alpha = 0$, $\Phi_0^m(H \cap \mathfrak{D}(\Phi_0^m)) = \mathfrak{Q}_0^m(H)$.*
- (iii) *When $0 < \alpha < 1$, $\Phi_\alpha^m(H \cap \mathfrak{D}(\Phi_\alpha^m)) = \mathfrak{Q}_\alpha^m(H) \cap \mathcal{C}_\alpha(\mathbb{R}^d)$.*
- (iv) *When $\alpha = 1$, $\Phi_1^m(H \cap \mathfrak{D}(\Phi_1^m)) = \mathfrak{Q}_1^m(H) \cap \mathcal{C}_1^*(\mathbb{R}^d)$.*
- (v) *When $1 < \alpha < 2$, $\Phi_\alpha^m(H \cap \mathfrak{D}(\Phi_\alpha^m)) = \mathfrak{Q}_\alpha^m(H) \cap \mathcal{C}_\alpha^0(\mathbb{R}^d)$.*

Proof. (i) It is proved in a similar way to (v).

(ii) We prove the statement by induction. The case for $m = 0$ comes from Lemma 4.1 of Barndorff-Nielsen et al. (2006) and Theorem 3.4 (i) with $\alpha = 0$ of this paper. Now assume that the statement is valid for $m - 1$ with $m \geq 2$ in place of m and let us prove $\Phi_0^m(H \cap \mathfrak{D}(\Phi_0^m)) = \mathfrak{Q}_0^m(H)$. If we put $H' := \Phi_0(H \cap \mathfrak{D}(\Phi_0))$, then it is equal to $\mathfrak{Q}_0(H)$ by the statement for $m = 0$ and thus it is c.c.s.s. Applying the assumption of induction to H' instead of H , we have that $\Phi_0^{m-1}(\Phi_0(H \cap \mathfrak{D}(\Phi_0)) \cap \mathfrak{D}(\Phi_0^{m-1})) = \mathfrak{Q}_0^m(H)$. Since it is easy to see that $\Phi_0(H \cap \mathfrak{D}(\Phi_0)) \cap \mathfrak{D}(\Phi_0^{m-1}) = \Phi_0(H \cap \mathfrak{D}(\Phi_0^m))$, it follows that $\Phi_0^m(H \cap \mathfrak{D}(\Phi_0^m)) = \mathfrak{Q}_0^m(H)$.

(iii) It is proved in a similar way to (v).

(iv) We prove the statement by induction. Let us prove the case for $m = 1$. We first show that $\Phi_1(H \cap \mathfrak{D}(\Phi_1)) \subset \mathfrak{Q}_1(H) \cap \mathcal{C}_1^*(\mathbb{R}^d)$. If $\mu \in \Phi_1(H \cap \mathfrak{D}(\Phi_1))$, then $\mu = \Phi_1(\mu_0)$ for some $\mu_0 \in H \cap \mathfrak{D}(\Phi_1)$. We have, for any $b > 1$ and $z \in \mathbb{R}^d$,

$$C_\mu(z) - bC_\mu(b^{-1}z) = \int_0^{b-1} C_{\mu_0}((1+t)^{-1}z) dt = C_{\rho_b}(z),$$

where

$$\rho_b = \mathcal{L} \left(\int_0^{b-1} (1+t)^{-1} dX_t^{(\mu_0)} \right).$$

Since H is c.c.s.s., $\rho_b \in H$ for all $b > 1$. Then it follows from Theorem 3.4 (i) that $\mu \in \mathfrak{Q}_1(H)$. Since $\mu \in \mathfrak{R}(\Phi_1) \subset \mathcal{C}_1^*(\mathbb{R}^d)$, we have $\mu \in \mathfrak{Q}_1(H) \cap \mathcal{C}_1^*(\mathbb{R}^d)$. We next show that $\Phi_1(H \cap \mathfrak{D}(\Phi_1)) \supset \mathfrak{Q}_1(H) \cap \mathcal{C}_1^*(\mathbb{R}^d)$. If $\mu \in \mathfrak{Q}_1(H) \cap \mathcal{C}_1^*(\mathbb{R}^d)$, then $\mu \in L^{(1)}(\mathbb{R}^d) \cap \mathcal{C}_1^*(\mathbb{R}^d)$ and hence $\mu = \Phi_1(\mu_0)$ for some $\mu_0 \in \mathfrak{D}(\Phi_1)$. On the other hand, due to Theorem 3.4 (i), for each $b > 1$, there is $\rho_b \in H$ satisfying (1.1) with $\alpha = 1$. Then, it follows that

$$(4.3) \quad \begin{aligned} C_{\mu_0}(z) &= \lim_{b \downarrow 1} \frac{1}{b-1} \int_0^{b-1} C_{\mu_0}((1+t)^{-1}z) dt \\ &= \lim_{b \downarrow 1} \frac{1}{b-1} \{C_\mu(z) - bC_\mu(b^{-1}z)\} = \lim_{b \downarrow 1} \frac{1}{b-1} C_{\rho_b}(z). \end{aligned}$$

This entails $\mu_0 \in H$ since H is c.c.s.s. Then $\mu = \Phi_1(\mu_0) \in \Phi_1(H \cap \mathfrak{D}(\Phi_1))$. Therefore the case for $m = 0$ is proved. Now assume that the statement is valid for $m - 1$ with $m \geq 2$ in place of m and let us prove $\Phi_1^m(H \cap \mathfrak{D}(\Phi_1^m)) = \mathfrak{Q}_1^m(H) \cap \mathcal{C}_1^*(\mathbb{R}^d)$. We first show that $\Phi_1^m(H \cap \mathfrak{D}(\Phi_1^m)) \subset \mathfrak{Q}_1^m(H) \cap \mathcal{C}_1^*(\mathbb{R}^d)$. If $\mu \in \Phi_1^m(H \cap \mathfrak{D}(\Phi_1^m))$, then $\mu = \Phi_1^m(\mu_0)$ for some $\mu_0 \in H \cap \mathfrak{D}(\Phi_1^m)$. We have, for any $b > 1$ and $z \in \mathbb{R}^d$,

$$C_\mu(z) - bC_\mu(b^{-1}z) = \int_0^{b-1} C_{\Phi_1^{m-1}(\mu_0)}((1+t)^{-1}z) dt = C_{\rho_b}(z),$$

where

$$\rho_b = \mathcal{L} \left(\int_0^{b-1} (1+t)^{-1} dX_t^{(\Phi_1^{m-1}(\mu_0))} \right).$$

Since $\Phi_1^{m-1}(\mu_0) \in \mathfrak{Q}_1^{m-1}(H) \cap \mathcal{C}_1^*(\mathbb{R}^d)$ by the assumption of induction and $\mathfrak{Q}_1^{m-1}(H)$ is c.c.s.s., we have $\rho_b \in \mathfrak{Q}_1^{m-1}(H)$ for each $b > 1$. Then, $\mu \in \mathfrak{Q}_1^m(H)$ due to Proposition 3.10 (ii). Since $\mu \in \mathfrak{R}(\Phi_1^m) \subset \mathfrak{R}(\Phi_1) \subset \mathcal{C}_1^*(\mathbb{R}^d)$, we have $\mu \in \mathfrak{Q}_1^m(H) \cap \mathcal{C}_1^*(\mathbb{R}^d)$. We next show that $\Phi_1^m(H \cap \mathfrak{D}(\Phi_1^m)) \supset \mathfrak{Q}_1^m(H) \cap \mathcal{C}_1^*(\mathbb{R}^d)$. If $\mu \in \mathfrak{Q}_1^m(H) \cap \mathcal{C}_1^*(\mathbb{R}^d)$, then $\mu \in L^{(1)}(\mathbb{R}^d) \cap \mathcal{C}_1^*(\mathbb{R}^d)$ since $\mathfrak{Q}_1^m(H) \subset \mathfrak{Q}_1^m(I(\mathbb{R}^d)) = L_{m-1}^{(1)}(\mathbb{R}^d) \subset L^{(1)}(\mathbb{R}^d)$. Hence $\mu = \Phi_1^m(\mu_0)$ for some $\mu_0 \in \mathfrak{D}(\Phi_1)$. On the other hand, due to Proposition 3.10 (ii), for each $b > 1$, there is $\rho_b \in \mathfrak{Q}_1^{m-1}(H)$ satisfying (1.1). Then, (4.3) holds. Since $\mathfrak{Q}_1^{m-1}(H)$ is c.c.s.s., $\mu_0 \in \mathfrak{Q}_1^{m-1}(H)$. Noting that $\mu_0 \in \mathfrak{D}(\Phi_1) = I_1^*(\mathbb{R}^d)$, we have $\mu_0 \in$

$\mathfrak{Q}_1^{m-1}(H) \cap I_1^*(\mathbb{R}^d)$. Since $\mathfrak{Q}_1^{m-1}(H) \subset \mathfrak{Q}_1^{m-1}(I(\mathbb{R}^d)) = L_{m-2}^{(1)}(\mathbb{R}^d) \subset L^{(1)}(\mathbb{R}^d)$, we have that $\mu_0 = \mu_{0(A_0, \nu_0, \gamma_0)} \in L^{(1)}(\mathbb{R}^d) \cap \mathcal{C}_1(\mathbb{R}^d)$. Therefore ν_0 has the polar decomposition as follows:

$$\nu_0(B) = \int_S \lambda_0(d\xi) \int_0^\infty \mathbb{1}_B(r\xi) r^{-2} k_{0,\xi}(r) dr, \quad B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}),$$

where $k_{0,\xi}(r)$ is right-continuous and nonincreasing in $r \in (0, \infty)$ and measurable in $\xi \in S$, and satisfies $\lim_{r \rightarrow \infty} k_{0,\xi}(r) = 0$ for each $\xi \in S$. Then Lemma 5.1 and its proof of Maejima et al. (2010) yield that

$$\nu_0(B) = \int_0^1 \nu_1(s^{-1}B) s^{-2} ds, \quad B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}), \quad \int_{\mathbb{R}^d} (|x|^2 \wedge |x|) \nu_1(dx) < \infty,$$

with

$$\nu_1(B) = - \int_S \lambda_0(d\xi) \int_0^\infty \mathbb{1}_B(r\xi) r^{-1} dk_{0,\xi}(r), \quad B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}).$$

Taking into account that $\mu_0 \in I_1^*(\mathbb{R}^d)$, we have $\int_{\mathbb{R}^d} x \mu_0(dx) = 0$, which is equivalent to that

$$\begin{aligned} \gamma_0 &= - \int_{\mathbb{R}^d} \frac{x|x|^2}{1+|x|^2} \nu_0(dx) = - \int_0^1 s^{-2} ds \int_{\mathbb{R}^d} \frac{sx|sx|^2}{1+|sx|^2} \nu_1(dx) \\ &= \int_0^1 s ds \int_S \xi \lambda_0(d\xi) \int_0^\infty \frac{r^2}{1+s^2 r^2} dk_{0,\xi}(r). \end{aligned}$$

This yields $\mu_0 \in \mathcal{C}_1^*(\mathbb{R}^d)$ and hence $\mu_0 \in \mathfrak{Q}_1^{m-1}(H) \cap \mathcal{C}_1^*(\mathbb{R}^d)$. By the assumption of induction, we have $\mu_0 = \Phi_1^{m-1}(\mu_2)$ for some $\mu_2 \in H \cap \mathfrak{D}(\Phi_1^{m-1})$. Then $\mu = \Phi_1(\mu_0) = \Phi_1^m(\mu_2) \in \Phi_1^m(H \cap \mathfrak{D}(\Phi_1^m))$.

(v) We prove the statement by induction. Let us prove the case for $m = 1$. We first show that $\Phi_\alpha(H \cap \mathfrak{D}(\Phi_\alpha)) \subset \mathfrak{Q}_\alpha(H) \cap \mathcal{C}_\alpha^0(\mathbb{R}^d)$. If $\mu \in \Phi_\alpha(H \cap \mathfrak{D}(\Phi_\alpha))$, then $\mu = \Phi_\alpha(\mu_0)$ for some $\mu_0 \in H \cap \mathfrak{D}(\Phi_\alpha)$. We have, for any $b > 1$ and $z \in \mathbb{R}^d$,

$$C_\mu(z) - b^\alpha C_\mu(b^{-1}z) = \int_0^{(b^\alpha-1)/\alpha} C_{\mu_0}((1+\alpha t)^{-1/\alpha} z) dt = C_{\rho_b}(z),$$

where

$$\rho_b = \mathcal{L} \left(\int_0^{(b^\alpha-1)/\alpha} (1+\alpha t)^{-1/\alpha} dX_t^{(\mu_0)} \right).$$

Since H is c.c.s.s., $\rho_b \in H$ for all $b > 1$. Then it follows from Theorem 3.4 (i) that $\mu \in \mathfrak{Q}_\alpha(H)$. Since $\mu \in \mathfrak{R}(\Phi_\alpha) \subset \mathcal{C}_\alpha^0(\mathbb{R}^d)$, we have $\mu \in \mathfrak{Q}_\alpha(H) \cap \mathcal{C}_\alpha^0(\mathbb{R}^d)$. We next show that $\Phi_\alpha(H \cap \mathfrak{D}(\Phi_\alpha)) \supset \mathfrak{Q}_\alpha(H) \cap \mathcal{C}_\alpha^0(\mathbb{R}^d)$. If $\mu \in \mathfrak{Q}_\alpha(H) \cap \mathcal{C}_\alpha^0(\mathbb{R}^d)$, then $\mu \in L^{(\alpha)}(\mathbb{R}^d) \cap \mathcal{C}_\alpha^0(\mathbb{R}^d)$ and hence $\mu = \Phi_\alpha(\mu_0)$ for some $\mu_0 \in \mathfrak{D}(\Phi_\alpha)$. On the other

hand, due to Theorem 3.4 (i), for each $b > 1$, there is $\rho_b \in H$ satisfying (1.1). Then, it follows that

$$(4.4) \quad \begin{aligned} C_{\mu_0}(z) &= \lim_{b \downarrow 1} \frac{\alpha}{b^\alpha - 1} \int_0^{(b^\alpha - 1)/\alpha} C_{\mu_0}((1 + \alpha t)^{-1/\alpha} z) dt \\ &= \lim_{b \downarrow 1} \frac{\alpha}{b^\alpha - 1} \{C_\mu(z) - b^\alpha C_\mu(b^{-1}z)\} = \lim_{b \downarrow 1} \frac{\alpha}{b^\alpha - 1} C_{\rho_b}(z). \end{aligned}$$

This entails $\mu_0 \in H$ since H is c.c.s.s. Then $\mu = \Phi_\alpha(\mu_0) \in \Phi_\alpha(H \cap \mathfrak{D}(\Phi_\alpha))$. Therefore the case for $m = 0$ is proved. Now assume that the statement is valid for $m - 1$ with $m \geq 2$ in place of m and let us prove $\Phi_\alpha^m(H \cap \mathfrak{D}(\Phi_\alpha^m)) = \mathfrak{Q}_\alpha^m(H) \cap \mathcal{C}_\alpha^0(\mathbb{R}^d)$. We first show that $\Phi_\alpha^m(H \cap \mathfrak{D}(\Phi_\alpha^m)) \subset \mathfrak{Q}_\alpha^m(H) \cap \mathcal{C}_\alpha^0(\mathbb{R}^d)$. If $\mu \in \Phi_\alpha^m(H \cap \mathfrak{D}(\Phi_\alpha^m))$, then $\mu = \Phi_\alpha^m(\mu_0)$ for some $\mu_0 \in H \cap \mathfrak{D}(\Phi_\alpha^m)$. We have, for any $b > 1$ and $z \in \mathbb{R}^d$,

$$C_\mu(z) - b^\alpha C_\mu(b^{-1}z) = \int_0^{(b^\alpha - 1)/\alpha} C_{\Phi_\alpha^{m-1}(\mu_0)}((1 + \alpha t)^{-1/\alpha} z) dt = C_{\rho_b}(z),$$

where

$$\rho_b = \mathcal{L} \left(\int_0^{(b^\alpha - 1)/\alpha} (1 + \alpha t)^{-1/\alpha} dX_t^{(\Phi_\alpha^{m-1}(\mu_0))} \right).$$

Since $\Phi_\alpha^{m-1}(\mu_0) \in \mathfrak{Q}_\alpha^{m-1}(H) \cap \mathcal{C}_\alpha^0(\mathbb{R}^d)$ by the assumption of induction and $\mathfrak{Q}_\alpha^{m-1}(H)$ is c.c.s.s., we have $\rho_b \in \mathfrak{Q}_\alpha^{m-1}(H)$ for each $b > 1$. Then, $\mu \in \mathfrak{Q}_\alpha^m(H)$ due to Proposition 3.10 (ii). Since $\mu \in \mathfrak{R}(\Phi_\alpha^m) \subset \mathfrak{R}(\Phi_\alpha) \subset \mathcal{C}_\alpha^0(\mathbb{R}^d)$, we have $\mu \in \mathfrak{Q}_\alpha^m(H) \cap \mathcal{C}_\alpha^0(\mathbb{R}^d)$. We next show that $\Phi_\alpha^m(H \cap \mathfrak{D}(\Phi_\alpha^m)) \supset \mathfrak{Q}_\alpha^m(H) \cap \mathcal{C}_\alpha^0(\mathbb{R}^d)$. If $\mu \in \mathfrak{Q}_\alpha^m(H) \cap \mathcal{C}_\alpha^0(\mathbb{R}^d)$, then $\mu \in L^{(\alpha)}(\mathbb{R}^d) \cap \mathcal{C}_\alpha^0(\mathbb{R}^d)$ since $\mathfrak{Q}_\alpha^m(H) \subset \mathfrak{Q}_\alpha^m(I(\mathbb{R}^d)) = L_{m-1}^{(\alpha)}(\mathbb{R}^d) \subset L^{(\alpha)}(\mathbb{R}^d)$. Hence $\mu = \Phi_\alpha(\mu_0)$ for some $\mu_0 \in \mathfrak{D}(\Phi_\alpha)$. On the other hand, due to Proposition 3.10 (ii), for each $b > 1$, there is $\rho_b \in \mathfrak{Q}_\alpha^{m-1}(H)$ satisfying (1.1). Then, (4.4) holds. Since $\mathfrak{Q}_\alpha^{m-1}(H)$ is c.c.s.s., $\mu_0 \in \mathfrak{Q}_\alpha^{m-1}(H)$. Noting that $\mu_0 \in \mathfrak{D}(\Phi_\alpha) = I_\alpha^0(\mathbb{R}^d) \subset \mathcal{C}_\alpha^0(\mathbb{R}^d)$, we have $\mu_0 \in \mathfrak{Q}_\alpha^{m-1}(H) \cap \mathcal{C}_\alpha^0(\mathbb{R}^d)$. By the assumption of induction, we have $\mu_0 = \Phi_\alpha^{m-1}(\mu_1)$ for some $\mu_1 \in H \cap \mathfrak{D}(\Phi_\alpha^{m-1})$. Then $\mu = \Phi_\alpha(\mu_0) = \Phi_\alpha^m(\mu_1) \in \Phi_\alpha^m(H \cap \mathfrak{D}(\Phi_\alpha^m))$. \square

Let $\alpha < 2$ and let $H \subset I(\mathbb{R}^d)$ be c.c.s.s. Then it follows from Proposition 3.10 (iii) and the theorem above that $\Phi_\alpha^m(H \cap \mathfrak{D}(\Phi_\alpha^m))$, $m \in \mathbb{N}$ are nested subclasses of $\Phi_\alpha(H \cap \mathfrak{D}(\Phi_\alpha))$. Using the results in the previous section, we obtain the limit $\lim_{m \rightarrow \infty} \Phi_\alpha^m(H \cap \mathfrak{D}(\Phi_\alpha^m)) = \bigcap_{m=1}^\infty \Phi_\alpha^m(H \cap \mathfrak{D}(\Phi_\alpha^m))$ as follows.

Corollary 4.2. *Let $H \subset I(\mathbb{R}^d)$ be c.c.s.s.*

(i) *If $\alpha < 0$ and $H \supset S(\mathbb{R}^d)$, then*

$$\lim_{m \rightarrow \infty} \Phi_\alpha^m(H) = \mathfrak{Q}_\alpha^\infty(H) = \overline{S(\mathbb{R}^d)}.$$

(ii) If $\alpha = 0$ and $H \supset S(\mathbb{R}^d)$, then

$$\lim_{m \rightarrow \infty} \Phi_0^m(H \cap \mathfrak{D}(\Phi_0^m)) = \mathfrak{Q}_0^\infty(H) = \overline{S(\mathbb{R}^d)}.$$

(iii) If $0 < \alpha < 1$ and $H \supset \bigcup_{\beta \in [\alpha, 2]} S_\beta(\mathbb{R}^d)$, then

$$\begin{aligned} \lim_{m \rightarrow \infty} \Phi_\alpha^m(H \cap \mathfrak{D}(\Phi_\alpha^m)) &= \mathfrak{Q}_\alpha^\infty(H) \cap \mathcal{C}_\alpha(\mathbb{R}^d) \\ &= \overline{\bigcup_{\beta \in [\alpha, 2]} S_\beta(\mathbb{R}^d)} \cap \mathcal{C}_\alpha(\mathbb{R}^d) = L_\infty^{(\alpha)}(\mathbb{R}^d) \cap \mathcal{C}_\alpha(\mathbb{R}^d). \end{aligned}$$

(iv) If $\alpha = 1$ and $H \supset \bigcup_{\beta \in [1, 2]} S_\beta(\mathbb{R}^d)$, then

$$\begin{aligned} \lim_{m \rightarrow \infty} \Phi_1^m(H \cap \mathfrak{D}(\Phi_1^m)) &= \mathfrak{Q}_1^\infty(H) \cap \mathcal{C}_1^*(\mathbb{R}^d) \\ &= \overline{\bigcup_{\beta \in [1, 2]} S_\beta(\mathbb{R}^d)} \cap \mathcal{C}_1^*(\mathbb{R}^d) = L_\infty^{(1)}(\mathbb{R}^d) \cap \mathcal{C}_1^*(\mathbb{R}^d). \end{aligned}$$

(v) If $1 < \alpha < 2$ and $H \supset \bigcup_{\beta \in [\alpha, 2]} S_\beta(\mathbb{R}^d)$, then

$$\begin{aligned} \lim_{m \rightarrow \infty} \Phi_\alpha^m(H \cap \mathfrak{D}(\Phi_\alpha^m)) &= \mathfrak{Q}_\alpha^\infty(H) \cap \mathcal{C}_\alpha^0(\mathbb{R}^d) \\ &= \overline{\bigcup_{\beta \in [\alpha, 2]} S_\beta(\mathbb{R}^d)} \cap \mathcal{C}_\alpha^0(\mathbb{R}^d) = L_\infty^{(\alpha)}(\mathbb{R}^d) \cap \mathcal{C}_\alpha^0(\mathbb{R}^d). \end{aligned}$$

Remark 4.3. Let Φ_f be a stochastic integral mapping defined by (2.1). It is an interesting problem to characterize the limit $\lim_{m \rightarrow \infty} \mathfrak{R}(\Phi_f^{m+1})$ as in Maejima and Sato (2009). Corollary 4.2 can be applied to this problem as follows. Assume that Φ_f is decomposed in the form that $\Phi_f = \Phi_\alpha \circ \Phi_g = \Phi_g \circ \Phi_\alpha$ for some $\alpha \in (-\infty, 2)$ and some stochastic integral mapping Φ_g . Then $\mathfrak{R}(\Phi_f^m) = \Phi_\alpha^m(\mathfrak{R}(\Phi_g^m) \cap \mathfrak{D}(\Phi_\alpha^m))$, so that $\Phi_\alpha^m(H \cap \mathfrak{D}(\Phi_\alpha^m)) \subset \mathfrak{R}(\Phi_f^m) \subset \mathfrak{R}(\Phi_\alpha^m)$, where $H = \lim_{m \rightarrow \infty} \mathfrak{R}(\Phi_g^{m+1})$. If H fulfills the conditions in Corollary 4.2, then we have $\lim_{m \rightarrow \infty} \mathfrak{R}(\Phi_f^{m+1}) = \lim_{m \rightarrow \infty} \mathfrak{R}(\Phi_\alpha^{m+1})$. An example of this application is found in Maejima and Ueda (2009b). This is why we consider nested classes of $L^{(\alpha)}(\mathbb{R}^d)$ based on not only $I(\mathbb{R}^d)$ but also general c.c.s.s. $H \subset I(\mathbb{R}^d)$.

Let $\mu \in L_\infty(\mathbb{R}^d) = L_\infty^{(0)}(\mathbb{R}^d)$, and let Γ and λ_β be the measures in Theorem 3.18 with $\alpha = 0$. We call Γ the Γ -measure of $\mu \in L_\infty(\mathbb{R}^d)$, sometimes denoted by Γ^μ . We also write λ_β^μ for λ_β . For a set $A \in \mathcal{B}((0, 2))$, let $L_\infty^A(\mathbb{R}^d)$ denote the class of $\mu \in L_\infty(\mathbb{R}^d)$ with Γ^μ satisfying $\Gamma^\mu((0, 2) \setminus A) = 0$. Note that $L_\infty^{(\alpha)}(\mathbb{R}^d) = L_\infty^{[\alpha, 2]}(\mathbb{R}^d)$ for $\alpha \in (0, 2)$ due to Theorem 3.18.

Lemma 4.4. Let $0 < \alpha < 2$. We have $L_\infty^{(\alpha)}(\mathbb{R}^d) \cap \mathcal{C}_\alpha(\mathbb{R}^d) = L_\infty^{(\alpha, 2)}(\mathbb{R}^d)$.

Proof. Let $\mu \in L_\infty^{(\alpha)}(\mathbb{R}^d) = L_\infty^{[\alpha,2]}(\mathbb{R}^d)$ with Lévy measure ν . Then,

$$\nu(B) = \int_{[\alpha,2]} \Gamma^\mu(d\beta) \int_S \lambda_\beta^\mu(d\xi) \int_0^\infty \mathbb{1}_B(r\xi) r^{-\beta-1} dr, \quad B \in \mathcal{B}(\mathbb{R}^d).$$

Since

$$r^\alpha \int_{|x|>r} \nu(dx) = r^\alpha \int_{[\alpha,2]} \Gamma^\mu(d\beta) \int_r^\infty u^{-\beta-1} du = \int_{[\alpha,2]} \beta^{-1} r^{\alpha-\beta} \Gamma^\mu(d\beta),$$

it follows from the bounded convergence theorem that

$$\lim_{r \rightarrow \infty} r^\alpha \int_{|x|>r} \nu(dx) = \alpha^{-1} \Gamma^\mu(\{\alpha\}).$$

Thus $\mu \in \mathcal{C}_\alpha(\mathbb{R}^d)$ if and only if $\Gamma^\mu(\{\alpha\}) = 0$ under the condition $\mu \in L_\infty^{(\alpha)}(\mathbb{R}^d) = L_\infty^{[\alpha,2]}(\mathbb{R}^d)$. \square

Using the lemma above, we have the following.

Theorem 4.5. *Let $H \subset I(\mathbb{R}^d)$ be c.c.s.s.*

(i) *If $0 < \alpha < 1$ and $H \supset \bigcup_{\beta \in [\alpha,2]} S_\beta(\mathbb{R}^d)$, then*

$$\lim_{m \rightarrow \infty} \Phi_\alpha^m(H \cap \mathfrak{D}(\Phi_\alpha^m)) = L_\infty^{(\alpha,2)}(\mathbb{R}^d).$$

(ii) *If $\alpha = 1$ and $H \supset \bigcup_{\beta \in [1,2]} S_\beta(\mathbb{R}^d)$, then*

$$\lim_{m \rightarrow \infty} \Phi_1^m(H \cap \mathfrak{D}(\Phi_1^m)) = \left\{ \mu = \mu_{(A,\nu,\gamma)} \in L_\infty^{(1,2)}(\mathbb{R}^d) : \lim_{\varepsilon \downarrow 0} \int_{(1,2)} B\left(\frac{3-\beta}{2}, \frac{\beta+1}{2}\right) \frac{1-\varepsilon^{\beta-1}}{\beta-1} \Gamma^\mu(d\beta) \int_S \xi \lambda_\beta^\mu(d\xi) = -\gamma \right\}.$$

(iii) *If $1 < \alpha < 2$ and $H \supset \bigcup_{\beta \in [\alpha,2]} S_\beta(\mathbb{R}^d)$, then*

$$\lim_{m \rightarrow \infty} \Phi_\alpha^m(H \cap \mathfrak{D}(\Phi_\alpha^m)) = L_\infty^{(\alpha,2)}(\mathbb{R}^d) \cap I_1^0(\mathbb{R}^d).$$

Proof. The statements (i) and (iii) come from Corollary 4.2 and Lemma 4.4.

Let us prove the statement (ii). Suppose $\mu \in L_\infty^{(1,2)}(\mathbb{R}^d)$ with Lévy measure ν . Let $c := \int_{\mathbb{R}^d} (|x|^2 \wedge 1) \nu(dx)$. Note that $L_\infty^{(1,2)}(\mathbb{R}^d) \subset L_\infty^{[1,2]}(\mathbb{R}^d) = L_\infty^{(1)}(\mathbb{R}^d) \subset L^{(1)}(\mathbb{R}^d)$. Let λ and $k_\xi(r)$ be the ones in Theorem 3.14 with $\alpha = 1$. It follows from Theorem 3.16 and Lemma 3.17 with $\alpha = 1$ that

$$k_\xi(r) = \int_{[1,2]} r^{1-\beta} \Gamma_\xi(d\beta), \quad \int_{[1,2]} \left(\frac{1}{\beta} + \frac{1}{2-\beta} \right) \Gamma_\xi(d\beta) = c,$$

where $\Gamma_\xi, \xi \in S$, are the measures in Lemma 3.17 with $\alpha = 1$. Choosing a $[1, 2)$ -valued random variable X and an S -valued random variable Y with joint distribution

$$P(X \in d\beta, Y \in d\xi) := c^{-1} \left(\frac{1}{\beta} + \frac{1}{2-\beta} \right) \lambda(d\xi) \Gamma_\xi(d\beta),$$

we have

$$\Gamma^\mu(d\beta) = c \left(\frac{1}{\beta} + \frac{1}{2-\beta} \right)^{-1} P(X \in d\beta), \quad \lambda_\beta^\mu(d\xi) = P(Y \in d\xi \mid X = \beta) \quad \Gamma^\mu\text{-a.e. } \beta$$

from the uniqueness of Γ^μ and λ_β^μ . Since $\Gamma^\mu(\{1\}) = 0$, it follows that $\Gamma_\xi(\{1\}) = 0$ λ -a.e. $\xi \in S$. Then we have

$$\begin{aligned} & - \int_\varepsilon^1 t dt \int_S \xi \lambda(d\xi) \int_0^\infty \frac{r^2}{1+t^2 r^2} dk_\xi(r) \\ &= - \int_\varepsilon^1 t dt \int_S \xi \lambda(d\xi) \int_{(1,2)} \Gamma_\xi(d\beta) \int_0^\infty \frac{r^2}{1+t^2 r^2} dr^{1-\beta} \\ &= \int_\varepsilon^1 t dt \int_S \xi \lambda(d\xi) \int_{(1,2)} (\beta-1) \Gamma_\xi(d\beta) \int_0^\infty \frac{r^{2-\beta}}{1+t^2 r^2} dr \\ &= \int_\varepsilon^1 dt \int_S \xi \lambda(d\xi) \int_{(1,2)} (\beta-1) t^{\beta-2} \Gamma_\xi(d\beta) \int_0^\infty \frac{s^{2-\beta}}{1+s^2} ds, \end{aligned}$$

which is, by 3.251.2 in Gradshteyn and Ryzhik (2007), equal to

$$\begin{aligned} & \int_\varepsilon^1 dt \int_S \xi \lambda(d\xi) \int_{(1,2)} \frac{\beta-1}{2} B\left(\frac{3-\beta}{2}, \frac{\beta-1}{2}\right) t^{\beta-2} \Gamma_\xi(d\beta) \\ &= \int_\varepsilon^1 dt \int_S \xi \lambda(d\xi) \int_{(1,2)} B\left(\frac{3-\beta}{2}, \frac{\beta+1}{2}\right) t^{\beta-2} \Gamma_\xi(d\beta) \\ &= \int_\varepsilon^1 dt \int_{(1,2)} B\left(\frac{3-\beta}{2}, \frac{\beta+1}{2}\right) t^{\beta-2} \Gamma^\mu(d\beta) \int_S \xi \lambda_\beta^\mu(d\xi) \\ &= \int_{(1,2)} B\left(\frac{3-\beta}{2}, \frac{\beta+1}{2}\right) \frac{1-\varepsilon^{\beta-1}}{\beta-1} \Gamma^\mu(d\beta) \int_S \xi \lambda_\beta^\mu(d\xi). \end{aligned}$$

Thus, under the condition $\mu \in L_\infty^{(1,2)}(\mathbb{R}^d)$,

$$\lim_{\varepsilon \downarrow 0} \int_\varepsilon^1 t dt \int_S \xi \lambda(d\xi) \int_0^\infty \frac{r^2}{1+t^2 r^2} dk_\xi(r) = \gamma$$

if and only if

$$\lim_{\varepsilon \downarrow 0} \int_{(1,2)} B\left(\frac{3-\beta}{2}, \frac{\beta+1}{2}\right) \frac{1-\varepsilon^{\beta-1}}{\beta-1} \Gamma^\mu(d\beta) \int_S \xi \lambda_\beta^\mu(d\xi) = -\gamma.$$

This completes the proof due to Corollary 4.2 and Lemma 4.4. \square

Letting $H = I(\mathbb{R}^d)$ in Theorems 4.1 and 4.5 and Corollary 4.2, we have the following two theorems.

Theorem 4.6. *Let $m \in \mathbb{Z}_+$.*

- (i) *When $\alpha \leq 0$, $\mathfrak{R}(\Phi_\alpha^{m+1}) = L_m^{(\alpha)}(\mathbb{R}^d)$.*
- (ii) *When $0 < \alpha < 1$, $\mathfrak{R}(\Phi_\alpha^{m+1}) = L_m^{(\alpha)}(\mathbb{R}^d) \cap \mathcal{C}_\alpha(\mathbb{R}^d)$.*
- (iii) *When $\alpha = 1$, $\mathfrak{R}(\Phi_1^{m+1}) = L_m^{(1)}(\mathbb{R}^d) \cap \mathcal{C}_1^*(\mathbb{R}^d)$.*
- (iv) *When $1 < \alpha < 2$, $\mathfrak{R}(\Phi_\alpha^{m+1}) = L_m^{(\alpha)}(\mathbb{R}^d) \cap \mathcal{C}_\alpha^0(\mathbb{R}^d)$.*

Theorem 4.7. (i) *When $\alpha \leq 0$,*

$$\lim_{m \rightarrow \infty} \mathfrak{R}(\Phi_\alpha^{m+1}) = L_\infty(\mathbb{R}^d).$$

(ii) *When $0 < \alpha < 1$,*

$$\lim_{m \rightarrow \infty} \mathfrak{R}(\Phi_\alpha^{m+1}) = L_\infty^{(\alpha,2)}(\mathbb{R}^d).$$

(iii) *When $\alpha = 1$,*

$$\lim_{m \rightarrow \infty} \mathfrak{R}(\Phi_1^{m+1}) = \left\{ \mu = \mu_{(A,\nu,\gamma)} \in L_\infty^{(1,2)}(\mathbb{R}^d) : \lim_{\varepsilon \downarrow 0} \int_{(1,2)} B\left(\frac{3-\beta}{2}, \frac{\beta+1}{2}\right) \frac{1-\varepsilon^{\beta-1}}{\beta-1} \Gamma^\mu(d\beta) \int_S \xi \lambda_\beta^\mu(d\xi) = -\gamma \right\}.$$

(iv) *When $1 < \alpha < 2$,*

$$\lim_{m \rightarrow \infty} \mathfrak{R}(\Phi_\alpha^{m+1}) = L_\infty^{(\alpha,2)}(\mathbb{R}^d) \cap I_1^0(\mathbb{R}^d).$$

Remark 4.8. The two theorems above in the case $\alpha = 0$ are well-known results. Also, Theorem 4.7 in the case $-1 \leq \alpha < 0$ is already proved in Example 3.5 (5) of Maejima and Sato (2009). Mappings having the same iterated limits as those of Φ_α , $\alpha \in (0, 2)$, were already found by Sato (2007–2009).

5. A SUPPLEMENTARY REMARK

Theorems 3.20 and 4.7 have given us the limits of the nested subclasses in terms of limit theorems and mappings, respectively, where, the forms of the limits look quite dependent on α . However, if we do not care explicit forms of the classes, we can unify the expressions of the results into one expression as follows. The first one is a restatement of Theorem 3.20.

Theorem 5.1. *Let $\alpha \in \mathbb{R}$. Then $L_\infty^{(\alpha)}(\mathbb{R}^d) = L^{(\alpha)}(\mathbb{R}^d) \cap \overline{S(\mathbb{R}^d)}$.*

Proof. (iii) and (iv) of Proposition 3.13 assure the statement for $\alpha \geq 2$. If $\alpha \leq 0$, Proposition 3.13 (i) and Theorem 3.20 yields the statement. Let $0 < \alpha < 2$. Then Propositions 3.12 (iii) and 3.20 (ii) yields that $L_\infty^{(\alpha)}(\mathbb{R}^d) \subset L^{(\alpha)}(\mathbb{R}^d) \cap \overline{S(\mathbb{R}^d)}$. Let $\mu \in L^{(\alpha)}(\mathbb{R}^d) \cap \overline{S(\mathbb{R}^d)}$ with Lévy measure ν . Since $\mu \in \overline{S(\mathbb{R}^d)}$, we have

$$\nu(B) = \int_{(0,2)} \Gamma^\mu(d\beta) \int_S \lambda_\beta^\mu(d\xi) \int_0^\infty \mathbb{1}_B(r\xi) r^{-\beta-1} dr, \quad B \in \mathcal{B}(\mathbb{R}^d).$$

Since $\mu \in L^{(\alpha)}(\mathbb{R}^d)$, we have that for all $\alpha' \in (0, \alpha)$, $\int_{|x|>1} |x|^{\alpha'} \nu(dx) < \infty$. Then

$$\int_{(0,2)} \Gamma^\mu(d\beta) \int_1^\infty r^{\alpha'-\beta-1} dr < \infty,$$

which entails $\Gamma^\mu((0, \alpha']) = 0$ for all $\alpha' \in (0, \alpha)$. Therefore $\Gamma^\mu((0, \alpha)) = 0$. It follows from Theorem 3.18 (ii) that $\mu \in L_\infty^{(\alpha)}(\mathbb{R}^d)$. \square

Using the theorem above, we have the following, which is a restatement of Theorem 4.7.

Theorem 5.2. *Let $\alpha \in \mathbb{R}$. Then*

$$(5.1) \quad \lim_{m \rightarrow \infty} \mathfrak{R}(\Phi_\alpha^{m+1}) = \mathfrak{R}(\Phi_\alpha) \cap \overline{S(\mathbb{R}^d)}.$$

Proof. If $\alpha \geq 2$, then $\lim_{m \rightarrow \infty} \mathfrak{R}(\Phi_\alpha^{m+1}) = \mathfrak{R}(\Phi_\alpha) \cap \overline{S(\mathbb{R}^d)} = \{\delta_0\}$. Combining Theorem 5.1 with (4.2) and Theorem 4.6, we have the statement for $\alpha < 2$. \square

Remark 5.3. Many known mappings satisfies (5.1), (see, e.g. Maejima and Sato (2009)). However, some mappings does not fulfill (5.1), (see, e.g. Maejima and Ueda (2009c)).

REFERENCES

- O.E. Barndorff-Nielsen, M. Maejima and K. Sato. Some classes of multivariate infinitely divisible distributions admitting stochastic integral representations. *Bernoulli*, 12(1):1–33, 2006.
- I.S. Gradshteyn and I.M. Ryzhik. *Table of Integrals, Series and Products. 7th ed.* Elsevier, 2007.
- A.M. Iksanov, Z.J. Jurek and B.M. Schreiber. A new factorization property of the selfdecomposable probability measures. *Ann. Probab.*, 32(2):1356–1369, 2004.
- Z.J. Jurek. Limit distributions for sums of shrunken random variables. *Diss. Math.*, 185, 1981.

- Z.J. Jurek. Relations between the s -selfdecomposable and selfdecomposable measures. *Ann. Probab.*, 13:592–608, 1985.
- Z.J. Jurek. Random integral representations for classes of limit distributions similar to Lévy class L_0 . *Probab. Theory Relat. Fields*, 78(3):473–490, 1988.
- Z.J. Jurek. Random integral representations for classes of limit distributions similar to Lévy class L_0 . II. *Nagoya Math. J.*, 114:53–64, 1989.
- Z.J. Jurek. Random integral representations for classes of limit distributions similar to Lévy class L_0 . III. R.M. Dudley et al. (ed.), *Probability in Banach Spaces*, 8: Birkhäuser, 137–151, 1992.
- Z.J. Jurek. The random integral representation hypothesis revisited: new classes of s -selfdecomposable laws. N.M. Chuong, et al. (ed.), *Abstract and Applied Analysis*. World Scientific. 495–514, 2004.
- Z.J. Jurek and B.M. Schreiber. Fourier transforms of measures from the classes \mathcal{U}_β , $-2 < \beta \leq -1$. *J. Multivariate Anal.*, 41(2):194–211, 1992.
- M. Maejima, M. Matsui, and M. Suzuki. Classes of infinitely divisible distributions on \mathbb{R}^d related to the class of selfdecomposable distributions. To appear in *Tokyo J. Math.*, 2010.
- M. Maejima and K. Sato. The limits of nested subclasses of several classes of infinitely divisible distributions are identical with the closure of the class of stable distributions. *Probab. Theory Relat. Fields*, 145(1-2):119–142, 2009.
- M. Maejima and Y. Ueda. α -selfdecomposable distributions and related Ornstein-Uhlenbeck type processes. Submitted, 2009a.
- M. Maejima and Y. Ueda. Compositions of mappings of infinitely divisible distributions with applications to finding the limits of some nested subclasses. Submitted, 2009b.
- M. Maejima and Y. Ueda. Stochastic integral characterizations of semi-selfdecomposable distributions and related Ornstein-Uhlenbeck type processes. *Commun. Stoch. Anal.*, 3(3):349–367, 2009c.
- A. Rocha-Arteaga and K. Sato. *Topics in Infinitely Divisible Distributions and Lévy Processes*. Aportaciones Matemáticas, Investigación 17, Sociedad Matemática Mexicana, 2003.
- K. Sato. Class L of multivariate distributions and its subclasses. *J. Multivariate Anal.*, 10:207–232, 1980.

- K. Sato. *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press, 1999.
- K. Sato. Stochastic integrals in additive processes and application to semi-Lévy processes. *Osaka J. Math.*, 41(1):211–236, 2004.
- K. Sato. Additive processes and stochastic integrals. *Ill. J. Math.*, 50(1-4):825–851, 2006a.
- K. Sato. Two families of improper stochastic integrals with respect to Lévy processes. *ALEA, Lat. Am. J. Probab. Math. Stat.*, 1:47–87, 2006b.
- K. Sato. Memos privately communicated. 2007–2009.