Erratum to: Stickelberger ideals and Fitting ideals of class groups for abelian number fields

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In the proofs of Theorems 3.5, 3.6 in [2], we used Lemma 6.3 in [1], which bounds the index of the Fitting ideal in terms of the class number. But the proof of the lemma is not correct, more precisely, an exact sequence in the proof of Lemma 6.4 in [1] (on page 64 line 19), does not hold in general. We thank Takeshi Tsuji for giving us some comments on this.

In this erratum, we give, without using Lemma 6.3 in [1], a proof of Theorems 3.5, 3.6 in [2] (and also of Theorems 0.4, 0.6 in [1] whose proof used Lemma 6.3 in [1]).

For a discrete valuation ring R, we denote by ord_R the normalized additive valuation of R such that $\mathrm{ord}_R(\pi) = 1$ for a uniformizer π. For an R-module M, $\mathrm{length}_R(M)$ denotes the length of M as an R-module. Let K/k be an abelian extension as in §3 in [2]. Recall that we decomposed $\mathrm{Gal}(K/k) = \Delta_K \times \Gamma_K$ where $\#\Delta_K$ is prime to p, and Γ_K is a p-group. For an arbitrary character ψ of Γ_K we denote by K_ψ the fixed subfield of K by $\mathrm{Ker}\,\psi$, and by O_ψ the discrete valuation ring $\mathbf{Z}_p[\mathrm{Image}\,\psi]$ on which Γ_K acts via ψ. Define $A^\psi := (A_{K_\psi})^\psi = A_{K_\psi} \otimes_{\mathbf{Z}_p[\Gamma_K]} O_\psi$, which is an $O_\psi[\Delta_K]$-module. We also use the notation ψ for the ring homomorphisms induced by ψ; $\psi : \mathbf{Z}_p[\mathrm{Gal}(K/k)] \to O_\psi[\Delta_K]$, $\psi : \mathbf{Z}_p[\Gamma_K] \to O_\psi$, etc. For a character χ of Δ_K and $\mathbf{Z}_p[\mathrm{Gal}(K/k)]$-module M, we also use the notation M^χ, defined by $M^\chi = M \otimes_{\mathbf{Z}_p[\Delta_K]} O_\chi$, which is now an $O_\chi[\Gamma_K]$-module. For a character χ of Δ_K and a character ψ of Γ_K, we regard $\chi\psi$ as a character of $\mathrm{Gal}(K/k)$, and define $A^{\chi\psi}$ by $A^{\chi\psi} = A^\psi \otimes_{O_\psi[\Delta_K]} O_{\chi\psi} = (A_{K_\psi})^\psi \otimes_{O_\psi[\Delta_K]} O_{\chi\psi}$, which is an $O_{\chi\psi}$-module. This module also coincides with $(A_{K_\psi})^\chi \otimes_{O_\chi[\Gamma_K]} O_{\chi\psi}$.

We begin with the following proposition.

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Proposition 0.1. *We assume that an abelian extension K/k satisfies the same conditions as that of Proposition 3.4 in [2]; namely, we assume that for every prime $\mathfrak{p}$ of k above p, $\mathfrak{p}$ is unramified in K and the ramification index of $\mathfrak{p}$ over $\mathbf{Q}$ is odd, and that the μ-invariant of the cyclotomic $\mathbf{Z}_p$-extension of K is zero, and that K/k satisfies the condition* (A_p) *(The condition* (A_p) *means that Γ_K is of the form $\Gamma_K = P_{\mathfrak{l}_1} \times \cdots \times P_{\mathfrak{l}_r}$ where $\mathfrak{l}_i$'s are all ramifying primes of k in K/k and $P_{\mathfrak{l}_i}$ is the p-component of the inertia group of $\mathfrak{l}_i$ for each i). In addition, we also assume that K does not contain a primitive p-th root of unity. Let ψ be a character of Γ_K such that for every i with $1 \leq i \leq r$, $\psi|_{P_{\mathfrak{l}_i}}$, the restriction of ψ to $P_{\mathfrak{l}_i}$, is injective (faithful). Note that K/K_ψ is an unramified extension.*

For any $\psi \in \widehat{\Gamma}_K$ as above and any odd character $\chi \in \widehat{\Delta}_K$, we have

$$\mathrm{length}_{O_{\chi\psi}} A^{\chi\psi} = \mathrm{ord}_{O_{\chi\psi}} \left(L(0, (\chi\psi)^{-1}) \right).$$

In particular, we also have

$$\psi \left(\mathrm{Fitt}_{O_\chi[\Gamma_K]}(A^\chi_{K_\chi}) \right) = \left(\psi(\theta^\chi_{K_\chi/k}) \right).$$

where K_χ is the fixed subfield of K by $\mathrm{Ker}\,\chi$.

Proof. By Proposition 3.4 in [2], we have

$$\mathrm{Fitt}_{O_\chi[\Gamma_K]}(A^\chi_K) \subset \left(\Theta_{K/k} \otimes \mathbf{Z}_p \right)^\chi \tag{1}$$

for all odd characters $\chi \in \widehat{\Delta}_K$. Since $[K : K_\chi]$ is prime to p, as we mentioned in the second paragraph of page 565 in [2], the usual norm argument shows that $A^\chi_K \simeq A^\chi_{K_\chi}$ and $\left(\Theta_{K/k} \otimes \mathbf{Z}_p \right)^\chi = \left(\Theta_{K_\chi/k} \otimes \mathbf{Z}_p \right)^\chi$. Furthermore, since $\Gamma_{K_\chi} = \Gamma_K$ and $K_\chi/K_{\chi\psi}$ is an unramified extension, we have $A^{\chi\psi}_{K_\chi} \simeq A^{\chi\psi}_{K_{\chi\psi}}$ by Lemma 1.2 in [2] and $\left(\Theta_{K_\chi/k} \otimes \mathbf{Z}_p \right)^{\chi\psi} = \left(\theta^{\chi\psi}_{K_{\chi\psi}/k} \right) = \left(L(0, (\chi\psi)^{-1}) \right)$. Therefore, taking ψ of (1), we have, for all odd characters $\chi \in \widehat{\Delta}_K$, $\mathrm{Fitt}_{O_{\chi\psi}}(A^{\chi\psi}) \subset \left(L(0, (\chi\psi)^{-1}) \right)$, equivalently

$$\mathrm{length}_{O_{\chi\psi}}(A^{\chi\psi}) \geq \mathrm{ord}_{O_{\chi\psi}} \left(L(0, (\chi\psi)^{-1}) \right). \tag{2}$$

Changing subscripts, we may assume that $P_{\mathfrak{l}_1} \simeq \mathbf{Z}/p^{n_1}\mathbf{Z}, \ldots, P_{\mathfrak{l}_r} \simeq \mathbf{Z}/p^{n_r}\mathbf{Z}$ and $n_1 \geq \ldots \geq n_r$. We put $F = K_\psi$, and $K_{(\Delta)} = K^{\Gamma_K}$, the fixed subfield of K by Γ_K. Since $\mathrm{Image}\,\psi = \mu_{p^{n_1}} \subset \overline{\mathbf{Q}}_p^\times$, we have $\Gamma_F := \mathrm{Gal}(F/K_{(\Delta)}) \simeq \mathbf{Z}/p^{n_1}\mathbf{Z}$. Let γ be a generator of Γ_F and F_1 the fixed subfield of F by $\langle \gamma^{p^{n_1-1}} \rangle$, namely the unique subfield of $F/K_{(\Delta)}$ such that $[F : F_1] = p$. We will show that the following sequence is exact:

$$0 \longrightarrow A^\chi_{F_1} \longrightarrow A^\chi_F \longrightarrow A^{\chi\psi}_F \longrightarrow 0.$$

In fact, since F does not contain a primitive p-th root of unity by our assumption, the natural map $A^-_{F_1} \to A^-_F$ is injective, and therefore the map $A^\chi_{F_1} \to A^\chi_F$ is also injective. By the definition of ψ-quotient, we have $A^{\chi\psi}_F \simeq A^\chi_F/(1 + \gamma^{p^{n_1-1}} + \gamma^{2p^{n_1-1}} + \cdots + \gamma^{(p-1)p^{n_1-1}})A^\chi_F = A^\chi_F/N_{\mathrm{Gal}(F/F_1)}A^\chi_F$, which implies that the map $A^\chi_F \to A^{\chi\psi}_F$ is surjective. Since the norm map $A^-_F \to A^-_{F_1}$ is surjective (cf. Lemma 1.2 in [2]), we have $\mathrm{Image}(A^{\chi}_{F_1} \to A^{\chi}_{F}) = N_{\mathrm{Gal}(F/F_1)} A^{\chi}_{F}$. This shows the exactness at the middle term of the above sequence.

Since $A^{\chi\psi} \simeq A_F^{\chi\psi}$, the above exact sequence implies that

$$\mathrm{length}_{O_\chi}(A^{\chi\psi}) = \mathrm{length}_{O_\chi}(A^{\chi}_{F}) - \mathrm{length}_{O_\chi}(A^{\chi}_{F_1}).$$

For two characters $\chi_1, \chi_2 \in \widehat{\Delta}_K$, we denote $\chi_1 \sim \chi_2$ if these are $\mathbf{Q}_p$-conjugate (cf. page 555 in [2]). Until the end of the proof of this proposition, we use the following notation: $\sum_\chi$ (resp. $\sum_{\chi/\sim}$) means the sum which is taken over all odd characters of Δ_K (resp. taken over the equivalence classes of all odd characters of Δ_K). We also use the notations $\bigoplus_\chi$, $\bigoplus_{\chi/\sim}$ in a similar way. We have the isomorphisms $A^-_F \simeq \bigoplus_{\chi/\sim} A^\chi_F$ and $A^-_{F_1} \simeq \bigoplus_{\chi/\sim} A^\chi_{F_1}$ because the order of Δ_K is prime to p. Put $f_\chi := [\mathbf{Q}_p(\mathrm{Image}\,\chi) : \mathbf{Q}_p]$ for each odd character $\chi \in \widehat{\Delta}_K$, and denote by ord_p the normalized valuation on $\overline{\mathbf{Q}}_p$ such that $\mathrm{ord}_p(p) = 1$. By the analytic class number formula, we have

$$\begin{aligned}
&\sum_{\chi/\sim} f_\chi \,\mathrm{length}_{O_{\chi\psi}}(A^{\chi\psi}) \\
&= \sum_{\chi/\sim} f_\chi \,\mathrm{length}_{O_\chi}(A^{\chi\psi}) = \sum_{\chi/\sim} f_\chi \Big(\mathrm{length}_{O_\chi}(A^\chi_F) - \mathrm{length}_{O_\chi}(A^\chi_{F_1})\Big) \\
&= \sum_{\chi/\sim} \big(\mathrm{ord}_p(\sharp A^\chi_F) - \mathrm{ord}_p(\sharp A^\chi_{F_1})\big) = \mathrm{ord}_p(\sharp A^-_F) - \mathrm{ord}_p(\sharp A^-_{F_1}) \\
&= \sum_{\chi} \sum_{\substack{i=1 \\ (i,p)=1}}^{p^{n_1}} \mathrm{ord}_p(L(0, (\chi\psi^i)^{-1})) = \sum_{\chi/\sim} f_\chi (p-1)p^{n_1-1}\,\mathrm{ord}_p(L(0, (\chi\psi)^{-1})) \\
&= \sum_{\chi/\sim} f_\chi \,\mathrm{ord}_{O_{\chi\psi}}(L(0, (\chi\psi)^{-1})).
\end{aligned}$$

This equality and the inequality (2) show that

$$\mathrm{length}_{O_{\chi\psi}}(A^{\chi\psi}) = \mathrm{ord}_{O_{\chi\psi}}\big(L(0, (\chi\psi)^{-1})\big)$$

for each odd character $\chi \in \widehat{\Delta}_K$.

Corollary 3.3 in [2]. In the proof of this corollary, Lemma 6.3 in [1] was not used, so the proof need not be changed. But it refers to Theorem 0.4 in [1] whose proof used Lemma 6.3 in [1]. So we give a correct proof of Theorem 0.4 here.

Proof of Theorem 0.4 in [1]. Let K be an imaginary abelian number field such that no prime of K^+ above p splits in K/K^+. It is enough to show the equalities

$$\mathrm{Fitt}_{O_\chi[\Gamma_K]}(A^\chi_K) = (\Theta_{K/\mathbf{Q}} \otimes \mathbf{Z}_p)^\chi \tag{3}$$

for all odd characters $\chi \in \widehat{\Delta}_K$. As we mentioned in the proof of Proposition 0.1, we know that $A^\chi_K \simeq A^\chi_{K_\chi}$ and $(\Theta_{K/k} \otimes \mathbf{Z}_p)^\chi = (\Theta_{K_\chi/k} \otimes \mathbf{Z}_p)^\chi$. Therefore, the equality (3) is equivalent to the equality

$$\mathrm{Fitt}_{O_\chi[\Gamma_K]}(A^\chi_{K_\chi}) = (\Theta_{K_\chi/\mathbf{Q}} \otimes \mathbf{Z}_p)^\chi. \tag{4}$$

For $\chi \neq \omega$, by Corollary 0.10 in [1] and Lemma 3.2 in [2], we obtain the equality (4) using the same argument as in the paragraph following Lemma 3.2 in [2] (on page 562). For $\chi = \omega$, by Lemma 2.3 in [1], we may assume that $K_\omega/\mathbf{Q}$ satisfies the condition (A), and can apply Proposition 2.4 (2) in [2] to get the equality (4).

Proof of Theorem 3.6 in [2]. Let K/k be an abelian extension satisfying the same conditions as that of Theorem 3.6 in [2], which are exactly the same as the conditions of Proposition 0.1 above. We have to prove that

$$\mathrm{Fitt}_{O_\chi[\Gamma_K]}(A_K^\chi) = \left(\Theta_{K/k} \otimes \mathbf{Z}_p\right)^\chi$$

for all odd characters $\chi \in \widehat{\Delta}_K$. Each of these equalities is equivalent to the equality

$$\mathrm{Fitt}_{O_\chi[\Gamma_K]}(A_{K_\chi}^\chi) = \left(\Theta_{K_\chi/k} \otimes \mathbf{Z}_p\right)^\chi$$

as we mentioned above. By Proposition 3.4 in [2], we have

$$\mathrm{Fitt}_{O_\chi[\Gamma_K]}(A_{K_\chi}^\chi) \subset \left(\Theta_{K_\chi/k} \otimes \mathbf{Z}_p\right)^\chi \tag{5}$$

for all odd characters $\chi \in \widehat{\Delta}_K$ (Note that all χ's are different from the Teichmüller character because $\mu_p \not\subset K$ by our assumption).

We show the other inclusion

$$\left(\Theta_{K_\chi/k} \otimes \mathbf{Z}_p\right)^\chi \subset \mathrm{Fitt}_{O_\chi[\Gamma_K]}(A_{K_\chi}^\chi). \tag{6}$$

By the definition of Stickelberger ideal, we know easily that $\left(\Theta_{K_\chi/k} \otimes \mathbf{Z}_p\right)^\chi$ is generated by $\{\nu_{K_\chi/F}(\theta_{F/k}^\chi) \,|\, F \in \mathscr{M}_{K_\chi/k}\}$ (see page 47 in [1] for the definition of $\mathscr{M}_{K_\chi/k}$). Since $[K_{(\Delta)} : k]$ is prime to p, we know that $\theta_{K_{(\Delta)}/k}^\chi \in \mathrm{Fitt}_{O_\chi[\Gamma_{K_{(\Delta)}}]}(A_{K_{(\Delta)}}^\chi)$. By induction, we may assume that $\theta_{F/k}^\chi \in \mathrm{Fitt}_{O_\chi[\Gamma_F]}(A_F^\chi)$ for any $F \in \mathscr{M}_{K_\chi/k}$, $F \neq K_\chi$. We have to show that $\nu_{K_\chi/F}(\theta_{F/k}^\chi) \in \mathrm{Fitt}_{O_\chi[\Gamma_K]}(A_{K_\chi}^\chi)$. By the condition ($\mathrm{A}_p$), we can reduce to the case that there is a prime v of k such that the only primes above v are ramified in K_χ/F. In this case, the inclusion $\nu_{K_\chi/F}\left(\mathrm{Fitt}_{O_\chi[\Gamma_F]}(A_F^\chi)\right) \subset \mathrm{Fitt}_{O_\chi[\Gamma_K]}(A_{K_\chi}^\chi)$ can be proved by the same argument as that on page 62 in [1]. This proves that $\nu_{K_\chi/F}(\theta_{F/k}^\chi) \in \mathrm{Fitt}_{O_\chi[\Gamma_K]}(A_{K_\chi}^\chi)$.

Thus it suffices to prove $\theta_{K_\chi/k} \in \mathrm{Fitt}_{O_\chi[\Gamma_K]}(A_{K_\chi}^\chi)$. Since K_χ/k satisfies the assumptions of Proposition 0.1, applying this proposition, for the character $\psi \in \widehat{\Gamma}_K$ as in the proposition we have $\psi(\theta_{K_\chi/k}^\chi) = \psi(x)$ for some $x \in \mathrm{Fitt}_{O_\chi[\Gamma_K]}(A_{K_\chi}^\chi)$. By the inclusion (5), we can write $x = \alpha\theta_{K_\chi/k}^\chi + y$ for some $\alpha \in O_\chi[\Gamma_K]$ and $y \in \langle\{\nu_{K_\chi/F}(\theta_{F/k}^\chi) \,|\, F \in \mathscr{M}_{K_\chi/k},\, F \neq K_\chi\}\rangle_{O_\chi[\Gamma_K]}$. Since ψ is faithful on $P_{\mathfrak{l}_i}$ for all i, we have $\psi(y) = 0$, and therefore $\psi(\theta_{K_\chi/k}^\chi) = \psi(x) = \psi(\alpha)\psi(\theta_{K_\chi/k}^\chi)$. Since $\psi(\theta_{K_\chi/k}^\chi) = L(0, (\chi\psi)^{-1}) \neq 0$, we obtain $\psi(\alpha) = 1$. This implies that α is a unit of the local ring $O_\chi[\Gamma_K]$. Note that $y \in \mathrm{Fitt}_{O_\chi[\Gamma_K]}(A_{K_\chi}^\chi)$ as we showed in the previous paragraph. We conclude that $\theta_{K_\chi/k}^\chi = \alpha^{-1}(x - y) \in \mathrm{Fitt}_{O_\chi[\Gamma_K]}(A_{K_\chi}^\chi)$. This completes the proof of the inclusion (6).

Proof of Theorem 3.5 in [2]. Let $K/\mathbf{Q}$ be an abelian extension satisfying the same conditions as that of Theorem 3.5. Namely, we assume that $K/\mathbf{Q}$ satisfies the condition (A) (see page 554 in [2]), and that p is tamely ramified in K. We have to show (4) for all odd characters $\chi \in \widehat{\Delta}_K$.

For an odd character χ such that p is ramified in $K_{(\Delta),\chi}/\mathbf{Q}$, the argument explained in the 4th paragraph on page 565 in [2] (in the proof of Theorem 3.5) shows that the equality (4) holds.

For an odd character χ such that p is unramified in $K_{(\Delta),\chi}/\mathbf{Q}$, we have one-sided inclusion $\mathrm{Fitt}_{O_\chi[\Gamma_K]}(A_{K_\chi}^\chi) \subset (\Theta_{K_\chi} \otimes \mathbf{Z}_p)^\chi$ as we explained in the 3rd paragraph on page 565 in [2]. Since $\mu_p \not\subset K_\chi$ in this case, we can apply Proposition 0.1 to the abelian extension $K_\chi/\mathbf{Q}$ and obtain the other inclusion, using the same argument as the proof of Theorem 3.6 we just gave above.

Proof of Theorem 0.6 in [1]. Lemma 6.3 in [1] was used only to prove Theorems 0.4 and 0.6 in [1] by the first named author. We gave a correct proof of Theorem 0.4 above. Theorem 0.6 in [1] follows from Theorem 3.5 in [2] (as a special case). So both Theorems 0.4 and 0.6 in [1] are now proved.

Remark. The proof given in this erratum was obtained in 2012. We explained this proof to several people at that time. But since we were asked similar questions several times, we have decided to publish this erratum.

References

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