Notes on the dual of the ideal class groups of CM-fields

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Résumé

Dans cet article, pour une extension abélienne K/k de corps de nombres de type CM, nous proposons une conjecture qui décrit complètement l'idéal de Fitting de la partie moins du dual de Pontryagin du groupe de classes de rayon T de K, pour un ensemble T d'idéaux premiers, comme un $\mathrm{Gal}(K/k)$-module. Nous soulignons que nous considérons ici le groupe de classes au sens propre, sans laisser de côté les idéaux ramifiés (l'objet que nous étudions *n'est pas* le quotient du groupe de classes par le sous-groupe engendré par les classes des idéaux premiers ramifiés). Nous prouvons que notre conjecture est une conséquence de la conjecture de nombres de Tamagawa équivariante, et prouvons la version de notre conjecture en théorie d'Iwasawa.

Abstract

In this paper, for a CM abelian extension K/k of number fields, we propose a conjecture which describes completely the Fitting ideal of the minus part of the Pontryagin dual of the T-ray class group of K for a set T of primes as a $\mathrm{Gal}(K/k)$-module. Here, we emphasize that we consider the full class group, and do not throw away the ramifying primes (the object we study is *not* the quotient of the class group by the subgroup generated by the classes of ramifying primes). We prove that our conjecture is a consequence of the equivariant Tamagawa number conjecture, and also prove the Iwasawa theoretic version of our conjecture.

1 Introduction

It is an important and central theme in number theory to pursue the relationship between the arithmetic objects such as class groups of number fields and the analytic objects such as values of L-functions. Let k be a totally real number field and K/k a finite abelian extension with Galois group $G = \mathrm{Gal}(K/k)$ such that K is a CM-field. Then the Stickelberger element $\theta_{K,S}(0)$ (for the definition, see (3.1)) is related to the class group of K, which we regard as a G-module. For example, Brumer's conjecture says that, roughly speaking, the Stickelberger element is in the annihilator of the class group. It is also in the Fitting ideal of the class group in several cases, and the determination of the Fitting ideal of the class group is an important subject in Iwasawa theory ([18]). If $k = \mathbb{Q}$, it was proven that the Fitting ideal of the class group of K is equal to the Stickelberger ideal (except the 2-component, see [21]). However, for a general totally real field k, the Pontryagin dual of the class group is the right object to study the Fitting ideal (see [13]). In [11] Greither determined the Fitting ideal of the dual of the class group assuming the equivariant Tamagawa number conjecture and that the group of roots of unity is cohomologically trivial.

For any finite set S of primes of k we denote by S_K the set of primes of K above S. For a finite set T of primes of k that are unramified in K, let Cl_K^T be the $(\Pi_{w \in T_K} w)$-ray class group of K. In this paper we study Cl_K^T and generalize the main result in [11] to Cl_K^T (see Corollary 3.7). We note that we do not assume the cohomological triviality of the group of roots of unity as in [11].

Let S be a finite set of primes of k containing all infinite primes and ramifying primes in K such that $S \cap T = \emptyset$. We denote by $Cl_{K,S}^T$ the quotient of Cl_K^T by the subgroup generated by the classes of finite primes in S_K. Burns, Sano and the author proved as a special case of Theorem 1.5 (i) in [5] that the equivariant Tamagawa number conjecture ("eTNC" in short) implies that the Fitting ideal of a certain Selmer module (see Remark 2.2) is generated by the Stickelberger element $\theta_{K,S}^T$, and $Cl_{K,S}^T$ appears as a subgroup of the Selmer module. Since $Cl_{K,S}^T$ is a subgroup, this does not give information on the Fitting ideal of $Cl_{K,S}^T$ in general. Also, $Cl_{K,S}^T$ is smaller than the full class group Cl_K^T which we want to study.

In order to overcome these difficulties we use the beautiful ideas in Greither's paper [11]. An important idea in [11] which we also use here is to use "the local modules" W_{K_w}, W_v by Gruenberg and Weiss [16], which we will introduce in §2. In this sense, this paper heavily relies on the ideas in [11]. A new idea in this paper is to consider a Tate sequence using linear duals $M^\circ = \mathrm{Hom}(M, \mathbb{Z})$ of modules M (see the exact sequences in Proposition 2.3 and Proposition 2.4).

In §2 we introduce homomorphisms ψ_S, ψ for a general Galois extension of number fields. The Pontryagin dual of Cl_K^T appears as the cokernel of the linear dual ψ° of ψ (see Proposition 2.4). The complex $\mathfrak{A} \xrightarrow{\psi_S} \mathfrak{B}$ represents $R\Gamma_T(\mathcal{O}_{K,S}, \mathbb{G}_m)$ in Burns, Sano and the author [5]. We compare in §3.2 the two homomorphisms ψ_S°, ψ° in order to get information on Cl_K^T.

In §3.2 we propose Conjecture 3.2 which describes completely the Fitting ideal of the minus part of the Pontryagin dual of Cl_K^T, and prove it assuming Conjecture 3.4 which is a conjecture on the homomorphisms ψ_S. We show that eTNC implies Conjecture 3.4 (see Proposition 3.5), so also implies Conjecture 3.2 (see Corollary 3.7). We use eTNC in the style of [5] in Proposition 3.5.

In §4 we also prove, without assuming eTNC, Theorem 4.4 which is the Iwasawa theoretic version of Conjecture 3.2, and which determines completely the Fitting ideal of the T-modified Iwasawa modules. Theorem 4.4 can be regarded as a refinement of a result by Greither and Popescu [15], and a generalization of a result in [19] (see Remark 4.5).

There have been so many works related to the subject of this paper, and it is impossible to mention all of them here. Burns in [2] (see the proof of Corollary 3.11 in [2]) and Burns, Sano and the author in [5] Corollary 1.14 proved that the (dualizing) T-modified Stickelberger element $(\theta_{K,S}^T)^\#$ belongs to the Fitting ideal of the Pontryagin dual of Cl_K^T, assuming eTNC (for the involution $x^\#$, see the paragraph before Conjecture 3.2). A simple proof of this fact can be also found in the exposition [20] (see Corollary 4.5 (2) in [20]). A. Nickel proved that eTNC implies the p-component of the belonging of $\theta_{K,S}^T$ to the Fitting ideal of Cl_K^T if K/k satisfies several conditions, one of which is that all p-adic primes are almost tame in K/k (see [22] Theorem 5 and [23] Corollary 5.7). Note that such belonging does not hold in general (see [13]).

This paragraph is added in proof. After this paper was accepted to be published in this journal, a great progess was made by S. Dasgupta and M. Kakde. They have recently proved the strong Brumer-Stark conjecture, and proceeded even more; they have proved Conjecture 3.2 in this paper *unconditionally* in their final version [8].

Concerning the eTNC, a famous theorem by Burns and Greither [3] says that it holds when $k = \mathbb{Q}$. For the conditions which imply the eTNC for K/k with totally real k, see also [22] Theorem 4, [2] Corollary 3.8, and [6] Theorem 1.1.

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2 T-class groups of number fields as Galois modules

2.1 A homomorphism $\psi_S : \mathfrak{A} \longrightarrow \mathfrak{B}$

In this subsection we suppose that K/k is a finite Galois extension of number fields with $G = \mathrm{Gal}(K/k)$.

The goal of this subsection is to define two $\mathbb{Z}[G]$-modules $\mathfrak{A}$, $\mathfrak{B}$, and a homomorphism $\psi_S : \mathfrak{A} \longrightarrow \mathfrak{B}$, which represents $R\Gamma_T(\mathcal{O}_{K,S}, \mathbb{G}_m)$ in [5].

For any finite set S of primes of k we denote by S_K the set of primes of K above S. Let S_∞ be the set of all infinite primes of k. For any finite set S of primes of k such that $S \supset S_\infty$ we denote by $\mathcal{O}_{K,S}$ the subring of K consisting of integral elements outside S. The integer ring $\mathcal{O}_{K,S_\infty}$ is denoted by $\mathcal{O}_K$.

We take and fix a finite set T of finite primes of k that are unramified in K such that $(\mathcal{O}_K^T)^\times = \{x \in \mathcal{O}_K^\times \mid x \equiv 1 \pmod{w} \text{ for all primes } w \text{ above } T\}$ is $\mathbb{Z}$-torsion free.

For a finite set S of primes of k such that $S \supset S_\infty$ and $S \cap T = \emptyset$, we define

$$(\mathcal{O}_{K,S}^T)^\times = \{x \in \mathcal{O}_{K,S}^\times \mid x \equiv 1 \pmod{w} \text{ for all primes } w \in T_K\}$$

and $Cl_{K,S}^T$ to be the ray class group of $\mathcal{O}_{K,S}$ modulo $\Pi_{w \in T_K} w$.

We define a subgroup $J_{K,S}^T$ of the idèle group of K by

$$J_{K,S}^T = \prod_{w \in T_K} U_{K_w}^1 \times \prod_{w \notin (S \cup T)_K} U_{K_w} \times \prod_{w \in S_K} K_w^\times.$$

Let $S_{\mathrm{ram}}(K/k)$ be the set of all ramifying finite primes in K/k. From now on we fix a finite set S of primes of k such that $S \supset S_\infty \cup S_{\mathrm{ram}}(K/k)$ and $S \cap T = \emptyset$.

We also take a finite set S' of primes of k such that (i) $S' \supset S$, (ii) $Cl_{K,S'}^T = 0$ and that (iii) the decomposition groups G_v of v for all $v \in S'$ generate G.

Let C_K be the idèle class group of K. By definitions, we have an exact sequence

$$0 \longrightarrow (\mathcal{O}_{K,S}^T)^\times \longrightarrow J_{K,S}^T \longrightarrow C_K \longrightarrow Cl_{K,S}^T \longrightarrow 0. \tag{2.1}$$

From our assumption (ii) above, we also have an exact sequence

$$0 \longrightarrow (\mathcal{O}_{K,S'}^T)^\times \longrightarrow J_{K,S'}^T \longrightarrow C_K \longrightarrow 0$$

for S'.

For any group $\mathcal{G}$, we denote by $\Delta\mathcal{G}$ the augmentation ideal in $\mathbb{Z}[\mathcal{G}]$. For a prime w of K, we denote by G_w, I_w the decomposition subgroup and the inertia subgroup of w in G. We consider V_{K_w} the extension of ΔG_w by $K_w^\times$ corresponding to the local fundamental class (see Gruenberg and Weiss [16], Ritter and Weiss [24], Greither [11]); $0 \longrightarrow K_w^\times \longrightarrow V_{K_w} \longrightarrow \Delta G_w \longrightarrow 0$. If w is a finite prime, we define W_{K_w} by $W_{K_w} = \mathrm{Coker}(U_{K_w} \longrightarrow V_{K_w})$. Thus we have an exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow W_{K_w} \longrightarrow \Delta G_w \longrightarrow 0.$$

More explicitly, as in Ritter and Weiss [24] §3 and Greither [11] (23) on page 1412, one can write

$$W_{K_w} = \mathrm{Ker}(\Delta G_w \times \mathbb{Z}[G_w/I_w] \longrightarrow \mathbb{Z}[G_w/I_w])$$

where the above homomorphism is defined by $(x, y) \mapsto \overline{x} + (\mathcal{F}_w^{-1} - 1)y$ with $\overline{x} = x \bmod I_w$ and the Frobenius $\mathcal{F}_w$ of w in G_w/I_w. Note that we are using a homomorphism which is slightly different from (23) in [11]. This modification is necessary to get good bases of $\mathfrak{B}$ and $W^\circ_{S_\infty} \otimes \mathbb{Q}$ later. We note that if w is unramified in K/k, $I_w = 0$ and the projection to the second component $(x, y) \mapsto y$ gives an isomorphism $W_{K_w} \simeq \mathbb{Z}[G_w]$.

We put

$$V_{S'}^T = \prod_{w \in T_K} U^1_{K_w} \times \prod_{w \notin (S' \cup T)_K,} U_{K_w} \times \prod_{w \in (S')_K} V_{K_w},$$

and $W_{S'} = V_{S'}^T / J_{K,S'}^T$, $W_S = V_{S'}^T / J_{K,S}^T$. So we have $W_{S'} = \prod_{w \in S'_K} \Delta G_w$, and

$$W_S = \prod_{w \in S_K} \Delta G_w \times \prod_{w \in (S' \setminus S)_K} W_{K_w} = \prod_{w \in S_K} \Delta G_w \times \prod_{w \in (S' \setminus S)_K} \mathbb{Z}[G_w].$$

where we used the isomorphisms $W_{K_w} \simeq \mathbb{Z}[G_w]$ for $w \in (S' \setminus S)_K$ which we defined in the previous paragraph to get the second equality (note that primes in $S' \setminus S$ are unramified).

Let

$$0 \longrightarrow C_K \longrightarrow \mathfrak{O} \longrightarrow \Delta G \longrightarrow 0$$

be the extension corresponding to the global fundamental class as in [11], and consider the commutative diagram of exact sequences;

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J_{K,S'}^T & \longrightarrow & V_{S'}^T & \longrightarrow & W_{S'} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C_K & \longrightarrow & \mathfrak{O} & \longrightarrow & \Delta G & \longrightarrow & 0. \end{array}$$

The conditions (ii) and (iii) imply that the left and right vertical maps in the diagram are surjective (see also the exact sequence (2.1)). Therefore, the central vertical map is also surjective (see [11] page 1409). We next consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J^T_{K,S} & \longrightarrow & V^T_{S'} & \longrightarrow & W_S & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C_K & \longrightarrow & \mathfrak{O} & \longrightarrow & \Delta G & \longrightarrow & 0, \end{array}$$

which is obtained by replacing $J^T_{K,S'}$ by $J^T_{K,S}$. We put $A = \mathrm{Ker}(V^T_{S'} \longrightarrow \mathfrak{O})$ and $W'_S = \mathrm{Ker}(W_S \longrightarrow \Delta G)$. By the exact sequence (2.1) and the snake lemma, we have an exact sequence

$$0 \longrightarrow (\mathcal{O}^T_{K,S})^\times \longrightarrow A \longrightarrow W'_S \longrightarrow Cl^T_{K,S} \longrightarrow 0. \tag{2.2}$$

We put $\mathfrak{B} = \prod_{w\in(S')_K} \mathbb{Z}[G_w]$, and regard W_S as a submodule of $\mathfrak{B}$. By definition $\mathfrak{B}/W_S \simeq \prod_{w\in S_K} \mathbb{Z}$. The map $W_S \longrightarrow \Delta G$ can be extended to $\mathfrak{B} \longrightarrow \mathbb{Z}[G]$. Since S is non-empty, this is surjective. We denote by B the kernel of this homomorphism. Now we have a commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & W'_S & \longrightarrow & W_S & \longrightarrow & \Delta G & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B & \longrightarrow & \mathfrak{B} & \longrightarrow & \mathbb{Z}[G] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathfrak{X}_{K,S} & \longrightarrow & \prod_{w\in S_K}\mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

where $\mathfrak{X}_{K,S}$ is the kernel of the homomorphism $\prod_{w\in S_K}\mathbb{Z} \longrightarrow \mathbb{Z}$.

The map $A \longrightarrow W'_S$ obtained above defines a map $\psi_S : A \longrightarrow B$ by regarding W'_S as a submodule of B. We define $\mathcal{H}^T_{K,S}$ to be the cokernel of $\psi_S : A \longrightarrow B$. Thus we have obtained the following.

Proposition 2.1. *The homomorphism $\psi_S : A \longrightarrow B$ obtained above has kernel $(\mathcal{O}^T_{K,S})^\times$ and cokernel $\mathcal{H}^T_{K,S}$ for which, we have an exact sequence*

$$0 \longrightarrow Cl^T_{K,S} \longrightarrow \mathcal{H}^T_{K,S} \longrightarrow \mathfrak{X}_{K,S} \longrightarrow 0.$$

The module B is a finitely generated free $\mathbb{Z}[G]$-module.

Remark 2.2. The module $\mathcal{H}_{K,S}^T$ is isomorphic to the module $\mathcal{S}_{S,T}^{\mathrm{tr}}(\mathbb{G}_{m/K})$ constructed in [5] Definition 2.6 by Burns, Sano and the author. This module is also regarded as the "Weil étale cohomology group $H_T^2(\mathcal{O}_{K,S}, \mathbb{Z}(1))$". We note that the assumption $S \supset S_{\mathrm{ram}}(K/k)$ is important to get this Proposition.

Note that the middle horizontal exact sequence in the diagram before Proposition 2.1 splits. So we have an isomorphism $\mathfrak{B} \simeq B \oplus \mathbb{Z}[G]$. Therefore, putting $\mathfrak{A} = A \oplus \mathbb{Z}[G]$, we can define $\mathfrak{A} \longrightarrow \mathfrak{B}$ which is an extension of $A \longrightarrow B$, and whose kernel and cokernel coincide with the kernel and cokernel of $\psi_S : A \longrightarrow B$, respectively. We denote this map also by $\psi_S : \mathfrak{A} \longrightarrow \mathfrak{B}$.

For any $\mathbb{Z}[G]$-module M, we denote the linear dual by $M^\circ = \mathrm{Hom}(M, \mathbb{Z})$, and the Pontryagin dual by $M^\vee = \mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z})$ We endow them with the contragredient action of G. Taking the linear dual of $\psi_S : \mathfrak{A} \longrightarrow \mathfrak{B}$, we have $\psi_S^\circ : \mathfrak{B}^\circ \longrightarrow \mathfrak{A}^\circ$, whose cokernel we denote by $\mathcal{S}_{K,S}^T$. Of course, this is isomorphic to the cokernel of $B^\circ \longrightarrow A^\circ$.

Proposition 2.3. *The kernel of $\psi_S^\circ : \mathfrak{B}^\circ \longrightarrow \mathfrak{A}^\circ$ is isomorphic to $\mathfrak{X}_{K,S}^\circ$, and the cokernel $\mathcal{S}_{K,S}^T$ sits in an exact sequence*

$$0 \longrightarrow (Cl_{K,S}^T)^\vee \longrightarrow \mathcal{S}_{K,S}^T \longrightarrow ((\mathcal{O}_{K,S}^T)^\times)^\circ \longrightarrow 0.$$

This module $\mathcal{S}_{K,S}^T$ is isomorphic to the module $\mathcal{S}_{S,T}(\mathbb{G}_{m/K})$ in [5] Definition 2.1. One can regard $0 \longrightarrow \mathfrak{X}_{K,S}^\circ \longrightarrow \mathfrak{B}^\circ \longrightarrow \mathfrak{A}^\circ \longrightarrow \mathcal{S}_{K,S}^T \longrightarrow 0$ as a (linear dual version of) Tate sequence.

Proof. We denote by M the image of $\psi_S : \mathfrak{A} \longrightarrow \mathfrak{B}$. Then $0 \longrightarrow M^\circ \longrightarrow \mathfrak{A}^\circ \longrightarrow ((\mathcal{O}_{K,S}^T)^\times)^\circ \longrightarrow 0$ and

$$0 \longrightarrow \mathfrak{X}_{K,S}^\circ \longrightarrow \mathfrak{B}^\circ \longrightarrow M^\circ \longrightarrow \mathrm{Ext}^1(\mathcal{H}_{K,S}^T, \mathbb{Z}) = (Cl_{K,S}^T)^\vee \longrightarrow 0$$

are both exact since the torsion part of $\mathcal{H}_{K,S}^T$ is $Cl_{K,S}^T$ and the quotient $\mathcal{H}_{K,S}^T / Cl_{K,S}^T$ is isomorphic to $\mathfrak{X}_{K,S}$. Thus ψ_S° has kernel isomorphic to $\mathfrak{X}_{K,S}^\circ$. Also, concerning the cokernel $\mathcal{S}_{K,S}^T$, we get the exact sequence in Proposition 2.3 from the above two exact sequences. □

2.2 A homomorphism ψ

From now on we assume that K/k is a finite abelian extension such that k is totally real and K is a CM-field as in §1. We will define a homomorphism $\psi^\circ : (W_{S_\infty}^\circ)_{\mathbb{Z}_p}^- \longrightarrow (\mathfrak{A}_{\mathbb{Z}_p}^\circ)^-$ and study it.

We consider J^T_{K,S_∞}, and get an exact sequence

$$0 \longrightarrow (\mathcal{O}_K^T)^\times \longrightarrow J^T_{K,S_\infty} \longrightarrow C_K \longrightarrow Cl_K^T \longrightarrow 0 \tag{2.3}$$

from definitions. We define $W_{S_\infty} = V^T_{S'}/J^T_{K,S_\infty}$, so

$$W_{S_\infty} = \prod_{w \in (S' \setminus S_\infty)_K} W_{K_w} \times \prod_{w \in (S_\infty)_K} \Delta G_w.$$

From the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J^T_{K,S_\infty} & \longrightarrow & V^T_{S'} & \longrightarrow & W_{S_\infty} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C_K & \longrightarrow & \mathfrak{O} & \longrightarrow & \Delta G & \longrightarrow & 0, \end{array}$$

defining $W'_{S_\infty} = \mathrm{Ker}(W_{S_\infty} \longrightarrow \Delta G)$, we have an exact sequence

$$0 \longrightarrow (\mathcal{O}_K^T)^\times \longrightarrow A \longrightarrow W'_{S_\infty} \longrightarrow Cl_K^T \longrightarrow 0 \tag{2.4}$$

as we got the exact sequence (2.2) in the previous subsection.

Now we take an odd prime number p, and study the p-components of the above modules. For any $\mathbb{Z}[G]$-module M we write $M_{\mathbb{Z}_p} = M \otimes \mathbb{Z}_p$ and denote by $M^-_{\mathbb{Z}_p}$ the minus part of $M_{\mathbb{Z}_p}$ (which consists of elements on which the complex conjugation acts as -1).

Since $(\Delta G)^-_{\mathbb{Z}_p} = \mathbb{Z}_p[G]^-$, the sequence $0 \longrightarrow (W'_{S_\infty})^-_{\mathbb{Z}_p} \longrightarrow (W_{S_\infty})^-_{\mathbb{Z}_p} \longrightarrow (\Delta G)^-_{\mathbb{Z}_p} \longrightarrow 0$ splits as an exact sequence of $\mathbb{Z}_p[G]^-$-modules. Therefore, putting $\mathfrak{A} = A \oplus \mathbb{Z}[G]$ as in the previous subsection, we can construct a map

$$\psi : \mathfrak{A}^-_{\mathbb{Z}_p} \longrightarrow (W_{S_\infty})^-_{\mathbb{Z}_p}$$

which is an extension of $A^-_{\mathbb{Z}_p} \longrightarrow (W'_{S_\infty})^-_{\mathbb{Z}_p}$, whose kernel is $((\mathcal{O}_K^T)^\times_{\mathbb{Z}_p})^-$, and whose cokernel is $((Cl_K^T)_{\mathbb{Z}_p})^-$. Since $(\mathcal{O}_K^T)^\times$ is torsion free, we have $((\mathcal{O}_K^T)^\times_{\mathbb{Z}_p})^- = 0$. Therefore, we have an exact sequence

$$0 \longrightarrow \mathfrak{A}^-_{\mathbb{Z}_p} \xrightarrow{\psi} (W_{S_\infty})^-_{\mathbb{Z}_p} \longrightarrow ((Cl_K^T)_{\mathbb{Z}_p})^- \longrightarrow 0. \tag{2.5}$$

Taking the linear dual of the exact sequence (2.5), we obtain

$$0 \longrightarrow (W^\circ_{S_\infty})^-_{\mathbb{Z}_p} \xrightarrow{\psi^\circ} (\mathfrak{A}^\circ)^-_{\mathbb{Z}_p} \longrightarrow ((Cl_K^T)^\vee_{\mathbb{Z}_p})^- \longrightarrow 0$$

because $\mathrm{Ext}^1_{\mathbb{Z}_p}(((Cl_K^T)_{\mathbb{Z}_p})^-, \mathbb{Z}_p) = ((Cl_K^T)^\vee_{\mathbb{Z}_p})^-$.

For an infinite prime $v \in S_\infty$ we consider $\Delta_v = \bigoplus_{w|v} \Delta G_w$. Here and from now on, we use the notation $\bigoplus$ instead of $\prod$. Since the complex conjugation ρ acts as -1 on ΔG_w, $(\Delta_v)^-_{\mathbb{Z}_p} = (\bigoplus_{w|v} \Delta G_w)^-_{\mathbb{Z}_p}$ is a free $\mathbb{Z}_p[G]^-$-module of rank 1. Choosing a prime w above v and taking $e_v \in (\Delta_v)^-_{\mathbb{Z}_p}$ whose w-component is $\frac{1-\rho}{2}$ and other components are zero where ρ is the complex conjugation, we have an equality $(\Delta_v)^-_{\mathbb{Z}_p} = \mathbb{Z}_p[G]^- e_v$.

For a finite prime v in S', we put $W_v = \bigoplus_{w|v} W_{K_w}$. For $w|v$, by the description of W_{K_w} mentioned in the previous subsection, we can show that $W^\circ_{K_w}$ is isomorphic to the quotient of $\mathbb{Z}[G_w]/(N_{G_w}) \oplus \mathbb{Z}[G_w/I_w]$ by the submodule generated by $(N_{I_w}x, (\mathcal{F}_w - 1)(x))$ for all $x \in \mathbb{Z}[G/I_w]$ (see (24) on page 1412 in [11]). In this way we regard $W^\circ_{K_w}$ as a quotient of $\mathbb{Z}[G_w] \oplus \mathbb{Z}[G_w]$. The natural map $\mathbb{Z}[G_w] \oplus \mathbb{Z}[G_w] \longrightarrow W^\circ_{K_w}$ induces $c_w : \mathbb{Q}[G_w] \oplus \mathbb{Q}[G_w] \longrightarrow W^\circ_{K_w} \otimes \mathbb{Q}$. By Greither [11] Lemma 6.1 $c_w((1,1))$ is a basis of $W^\circ_{K_w} \otimes \mathbb{Q}$;

$$\mathbb{Q}[G_w] c_w((1,1)) = W^\circ_{K_w} \otimes \mathbb{Q}\,. \tag{2.6}$$

Since we slightly modified the homomorphism used in the definition of W_{K_w} as we mentioned in the previous subsection, we give a proof of (2.6). Since G is abelian, $\mathcal{F}_w$ and I_w are independent of the choice of w above v, so we write I_v and $\mathcal{F}_v$ for them. Put

$$g_v = 1 - \mathcal{F}_v + \#I_v. \tag{2.7}$$

This is a nonzero divisor in $\mathbb{Q}[G_w]$. Since

$$(0, -g_v) = (N_{I_v}, \mathcal{F}_v - 1) - (N_{I_v}, \#I_v),$$

$c_w((N_{I_v}, \mathcal{F}_v - 1)) = 0$ and $c_w((N_{I_v}, N_{I_v})) = c_w((N_{I_v}, \#I_v))$, we have

$$c_w((0,1)) = g_v^{-1} N_{I_v} c_w((1,1)) \tag{2.8}$$

in $W^\circ_{K_w} \otimes \mathbb{Q}$. This shows that both $c_w((0,1))$ and $c_w((1,0))$ are in the space generated by $c_w((1,1))$ and we get $\mathbb{Q}[G_w] c_w((1,1)) = W^\circ_{K_w} \otimes \mathbb{Q}$.

Thus, by fixing w above v and using c_w, we have an isomorphism $\mathbb{Q}[G] \simeq \bigoplus_{w|v} \mathbb{Q}[G_w]$ and a homomorphism

$$c_v : \mathbb{Q}[G] \oplus \mathbb{Q}[G] \longrightarrow \bigoplus_{w|v} W^\circ_{K_w} \otimes \mathbb{Q} = W^\circ_v \otimes \mathbb{Q}\,.$$

We define

$$e_v = c_v((1,1)) \in W^\circ_v \otimes \mathbb{Q}, \tag{2.9}$$

which is a basis of $W_v^\circ \otimes \mathbb{Q}$.

In this way we get a basis $(e_v)_{v\in S'}$ of a free $\mathbb{Q}[G]$-module $W_{S_\infty}^\circ \otimes \mathbb{Q}$ of rank $\#S'$.

For a finite prime $v \in S'$ we consider the equality (2.6). Since $W_{K_w}^\circ$ is generated by $c_w((1,1))$ and $c_w((0,1))$, using (2.8), we have

$$W_{K_w}^\circ = (1, \frac{1}{g_v} N_{I_v})\, \mathbb{Z}[G_w] c_w((1,1)).$$

Therefore, we get

$$W_v^\circ = (1, \frac{1}{g_v} N_{I_v})\, \mathbb{Z}[G] e_v$$

where $g_v = 1 - \mathcal{F}_v + \#I_v$ as in (2.7).

Put

$$h_v = (1 - \frac{N_{I_v}}{\#I_v}) + \frac{N_{I_v}}{\#I_v} g_v \tag{2.10}$$

which is a nonzero divisor of $\mathbb{Q}[G]$ as in Lemma 8.3 in Greither [11]. Then by this lemma we have

$$(1, \frac{1}{g_v} N_{I_v})\, \mathbb{Z}[G] = h_v^{-1} (N_{I_v}, 1 - \frac{N_{I_v}}{\#I_v} \mathcal{F}_v)\, \mathbb{Z}[G]$$

because $h_v = 1 - \frac{N_{I_v}}{\#I_v} \mathcal{F}_v + N_{I_v}$. Therefore, we have

$$W_v^\circ = (1, \frac{1}{g_v} N_{I_v})\, \mathbb{Z}[G] e_v = h_v^{-1} (N_{I_v}, 1 - \frac{N_{I_v}}{\#I_v} \mathcal{F}_v) e_v \tag{2.11}$$

and an isomorphism

$$W_v^\circ \simeq (N_{I_v}, 1 - \frac{N_{I_v}}{\#I_v} \mathcal{F}_v)\, \mathbb{Z}[G].$$

We note that if v is unramified, $(N_{I_v}, 1 - \frac{N_{I_v}}{\#I_v} \mathcal{F}_v)\, \mathbb{Z}[G] = (1, 1 - \mathcal{F}_v)\, \mathbb{Z}[G] = \mathbb{Z}[G]$. Therefore, recalling that $S_{\mathrm{ram}}(K/k)$ is the set of finite primes that are ramified in K, we have an isomorphism

$$\bigoplus_{v \in S' \setminus S_\infty} W_v^\circ \simeq \bigoplus_{v \in S_{\mathrm{ram}}(K/k)} (N_{I_v}, 1 - \frac{N_{I_v}}{\#I_v} \mathcal{F}_v)\, \mathbb{Z}[G] \oplus \bigoplus_{v \in S' \setminus (S_\infty \cup S_{\mathrm{ram}}(K/k))} \mathbb{Z}[G].$$

Thus we get information of the Galois module structure of $W_{S_\infty}^\circ$. We have obtained

Proposition 2.4.

$$0 \longrightarrow (W^{\circ}_{S_\infty})^-_{\mathbb{Z}_p} \xrightarrow{\psi^\circ} (\mathfrak{A}^{\circ}_{\mathbb{Z}_p})^- \longrightarrow ((Cl^T_K)^\vee_{\mathbb{Z}_p})^- \longrightarrow 0 \tag{2.12}$$

is exact. Here, $(\mathfrak{A}^{\circ}_{\mathbb{Z}_p})^-$ *is a free* $\mathbb{Z}_p[G]^-$*-module of rank* $\#S'$*, and*

$$(W^{\circ}_{S_\infty})^-_{\mathbb{Z}_p} \simeq \bigoplus_{v \in S' \setminus S_{\mathrm{ram}}(K/k)} \mathbb{Z}_p[G]^- \oplus \bigoplus_{v \in S_{\mathrm{ram}}(K/k)} (N_{I_v}, 1 - \frac{N_{I_v}}{\# I_v}\mathcal{F}_v)\, \mathbb{Z}_p[G]^-. \tag{2.13}$$

Proof. The exactness of the sequence (2.12) and the isomorphism (2.13) were already proved before this proposition. Since A is torsion free and cohomologically trivial, $\mathfrak{A}^-_{\mathbb{Z}_p}$ is also cohomologically trivial. Note that $\mathfrak{A}^-_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is isomorphic to $(W^\circ)^-_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ which is free of rank $\#S'$ over $\mathbb{Q}_p[G]^-$. So $\mathfrak{A}^-_{\mathbb{Z}_p}$ is a free $\mathbb{Z}_p[G]^-$-module of rank $\#S'$. □

3 Fitting ideals

3.1 Stickelberger ideals and a conjecture on Fitting ideals

Let K/k be a finite abelian CM-extension, and G, T,...be as in §2.2. We will first define a certain Stickelberger ideal $\Theta^T(K) \subset \mathbb{Z}[G]$.

For a character χ of G, we write $L(s, \chi)$ for the primitive L-function for χ; this function omits exactly the Euler factors of primes dividing the conductor of χ. We define

$$\omega^T = \sum_{\chi \in \hat{G}} L_T(0, \chi^{-1}) \epsilon_\chi \in \mathbb{Q}[G]$$

where

$$L_T(s, \chi) = (\prod_{v \in T} (1 - \chi(\mathcal{F}_v) N(v)^{1-s})) L(s, \chi)$$

is the T-modified L-function and $\epsilon_\chi = (\#G)^{-1} \sum_{\sigma \in G} \chi(\sigma)\sigma^{-1}$ is the idempotent of the χ-component. We know that $\omega^T \in \mathbb{Q}[G]$ by [25].

As in the previous section we denote by $S_{\mathrm{ram}}(K/k)$ the set of all ramifying finite primes in K/k. For $v \in S_{\mathrm{ram}}(K/k)$ we define a $\mathbb{Z}[G]$-module U_v in $\mathbb{Q}[G]$ by

$$U_v = (N_{I_v}, 1 - \frac{N_{I_v}}{\# I_v} \mathcal{F}_v^{-1})\, \mathbb{Z}[G] \subset \mathbb{Q}[G].$$

We define the Stickelberger ideal $\Theta^T(K)$ by

$$\Theta^T(K) = (\prod_{v \in S_{\mathrm{ram}}(K/k)} U_v)\omega^T$$

(cf. the definition of $\mathrm{SKu}'(K/k)$ in Greither [10] §2).

Proposition 3.1. *This Stickelberger ideal $\Theta^T(K)$ is in $\mathbb{Z}[G]$, namely it is an ideal of $\mathbb{Z}[G]$.*

Proof. This is essentially obtained in Greither [10] §2. In fact, our $\Theta^T(K)$ is the T-modified version of $\mathrm{SKu}'(K/k)$ in [10] §2, and one can show by the argument of Proposition 2.4 in [10] that it is equal to the T-modified version of $\mathrm{SKu}'_1(K/k)$ which can be seen to be integral. But here, we give a slightly different proof for the convenience of readers.

For an intermediate field F of K/k and a finite set S of finite primes that contains all ramifying primes in F, we define the equivariant zeta function $\theta_{F,S}(s)$ by

$$\theta_{F,S}(s) = \prod_{\chi \in \hat{\mathrm{Gal}(F/k)}} L_S(s, \chi^{-1})\epsilon_\chi \tag{3.1}$$

where $L_S(s,\chi)$ is the L-function obtained by removing the Euler factors for all $v \in S$. We consider its T-modification

$$\theta^T_{F,S}(s) = (\prod_{v \in T}(1 - \mathcal{F}_v^{-1} N(v)^{1-s}))\theta_{F,S}(s)$$

and the (S,T)-Stickelberger element

$$\theta^T_{F,S} = \theta^T_{F,S}(0) = (\prod_{v \in T}(1 - \mathcal{F}_v^{-1} N(v)))\theta_{F,S}(0). \tag{3.2}$$

It is known by Deligne and Ribet [9] and Cassou-Noguès [7] that $\theta^T_{F,S} \in \mathbb{Z}[\mathrm{Gal}(F/k)]$.

We put $S_{\mathrm{r}} = S_{\mathrm{ram}}(K/k)$. For a subset J of S_{r} we define K_J to be the maximal subextension of k in K that are unramified at all primes in J. Namely K_J is the fixed subfield of the subgroup of G generated by I_v for all $v \in J$. If J is empty, we take $K_J = K$. We put $N_J = \prod_{v \in J} N_{I_v} \in \mathbb{Z}[G]$. Then the multiplication by N_J defines a homomorphism

$$\nu_J : \mathbb{Z}[\mathrm{Gal}(K_J/k)] \longrightarrow \mathbb{Z}[G].$$

Note that this is not a norm homomorphism for K/K_J but the multiplication by some constant of the norm homomorphism. We have

$$\nu_J(\theta^T_{F_J,S_{\mathrm{r}}\setminus J}) = \prod_{v\in J} N_{I_v} \prod_{v\in S_{\mathrm{r}}\setminus J}(1-\frac{N_{I_v}}{\# I_v}\mathcal{F}_v^{-1})\omega^T.$$

This equality can be proved by comparing the χ-components of both sides for each character χ of G (see, for example, Lemma 2.1 in [18]).

This equality shows that $\Theta^T(K)$ is generated by $\nu_J(\theta^T_{F_J,S_{\mathrm{r}}\setminus J})$ for all subsets J of S_{r}. In particular, we obtain $\Theta^T(K)\subset\mathbb{Z}[G]$. This completes the proof. □

For any group ring $R[G]$ we denote by $x\mapsto x^{\#}$ the involution $R[G]\longrightarrow R[G]$ induced by $\sigma\mapsto\sigma^{-1}$ for all $\sigma\in G$.

Conjecture 3.2. *Put* $R=\mathbb{Z}[1/2][G]^-$, $((Cl_K^T)')^\vee = ((Cl_K^T\otimes\mathbb{Z}[1/2])^-)^\vee$, *and* $\Theta^T(K)'=(\Theta^T(K)\otimes\mathbb{Z}[1/2])^-\subset R$. *Then*

$$\mathrm{Fitt}_R(((Cl_K^T)')^\vee) = (\Theta^T(K)')^{\#}$$

holds true.

Now we study this conjecture, using Proposition 2.4. Consider the $\mathbb{Q}_p[G]^-$-homomorphism

$$\psi^\circ : (W^\circ_{S_\infty}\otimes\mathbb{Q}_p)^- \longrightarrow (\mathfrak{A}^\circ\otimes\mathbb{Q}_p)^- \ .$$

For a finite prime v in S' let e_v be as in (2.9) (see also (2.8)). For an infinite prime v we also defined e_v of $W^\circ_{S_\infty}\otimes\mathbb{Q}$ in §2.2. We also write e_v for the minus component of e_v, and take a basis $(e_v)_{v\in S'}$ of $(W^\circ_{S_\infty}\otimes\mathbb{Q}_p)^-$.

We consider $\det\psi^\circ\in\mathbb{Q}[G]^-$ with respect to the basis $(e_v)_{v\in S'}$ and a basis of $(\mathfrak{A}^\circ\otimes\mathbb{Z}_p)^-$ which is a free $\mathbb{Z}_p[G]^-$-module of rank $\# S'$ by Proposition 2.4. Then $\det\psi^\circ$ is determined up to unit of $\mathbb{Z}_p[G]^-$, and is a nonzero divisor of $\mathbb{Q}_p[G]^-$.

Recall that $h_v\in\mathbb{Q}[G]$ was defined in (2.10) (see also (2.7)).

Theorem 3.3. *For any odd prime number p we have*

$$\mathrm{Fitt}_{\mathbb{Z}_p[G]^-}(((Cl_K^T)^\vee)^-_{\mathbb{Z}_p}) = ((\prod_{v\in S_{\mathrm{ram}}(K/k)} U_v^{\#})(\prod_{v\in S'\setminus S_\infty} h_v^{-1}))^-_{\mathbb{Z}_p}\det\psi^\circ$$

where $\det\psi^\circ$ *is taken with respect to* $(e_v)_{v\in S'}$ *and a basis of* $(\mathfrak{A}^\circ\otimes\mathbb{Z}_p)^-$.

Proof. We use the presentation of $((Cl_K^T)^\vee)^-_{\mathbb{Z}_p}$ in Proposition 2.4. For a finite prime $v \in S'$ we proved in (2.11) that

$$W_v^\circ = h_v^{-1} U_v^{\#}\, \mathbb{Z}[G] e_v.$$

Therefore, the minus part of $(W^\circ_{S_\infty})_{\mathbb{Z}_p}$ can be written as

$$(W^\circ_{S_\infty})^-_{\mathbb{Z}_p} = \bigoplus_{v \in S_\infty} \mathbb{Z}_p[G]^- e_v \oplus \bigoplus_{v \in S' \setminus S_\infty} (h_v^{-1} U_v^{\#})^-_{\mathbb{Z}_p} e_v.$$

It follows from the exact sequence (2.12) that

$$\mathrm{Fitt}_{\mathbb{Z}_p[G]^-}(((Cl_K^T)^\vee)^-_{\mathbb{Z}_p}) = (\prod_{v \in S' \setminus S_\infty} h_v^{-1} U_v^{\#})^-_{\mathbb{Z}_p} \det \psi^\circ$$

If v is unramified, we have $U_v = \mathbb{Z}[G]$, which implies the conclusion of Theorem 3.3. □

By Theorem 3.3, we know that Conjecture 3.2 is equivalent to

$$(\prod_{v \in S' \setminus S_\infty} h_v^{-1}) \det \psi^\circ \cdot \mathbb{Z}_p[G]^- = (\omega^T\, \mathbb{Z}_p[G]^-)^{\#} \tag{3.3}$$

for all odd p.

3.2 A conjecture on $\det \psi_S$

For a finite set S of primes such that $S_\infty \cup S_{\mathrm{ram}}(K/k) \subset S \subset S'$, we consider the homomorphism $\psi_S : \mathfrak{A} \longrightarrow \mathfrak{B}$ which was constructed in §2.1, and study its determinant $\det \psi_S$.

Since we defined $\mathfrak{B}$ by $\mathfrak{B} = \bigoplus_{w \in (S')_K} \mathbb{Z}[G_w]$, fixing a prime w above v, we have $\mathfrak{B} = \bigoplus_{v \in S'} \mathbb{Z}[G]$. For each v, we take a canonical basis $(e_v^{\mathfrak{B}})_{v \in S'}$ of $\mathfrak{B}$ where $e_v^{\mathfrak{B}}$ is the element whose v-component is 1 and other components are zero.

We write $\mathfrak{A}_K$, $\mathfrak{B}_K$ for $\mathfrak{A}$, $\mathfrak{B}$ in order to clarify the field over which these modules are defined. For modules W_S,A, B,... and for an intermediate field F of K/k, we write $W_{F,S}$, A_F, B_F,...for the corresponding modules for F. Let $\psi_{F,S} : \mathfrak{A}_F \longrightarrow \mathfrak{B}_F$ denote the ψ_S for F. For $\mathfrak{B}_F$ we use the canonical basis $(e^{\mathfrak{B}}_{F,v})_{v \in S'}$ which is defined similar to the above $(e_v^{\mathfrak{B}})_{v \in S'} = (e^{\mathfrak{B}}_{K,v})_{v \in S'}$. More precisely, recalling that for $v \in S'$ we fixed a prime w of K above v, we define $e^{\mathfrak{B}}_{F,v}$ by using the prime w_F of F below w. In other words, $e^{\mathfrak{B}}_{F,v}$ is the image of $e^{\mathfrak{B}}_{K,v}$ under the canonical homomorphism $\mathfrak{B}_K \longrightarrow \mathfrak{B}_F$.

In order to compare $\psi_{F,S}$ for several F and S below, it is convenient to remove the ambiguity of the definition of this map (recall that $\psi_{F,S}$ was defined as an extension of $\psi_{F,S} : A_F \longrightarrow B_F$). We take and fix an infinite prime $v_\infty \in S$, and define $\psi_{F,S} : \mathfrak{A}_F = A_F \oplus \mathbb{Z}[\mathrm{Gal}(F/k)] \longrightarrow \mathfrak{B}_F$ by $\psi_{F,S}((0,1)) = e^{\mathfrak{B}}_{F,v_\infty}$.

For S such that $S_\infty \cup S_{\mathrm{ram}}(F/k) \subset S \subset S'$, we define $\theta^T_{F,S} \in \mathbb{Z}[\mathrm{Gal}(F/k)]$ as in (3.2).

Conjecture 3.4. *Put $r = \#S'$. The module $\mathfrak{A}_K$ is a free $\mathbb{Z}[G]$-module of rank r with a basis $(e^{\mathfrak{A}}_{K,i})_{1\leq i\leq r}$ such that for any intermediate field F of K/k and for any S such that $S_\infty \cup S_{\mathrm{ram}}(F/k) \subset S \subset S'$, we have*

$$\det(\psi_{F,S}) = \theta^T_{F,S}$$

Here, we define a basis $(e^{\mathfrak{A}}_{F,i})_{1\leq i\leq r}$ of $\mathfrak{A}_F$ as the image of $(e^{\mathfrak{A}}_{K,i})_{1\leq i\leq r}$ under the natural map $\mathfrak{A}_K \longrightarrow \mathfrak{A}_F$, and $\det(\psi_{F,S})$ is taken with respect to the bases $(e^{\mathfrak{A}}_{F,i})_{1\leq i\leq r}$ of $\mathfrak{A}_F$ and $(e^{\mathfrak{B}}_{F,v})_{v\in S'}$ of $\mathfrak{B}_F$.

We note that $\det(\psi_{F,S}) = \theta^T_{F,S}$ in Conjecture 3.4 is not an equality of ideals, but of elements in $\mathbb{Z}[\mathrm{Gal}(F/k)]$. Also, this conjecture asserts the existence of a good basis which can be used for any F and S. This equivariant statement would remind one of the equivariant Tamagawa number conjecture. In fact,

Proposition 3.5. *The equivariant Tamagawa number conjecture for K/k (eTNC in short) implies Conjecture 3.4.*

Proof. We use the notation and terminology in [5]. Let $R\Gamma_T((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbb{G}_m)$ be the complex defined in §2.2 in [5]. We use Conjecture 3.6 in [5] as eTNC, which claims that there is an element $z_{K/k,S,T}$ which is a basis of $\det_G R\Gamma_T((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbb{G}_m)$ as a $\mathbb{Z}[G]$-module such that $\vartheta_{\lambda_{K,S}}(z_{K/k,S,T}) = \theta^{T\,*}_{K/k,S}(0)$ where

$$\vartheta_{\lambda_{K,S}} : \det_G R\Gamma_T((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbb{G}_m) \otimes \mathbb{R} \xrightarrow{\simeq} \mathbb{R}[G]$$

is the isomorphism defined by using the Dirichlet regulator, $\det_G C^\bullet$ is the determinant module of the complex $C^\bullet$, and $\theta^{T\,*}_{K/k,S}(0)$ is the leading term of (S,T)-modified equivariant zeta function $\theta^T_{K/k,S}(s)$ at $s = 0$ (see §3 in [5]). We assume this conjecture. Since the complex $R\Gamma_T((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbb{G}_m)$ is represented by $\mathfrak{A}_K \xrightarrow{\psi_S} \mathfrak{B}_K$, $\mathfrak{A}_K$ is a free $\mathbb{Z}[G]$-module (see, for example, Lemma 3.2 in [1]). Also, since we fixed a basis of $\mathfrak{B}_K$, $z_{K/k,S,T}$ yields a basis of $\mathfrak{A}_K$ up to base change of determinant 1. We take such a basis $(e^{\mathfrak{A}}_{K,i})_{1\leq i\leq r}$, and use it from now on. By definition, we have $\det(\psi_{K,S}) = \theta^T_{K/k,S}(0) = \theta^T_{K/k,S}$. Also, for an intermediate field F, we know that the zeta element $z_{F/k,S,T}$ is the image of $z_{K/k,S,T}$. This shows that

$$\det(\psi_{F,S}) = \theta^T_{F/k,S}(0) = \theta^T_{F/k,S}.$$

Suppose that v is in $S \setminus S_{\mathrm{ram}}(F/k)$, and put $S'' = S \setminus \{v\}$. We will next prove $\det(\psi_{F,S''}) = \theta^T_{F/k,S''}$.

We first suppose that v splits completely in F. We put $Y_v = \bigoplus_{w\in\{v\}_F} \mathbb{Z}$. Recall that we fixed in the beginning of this subsection a prime w_F of F above v in S' when we defined $e^{\mathfrak{B}}_{F,v} \in \mathfrak{B}_F$. Since v splits completely in F, the prime w_F gives a basis of Y_v as a free $\mathbb{Z}[\mathrm{Gal}(F/k)]$ of rank 1. We have a distinguished triangle (see (18) in §3.2 in [5])

$$R\Gamma_T((\mathcal{O}_{F,S''})_{\mathcal{W}}, \mathbb{G}_m) \longrightarrow R\Gamma_T((\mathcal{O}_{F,S})_{\mathcal{W}}, \mathbb{G}_m) \longrightarrow Y_v[-1] \oplus Y_v[-2].$$

Put $\mathcal{Y}_v = Y_v[-1] \oplus Y_v[-2]$. We define a basis u of $\det \mathcal{Y}_v$, using the basis of Y_v we explained above.

Using the equality

$$\det R\Gamma_T((\mathcal{O}_{F,S})_{\mathcal{W}}, \mathbb{G}_m) = \det R\Gamma_T((\mathcal{O}_{F,S''})_{\mathcal{W}}, \mathbb{G}_m) \otimes \det \mathcal{Y}_v\ ,$$

we write $z_{F/k,S,T} = z \otimes u$ for some $z \in \det R\Gamma_T((\mathcal{O}_{F,S''})_{\mathcal{W}}, \mathbb{G}_m)$. Then we know $z = z_{F/k,S'',T}$. In fact, from the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\mathcal{O}^T_{F,S''})^\times \otimes \mathbb{R} & \longrightarrow & (\mathcal{O}^T_{F,S})^\times \otimes \mathbb{R} & \longrightarrow & Y_v \otimes \mathbb{R} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \lambda_{K,S''}} & & \downarrow{\scriptstyle \lambda_{K,S}} & & \downarrow{\scriptstyle \log N(v)} & & \\ 0 & \longrightarrow & \mathfrak{X}_{F,S''} \otimes \mathbb{R} & \longrightarrow & \mathfrak{X}_{F,S} \otimes \mathbb{R} & \longrightarrow & Y_v \otimes \mathbb{R} & \longrightarrow & 0 \end{array}$$

of exact sequences where the first two vertical arrows are regulator maps ($\lambda_{K,S}(a) = -\sum_{w\in S_K} \log|a|_w w$ and $\lambda_{K,S''}$ is defined similarly) and the right-most map is the multiplication by $\log N(v)$, we get

$$\vartheta_{\lambda_{F,S}}(z_{F/k,S,T}) = \vartheta_{\lambda_{F,S''}}(z) \log N(v).$$

This shows that

$$\vartheta_{\lambda_{F,S''}}(z) = \theta^{T\,*}_{F/k,S}(0)(\log N(v))^{-1} = \theta^{T\,*}_{F/k,S''}(0)$$

from which we deduce $z = z_{F/k,S'',T}$. This fact means the following. The complex $R\Gamma_T((\mathcal{O}_{F,S''})_{\mathcal{W}}, \mathbb{G}_m)$ is represented by $\mathfrak{A}_F \xrightarrow{\psi_{S''}} \mathfrak{B}_F$, and the basis $(e^{\mathfrak{A}}_{F,i})_{1\leq i\leq r}$ of $\mathfrak{A}_F$ and the basis $(e^{\mathfrak{B}}_{F,v})_{v\in S'}$ of $\mathfrak{B}_F$ yield an element z in

$\det R\Gamma_T((\mathcal{O}_{F,S''})_{\mathcal{W}}, \mathbb{G}_m)$. In this situation we have shown $z = z_{F/k,S'',T}$. Therefore, in particular, $\det(\psi_{F,S''}) = \theta^T_{F/k,S''}$ holds.

Next, we consider a general v. For an element $x \in \mathbb{Q}[\mathrm{Gal}(F/k)]$ and a character χ of $\mathrm{Gal}(F/k)$, we denote by $\epsilon_\chi = \epsilon_{F,\chi}$ the idempotent of the χ-component for $\mathrm{Gal}(F/k)$, and write $x^\chi = \epsilon_\chi x$ which is an element of the χ-component of $\mathbb{Q}(\mu_m)[\mathrm{Gal}(F/k)]$ where $m = \#\,\mathrm{Gal}(F/k)$. In order to prove $\det(\psi_{F,S''}) = \theta^T_{F/k,S''}$, it suffices to show the equality $\det(\psi_{F,S''})^\chi = (\theta^T_{F/k,S''})^\chi$ for all characters χ of $\mathrm{Gal}(F/k)$.

Note that v is unramified in F. If $\chi(\mathcal{F}_v) = 1$, then we can prove this equality by the same argument as when v splits completely. So we assume $\chi(\mathcal{F}_v) \neq 1$.

The images of $\psi_{F,S}$, $\psi_{F,S''}$ are in $W_{F,S}$, $W_{F,S''}$, respectively. The difference between $W_{F,S}$ and $W_{F,S''}$ lies only on the v-component; the former is $\Delta_{F,v} = \bigoplus_{w|v} \Delta G_w(F/k)$ and the latter is $W_{F,v} \simeq \mathbb{Z}[\mathrm{Gal}(F/k)]$ which is defined by $(x, y) \mapsto y$. If (x, y) is in $W_{F,v}$, then $x = (1 - \mathcal{F}_v^{-1})y$ by definition. Therefore, the natural map $W_{F,S''} \longrightarrow W_{F,S}$ is the multiplication by $1 - \mathcal{F}_v^{-1}$ on the v-component and the identity on other components. Let $\phi_v : \mathfrak{B}_F \longrightarrow \mathfrak{B}_F$ be the map which is the multiplication by $1 - \mathcal{F}_v^{-1}$ on the v-component and the identity on other components. Then we have

$$\psi_{F,S} = \phi_v \circ \psi_{F,S''}.$$

Since $\det \phi_v = 1 - \mathcal{F}_v^{-1}$, we get

$$\det(\psi_{F,S})^\chi = (1 - \chi(\mathcal{F}_v)^{-1}) \det(\psi_{F,S''})^\chi.$$

Therefore, the equality $\det(\psi_{F,S})^\chi = (\theta^T_{F/k,S})^\chi$ we obtained above implies $\det(\psi_{F,S''})^\chi = (\theta^T_{F/k,S''})^\chi$. Now we have obtained the equality for all χ-components, so we get

$$\det(\psi_{F,S''}) = \theta^T_{F/k,S''}.$$

By induction on $\#(S' \setminus S)$, starting from $S = S'$ and applying the above argument, we obtain for any S and any F

$$\det(\psi_{F,S}) = \theta^T_{F/k,S}.$$

□

It is also easily checked by the argument in the above proof that Conjecture 3.4 implies the eTNC, namely the existence of $z_{K/k,S,T}$.

We assume Conjecture 3.4, so the existence of a basis $(e^{\mathfrak{A}}_{K,i})_{1\le i\le r}$ of $\mathfrak{A}_K$. We denote by $(e^{\mathfrak{A}^\circ}_{K,i})_{1\le i\le r}$ the dual basis of $\mathfrak{A}^\circ_K$. We next study the homomorphism

$$\psi^\circ : (W^\circ_{K,S_\infty}\otimes\mathbb{Q}_p)^- \longrightarrow (\mathfrak{A}^\circ_K\otimes\mathbb{Q}_p)^- \ .$$

in Proposition 2.4. We take a basis $(e_v)_{v\in S'}$ of $W^\circ_{S_\infty}$ as in Theorem 3.3, and $(e^{\mathfrak{A}^\circ}_{K,i})_{1\le i\le r}$ as a basis of $(\mathfrak{A}^\circ_K\otimes\mathbb{Q}_p)^-$ to study $\det\psi^\circ\in\mathbb{Q}_p[G]^-$.

Theorem 3.6. *We assume Conjecture 3.4.*
(1) *We have*

$$\det\psi^\circ = (\omega^T)^\# \prod_{v\in S'\setminus S_\infty} h_v$$

where $\det\psi^\circ$ *is taken with respect to the bases* $(e_v)_{v\in S'}$ *and* $(e^{\mathfrak{A}^\circ}_{K,i})_{1\le i\le r}$*, and* h_v *was defined in (2.10).*
(2) *Conjecture 3.2 holds, namely*

$$\mathrm{Fitt}_{\mathbb{Z}[1/2]}(((Cl^T_K)')^\vee) = (\Theta^T(K)')^\#.$$

Proof. Theorem 3.6 (2) is a consequence of Theorems 3.3 and 3.6 (1) (see also (3.3)). So it suffices to prove Theorem 3.6 (1). To do this, we prove

$$(\det\psi^\circ)^\chi = ((\omega^T)^\# \prod_{v\in S'\setminus S_\infty} h_v)^\chi = L_T(0,\chi) \prod_{v\in S'\setminus S_\infty} h_v^\chi$$

for any character χ of G where we denote the χ-component $\epsilon_\chi x$ by x^χ for any element x in $\mathbb{Q}_p[G]^-$.

We use the notation in §2.2. Suppose that v is a finite prime in S', and w is the prime we fixed above v. It follows from (2.8) that $c_w((0,1)) = g_v^{-1}N_{I_v}c_w((1,1))$ and

$$c_w((1,0)) = c_w((1,1)) - c_w((0,1)) = (1-g_v^{-1}N_{I_v})c_w((1,1)). \qquad (3.4)$$

Let K_χ/k be the intermediate field of K/k corresponding to $\mathrm{Ker}\,\chi$. We put $F = K_\chi$, $S_\chi = S_\infty\cup S_{\mathrm{ram}}(F/k) = S_\infty\cup S_{\mathrm{ram}}(K_\chi/k)$, and consider $\psi_{F,S_\chi} : \mathfrak{A}_F \longrightarrow \mathfrak{B}_F$ and its dual $\psi^\circ_{F,S_\chi} : \mathfrak{B}^\circ_F \longrightarrow \mathfrak{A}^\circ_F$ with which we compare $\psi^\circ : (W^\circ_{K,S_\infty}\otimes\mathbb{Q}_p)^- \longrightarrow (\mathfrak{A}^\circ_K\otimes\mathbb{Q}_p)^-$.

Consider a homomorphism

$$\iota : \mathfrak{B}^\circ_F \xrightarrow{\alpha} W^\circ_{F,S_\infty}\otimes\mathbb{Q} \xrightarrow{\beta} W^\circ_{K,S_\infty}\otimes\mathbb{Q}$$

where α is induced by the natural inclusion $W_{F,S_\infty} \subset \mathfrak{B}_F$ and β is induced by the canonical homomorphism $W_{K,S_\infty} \longrightarrow W_{F,S_\infty}$. Since $\det\psi^\circ$, $\det\psi^\circ_{F,S_\chi}$ are defined by using the basis $(e_v)_{v\in S'}$, $(e^{\mathfrak{B}^\circ}_{F,v})_{v\in S'}$, respectively, we compare the image under the homomorphism ι of the dual basis $(e^{\mathfrak{B}^\circ}_{K,v})_{v\in S'}$ of $\mathfrak{B}^\circ_F$, obtained from $(e^{\mathfrak{B}}_{F,v})_{v\in S'}$, with the basis $(e_v)_{v\in S'}$ of $W^\circ_{K,S_\infty}\otimes\mathbb{Q}$. (Note that since $\mathfrak{B}_F$ was constructed from $W_{F,S}$, it depends on S though the notation does not carry S. In our case above, $S = S_\chi$.)

We denote by w' the prime of F below w that is the prime we fixed of K above v. Put $H_w = \mathrm{Gal}(K_w/F_{w'})$, and $G_{w'} = \mathrm{Gal}(F_{w'}/k)$, then since $G_w = \mathrm{Gal}(K_w/k_v)$, we have $G_{w'} = G_w/H_w$.

Suppose at first v is in $S_\chi = S_{\mathrm{ram}}(F/k)$. We will prove

$$\iota(e^{\mathfrak{B}^\circ}_{F,v}) = N_{H_w}(1 - g_v^{-1}N_{I_v})e_v. \tag{3.5}$$

Since w' is ramified, the w'-component of the natural map $W_{F,S_\chi} \longrightarrow \mathfrak{B}_F$ is $W_{F_{w'}} \longrightarrow \mathbb{Z}[G_w]; (x,y) \mapsto x$. Let

$$c_{w'} : \mathbb{Q}[G_{w'}] \oplus \mathbb{Q}[G_{w'}] \longrightarrow W^\circ_{F_{w'}} \otimes \mathbb{Q}$$

be the homomorphism obtained by applying the definition of c_w in §2.2 to w'. We consider the natural map $W_{F,S_\infty} \longrightarrow \mathfrak{B}_F$ and its dual $\alpha : \mathfrak{B}^\circ_F \longrightarrow W^\circ_{F,S_\infty} \subset W^\circ_{F,S_\infty} \otimes \mathbb{Q}$. Then the w'-component $\alpha_{w'}$ of α,

$$\alpha_{w'} : \mathbb{Z}[G_{w'}] \longrightarrow W^\circ_{F_{w'}} \otimes \mathbb{Q}$$

is described as $\alpha_{w'}(1) = c_{w'}((1,0))$ by what we explained above and the definitions of the modules. Since the diagram

$$\begin{array}{ccc} \mathbb{Z}[G_{w'}] & \xrightarrow{\alpha_{w'}} & W^\circ_{F_{w'}} \otimes \mathbb{Q} \\ \big\downarrow{\scriptstyle N_{H_w}} & & \big\downarrow \\ \mathbb{Z}[G_w] & \xrightarrow{p_1} & W^\circ_{K_w} \otimes \mathbb{Q} \end{array}$$

is commutative where the bottom map p_1 is $p_1(x) = c_w((x,0))$, the w'-component of $\iota = \beta \circ \alpha$ can be described as

$$\mathbb{Z}[G_{w'}] \longrightarrow W^\circ_{K_w} \otimes \mathbb{Q};\ 1 \mapsto N_{H_w} c_w((1,0)) = N_{H_w}(1 - g_v^{-1}N_{I_v})c_w((1,1))$$

where we used (3.4) to get the last equality. This shows that

$$\iota(e^{\mathfrak{B}^\circ}_{F,v}) = N_{H_w}(1 - g_v^{-1}N_{I_v})e_v,$$

which completes the proof of (3.5).

Since v is ramified, taking the χ-component (multiplying (3.5) by ϵ_χ), we get

$$\iota(e^{\mathfrak{B}^\circ}_{F,v}\epsilon_{F,\chi}) = e_v\epsilon_\chi \tag{3.6}$$

where $\epsilon_{F,\chi} = \#\,\mathrm{Gal}(F/k)^{-1}\sum_{\sigma\in\mathrm{Gal}(F/k)}\chi(\sigma)\sigma^{-1}$ is the idempotent of the χ-component of the group ring for $\mathrm{Gal}(F/k)$.

Next, suppose that v is unramified in $F = K_\chi$. This time v is not in S_χ, so the w'-component of $W_{F,S_\chi} \longrightarrow \mathfrak{B}_F$ is $W_{F_{w'}} \longrightarrow \mathbb{Z}[G_w]; (x,y) \mapsto y$. Therefore, $\alpha_{w'} : \mathbb{Z}[G_{w'}] \longrightarrow W^\circ_{F_{w'}} \otimes \mathbb{Q}$ is described as

$$\alpha_{w'}(1) = c_{w'}((0,1)).$$

Since v is unramified, I_v is in H_w. We note that the map $x \mapsto c_w((0,1))$ factors through $\mathbb{Z}[G_w/I_w]$. We denote this map $\mathbb{Z}[G_w/I_w] \longrightarrow W^\circ_{K_w}\otimes\mathbb{Q}$ by p_2. Then the diagram

$$\begin{array}{ccc}
\mathbb{Z}[G_{w'}] & \xrightarrow{\alpha_{w'}} & W^\circ_{F_{w'}}\otimes\mathbb{Q} \\
\Big\downarrow N_{H_w/I_v} & & \Big\downarrow \\
\mathbb{Z}[G_w/I_v] & \xrightarrow{p_2} & W^\circ_{K_w}\otimes\mathbb{Q}
\end{array}$$

is commutative. Thus the w'-component of $\iota = \beta\circ\alpha$, $\mathbb{Z}[G_{w'}] \longrightarrow W^\circ_{K_w}\otimes\mathbb{Q}$ is described as

$$1 \mapsto N_{H_w/I_v}c_w((0,1)) = N_{H_w/I_v}N_{I_v}g_v^{-1}c_w((1,1)) = N_{H_w}g_v^{-1}c_w((1,1))$$

where we used (2.8) to get the first equality. This implies that

$$\iota(e^{\mathfrak{B}^\circ}_{F,v}) = N_{H_w}g_v^{-1}e_v. \tag{3.7}$$

Multiplying ϵ_χ, we now get

$$\iota(e^{\mathfrak{B}^\circ}_{F,v}\epsilon_{F,\chi}) = g_v^{-1}e_v\epsilon_\chi. \tag{3.8}$$

Recall that $\det\psi^\circ$, $\det\psi^\circ_{F,S_\chi}$ are computed by using the basis $(e_v)_{v\in S'}$, $(e^{\mathfrak{B}^\circ}_{F,v})_{v\in S'}$, respectively. Therefore, it follows from (3.6) and (3.8) that

$$\det(\psi^\circ_{F,S_\chi})^\chi = ((\prod_{S'\setminus(S_\infty\cup S_\chi)} g_v^{-1})\det(\psi^\circ))^\chi = ((\prod_{S'\setminus S_\infty} h_v^{-1})\det(\psi^\circ))^\chi.$$

To get the last equality, we used (2.10). Using Conjecture 3.4, we obtain

$$\begin{aligned}\det(\psi^\circ)^\chi &= ((\prod_{S'\setminus S_\infty} h_v)\det(\psi^\circ_{F,S_\chi}))^\chi = ((\prod_{S'\setminus S_\infty} h_v)(\theta^T_{F,S_\chi})^\#)^\chi \\ &= L_T(0,\chi)\prod_{v\in S'\setminus S_\infty} h_v^\chi.\end{aligned}$$

This holds for all characters χ of G, so we get the desired equality in Theorem 3.6 (1). ☐

Corollary 3.7. *The equivariant Tamagawa number conjecture for K/k implies Conjecture 3.2* [1].

Proof. This follows from Theorem 3.6 (2) and Proposition 3.5. ☐

4 Cyclotomic $\mathbb{Z}_p$-extensions

Let K_∞/K be the cyclotomic $\mathbb{Z}_p$-extension and K_n the n-th layer. Put $\Lambda = \mathbb{Z}_p[[\mathrm{Gal}(K_\infty/k)]]$. We first take the projective limit of the sequence (2.12) in Proposition 2.4.

We denote by $S_{\mathrm{ram}} = S_{\mathrm{ram}}(K_\infty/k)$ the set of all finite primes of k ramifying in K_∞. The set S_p of all primes above p is contained in S_{ram}. We put $S_{\mathrm{ram}}^{\mathrm{non}\ p} = S_{\mathrm{ram}} \setminus S_p$. We take S' which satisfies the conditions in §2.1 for K/k and which satisfies $S' \supset S_{\mathrm{ram}}$.

We consider W_{K_n,S_∞} which is W_{S_∞} in §2.2 for K_n. Let w be a prime of K_∞. We also denote by w the prime of K_n below w and consider $W_{K_{n,w}}$. We define

$$W(K_\infty/k)^\circ_{\mathbb{Z}_p} = \varprojlim (W^\circ_{K_n,S_\infty}\otimes\mathbb{Z}_p),$$
$$W_w(K_\infty/k)^\circ_{\mathbb{Z}_p} = \varprojlim (W^\circ_{K_{n,w}}\otimes\mathbb{Z}_p)$$

and $W_v(K_\infty/k)^\circ_{\mathbb{Z}_p} = \bigoplus_{w|v} W_w(K_\infty/k)^\circ_{\mathbb{Z}_p}$ for a finite prime v of k.

We first consider a prime w above p. Suppose that n is sufficiently large such that K_∞/K_n is totally ramified at all primes above p. Consider the canonical exact sequences for $W_{K_{n,w}}$ and $W_{K_{n+1,w}}$ (see (1.4) in [16]). Then we have a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & W_{K_{n,w}} & \longrightarrow & \Delta D_w(K_n/k) & \longrightarrow & 0 \\ & & \downarrow\xi & & \downarrow & & \downarrow\nu & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & W_{K_{n+1,w}} & \longrightarrow & \Delta D_w(K_{n+1}/k) & \longrightarrow & 0 \end{array}$$

[1] See the comment in the end of §1 on the recent work by Dasgupta and Kakde [8].

where $D_w(K_n/k)$ is the decomposition subgroup of w in $\mathrm{Gal}(K_n/k)$, ξ is the multiplication by p and ν is the norm map. The above commutative diagram shows that for a prime w of K_∞ above p,

$$W_w(K_\infty/k)^\circ_{\mathbb{Z}_p} = \varprojlim \mathbb{Z}_p[D_w(K_n/k)]/(N_{D_w(K_n/k)}) = \mathbb{Z}_p[[D_w(K_\infty/k)]].$$

Therefore, $W_v(K_\infty/k)^\circ_{\mathbb{Z}_p} = \bigoplus_{w|v} W_w(K_\infty/k)^\circ_{\mathbb{Z}_p}$ is a free Λ-module of rank 1 for $v \in S_p$.

We use the notation in §2.2. Let $c_w : \mathbb{Z}[G_w(K_n/k)] \oplus \mathbb{Z}[G_w(K_n/k)] \longrightarrow W^\circ_{K_n,w}$ be the map obtained by applying to K_n/k the definition for K/k before Proposition 2.4 in §2.2. Taking the projective limit, we have a map

$$c_w : \mathbb{Z}_p[[D_w(K_\infty/k)]] \oplus \mathbb{Z}_p[[D_w(K_\infty/k)]] \longrightarrow W_w(K_\infty/k)^\circ_{\mathbb{Z}_p}.$$

What we have shown in the previous paragraph, means that $c_w((1,0))$ generates $W_w(K_\infty/k)^\circ_{\mathbb{Z}_p}$, namely $W_w(K_\infty/k)^\circ_{\mathbb{Z}_p} = \mathbb{Z}_p[[D_w(K_\infty/k)]]c_w((1,0))$.

Fixing w above v, we have a map

$$c_v : \mathbb{Z}_p[[\mathrm{Gal}(K_\infty/k)]] \oplus \mathbb{Z}_p[[\mathrm{Gal}(K_\infty/k)]] = \Lambda \oplus \Lambda \longrightarrow W_v(K_\infty/k)^\circ_{\mathbb{Z}_p}.$$

We put $e'_v = c_v((1,0))$. Then we have $W_v(K_\infty/k)^\circ_{\mathbb{Z}_p} = \Lambda e'_v$.

Next, suppose that v is a non p-adic finite prime. Note that the inertia group $I_v(K_\infty/k)$ of $\mathrm{Gal}(K_\infty/k)$ coincides with the inertia group $I_v(K/k)$ of $\mathrm{Gal}(K/k)$. We denote it by I_v.

We define c_v as above and also define $e'_v = c_v((1,0))$. Then the map $\Lambda \longrightarrow W_v(K_\infty/k)^\circ_{\mathbb{Z}_p}$, $a \mapsto ae'_v$ is injective because $\mathcal{F}_v - 1$ is a nonzero divisor in Λ.

Let $\mathcal{R}$ be the total quotient ring of Λ. Then $\mathcal{R} \longrightarrow W_v(K_\infty/k)^\circ_{\mathbb{Z}_p} \otimes \mathcal{R}$ which is defined by $a \mapsto ae'_v$ is bijective. Since $W_v(K_\infty/k)^\circ_{\mathbb{Z}_p}$ is generated by $c_v((1,0))$ and $c_v((0,1))$, we have

$$W_v(K_\infty/k)^\circ_{\mathbb{Z}_p} = (1, \frac{N_{I_v}}{1-\mathcal{F}_v})\Lambda e'_v. \tag{4.1}$$

We now suppose that v is an infinite prime. Then the v-component of W_{K_n} is canonically isomorphic to $\mathbb{Z}[\mathrm{Gal}(K_n/k)]$ (after fixing a prime w above v). We took a generator e_v of $(W^\circ_{K_n,v})^-_{\mathbb{Z}_p}$ in §2.2. We define e'_v to be the projective limit of e_v as $n \to \infty$. So in this case we have $W_v(K_\infty/k)^\circ_{\mathbb{Z}_p} = \Lambda e'_v$. Thus if v is p-adic or infinite,

$$W_v(K_\infty/k)^\circ_{\mathbb{Z}_p} = \Lambda e'_v$$

holds.

We regard e'_v as an element of $W(K_\infty/k)^\circ_{\mathbb{Z}_p}$ (by defining that the v'-component of e'_v is zero for all $v' \neq v$). Then $(e'_v)_{v\in S'}$ is a basis of a free $\mathcal{R}$-module $W(K_\infty/k)^\circ_{\mathbb{Z}_p} \otimes \mathcal{R}$.

We first note that $\mathrm{Coker}(V^T_{K_n,S'} \longrightarrow \mathfrak{O}_{K_n}) \otimes \mathbb{Z}_p = 0$ for any $n \geq 0$ where $V^T_{K_n,S'}$, $\mathfrak{O}_{K_n}$ are $V^T_{S'}$, $\mathfrak{O}$ for K_n. This can be checked as follows. Put $G_n = \mathrm{Gal}(K_n/K)$. Since the natural maps induce isomorphisms $(V^T_{K_n,S'})_{G_n} \simeq V^T_{K,S'}$ and $(\mathfrak{O}_{K_n})_{G_n} \simeq \mathfrak{O}_K$, the surjectivity of $V^T_{K,S'} \longrightarrow \mathfrak{O}_K$ implies $\mathrm{Coker}(V^T_{K_n,S'} \longrightarrow \mathfrak{O}_{K_n})_{G_n} = 0$. Therefore, Nakayama's lemma implies $\mathrm{Coker}(V^T_{K_n,S'} \longrightarrow \mathfrak{O}_{K_n}) \otimes \mathbb{Z}_p = 0$.

Thus we have exact sequences (2.12) in Proposition 2.4 for any K_n, and can take the projective limit.

Consider $\mathfrak{A}^\circ_{K_n}$ which is $\mathfrak{A}^\circ$ for K_n, and define

$$\mathfrak{A}^\circ(K_\infty/k)_{\mathbb{Z}_p} = \varprojlim(\mathfrak{A}^\circ_{K_n} \otimes \mathbb{Z}_p).$$

The minus part $\mathfrak{A}^\circ(K_\infty/k)^-_{\mathbb{Z}_p}$ is a free Λ^--module of finite rank. We put

$$Cl^T_{K_\infty,p} = \varinjlim(Cl^T_{K_n} \otimes \mathbb{Z}_p).$$

Taking the projective limit of the exact sequence (2.12), we have an exact sequence

$$0 \longrightarrow (W(K_\infty/k)^\circ_{\mathbb{Z}_p})^- \longrightarrow \mathfrak{A}^\circ(K_\infty/k)^-_{\mathbb{Z}_p} \longrightarrow ((Cl^T_{K_\infty,p})^\vee)^- \longrightarrow 0. \qquad (4.2)$$

Let W' be the Λ-submodule of $(W(K_\infty/k)^\circ_{\mathbb{Z}_p})^-$ generated by e'_v for all $v \in S'$. Then W' is a free Λ^--module. We write f for the restriction of the homomorphism $(W(K_\infty/k)^\circ_{\mathbb{Z}_p})^- \longrightarrow \mathfrak{A}^\circ(K_\infty/k)^-_{\mathbb{Z}_p}$ to W'. We consider $\det f$ with respect to the basis $(e'_v)_{v\in S'}$. So $\det f$ is determined up to $\Lambda^\times$.

Lemma 4.1. *Suppose that $f : W' \longrightarrow \mathfrak{A}^\circ(K_\infty/k)^-_{\mathbb{Z}_p}$ is the homomorphism defined above, and we take $\det f$ with respect to the basis $(e'_v)_{v\in S'}$. Then we have*

$$\mathrm{Fitt}_{\Lambda^-}(((Cl^T_{K_\infty,p})^\vee)^-) = (\prod_{v\in S'\setminus(S_\infty\cup S_p)} (1, \frac{N_{I_v}}{1-\mathcal{F}_v})) \det f$$

where $I_v = I_v(K/k)$ for each v.

Proof. If $v \in S_\infty \cup S_p$, we know $W_v(K_\infty/k)^\circ_{\mathbb{Z}_p} = \Lambda e'_v$. For $v \in S' \setminus (S_\infty \cup S_p)$, we have $W_v(K_\infty/k)^\circ_{\mathbb{Z}_p} = (1, \frac{N_{I_v}}{1-\mathcal{F}_v})\Lambda e'_v$ by (4.1). Therefore, we have

$$(W(K_\infty/k)^\circ_{\mathbb{Z}_p})^- = \bigoplus_{v \in S_\infty \cup S_p} \Lambda^- e'_v \oplus \bigoplus_{v \in S' \setminus (S_\infty \cup S_p)} (1, \frac{N_{I_v}}{1-\mathcal{F}_v})\Lambda^- e'_v.$$

Therefore, it follows from (4.2) that

$$\mathrm{Fitt}_{\Lambda^-}(((Cl^T_{K_\infty,p})^\vee)^-) = (\prod_{v \in S' \setminus (S_\infty \cup S_p)} (1, \frac{N_{I_v}}{1-\mathcal{F}_v})) \det f \ .$$

□

Our final task is to determine $\det f$.

For a finite set S which contains all ramifying primes in K_∞, we denote by $\theta^T_{K_n,S}$ the (S,T)-modified Stickelberger element as in (3.2), and by $\theta^T_{K_\infty,S}$ its projective limit (for $n \gg 0$) in Λ^-. We simply write $\theta^T_{K_\infty}$ when $S = S_{\mathrm{ram}}$. Also, for an intermediate CM-subfield F of K/k and the cyclotomic $\mathbb{Z}_p$-extension F_∞/F, we define $\theta^T_{F_\infty}$ to be $\theta^T_{F_\infty,S_{\mathrm{ram}}(F_\infty/k)}$ where $S_{\mathrm{ram}}(F_\infty/k)$ is the set of all ramifying primes in F_∞/k. We also use elements $\theta^{T\#}_{F_\infty,S}$, $\theta^{T\#}_{F_\infty}$,... where $\#$ is the involution of the group ring induced by $\sigma \mapsto \sigma^{-1}$ for elements σ in the group as in §3.1.

Lemma 4.2. *We assume $\mu = 0$ for K_∞/k. We have*

$$(\det f)\Lambda^- = \theta^{T\#}_{K_\infty,S'}\Lambda^-$$

as ideals of Λ^-.

Proof. We write $\mathfrak{A}^\circ = \mathfrak{A}^\circ(K_\infty/k)^-_{\mathbb{Z}_p}$, and $Cl^\vee = ((Cl^T_{K_\infty,p})^\vee)^-$. Since

$$(W_v(K_\infty/k)^\circ_{\mathbb{Z}_p}/\Lambda e'_v)^- \simeq \Lambda^-/(1-\mathcal{F}_v, \Delta I_v),$$

for $v \in S' \setminus (S_\infty \cup S_p)$, the exact sequence (4.2) yields an exact sequence

$$0 \longrightarrow \bigoplus_{v \in S' \setminus (S_\infty \cup S_p)} \Lambda^-/(1-\mathcal{F}_v, \Delta I_v) \longrightarrow \mathfrak{A}^\circ \,/\, \mathrm{Image}\, f \longrightarrow Cl^\vee \longrightarrow 0.$$

Since $\mathrm{Gal}(K_\infty/k)$ is abelian and an extension of $\mathbb{Z}_p$ by a finite abelian group, we can write $\mathrm{Gal}(K_\infty/k) \simeq G' \times \mathbb{Z}_p$ for some finite subgroup G'. Let K' be the field such that $\mathrm{Gal}(K_\infty/K') = \mathbb{Z}_p$, $\mathrm{Gal}(K'/k) = G'$, $K' \cap k_\infty = k$.

By taking $K = K'$ from the first, we may assume $K \cap k_\infty = k$. Then Λ is isomorphic to the power series ring $\mathbb{Z}_p[G][[t]]$.

For an odd character χ of G, we consider the χ-quotient only in the proof of this lemma. For a $\mathbb{Z}_p[G]$-module M and $\chi : G \longrightarrow \overline{\mathbb{Q}}_p^\times$ which is a character of G, whose image is in an algebraic closure of $\mathbb{Q}_p$, we define the χ-quotient $[M]_\chi$ by $[M]_\chi = M \otimes_{\mathbb{Z}_p[G]} \mathcal{O}_\chi$ where $\mathcal{O}_\chi = \mathbb{Z}_p[\mathrm{Image}\,\chi]$ on which G acts via χ. For an element x of M, the image of x in $[M]_\chi$ is denoted by x_χ.

Taking the χ-quotients of the above exact sequence, we get an exact sequence

$$\bigoplus_{v \in S' \setminus (S_\infty \cup S_p)} [\Lambda^-/(1 - \mathcal{F}_v, \Delta I_v)]_\chi \longrightarrow [\mathfrak{A}^\circ \,/\, \mathrm{Image}\, f]_\chi \longrightarrow [Cl^\vee]_\chi \longrightarrow 0.$$

The kernel of the first map is finite since

$$\bigoplus_{v \in S' \setminus (S_\infty \cup S_p)} [(\Lambda^-/(1 - \mathcal{F}_v, \Delta I_v)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p]_\chi \longrightarrow [(\mathfrak{A}^\circ \,/\, \mathrm{Image}\, f) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p]_\chi$$

is injective. We consider the characteristic ideals over $[\Lambda]_\chi = \mathcal{O}_\chi[[t]]$. We know $\mathrm{char}([\mathfrak{A}^\circ \,/\, \mathrm{Image}\, f]_\chi) = ((\det f)_\chi)$. If χ is trivial on I_v, we have $\mathrm{char}([\Lambda^-/(1 - \mathcal{F}_v, \Delta I_v)]_\chi) = ((1 - \mathcal{F}_v)_\chi)$. Otherwise, $[\Lambda^-/(1 - \mathcal{F}_v, \Delta I_v)]_\chi$ is finite.

Let K_χ be the intermediate field of K/k corresponding to $\mathrm{Ker}\,\chi$, and $K_{\chi\,\infty}$ its cyclotomic $\mathbb{Z}_p$-extension. Then the characteristic ideal of $[Cl^\vee]_\chi$ is generated by $(\theta^{T\,\#}_{K_{\chi\,\infty}})_\chi$ by the main conjecture proved by Wiles [26]. Therefore, the above exact sequence implies that

$$\mathrm{char}([\mathfrak{A}^\circ \,/\, \mathrm{Image}\, f]_\chi) = ((\det f)_\chi) = ((\prod_{\chi_{|I_v = 1}} (1 - \mathcal{F}_v)_\chi)(\theta^{T\,\#}_{K_{\chi\,\infty}})_\chi)$$

where v ranges over all primes in S' which are unramified in $K_{\chi\,\infty}$. Let

$$\mathrm{res}_{K_{\chi\,\infty}} : \Lambda = \mathbb{Z}_p[[\mathrm{Gal}(K_\infty/k)]] \longrightarrow \mathbb{Z}_p[[\mathrm{Gal}(K_{\chi\,\infty}/k)]]$$

be the restriction map. Since we know

$$\mathrm{res}_{K_{\chi\,\infty}}(\theta^{T\,\#}_{K_\infty, S'}) = \prod_{\chi_{|I_v = 1}} (1 - \mathcal{F}_v)\theta^{T\,\#}_{K_{\chi\,\infty}},$$

we obtain

$$((\det f)_\chi) = ((\theta^{T\,\#}_{K_\infty, S'})_\chi)$$

as ideals of $\mathcal{O}_\chi[[t]]$. Since this equality holds for any odd character χ of G, the conclusion of Lemma 4.2 follows from the next lemma. $\square$

Lemma 4.3. *Let a, b be elements of Λ such that the μ-invariants of a_χ and b_χ in $\mathcal{O}_\chi[[t]]$ are zero for any character χ of G. If $(a_\chi) = (b_\chi)$ holds as ideals of $\mathcal{O}_\chi[[t]]$ for all χ of G, we get $(a) = (b)$ as ideals of Λ.*

Proof. This lemma seems to be well-known, but we give here a proof. By Proposition 2.1 in [4] we may assume that a, b are distinguished polynomials in the sense of [4]. We write $a = bq + r$ for some $q \in \Lambda$ and some polynomial r whose degree is smaller than the degree of b. Here, Λ is semi-local, and the degree means the vector of the degree of each component (see [4] §2). The condition $(a_\chi) = (b_\chi)$ in $\mathcal{O}_\chi[[t]]$ implies $r_\chi = 0$ for any χ, so we have $r = 0$ and $(a) \subset (b)$. The converse is also true, and we get $(a) = (b)$. □

Now we can prove the main theorem in this section. Recall that $S_{\mathrm{ram}}^{\mathrm{non}\ p} = S_{\mathrm{ram}} \setminus S_p$.

Theorem 4.4. *Assuming $\mu = 0$* [2] *for K_∞/k, we have*

$$\mathrm{Fitt}_{\Lambda^-}(((Cl_{K_\infty,p}^T)^\vee)^-) = (\prod_{v \in S_{\mathrm{ram}}^{\mathrm{non}\ p}} (1, \frac{N_{I_v}}{1-\mathcal{F}_v}))\theta_{K_\infty}^{T\,\#} .$$

Proof. By Lemmas 4.1 and 4.2, we have

$$\begin{aligned}\mathrm{Fitt}_{\Lambda^-}(((Cl_{K_\infty,p}^T)^\vee)^-) &= (\prod_{v \in S' \setminus (S_\infty \cup S_p)} (1, \frac{N_{I_v}}{1-\mathcal{F}_v}))\det f \\ &= (\prod_{v \in S' \setminus (S_\infty \cup S_p)} (1, \frac{N_{I_v}}{1-\mathcal{F}_v}))\theta_{K_\infty,S'}^{T\,\#} .\end{aligned}$$

If v is unramified, we know $(1, \frac{N_{I_v}}{1-\mathcal{F}_v})(1-\mathcal{F}_v) = \Lambda$, so using

$$\theta_{K_\infty,S'}^{T\,\#} = \prod_{S' \setminus (S_\infty \cup S_{\mathrm{ram}})} (1-\mathcal{F}_v)\theta_{K_\infty}^{T\,\#},$$

we obtain

$$(\prod_{v \in S' \setminus (S_\infty \cup S_p)} (1, \frac{N_{I_v}}{1-\mathcal{F}_v}))\theta_{K_\infty,S'}^{T\,\#} = (\prod_{v \in S_{\mathrm{ram}}^{\mathrm{non}\ p}} (1, \frac{N_{I_v}}{1-\mathcal{F}_v}))\theta_{K_\infty}^{T\,\#}.$$

This completes the proof of Theorem 4.4. □

[2]Using the recent groundbreaking result [8] by Dasgupta and Kakde, H. Johnston and A. Nickel [17] proved the equivariant Iwasawa main conjecture *unconditionally*, namely without assuming $\mu = 0$. Using [17] (or [8] directly), we can remove the assumption $\mu = 0$.

Remark 4.5. (1) Greither and Popescu proved that $\theta_{K_\infty}^{T\#}$ is in the Fitting ideal of $((Cl_{K_\infty,p}^T)^\vee)^-$ in [15]. The above theorem gives a refinement in the sense that it gives a full description of the Fitting ideal.
(2) The author obtained a similar result for the non-Teichmüller character components of class groups with $T = \emptyset$, assuming Leopoldt's conjecture in [19] Theorem A.5. Theorem 4.4 implies Theorem A.5 in [19] without assuming Leopoldt's conjecture by choosing T suitably as a set of auxiliary primes. Thus Theorem 4.4 is also a generalization of the main result in the Appendix in [19].
(3) When we study the non-Teichmüller character components of the class groups (and the T-modified class groups), we saw that the duals of the class groups are suitable objects for studying their Galois module structure in our previous papers (see [19], [11], [10], [13], for example). One can see by Proposition 2.4 in this paper why the dual of the class group is relatively easier to handle than the class group itself. Concerning the study on the dual of the Teichmüller character components, see [14] and [12].

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