RIMS Kôkyûroku Bessatsu
Bx (201x), 000–000

Rubin-Stark elements and ideal class groups

By

Masato KURIHARA*

Abstract. This article is a survey of the Galois module structure of the class groups of number fields, and of their relation with the L-values. After we explain several classical results, e.g., the order of a character component of a class group, etc., we introduce the Rubin-Stark conjecture in Rubin [27], which asserts the existence of certain algebraic elements related to the L-values. We also explain some new properties of Rubin-Stark elements obtained in a joint work of the author with D. Burns and T. Sano [7], including the description of the Fitting ideals of class groups and certain cohomology groups. We also give several concrete examples.

.

§ 1. Introduction

§ 1.1. the class number formula

The history of the arithmetic meaning of the values of zeta functions began with the celebrated class number formula by Dirichlet in the early 19th century, which was later developed by Kummer and Dedekind. For a number field K, the Dedekind zeta function $\zeta_K(s)$ has a simple pole at $s = 1$ with residue

$$2^{r_1}(2\pi)^{r_2}\frac{h_K R_K}{w_K\sqrt{d_K}} \tag{1.1}$$

where h_K is the order of the ideal class group, R_K the regulator, w_K the number of roots of unity in K, and d_K the absolute value of the discriminant. By the functional equation, this is equivalent to the fact that $\zeta_K(s)$ has a zero at $s = 0$ of order $r_1 + r_2 - 1$ and

$$\lim_{s\to 0}\frac{\zeta_K(s)}{s^{r_1+r_2-1}} = -\frac{h_K R_K}{w_K}. \tag{1.2}$$

Received ****, 201x. Revised ****, 201x.
2010 Mathematics Subject Classification(s): 11R42, 11R29
Key Words: *ideal class group, L-value, Galois module structure, Stark conjecture, Rubin-Stark element*:
*Department of Mathematics, Keio University, 3-14-1 Hiyoshi, Kohoku-ku, Yokohama, 223-8522, Japan
e-mail: `kurihara@math.keio.ac.jp`

If K/k is an abelian extension, $\zeta_K(s)$ decomposes into L-functions, but it is not easy in general to obtain the corresponding formula which describes the arithmetic meaning of the values of L-functions.

§ 1.2. the order of a character component of a class group

Let us consider a simple and classical case that $K = \mathbb{Q}(\mu_p)$ is the field of p-th roots of unity for an odd prime number p, $k = \mathbb{Q}$, and $G = \mathrm{Gal}(K/k)$. Then we know that the Dedekind zeta function $\zeta_K(s)$ decomposes into Dirichlet L-functions; $\zeta_K(s) = \Pi_{\chi \in \hat{G}} L(s, \chi)$. Comparing the class number formulae of K and the maximal real subfield K^+, we have

$$2^{\frac{p-3}{2}} \frac{1}{p} h_K^- = \prod_{\chi(-1)=-1} L(0, \chi) \tag{1.3}$$

where $h_K^- = h_K/h_{K^+}$ and χ runs over all odd characters of G. Let $\mathrm{Cl}(K)$ be the ideal class group of K. We study the $\mathbb{Z}[G]$-module $\mathrm{Cl}(K)$ later, but in this subsection we only consider the p-component and put $A_K = \mathrm{Cl}(K) \otimes \mathbb{Z}_p$. Then A_K is decomposed as a $\mathbb{Z}_p[G]$-module into $A_K = \bigoplus_{\chi \in \hat{G}} A_K^\chi$ where χ is regarded as a character whose values are in $\mathbb{Z}_p^\times$, and A_K^χ is the subgroup consisting of elements x with $\sigma(x) = \chi(\sigma)x$ for all $\sigma \in G$. For the Teichmüller character ω which gives the action of G on the group μ_p of p-th roots of unity, we know $\mathrm{ord}_p\, L(0, \omega^{-1}) = -1$ and $A_K^\omega = 0$, so (1.3) suggests that

$$\# A_K^\chi = \#(\mathbb{Z}_p / L(0, \chi^{-1})) \tag{1.4}$$

for any odd χ with $\chi \neq \omega$. After Kummer, Stickelberger, Herbrand and Ribet, this final form on the order of the character component of the class group was proved by Mazur and Wiles ([23] Chapter 1 §10 Theorem 2) as a corollary of the main conjecture. We note that if $\chi = \omega^i$ such that i is odd and $1 < i < p-1$, we know $L(0, \chi^{-1}) \equiv B_{p-i}/(p-i)$ (mod p) where B_{p-i} is the Bernoulli number, so (1.4) implies the following famous Herbrand-Ribet's theorem;

$$A_K^{\omega^i} \neq 0 \Longleftrightarrow p | B_{p-i}.$$

For $a \in \mathbb{Z}$ which is prime to p, we define $\sigma_a \in G$ by $\sigma_a(\zeta_p) = \zeta_p^a$ where ζ_p is a primitive p-th root of unity. Let

$$\theta_{K/\mathbb{Q}} = \sum_{a=1}^{p-1} (\frac{1}{2} - \frac{a}{p}) \sigma_a^{-1} \in \mathbb{Q}[G]$$

be the Stickelberger element of $K/\mathbb{Q}$. Note that $\theta_{K/\mathbb{Q}}$ is in the minus part $\mathbb{Q}[G]^-$ which consists of elements on which the complex conjugation acts as -1. If χ is a character of G with $\chi \neq 1$, the image of $\theta_{K/\mathbb{Q}}$ under the ring homomorphism $\tilde{\chi} : \mathbb{Q}[G] \longrightarrow \mathbb{Q}_p$

that is defined by $\sigma \mapsto \chi(\sigma)$ is $L(0, \chi^{-1})$. We know $(c - \sigma_c)\theta_{K/\mathbb{Q}} \in \mathbb{Z}[G]$ for any $c \in \mathbb{Z}$ which is prime to $2p$. We take $g \in \mathbb{Z}$ such that g is prime to $2p$, σ_g is a generator of G, and p^2 does not divide $g - \omega(g)$. We also consider the decomposition $A_K = A_K^+ \oplus A_K^-$ where $A_K^{\pm}$ is the subgroup of elements on which the complex conjugation acts as ± 1 (A_K^+ equals A_{K^+}). Since $\mathbb{Z}_p[G]$ is a product of discrete valuation rings, the assertion (1.4) together with information on A_K^{ω} is equivalent to

$$\mathrm{Fitt}_{0,\mathbb{Z}_p[G]^-}(A_K^-) = ((g - \sigma_g)\theta_{K/\mathbb{Q}}) \tag{1.5}$$

where $\mathbb{Z}_p[G]^-$ is the ring consisting of elements of $\mathbb{Z}_p[G]$ on which the complex conjugation acts as -1, the left hand side is the initial Fitting ideal whose definition we explain in §5, and the right hand side is the principal ideal of $\mathbb{Z}_p[G]^-$ generated by $(g-\sigma_g)\theta_{K/\mathbb{Q}}$.

Next, we consider even characters χ. First of all, if $\chi = 1$, we know $A_K^1 = A_{\mathbb{Q}} = 0$. Let ζ_p be a primitive p-th root of unity and consider the cyclotomic unit $1 - \zeta_p$, which is a p-unit $\in O_K[1/p]^\times$. The group $O_K[1/p]^\times \otimes \mathbb{Z}_p$ decomposes into $O_K[1/p]^\times \otimes \mathbb{Z}_p = \bigoplus_\chi (O_K[1/p]^\times \otimes \mathbb{Z}_p)^\chi$, and we denote by $c_K^\chi \in (O_K[1/p]^\times \otimes \mathbb{Z}_p)^\chi$ the χ-component of $1-\zeta_p$. For an even character $\chi \neq 1$, we know c_K^χ is in $(O_K^\times \otimes \mathbb{Z}_p)^\chi = (O_K[1/p]^\times \otimes \mathbb{Z}_p)^\chi$. Since the cyclotomic unit is related to the L-values, the class number formula for $K^+ = \mathbb{Q}(\cos(2\pi/p))$ suggests

$$\# A_K^\chi = ((O_K^\times \otimes \mathbb{Z}_p)^\chi : \langle c_K^\chi \rangle) \tag{1.6}$$

for $\chi \neq 1$ where $\langle c_K^\chi \rangle$ is the $\mathbb{Z}_p$-submodule generated by c_K^χ. This was known as Gras' conjecture, and proved by Mazur and Wiles ([23] Chapter 1 §10 Theorem 1) also as a corollary of the main conjecture. Let $C_{K^+,p}$ be the intersection of the $\mathbb{Z}_p[G]$-module generated by $1 - \zeta_p$ and ζ_p with $O_{K^+}^\times \otimes \mathbb{Z}_p$ where $O_{K^+}^\times$ is the unit group of K^+. Then (1.6) is equivalent to

$$\mathrm{Fitt}_{0,\mathbb{Z}_p[G]^+}(A_K^+) = \mathrm{Fitt}_{0,\mathbb{Z}_p[G]^+}((O_{K^+}^\times \otimes \mathbb{Z}_p)/C_{K^+,p}) \ . \tag{1.7}$$

§ 1.3. an annihilation result

We introduce a classical theorem on the Galois module structure of the ideal class groups of cyclotomic fields, which was obtained in the 19-th century by Stickelberger after works by Gauss, Jacobi, Cauchy and Kummer. Consider a cyclotomic field $K = \mathbb{Q}(\mu_m)$ for some $m \in \mathbb{Z}_{>0}$ (we take m to be the conductor of K), and its Stickelberger element

$$\theta_{K/\mathbb{Q}} = \sum_{\substack{a=1 \\ (a,m)=1}}^{m} (\frac{1}{2} - \frac{a}{m})\sigma_a^{-1} \in \mathbb{Q}[G] \tag{1.8}$$

where $G = \mathrm{Gal}(\mathbb{Q}(\mu_m)/\mathbb{Q})$ and σ_a is the element of G such that $\sigma_a(\zeta) = \zeta^a$ for any $\zeta \in \mu_m$. Then for $c \in \mathbb{Z}$ which does not divide $2m$, $(c - \sigma_c)\theta_{K/\mathbb{Q}}$ is in $\mathbb{Z}[G]$, and Stickelberger's theorem asserts that

$$(c - \sigma_c)\theta_{K/\mathbb{Q}} \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(K)) \tag{1.9}$$

where $\mathrm{Ann}_R(M)$ is the annihilator ideal of an R-module M. This theorem is proved by using the prime decomposition of Gauss sums. More precisely, for a prime ideal $\mathfrak{p}$ which is prime to m, we can construct $\epsilon \in K^\times$ such that the prime decomposition of ϵ is $\mathfrak{p}^{(c-\sigma_c)\theta_{K/\mathbb{Q}}}$, using Gauss sums. The prime decomposition of a Gauss sum was important in the study of reciprocity laws, and was done by Kummer for $\mathbb{Q}(\mu_p)$ with a prime p. Stickelberger generalized it to general cyclotomic fields.

Stickelberger elements, cyclotomic units, and Gauss sums are typical examples of Rubin-Stark elements, which we will explain in §2.

In §4 we discuss a generalization of the formulae on the orders of the character components of the class group of the cyclotomic field in §1.2, and a generalization of the annihilation result by Stickelberger in §1.3, to general finite abelian extensions K/k.

Remark 1. (1) For any finitely presented R-module M, we know $\mathrm{Fitt}_{0,R}(M) \subset \mathrm{Ann}_R(M)$, so we have a natural question whether

$$(c - \sigma_c)\theta_{K/\mathbb{Q}} \in \mathrm{Fitt}_{0,\mathbb{Z}[G]}(\mathrm{Cl}(K))$$

holds for $K = \mathbb{Q}(\mu_m)$. This was proved in [19] except the 2-component, namely

$$(c - \sigma_c)\theta_{K/\mathbb{Q}} \in \mathrm{Fitt}_{0,\mathbb{Z}'[G]}(\mathrm{Cl}(K) \otimes \mathbb{Z}')$$

holds where $\mathbb{Z}' = \mathbb{Z}[1/2]$. In fact, we proved in [19] Theorem 0.1 that $\mathrm{Fitt}_{0,\mathbb{Z}'[G]}((\mathrm{Cl}(K)\otimes \mathbb{Z}')^-)$ coincides with the Stickelberger ideal of Iwasawa-Sinnott over $\mathbb{Z}'$ (cf. Sinnott [30]). In general, $\mathrm{Fitt}_{0,\mathbb{Z}[G]}(\mathrm{Cl}(K))$ is not a principal ideal.
(2) For a prime number ℓ which is prime to $2m$, we denote by $\mathrm{Cl}^{\{\ell\}}(K)$ the ray class group modulo $\Pi_{v|\ell}v$ of K. We define in §3 the Selmer module $\mathcal{S}_{S_\infty,\{\ell\}}(K)$ which contains $\mathrm{Cl}^{\{\ell\}}(K)^\vee$ as a submodule, where $\mathrm{Cl}^{\{\ell\}}(K)^\vee$ is the Pontrjagin dual of $\mathrm{Cl}^{\{\ell\}}(K)$. We will see in (4.1) in §4.3 a stronger result

$$((\ell - \sigma_\ell)\theta_{K/\mathbb{Q}})^\# \in \mathrm{Fitt}_{0,\mathbb{Z}[G]}(\mathcal{S}_{S_\infty,\{\ell\}}(K))$$

than (1.9) where $x \mapsto x^\#$ is the involution induced by $\sigma \mapsto \sigma^{-1}$ for $\sigma \in G$. This implies

$$(\ell - \sigma_\ell)\theta_{K/\mathbb{Q}} \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}^{\{\ell\}}(K)),$$

so implies (1.9) since $\mathrm{Cl}(K)$ is a quotient of $\mathrm{Cl}^{\{\ell\}}(K)$.

§ 2. Rubin-Stark conjecture

Let k be a number field or a function field of one variable over a finite field, and K/k a finite abelian extension with Galois group G. In this section we introduce and explain the Rubin-Stark conjecture formulated in Rubin [27]. A fundamental reference of Stark's conjecture is a book by Tate [31].

§ 2.1. (S,T)-modified L-functions

We denote by S_∞ the set of all infinite places of k, and by $S_{\mathrm{ram}}(K/k)$ (or simply by S_{ram} if there is no confusion) the set of all ramifying places of k in K. For a character χ of G and a non-empty finite set S of places which contains $S_\infty \cup S_{\mathrm{ram}}$, we define the S-truncated L-function by

$$L_S(s,\chi) = \prod_{v \notin S} (1 - \chi(\mathrm{Frob}_v) N v^{-s})^{-1},$$

which is analytically continued to the whole complex plane (except $s = 1$ in the case $\chi = 1$). Let G_v be the decomposition subgroup of v in G for $v \in S$. The L-function $L_S(s,\chi)$ has order $\#\{v \in S \mid \chi(G_v) = 1\}$ at $s = 0$ if $\chi \neq 1$, and order $\#S - 1$ if $\chi = 1$.

Suppose that T is a non-empty finite set of places of k such that $S \cap T = \emptyset$. The (S,T)-modified L-function is defined by

$$\begin{aligned} L_{S,T}(s,\chi) &= \prod_{t \in T} (1 - \chi(\mathrm{Frob}_t) N t^{1-s}) \prod_{v \notin S} (1 - \chi(\mathrm{Frob}_v) N v^{-s})^{-1} \\ &= (\prod_{t \in T} (1 - \chi(\mathrm{Frob}_t) N t^{1-s})) L_S(s,\chi). \end{aligned}$$

The first person who made this kind of modification is Euler. He computed the special values of the Riemann zeta function at negative integers, using this kind of modification. This L-function is now often used in the study of the Stark conjecture. An advantage of this modification is that $L_{S,T}(s,\chi)$ is holomorphic over $\mathbb{C}$. We consider the equivariant L-function

$$\theta_{K/k,S,T}(s) = \sum_{\chi \in \hat{G}} L_{S,T}(s,\chi^{-1}) e_\chi \in \mathbb{C}[G]$$

for each $s \in \mathbb{C}$ where $e_\chi = (\#G)^{-1} \Sigma_{\sigma \in G} \chi(\sigma)\sigma^{-1}$.

We denote by S_K, T_K the set of places of K above S, T, respectively. We define a ring $\mathcal{O}_{K,S}$ by

$$\mathcal{O}_{K,S} = \{a \in K \mid \mathrm{ord}_w(a) \geq 0 \text{ for all finite places } w \text{ of } K \text{ not contained in } S_K\}$$

where ord_w denotes the normalized additive valuation at w. We define the (S,T)-unit group of K by

$$\mathcal{O}_{K,S,T}^\times = \{a \in \mathcal{O}_{K,S}^\times \mid a \equiv 1 \pmod{w} \text{ for all } w \in T_K\},$$

which is a subgroup of $\mathcal{O}_{K,S}^{\times}$ of finite index. We take T such that $\mathcal{O}_{K,S,T}^{\times}$ is $\mathbb{Z}$-torsion-free. If k has positive characteristic, any non-empty T with $S \cap T = \emptyset$ satisfies this condition. When $\mathrm{char}(k) = 0$, the condition on T is satisfied if $S \cap T = \emptyset$ and T contains two primes of distinct residual characteristics, for example.

Suppose that $r \in \mathbb{Z}_{\geq 0}$ and $L_{S,T}(s,\chi)$ has a zero of order $\geq r$ at $s = 0$ for all $\chi \in \hat{G}$. We define

$$\theta_{K/k,S,T}^{(r)} = \lim_{s \to 0} s^{-r} \theta_{K/k,S,T}(s). \tag{2.1}$$

When $r = 0$, $\theta_{K/k,S,T}^{(0)} = \theta_{K/k,S,T}(0)$ is the (generalized) Stickelberger element, which we denote by $\theta_{K/k,S,T}$. We know ([27] Theorem 3.3)

$$\theta_{K/k,S,T} = \theta_{K/k,S,T}(0) \in \mathbb{Z}[G]$$

(an essential case is that k is totally real and K is a CM-field, and we need a theorem of Deligne and Ribet [11] or of P. Cassou-Noguès [8] to show the above). For general r, one can see that $\theta_{K/k,S,T}^{(r)} \in \mathbb{R}[G]$.

§ 2.2. regulator isomorphisms and Rubin-Stark elements

Let S be a finite set of places of k containing $S_\infty \cup S_{\mathrm{ram}}$. As above, we denote by S_K the set of places of K above S. We define $X_{K,S}$ to be the subgroup of the free abelian group on the set S_K comprising elements whose coefficients sum to zero, namely $X_{K,S} = \mathrm{Ker}(\bigoplus_{w \in S_K} \mathbb{Z} \longrightarrow \mathbb{Z})$. We define the Dirichlet regulator map

$$\lambda_{K,S} : \mathcal{O}_{K,S}^{\times} \otimes \mathbb{R} \xrightarrow{\simeq} X_{K,S} \otimes \mathbb{R}$$

by $\lambda_{K,S}(x) = -\sum_{w \in S_K} \log |x|_w w$, which is an isomorphism. We take T such that $\mathcal{O}_{K,S,T}^{\times}$ is $\mathbb{Z}$-torsion-free as in the previous subsection. For each place $v \in S$, we fix a place w of K above v. We write $S = \{v_0, v_1, ..., v_s\}$. We take $r \in \mathbb{Z}_{\geq 0}$ such that $r \leq \#\{v \in S \mid v \text{ splits completely in } K\}$. By convention, we assume that each place v_i with $1 \leq i \leq r$ splits completely in K and put $V = \{v_1, ..., v_r\}$ (in the case $r = 0$, we take V to be the empty set). Then $L_{S,T}(s,\chi)$ has a zero of order $\geq r$ at $s = 0$ for all $\chi \in \hat{G}$.

For a $\mathbb{Z}[G]$-module M, we denote by $\bigwedge_{\mathbb{Z}[G]}^{r} M$ the r-th exterior power of the $\mathbb{Z}[G]$-module M. The above regulator isomorphism induces an isomorphism

$$\lambda_{K,S} : (\bigwedge_{\mathbb{Z}[G]}^{r} \mathcal{O}_{K,S,T}^{\times}) \otimes \mathbb{R} \xrightarrow{\sim} (\bigwedge_{\mathbb{Z}[G]}^{r} X_{K,S}) \otimes \mathbb{R},$$

which we also denote by $\lambda_{K,S}$. We define the Rubin-Stark element

$$\epsilon_{K/k,S,T}^{V} \in (\bigwedge_{\mathbb{Z}[G]}^{r} \mathcal{O}_{K,S,T}^{\times}) \otimes \mathbb{R}$$

by

$$\lambda_{K,S}(\epsilon^V_{K/k,S,T}) = \theta^{(r)}_{K/k,S,T} \bigwedge_{i=1}^{r}(w_i - w_0). \tag{2.2}$$

Here, w_i is the place we fixed above v_i ($0 \leq i \leq r$). This element $\epsilon^V_{K/k,S,T}$ does not depend on the choice of $v_0 \in S \setminus V$.

When $r = 0$, $\lambda_{K,S}$ is the identity map on $(\bigwedge^0_{\mathbb{Z}[G]} \mathcal{O}^\times_{K,S,T}) \otimes \mathbb{R} = (\bigwedge^0_{\mathbb{Z}[G]} X_{K,S}) \otimes \mathbb{R} = \mathbb{R}[G]$ and $\epsilon^V_{K/k,S,T}$ is the Stickelberger element $\theta_{K/k,S,T}$.

Next, suppose that $k = \mathbb{Q}$ and $K = \mathbb{Q}(\mu_m)^+$ with conductor $m > 0$, and take $r = 1$ and $V = \{\infty\}$. We take a place w above ∞ and regard K as a subfield of $\mathbb{R}$ by the embedding w. Put $\zeta_m = e^{2\pi i/m} \in \mathbb{C}$. We know the classical formula

$$L'(0, \chi) = -\frac{1}{2} \sum_{\sigma \in G} \log |(1 - \zeta_m^\sigma)(1 - \zeta_m^{-\sigma})|_w \chi(\sigma)$$

for any character χ of G (Tate [31] p.79). We take $S = \{p \mid p \text{ divides } m\} \cup \{\infty\}$ and T to be a finite set of primes containing an odd prime such that $T \cap S = \emptyset$. Using the above formula, we can prove that

$$\lambda_{K,S}(c_T) = \theta^{(1)}_{K/k,S,T}(w - w_0)$$

where $c_T = (1 - \zeta_m)^{\delta_T}$ with $\delta_T = \prod_{\ell \in T}(1 - \ell\sigma_\ell^{-1})$ (see [31] p.79 and [25] §4.2). Note that since the complex conjugation fixes c_T, it is in K and we can show that $c_T \in \mathcal{O}^\times_{K,S,T}$. In this case, we have $\epsilon^V_{K/k,S,T} = c_T$.

Thus the Rubin-Stark elements can be regarded as generalization of the Stickelberger elements and of the cyclotomic units.

§ 2.3. Rubin's lattice

We need a theory of integral lattices in $\mathbb{R}[G]$-modules.

Let M be a $\mathbb{Z}[G]$-module. For any G-homomorphism $\varphi : M \longrightarrow \mathbb{Z}[G]$ and any $r \in \mathbb{Z}_{>0}$, we define

$$\varphi : \bigwedge^r_{\mathbb{Z}[G]} M \longrightarrow \bigwedge^{r-1}_{\mathbb{Z}[G]} M$$

by $m_1 \wedge \ldots \wedge m_r \mapsto \sum_{i=1}^r (-1)^{i-1}\varphi(m_i) m_1 \wedge \ldots \wedge m_{i-1} \wedge m_{i+1} \wedge \ldots \wedge m_r$ (we use the same notation φ for this homomorphism). We note that $\bigwedge^0_{\mathbb{Z}[G]} M = \mathbb{Z}[G]$.

For $\varphi_1,..,\varphi_j \in \mathrm{Hom}_G(M, \mathbb{Z}[G])$, we define

$$\varphi_1 \wedge \ldots \wedge \varphi_j : \bigwedge^r_{\mathbb{Z}[G]} M \longrightarrow \bigwedge^{r-j}_{\mathbb{Z}[G]} M$$

to be the composite homomorphism $\varphi_j \circ ... \circ \varphi_1$. Thus, for $\Phi \in \bigwedge^j \mathrm{Hom}_G(M, \mathbb{Z}[G])$ and $x \in \bigwedge^r M$, $\Phi(x) \in \bigwedge^{r-j} M$ is defined. If $j = r$, we know

$$(\varphi_1 \wedge ... \wedge \varphi_r)(m_1 \wedge ... \wedge m_r) = \det(\varphi_i(m_j)) \in \mathbb{Z}[G].$$

Let M be a finitely generated $\mathbb{Z}[G]$-module such that M is $\mathbb{Z}$-torsion-free. Then we define Rubin's lattice $\bigcap_G^r M$ by

$$\bigcap_G^r M = \{x \in (\bigwedge_{\mathbb{Z}[G]}^r M) \otimes \mathbb{Q} \mid \Phi(x) \in \mathbb{Z}[G] \text{ for all } \Phi \in \bigwedge^r \mathrm{Hom}_G(M, \mathbb{Z}[G])\}, \tag{2.3}$$

which is a lattice in $(\bigwedge_{\mathbb{Z}[G]}^r M) \otimes \mathbb{Q}$.

Note that $\bigcap_G^r M$ is bigger than the image of $\bigwedge_{\mathbb{Z}[G]}^r M$ in $(\bigwedge_{\mathbb{Z}[G]}^r M) \otimes \mathbb{Q}$, in general.

If $r = 0$, we have $\bigcap_G^0 M = \mathbb{Z}[G]$ by definition. If $r = 1$, we have $\bigcap_G^1 M = M$ ([27] Proposition 1.2).

§ 2.4. Rubin-Stark conjecture

We can interpret Stickelberger elements, cyclotomic units, and Gauss sums as Rubin-Stark elements. As a generalization of these elements, Rubin formulate in [27] the following beautiful conjecture.

Conjecture 2.1. (Rubin-Stark conjecture; see Rubin [27] Conjecture B) Let $\epsilon_{K/k,S,T}^V \in (\bigwedge_{\mathbb{Z}[G]}^r \mathcal{O}_{K,S,T}^\times) \otimes \mathbb{R}$ be the Rubin-Stark element defined in (2.2), and $\bigcap_G^r \mathcal{O}_{K,S,T}^\times$ be the Rubin's lattice for $\mathcal{O}_{K,S,T}^\times$ defined in (2.3). Then

$$\epsilon_{K/k,S,T}^V \in \bigcap_G^r \mathcal{O}_{K,S,T}^\times$$

would hold.

Remark 2. (1) The statement $\epsilon_{K/k,S,T}^V \in (\bigwedge_{\mathbb{Z}[G]}^r \mathcal{O}_{K,S,T}^\times) \otimes \mathbb{Q}$ is equivalent to the so called Stark's conjecture for characters χ whose L-functions $L_S(s, \chi)$ have order r at $s = 0$ (see Tate [31] Chap. I §5, and Rubin [27] Proposition 2.3). Rubin's beautiful idea is that the element related to the zeta values lies in some *integral* lattice, which is *a little bigger* than $\bigwedge_{\mathbb{Z}[G]}^r \mathcal{O}_{K,S,T}^\times$.
(2) In the case $r = 0$, $\epsilon_{K/k,S,T}^V = \theta_{K/k,S,T} \in \mathbb{Z}[G]$, so the Rubin-Stark conjecture holds. In the case $r = 1$, the Rubin-Stark conjecture is equivalent to the refined Stark conjecture formulated in Tate [31] Chap. IV §2 (see Rubin [27] Proposition 2.5).
(3) The system of Rubin-Stark elements becomes an Euler system of rank r (for the precise statement, see Rubin [27] Proposition 6.1). We note that it is important in the argument of Euler systems to know that the elements live in an integral lattice.

Remark 3. (1) Conjecture 2.1 is known to be true if $K = k$ and $[K : k] = 2$ (Rubin [27] Corollary 3.2 and Theorem 3.5). This can be verified by using the class number formulae for K and k.
(2) Conjecture 2.1 holds if $k = \mathbb{Q}$ or if K is a function field. In fact, in the case $k = \mathbb{Q}$, the equivariant Tamagawa number conjecture holds for K/k by Burns, Greither and Flach [6], [12], and Burns proved that the equivariant Tamagawa number conjecture implies the Rubin-Stark conjecture (Burns [2] Theorem A).

In the function field case, the leading term conjecture for K/k in Burns [2] was proved by Burns [3], and it implies the Rubin-Stark conjecture (Burns [2] Theorem A). We know that the equivariant Tamagawa number conjecture is equivalent to the leading term conjecture in the number field case. In this article, *we call* both the equivariant Tamagawa number conjecture and the leading term conjecture ETNC in the following.

In [7] §5, we gave a simplified proof that the ETNC implies the Rubin-Stark conjecture, by showing that the Rubin-Stark element $\epsilon^V_{K/k,S,T}$ is the image under a certain simple map of the zeta element whose existence is asserted by the ETNC.

Question 2.2. The Rubin-Stark conjecture asserts that

$$\{\Phi(\epsilon^V_{K/k,S,T}) \mid \Phi \in \bigwedge^r \mathrm{Hom}_G(M, \mathbb{Z}[G])\}$$

is an ideal of $\mathbb{Z}[G]$. Then, what is the arithmetic meaning of this ideal? We will answer this question in §4.

§ 3. Canonical $\mathbb{Z}$-structure of Selmer modules

In this section, we explain some integral cohomology groups introduced in [7]. We suppose that K/k, G are as in §2. We take finite sets S, T of places of k such that $S \supset S_\infty$ and $S \cap T = \emptyset$. We do not assume $S \supset S_{\mathrm{ram}}$ nor that $\mathcal{O}^\times_{K,S,T}$ is $\mathbb{Z}$-torsion-free in this section.

§ 3.1. two Selmer modules

Let S_K, T_K be the sets of places of K above S, T. We define

$$K^\times_T = \{x \in K^\times : \mathrm{ord}_w(x-1) > 0 \text{ for all } w \in T_K\}$$

and

$$\mathcal{S}_{S,T}(K) = \mathrm{Coker}(\prod_{w \notin S_K \cup T_K} \mathbb{Z} \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(K^\times_T, \mathbb{Z})) \tag{3.1}$$

where the above homomorphism is defined by

$$(x_w)_w \mapsto (a \mapsto \sum_{w \notin S_K \cup T_K} \mathrm{ord}_w(a) x_w).$$

Let $\mathrm{Cl}_S^T(K)$ be the ray class group of $\mathcal{O}_{K,S}$ modulo $\Pi_{w \in T_K} w$. Therefore, $\mathrm{Cl}_S^T(K)$ is the cokernel of the divisor map $K_T^\times \longrightarrow \bigoplus_{w \notin S_K \cup T_K} \mathbb{Z}$ by definition. When T is the empty set $\emptyset$, we write $\mathrm{Cl}_S(K)$ for $\mathrm{Cl}_S^\emptyset(K)$, which is equal to the class group $\mathrm{Pic}(\mathcal{O}_{K,S})$ of $\mathcal{O}_{K,S}$. We have an exact sequence

$$0 \longrightarrow \mathcal{O}_{K,S,T}^\times \longrightarrow \mathcal{O}_{K,S}^\times \longrightarrow \bigoplus_{w \in T_K} \kappa(w)^\times \longrightarrow \mathrm{Cl}_S^T(K) \longrightarrow \mathrm{Cl}_S(K) \longrightarrow 0$$

where $\kappa(w)$ is the residue field of w.

Using the definition of $\mathcal{S}_{S,T}(K)$, we can prove ([7] §2)

Proposition 3.1. *We have an exact sequence*

$$0 \longrightarrow \mathrm{Cl}_S^T(K)^\vee \longrightarrow \mathcal{S}_{S,T}(K) \longrightarrow \mathrm{Hom}_\mathbb{Z}(\mathcal{O}_{K,S,T}^\times, \mathbb{Z}) \longrightarrow 0 \tag{3.2}$$

of G-modules where $\mathrm{Cl}_S^T(K)^\vee$ is the Pontrjagin dual of $\mathrm{Cl}_S^T(K)$. In particular, $\mathcal{S}_{S,T}(K)$ is a finitely generated $\mathbb{Z}$-module.

Take $S = S_\infty$ and $T = \emptyset$. We write $\mathrm{Cl}(K)$ for the ideal class group of K, so $\mathrm{Cl}_{S_\infty}^\emptyset(K) = \mathrm{Cl}(K)$. For general T, we also use the notation $\mathrm{Cl}^T(K)$ which means $\mathrm{Cl}_{S_\infty}^T(K)$. Consider the Bloch-Kato Selmer group

$$H_f^1(K, \mathbb{Q}/\mathbb{Z}(1)) = \mathrm{Ker}(H^1(K, \mathbb{Q}/\mathbb{Z}(1)) \longrightarrow \prod_v H^1(K_v, \mathbb{Q}/\mathbb{Z}(1))/(\mathcal{O}_{K_v}^\times \otimes \mathbb{Q}/\mathbb{Z}))$$

where v runs over all finite primes. The Pontrjagin dual $H_f^1(K, \mathbb{Q}/\mathbb{Z}(1))^\vee$ is a $\hat{\mathbb{Z}}$-module, and

$$0 \longrightarrow \mathrm{Cl}(K)^\vee \longrightarrow H_f^1(K, \mathbb{Q}/\mathbb{Z}(1))^\vee \longrightarrow \mathrm{Hom}_\mathbb{Z}(\mathcal{O}_K^\times, \hat{\mathbb{Z}}) \longrightarrow 0 \tag{3.3}$$

is exact. Our Selmer module $\mathcal{S}_{S_\infty,\emptyset}(K)$ gives the canonical $\mathbb{Z}$-structure of the dual of the Bloch-Kato Selmer group $H_f^1(K, \mathbb{Q}/\mathbb{Z}(1))^\vee$.

We also introduce a G-module $\mathcal{S}_{S,T}^{\mathrm{tr}}(K)$ in the next subsection, which sits in an exact sequence

$$0 \longrightarrow \mathrm{Cl}_S^T(K) \longrightarrow \mathcal{S}_{S,T}^{\mathrm{tr}}(K) \longrightarrow X_{K,S} \longrightarrow 0\,. \tag{3.4}$$

§ 3.2. Weil étale cohomology

In [7] we define a Weil étale cohomology complex

$$C^{\bullet} = R\Gamma_{c,T}((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbb{Z}),$$

which is acyclic outside degrees one, two and three. For $i = 1$, we have $H^1(C^{\bullet}) = (\bigoplus_{w \in S_K} \mathbb{Z})/\Delta(\mathbb{Z})$ where Δ is the diagonal map, and for $i = 3$, we have $H^3(C^{\bullet}) = \mathrm{Hom}((K_T^{\times})_{\mathrm{tors}}, \mathbb{Q}/\mathbb{Z})$. For $i = 2$, we write $H^2(C^{\bullet}) = H^2_{c,T}((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbb{Z})$. We can prove that there is an isomorphism

$$H^2_{c,T}((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbb{Z}) \simeq \mathcal{S}_{S,T}(K). \tag{3.5}$$

In the function field case, let X_k be the proper smooth curve over a finite field, corresponding to k, and $(X_k)_W$ the Weil étale site on X_k defined by Lichtenbaum [21] §2. We denote by $j : \mathrm{Spec}(\mathcal{O}_{K,S}) \longrightarrow X_k$ the open immersion for $\mathcal{O}_{K,S}$. Suppose that $R\Gamma_c((\mathcal{O}_{K,S})_W, \mathbb{Z})$ is a complex which gives the Weil étale cohomology $H^i((X_k)_W, j_! \mathbb{Z})$. Then $R\Gamma_{c,T}((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbb{Z})$ sits in the distinguished triangle

$$R\Gamma_c((\mathcal{O}_{K,S})_W, \mathbb{Z}) \longrightarrow R\Gamma_{c,T}((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbb{Z}) \longrightarrow (\bigoplus_{w \in T_K} \kappa(w)^{\times})^{\vee}[-2].$$

In the number field case, the Weil étale site has not yet been defined, but we can define the complex $R\Gamma_{c,T}((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbb{Z})$ which gives the "correct Weil étale cohomology groups".

We define

$$D^{\bullet} = R\Gamma_T((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbb{G}_m) = R\,\mathrm{Hom}_{\mathbb{Z}}(R\Gamma_{c,T}((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbb{Z}), \mathbb{Z})[-2],$$

and

$$\mathcal{S}^{\mathrm{tr}}_{S,T}(K) := H^1(D^{\bullet}) = H^1_T((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbb{G}_m). \tag{3.6}$$

Then we can prove that $\mathcal{S}^{\mathrm{tr}}_{S,T}(K)$ sits in the exact sequence (3.4) (see [7] §2).

Suppose that S and T satisfy the condition in §2.2. Then $R\Gamma_T((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbb{G}_m)$ is perfect. Using it, we can formulate a statement which is equivalent to the ETNC. Namely, we can define a certain zeta element related to the *leading terms* of the (S,T)-modified L-functions at $s = 0$, and the ETNC is equivalent to the statement that the zeta element is a basis of $\det R\Gamma_T((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbb{G}_m)$ ([7] §3).

§ 4. Main results

In this section, we introduce some results on the Galois module structure of Selmer modules, obtained in [7]. For other results obtained in [7], see Sano's article [29] in this volume.

§ 4.1. main theorems

We recall our assumptions for the convenience of readers. Let K/k be a finite abelian extension of number fields or function fields of one variable over a finite field, and $G = \mathrm{Gal}(K/k)$. We suppose that S is a non-empty finite set of places, which contains $S_\infty \cup S_{\mathrm{ram}}$, and that T is a finite set of places of k such that $S \cap T = \emptyset$ and that $\mathcal{O}_{K,S,T}^\times$ is $\mathbb{Z}$-torsion-free. We assume that V is a proper subset of S with $r = \#V$ and each place of V splits completely in K. We denote by $\epsilon_{K/k,S,T}^V$ the Rubin-Stark element defined in (2.2) and use the Selmer modules $\mathcal{S}_{S,T}(K)$, $\mathcal{S}_{S,T}^{\mathrm{tr}}(K)$ defined in (3.1), (3.5), (3.6).

For a finitely presented R-module M, we denote by $\mathrm{Fitt}_{r,R}(M)$ the r-th Fitting ideal of M (see §5). The involution $\mathbb{Z}[G] \longrightarrow \mathbb{Z}[G]$ induced by $\sigma \mapsto \sigma^{-1}$ for $\sigma \in G$ is denoted by $x \mapsto x^\#$.

Theorem 4.1. *We assume that the ETNC is true for K/k (which implies by [2] that Conjecture 2.1 holds true). Then we have*

$$\mathrm{Fitt}_{r,\mathbb{Z}[G]}(\mathcal{S}_{S,T}(K)) = \{\Phi(\epsilon_{K/k,S,T}^V)^\# \mid \Phi \in \bigwedge^r \mathrm{Hom}_G(\mathcal{O}_{K,S,T}^\times, \mathbb{Z}[G])\}$$

and

$$\mathrm{Fitt}_{r,\mathbb{Z}[G]}(\mathcal{S}_{S,T}^{\mathrm{tr}}(K)) = \{\Phi(\epsilon_{K/k,S,T}^V) \mid \Phi \in \bigwedge^r \mathrm{Hom}_G(\mathcal{O}_{K,S,T}^\times, \mathbb{Z}[G])\} .$$

Though the above theorem deals with the r-th Fitting ideals, it can be regarded as a statement on the *initial* Fitting ideals of certain modules. We explain it for the statement on $\mathcal{S}_{S,T}^{\mathrm{tr}}(K)$. Put $Y_{K,V} = \bigoplus_{w \in V_K} \mathbb{Z}$, which is free of rank r over $\mathbb{Z}[G]$ and which we regard as a quotient of $X_{K,S}$. Let S' be the kernel of the surjective homomorphism $\mathcal{S}_{S,T}^{\mathrm{tr}}(K) \longrightarrow X_{K,S} \longrightarrow Y_{K,V}$. By the definition of the Fitting ideal, we have

$$\mathrm{Fitt}_{i,\mathbb{Z}[G]}(\mathcal{S}_{S,T}^{\mathrm{tr}}(K)) = 0 \quad \text{for any } i < r,$$

and $\mathrm{Fitt}_{r,\mathbb{Z}[G]}(\mathcal{S}_{S,T}^{\mathrm{tr}}(K)) = \mathrm{Fitt}_{0,\mathbb{Z}[G]}(S')$. Note that $\mathrm{Cl}_S^T(K) \subset S'$.

Since $\epsilon_{K/k,S,T}^V$ is an Euler system of rank r and it is very important to pursue annihilation results for class groups in the theory of Euler systems, we mention one more theorem, which is proved by using some argument in the proof of Theorem 4.1.

Theorem 4.2. *We assume that the ETNC is true for K/k. Then for any $\Phi \in \bigwedge^r \mathrm{Hom}_G(\mathcal{O}_{K,S,T}^\times, \mathbb{Z}[G])$ and for any $v \in S \setminus V$, we have*

$$\Phi(\epsilon_{K/k,S,T}^V) \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}_{S_\infty \cup V \cup \{v\}}^T(K)).$$

Moreover, if G is cyclic, we have

$$\Phi(\epsilon_{K/k,S,T}^V) \in \mathrm{Fitt}_{0,\mathbb{Z}[G]}(\mathrm{Cl}_{S_\infty \cup V \cup \{v\}}^T(K))$$

for any Φ and any v as above.

§ 4.2. examples – Cyclotomic fields

We apply Theorem 4.1 to two simple examples. The results in this section are also new. First of all, we take $k = \mathbb{Q}$ and $K = \mathbb{Q}(\mu_{p^n})$ for some odd prime p and some $n \in \mathbb{Z}_{>0}$. We take $S = \{\infty, p\}$ and $V = \emptyset$. Since the base field is $\mathbb{Q}$, the ETNC is known to be true. The condition on T means that $T \cap S = \emptyset$ and T contains at least an odd prime $\ell \neq p$. We use the notation in §§1,2, and put $\delta_T = \prod_{\ell \in T}(1 - \ell\sigma_\ell^{-1})$. We know

$$\begin{aligned}\epsilon^{\emptyset}_{\mathbb{Q}(\mu_{p^n})/\mathbb{Q},S,T} &= \theta_{\mathbb{Q}(\mu_{p^n})/\mathbb{Q},S,T} = \delta_T \theta_{\mathbb{Q}(\mu_{p^n})/\mathbb{Q}} \\ &= (\prod_{\ell \in T}(1 - \ell\sigma_\ell^{-1})) \sum_{a=1}^{p^n-1}(\frac{1}{2} - \frac{a}{p^n})\sigma_a^{-1} \in \mathbb{Z}[G].\end{aligned}$$

Let $j \in G = \mathrm{Gal}(\mathbb{Q}(\mu_{p^n})/\mathbb{Q}) = \mathrm{Gal}(K/k)$ be the complex conjugation. We have $X_{K,S} \simeq \bigoplus_{w|\infty} \mathbb{Z} \simeq \mathbb{Z}[G]/(1-j)$. Therefore, by the exact sequence (3.4) we get

$$\mathrm{Fitt}_{0,\mathbb{Z}[G]}(\mathcal{S}^{\mathrm{tr}}_{S,T}(K)) = (1-j)\,\mathrm{Fitt}_{0,\mathbb{Z}[G]}(\mathrm{Cl}^T_S(K)).$$

It follows from Theorem 4.1 that we have

Corollary 4.3. *Let $j \in G = \mathrm{Gal}(\mathbb{Q}(\mu_{p^n})/\mathbb{Q})$ be the complex conjugation. For any finite set T of places, which is disjoint from $S = \{\infty, p\}$, and which contains an odd prime, we have*

$$(1-j)\,\mathrm{Fitt}_{0,\mathbb{Z}[G]}(\mathrm{Cl}^T_S(\mathbb{Q}(\mu_{p^n}))) = (\theta_{\mathbb{Q}(\mu_{p^n})/\mathbb{Q},S,T})$$

where $\mathrm{Cl}^T_S(\mathbb{Q}(\mu_{p^n}))$ is the ray class group of $\mathbb{Z}[\mu_{p^n}][1/p]$ modulo T. In particular, for the full ideal class group $\mathrm{Cl}(\mathbb{Q}(\mu_{p^n}))$, (since $\mathrm{Cl}(\mathbb{Q}(\mu_{p^n}))$ is a quotient of $\mathrm{Cl}^T_S(\mathbb{Q}(\mu_{p^n}))$) we have

$$(c - \sigma_c)\theta_{\mathbb{Q}(\mu_{p^n})/\mathbb{Q}} \in (1-j)\,\mathrm{Fitt}_{0,\mathbb{Z}[G]}(\mathrm{Cl}(\mathbb{Q}(\mu_{p^n})))$$

where c is any integer prime to $2p$, and $\theta_{\mathbb{Q}(\mu_{p^n})/\mathbb{Q}}$ is the classical Stickelberger element defined in (1.8) with $m = p^n$.

Note that we are working over $\mathbb{Z}[G]$ (and do not neglect the 2-component). The left hand side of the equation in Corollary 4.3 gives the algebraic meaning of the ideal generated by the Stickelberger element. Also, Corollary 4.3 implies

$$(c - \sigma_c)\theta_{\mathbb{Q}(\mu_{p^n})/\mathbb{Q}} \in (1-j)\,\mathrm{Fitt}_{0,\mathbb{Z}[G]}(\mathrm{Cl}(\mathbb{Q}(\mu_{p^n}))) \subset \mathrm{Fitt}_{0,\mathbb{Z}[G]}(\mathrm{Cl}(\mathbb{Q}(\mu_{p^n}))).$$

Therefore, Corollary 4.3 affirmatively answers the question in Remark 1 (1), and gives a refinement of (1.9) in §1.3 in the case $m = p^n$.

Next, we consider $k = \mathbb{Q}$ and $K = \mathbb{Q}(\mu_{p^n})^+$. We take $S = \{\infty, p\}$ to be minimal again, and $V = \{\infty\}$. The condition on T is the same as in Corollary 4.3. We take $\zeta_{p^n} = e^{2\pi i/p^n} \in \mathbb{C}$, and put $\delta_T = \prod_{\ell \in T}(1 - \ell\sigma_\ell^{-1})$. Then the cyclotomic unit $(1-\zeta_{p^n})^{\delta_T}$ is in $\mathbb{R}$, and regarded as an element of $\mathbb{Q}(\mu_{p^n})^+$ by the embedding $\mathbb{Q}(\mu_{p^n})^+ \longrightarrow \mathbb{R}$ which corresponds to our choice of a place above ∞. We also get $(1-\zeta_{p^n})^{\delta_T} \in \mathcal{O}_{K,S,T}^\times$, and it is the Rubin-Stark element (see §2.2);

$$\epsilon^{\{\infty\}}_{\mathbb{Q}(\mu_{p^n})^+/\,\mathbb{Q},S,T} = (1-\zeta_{p^n})^{\delta_T}.$$

We have $X_{K,S} \simeq \bigoplus_{w|\infty} \mathbb{Z} \simeq \mathbb{Z}[G]$. Therefore, using the exact sequence (3.4), we get

$$\mathrm{Fitt}_{1,\mathbb{Z}[G]}(\mathcal{S}^{\mathrm{tr}}_{S,T}(K)) = \mathrm{Fitt}_{0,\mathbb{Z}[G]}(\mathrm{Cl}^T_S(K)).$$

On the other hand, put $c_T = \epsilon^{\{\infty\}}_{\mathbb{Q}(\mu_{p^n})^+/\,\mathbb{Q},S,T}$ and let $\langle c_T \rangle$ be the $\mathbb{Z}[G]$-submodule of $\mathcal{O}_{K,S,T}^\times$ generated by c_T. We have an exact sequence

$$0 \longrightarrow \mathrm{Hom}_G(\mathcal{O}_{K,S,T}^\times, \mathbb{Z}[G]) \longrightarrow \mathrm{Hom}_G(\langle c_T \rangle, \mathbb{Z}[G]) \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(\mathcal{O}_{K,S,T}^\times/\langle c_T\rangle, \mathbb{Z}) \longrightarrow 0.$$

Thus $\mathrm{Ext}^1_{\mathbb{Z}}(\mathcal{O}_{K,S,T}^\times/\langle c_T\rangle, \mathbb{Z})$ is isomorphic to

$$\mathbb{Z}[G]/\{\Phi(c_T)^\# \mid \Phi \in \mathrm{Hom}_G(\mathcal{O}_{K,S,T}^\times, \mathbb{Z}[G])\}.$$

Note that $\mathrm{Ext}^1_{\mathbb{Z}}(\mathcal{O}_{K,S,T}^\times/\langle c_T\rangle, \mathbb{Z}) = \mathrm{Hom}(\mathcal{O}_{K,S,T}^\times/\langle c_T\rangle, \mathbb{Q}/\mathbb{Z}) = (\mathcal{O}_{K,S,T}^\times/\langle c_T\rangle)^\vee$ where $(*)^\vee$ means the Pontrjagin dual. Since G is cyclic, $\mathrm{Fitt}_{0,\mathbb{Z}[G]}(M)^\# = \mathrm{Fitt}_{0,\mathbb{Z}[G]}(M^\vee)$ for any finite $\mathbb{Z}[G]$-module M (Proposition 1 in the appendix of [23]). Therefore, we have

$$\mathrm{Fitt}_{0,\mathbb{Z}[G]}(\mathcal{O}_{K,S,T}^\times/\langle c_T\rangle) = \{\Phi(c_T) \mid \Phi \in \mathrm{Hom}_G(\mathcal{O}_{K,S,T}^\times, \mathbb{Z}[G])\}.$$

From Theorem 4.1 we have

Corollary 4.4. *For any finite set T which is disjoint from $S = \{\infty, p\}$, and which contains an odd prime, we put $c_T = \epsilon^{\{\infty\}}_{\mathbb{Q}(\mu_{p^n})^+/\,\mathbb{Q},S,T} = (1-\zeta_{p^n})^{\delta_T}$. Then we have*

$$\begin{aligned}\mathrm{Fitt}_{0,\mathbb{Z}[G]}(\mathrm{Cl}^T_S(\mathbb{Q}(\mu_{p^n})^+)) &= \{\Phi(c_T) \mid \Phi \in \mathrm{Hom}_G(\mathcal{O}^\times_{\mathbb{Q}(\mu_{p^n})^+,S,T}, \mathbb{Z}[G])\} \\ &= \mathrm{Fitt}_{0,\mathbb{Z}[G]}(\mathcal{O}^\times_{\mathbb{Q}(\mu_{p^n})^+,S,T}/\langle c_T\rangle).\end{aligned}$$

Concerning the full ideal class group $\mathrm{Cl}(\mathbb{Q}(\mu_{p^n})^+)$, we have

$$\begin{aligned}\mathrm{Fitt}_{0,\mathbb{Z}[G]}(\mathrm{Cl}(\mathbb{Q}(\mu_{p^n})^+)) &= \{\Phi(c) \mid \Phi \in \mathrm{Hom}_G(\mathcal{O}^\times_{\mathbb{Q}(\mu_{p^n})^+}, \mathbb{Z}[G]) \text{ and } c \in \mathcal{C}_{\mathbb{Q}(\mu_{p^n})^+}\} \\ &= \mathrm{Fitt}_{0,\mathbb{Z}[G]}(\mathcal{O}^\times_{\mathbb{Q}(\mu_{p^n})^+}/\mathcal{C}_{\mathbb{Q}(\mu_{p^n})^+})\end{aligned}$$

where $\mathcal{C}_{\mathbb{Q}(\mu_{p^n})^+} = \langle 1-\zeta_{p^n}, \zeta_{p^n}\rangle \cap \mathcal{O}^\times_{\mathbb{Q}(\mu_{p^n})^+}$ is the group of cyclotomic units in $\mathcal{O}^\times_{\mathbb{Q}(\mu_{p^n})^+}$.

The statement on $\mathrm{Cl}(\mathbb{Q}(\mu_{p^n})^+)$ (the latter half) is a theorem of Cornacchia and Greither in [9] Theorem 1. It can be derived from the first half of Corollary 4.4, namely from the statement for $\mathrm{Cl}_S^T(\mathbb{Q}(\mu_{p^n})^+)$ (by considering all T). Thus Corollary 4.4 is a generalization of the theorem of Cornacchia and Greither.

§ 4.3. CM-extensions

We suppose that k is totally real, K is a CM-field, and consider the case $r = 0$, namely $V = \emptyset$. First of all, by definition

$$\epsilon^{\emptyset}_{K/k,S,T} = \theta_{K/k,S,T} \in \bigcap_G^0 \mathcal{O}^\times_{K,S,T} = \mathbb{Z}[G].$$

When $S = S_{\mathrm{ram}} \cup S_\infty$ and $T = \emptyset$, we simply write $\theta_{K/k}$ for $\theta_{K/k,S_{\mathrm{ram}}\cup S_\infty,\emptyset}$. It can be written as

$$\theta_{K/k} = \sum_{\sigma\in G} \zeta_{K/k}(0,\sigma)\sigma^{-1} \in \mathbb{Q}[G]$$

where $\zeta_{K/k}(s,\sigma) = \sum_{(\mathfrak{a},K/k)=\sigma}(N\mathfrak{a})^{-s}$ is the partial zeta function. We know

$$\theta_{K/k,S_{\mathrm{ram}}\cup S_\infty,T} = \delta_T\theta_{K/k}$$

for any T that is disjoint from $S = S_{\mathrm{ram}} \cup S_\infty$ where

$$\delta_T = \prod_{t\in T}(1 - Nt\,\mathrm{Frob}_t^{-1}).$$

We can prove that the natural homomorphism $\mathcal{S}_{S,T}(K) \longrightarrow \mathcal{S}_{S_\infty,T}(K)$ is surjective. It follows from Theorem 4.1 that the ETNC for K/k implies

$$\theta^{\#}_{K/k,S_{\mathrm{ram}}\cup S_\infty,T} = (\delta_T\theta_{K/k})^{\#} \in \mathrm{Fitt}_{0,\mathbb{Z}[G]}(\mathcal{S}_{S_\infty,T}(K)) \tag{4.1}$$

for T satisfying the conditions in the beginning of this section. By Proposition 3.1, $\mathrm{Cl}^T(K)^\vee$ is a subgroup of $\mathcal{S}_{S_\infty,T}(K)$. So it also implies

$$\delta_T\theta_{K/k} \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}^T(K)). \tag{4.2}$$

Namely, the ETNC implies (4.1) and (4.2). Thus, we can recover a known fact that the ETNC implies the Brumer-Stark conjecture (4.2).

Concerning more precise statements on the Fitting ideals, we can prove (see [7] Theorem 1.9 and Corollary 1.13)

Corollary 4.5. *Suppose that k is totally real, K is a CM-field and the ETNC holds for K/k.*
(1) *If G is cyclic,*

$$\theta_{K/k}\,\mathrm{Ann}_{\mathbb{Z}[G]}(\mu(K)) \subset \mathrm{Fitt}_{0,\mathbb{Z}[G]}(\mathrm{Cl}(K)).$$

(2) *Suppose that G is a general finite abelian group. Put $\mathbb{Z}' = \mathbb{Z}[1/2]$. Then for any T satisfying the above condition, we have*

$$(\delta_T\theta_{K/k})^\# \in \mathrm{Fitt}_{0,\mathbb{Z}'[G]}(((\mathrm{Cl}^T(K) \otimes \mathbb{Z}')^-)^\vee)$$

We first prove (2). It follows from Proposition 3.1 that $(\mathrm{Cl}^T(K)^\vee \otimes \mathbb{Z}')^- = (\mathcal{S}_{S_\infty,T}(K) \otimes \mathbb{Z}')^-$. Therefore, (4.1) implies

$$(\delta_T\theta_{K/k})^\# \in \mathrm{Fitt}_{0,\mathbb{Z}'[G]}(((\mathcal{S}_{S_\infty,T}(K) \otimes \mathbb{Z}')^-)^\vee) = \mathrm{Fitt}_{0,\mathbb{Z}'[G]}(((\mathrm{Cl}^T(K) \otimes \mathbb{Z}')^-)^\vee).$$

We prove (1), assuming that G is cyclic. We take $v \in S$ to be an infinite place. Then $\mathrm{Cl}^T_{S_\infty \cup \{v\}}(K) = \mathrm{Cl}^T_{S_\infty}(K) = \mathrm{Cl}^T(K)$, so Theorem 4.2 implies that $\delta_T\theta_{K/k} \in \mathrm{Fitt}_{0,\mathbb{Z}[G]}(\mathrm{Cl}^T(K))$. We obtain the conclusion of (1) from the above statement, considering all T satisfying our conditions.

Remark 4. The statement (2) in Corollary 4.5 can be proved in certain cases without assuming the ETNC. It was proved by Greither and Popescu ([14] Theorem 6.5) if the Iwasawa μ-invariant of the cyclotomic $\mathbb{Z}_p$-extension of K vanishes and no prime above p splits in K/K^+. More generally, if the μ-invariant vanishes and $\#\Sigma_p \le 1$ where

$$\Sigma_p = \{v : \text{a place of } k \text{ above } p \mid \text{ a place of } K^+ \text{ above } v \text{ splits in } K\},$$

we proved the minus part of the ETNC in [7] (see §4.5 below), so the conclusion of (2) holds unconditionally.

Remark 5. (1) In the terminology of [17] §0, both (SB) and (DSB) hold if G is cyclic and the ETNC for K/k is true. But if G is not cyclic, (1) in Corollary 4.5 does not hold, in general (cf. [13] §3, [17] Corollary 0.5).
(2) The statement (2) in Corollary 4.5 does not imply

$$(\delta_T\theta_{K/k})^\# \in \mathrm{Fitt}_{0,\mathbb{Z}[G]}(((\mathrm{Cl}(K) \otimes \mathbb{Z}')^-)^\vee)$$

because $((\mathrm{Cl}(K) \otimes \mathbb{Z}')^-)^\vee$ is not a quotient of $((\mathrm{Cl}^T(K) \otimes \mathbb{Z}')^-)^\vee$. Actually, the above statement also has counterexamples (cf. [17] Corollay 0.5, [20]).

§ 4.4. the order of a character component of class groups

Let χ be a one dimensional non-trivial character of the absolute Galois group of k. We denote by K_χ the field corresponding to the kernel of χ, so K_χ/k is a cyclic extension such that $\mathrm{Gal}(K_\chi/k) \simeq \mathrm{Image}\,\chi$. We suppose that χ is a p-adic character, namely the image of χ is in an algebraic closure of $\mathbb{Q}_p$, and assume that the order of χ is prime to p.

We first consider the case $K = K_\chi$, so $G = \mathrm{Gal}(K_\chi/k)$. For any $\mathbb{Z}_p[G]$-module M, we define M^χ by $M^\chi = M \otimes_{\mathbb{Z}_p[G]} \mathcal{O}_\chi$ where $\mathcal{O}_\chi = \mathbb{Z}_p[\mathrm{Image}\,\chi]$ is a $\mathbb{Z}_p[G]$-module on which G acts via χ. Since $\#G$ is prime to p, $\mathcal{O}_\chi$ is a direct summand of $\mathbb{Z}_p[G]$, and M^χ is a direct summand of M.

We take S to be minimal, namely $S = S_\infty \cup S_{\mathrm{ram}}$, and $V = \{v \in S \mid v \text{ splits completely in } K\}$, so take it "maximal". By definition, V consists of infinite places. We put $r = \#V$ and $(\bigcap_G^r \mathcal{O}_{K,S,T}^\times)_p^\chi = (\bigcap_G^r \mathcal{O}_{K,S,T}^\times \otimes \mathbb{Z}_p)^\chi$ where we take T as in the beginning of this section. Since $\#G$ is prime to p, we know

$$(\bigcap_G^r \mathcal{O}_{K,S,T}^\times)_p^\chi = (\bigwedge_{\mathbb{Z}[G]}^r \mathcal{O}_{K,S,T}^\times \otimes \mathbb{Z}_p)^\chi \simeq \mathcal{O}_\chi.$$

Since $(X_{K,S} \otimes \mathbb{Z}_p)^\chi \simeq \mathcal{O}_\chi^r$, we have $\mathrm{Fitt}_{r,\mathcal{O}_\chi}((\mathcal{S}_{S,T}^{\mathrm{tr}}(K) \otimes \mathbb{Z}_p)^\chi) = \mathrm{Fitt}_{0,\mathcal{O}_\chi}((\mathrm{Cl}^T(K) \otimes \mathbb{Z}_p)^\chi)$. Therefore, Theorem 4.1 implies

Corollary 4.6. *Let K/k be the extension corresponding to χ as above, and S, T, V as above. We assume that the ETNC holds for K/k. Then we have*

$$\#(\mathrm{Cl}^T(K) \otimes \mathbb{Z}_p)^\chi = ((\bigcap_G^r \mathcal{O}_{K,S,T}^\times)_p^\chi : \langle \epsilon_{K/k,S,T}^V \rangle)$$

where $\langle \epsilon_{K/k,S,T}^V \rangle$ is the $\mathcal{O}_\chi$-submodule generated by the image of $\epsilon_{K/k,S,T}^V$.

This corollary is a generalization of (1.4), (1.6) in §1.

We remark that Rubin proved in [27] Corollary 5.4 that if the Rubin-Stark conjecture holds for all finite sets S' which contain S, then we get the conclusion of Corollary 4.6.

Next we consider the case that p divides $\#G$ (a generalization of Gras's conjecture to this case is not studied in [27]). Let χ, K_χ be as above and K/k a finite abelian extension such that $K \supset K_\chi$ and K/K_χ is a p-extension. Put $\Gamma = \mathrm{Gal}(K/K_\chi)$ which is an abelian p-group, and $\Delta = \mathrm{Gal}(K_\chi/k)$ which is of order prime to p by our assumption on χ. For a $\mathbb{Z}_p[G]$-module M, we define $M^\chi = M \otimes_{\mathbb{Z}_p[\Delta]} \mathcal{O}_\chi$ where $\mathcal{O}_\chi = \mathbb{Z}_p[\mathrm{Image}\,\chi]$ is a $\mathbb{Z}_p[\Delta]$-module on which G acts via χ. Note that M^χ is not $M \otimes_{\mathbb{Z}_p[G]} \mathcal{O}_\chi$ and that M^χ is an $\mathcal{O}_\chi[\Gamma]$-module.

As a generalization of Corollary 4.6, from Theorem 4.1 we can prove (see [7] §9)

Corollary 4.7. *Let K/k, χ be as above. We assume the ETNC for K/k and that if a place v of k is ramified in K, v does not split competely in K_χ. Then we have*

$$\mathrm{Fitt}_{0,\mathcal{O}_\chi[\Gamma]}((\mathrm{Cl}^T(K) \otimes \mathbb{Z}_p)^\chi) = I_p(\epsilon_{K/k,S,T}^V)^\chi$$

where $I_p(\epsilon^V_{K/k,S,T}) = \{\Phi(\epsilon^V_{K/k,S,T}) \mid \Phi \in \bigwedge^r \mathrm{Hom}_G(\mathcal{O}^\times_{K,S,T}, \mathbb{Z}_p[G])\}$ *and* $I_p(\epsilon^V_{K/k,S,T})^\chi$ *is the* χ*-component* $I_p(\epsilon^V_{K/k,S,T}) \otimes_{\mathbb{Z}_p[\Delta]} \mathcal{O}_\chi$.

§ 4.5. the structure of ideal class groups

We do not have enough space in this article to explain the results on the structure of ideal class groups obtained in [7], so briefly introduce it. Theorem 4.1 gives information on $\mathrm{Fitt}_{0,\mathbb{Z}[G]}(\mathrm{Cl}^T_S(K))$. In [7] we also study the higher Fitting ideals $\mathrm{Fitt}_{i,\mathbb{Z}[G]}(\mathrm{Cl}^T_S(K))$. In [7] §7, we define the notion of relative higher Fitting ideal, and compute the relative higher Fitting ideals of the Selmer module $\mathcal{S}^{\mathrm{tr}}_{S,T}(K)$ from which we obtain information on $\mathrm{Fitt}_{i,\mathbb{Z}[G]}(\mathrm{Cl}^T_S(K))$. We state our structure theorem on the class groups in one of the simplest cases.

We assume that χ satisfies the conditions in the previous subsection. In particular, the order is prime to p. We further assume that k is totally real and χ is totally odd. We take $K = K_\chi$. For simplicity, we also assume $\chi \neq \omega$ where ω is the Teichmüller character. We take $T = \emptyset$, and consider $\epsilon^{\emptyset}_{K/k,S_\infty \cup S_{\mathrm{ram}},\emptyset} = \theta_{K/k}$. For $i \geq 0$, let $\Theta^\chi_{i,K} \subset \mathcal{O}_\chi$ be the higher Stickelberger ideal defined in [16] §3 by using $\theta^\chi_{F/k}$ for fields $F \supset K$.

We assume that the Iwasawa μ-invariant of the cyclotomic $\mathbb{Z}_p$-extension of K vanishes. As in Remark 4, put

$$\Sigma_p = \{v : \text{a place of } k \text{ above } p \mid \text{ a place of } K^+ \text{ above } v \text{ splits in } K\}.$$

If Σ_p is empty, it is known that the minus part of the ETNC for K/k holds. If $\#\Sigma_p = 1$, using a recent result on Gross's conjecture for the p-adic L-functions by Ventullo [32], which is a generalization of Darmon, Dasgupta and Pollack [10], we can prove that the minus part of the ETNC for K/k holds (cf. [7]).

Using the computation on the relative higher Fitting ideals of the Selmer module, our result in [7] on the conjecture by Mazur, Rubin and Sano ([22], [28]), and the result of Ventullo we mentioned, we obtain the following structure theorem ([7] §9).

Corollary 4.8. *Assume that* $\#\Sigma_p \leq 1$ *and the Iwasawa* μ*-invariant of* K *vanishes. Then we have*

$$\mathrm{Fitt}_{i,\mathcal{O}_\chi}((\mathrm{Cl}(K) \otimes \mathbb{Z}_p)^\chi) = \Theta^\chi_{i,K}$$

for all $i \geq 0$*. In particular, we have*

$$(\mathrm{Cl}(K) \otimes \mathbb{Z}_p)^\chi = \bigoplus_{i \geq 1} \Theta^\chi_{i,K}/\Theta^\chi_{i-1,K}.$$

When $k = \mathbb{Q}$, this corollary was essentially proved by Kolyvagin in [15] Theorem 7 and Rubin [26] Theorem 4.4 by using the Euler system of Gauss sums. For a totally real field k this corollary was proved by the author in [16] under the assumption that Σ_p is empty. Corollary 4.8 is a generalization of these theorems.

The case that p divides $\#G$ is also treated in [7] §9, but we do not explain it here.

§ 5. Appendix – Fitting ideals

Let R be a commutative ring and M be a finitely presented R-module. Suppose that $R^m \xrightarrow{\varphi} R^n \longrightarrow M \longrightarrow 0$ is exact and φ corresponds to the (n, m)-matrix A. Let r be a non-negative integer. The r-th Fitting ideal $\mathrm{Fitt}_{r,R}(M)$ is defined as follows. If $r < n$, $\mathrm{Fitt}_{r,R}(M)$ is the ideal of R generated by all $(n-r) \times (n-r)$ minors of A. If $r \geq n$, it is defined to be R. This definition does not depend on the choice of the presentation ([24] Chapter 3, Theorem 1). We have an increasing sequence of ideals

$$\mathrm{Fitt}_{0,R}(M) \subset \mathrm{Fitt}_{1,R}(M) \subset \mathrm{Fitt}_{2,R}(M) \subset \ldots .$$

We call $\mathrm{Fitt}_{0,R}(M)$ the initial Fitting ideal. By definition, we have

$$\mathrm{Fitt}_{0,R}(M) \subset \mathrm{Ann}_R(M).$$

Also, by definition we have $\mathrm{Fitt}_{i,R}(M) \subset \mathrm{Fitt}_{i,R}(N)$ for any $i \geq 0$ when there is a surjective homomorphism $M \longrightarrow N$ of R-modules. A good reference on Fitting ideals is a book by Northcott [24].

If R is a discrete valuation ring with maximal ideal m and M is a finitely generated R-module such that

$$M \simeq R^t \oplus R/m^{i_1} \oplus \ldots \oplus R/m^{i_s}$$

with $i_1 \leq \ldots \leq i_s$, we know $\mathrm{Fitt}_{0,R}(M) = \ldots = \mathrm{Fitt}_{t-1,R}(M) = 0$, $\mathrm{Fitt}_{t,R}(M) = m^{i_1+\ldots+i_s}$, $\mathrm{Fitt}_{t+j,R}(M) = m^{i_1+\ldots+i_{s-j}}$ for any j such that $0 \leq j \leq s-1$, and $\mathrm{Fitt}_{t+s,R}(M) = R$. Thus, knowing the structure of M is equivalent to knowing the higher Fitting ideals $\mathrm{Fitt}_{i,R}(M)$ for all $i \geq 0$. In this case, we have

$$M_{\mathrm{tors}} \simeq \bigoplus_{i \geq t+1} \mathrm{Fitt}_{i,R}(M)/\mathrm{Fitt}_{i-1,R}(M).$$

If R is semi-local and a product of discrete valuation rings, for example $R = \mathbb{Z}_p[G]$ where G is an abelian group whose order is prime to p, knowing the structure of all components of M is equivalent to knowing the higher Fitting ideals $\mathrm{Fitt}_{i,R}(M)$ for all $i \geq 0$.

Acknowledgement. The author would like to thank the editors of this volume and the referee for their careful reading of this manuscript and for their comments.

References

[1] S. Bloch and K. Kato, L-functions and Tamagawa numbers of motives, in the Grothendieck Festschrift Vol I, Progress in Math. Vol **86**, Birkhäuser (1990) 333-400.

[2] D. Burns, Congruences between derivatives of abelian L-functions at $s = 0$, Invent. math. **169** (2007) 451-499.

[3] D. Burns, On main conjectures of geometric Iwasawa theory and related conjectures, submitted for publication.

[4] D. Burns and M. Flach, On Galois structure invariants associated to Tate motives, Amer. J. Math. **120** (1998) 1343-1397.

[5] D. Burns and M. Flach, Tamagawa numbers for motives with (non-commutative) coefficients, Doc. Math. **6** (2001) 501-570.

[6] D. Burns and C. Greither, On the equivariant Tamagawa number conjecture for Tate motives, Invent. math. **153** (2003) 303-359.

[7] D. Burns, M. Kurihara and T. Sano, On arithmetic properties of zeta elements I, preprint 2014, arXiv:1407.6409.

[8] P. Cassou-Noguès, Valeurs aux entiers négatifs des fonctions zêta et fonctions zêta p-adiques, Invent. math. **51** (1979) 29-59.

[9] P. Cornacchia and C. Greither, Fitting ideals of class groups of real fields with prime power conductor, Journal of Number Theory **73** (1998) 459-471.

[10] H. Darmon, S. Dasgupta and R. Pollack, Hilbert modular forms and the Gross-Stark conjecture, Ann. Math., **174** (2011) 439-484.

[11] P. Deligne and K. Ribet, Values of abelian L-functions at negative integers over totally real fields, Invent. math. **59** (1980) 227-286.

[12] M. Flach, On the cyclotomic main conjecture for the prime 2, J. reine angew. Math. **661** (2011) 1-36.

[13] C. Greither and M. Kurihara, Stickelberger elements, Fitting ideals of class groups of CM fields, and dualisation, Math. Zeitschrift **260** (2008) 905-930.

[14] C. Greither, C. Popescu, An equivariant main conjecture in Iwasawa theory and applications, arXiv:1103.3069, to appear in J. Algebraic Geometry.

[15] Kolyvagin, V.A., Euler systems, The Grothendieck Festschrift Vol II (1990) 435-483.

[16] M. Kurihara, On the structure of ideal class groups of CM-fields, Documenta Mathematica, Extra Volume Kato (2003) 539-563.

[17] M. Kurihara, On stronger versions of Brumer's conjecture, Tokyo Journal of Math. **34** (2011) 407-428.

[18] M. Kurihara, Refined Iwasawa theory and Kolyvagin systems of Gauss sum type, Proceedings of the London Mathematical Society **104** (2012) 728-769.

[19] M. Kurihara and T. Miura, Stickelberger ideals and Fitting ideals of class groups for abelian number fields, Math. Annalen. **350** (2011) 549-575.

[20] M. Kurihara and T. Miura, Ideal class groups of CM-fields with non-cyclic Galois action, Tokyo Journal of Math. **35** (2012) 411-439.

[21] S. Lichtenbaum, The Weil-étale topology on schemes over finite fields, Compositio Math. **141** (2005) 689-702.

[22] B. Mazur and K. Rubin, Refined class number formula for $\mathbb{G}_m$, preprint 2013, arXiv: 1312.4053.

[23] B. Mazur and A. Wiles, Class fields of abelian extensions of $\mathbf{Q}$, Invent. math. **76** (1984) 179-330.

[24] D. G. Northcott, *Finite free resolutions*, Cambridge Univ. Press, Cambridge New York 1976.
[25] C. D. Popescu, Integral and p-adic refinements of the abelian Stark conjecture, In: Arithmetic of L-functions by C. Popescu, K. Rubin, and A. Silverberg, IAS/Park City Mathematics Series **18**, AMS (2011) 45-101.
[26] K. Rubin, Kolyvagin's system of Gauss sums, In: Arithmetic Algebraic Geometry, G. van der Geer et al eds, Progress in Math 89 (1991) 309-324.
[27] K. Rubin, A Stark conjecture 'over $\mathbb{Z}$' for abelian L-functions with multiple zeros, Ann. Inst. Fourier **46** (1996) 33-62.
[28] T. Sano, Refined abelian Stark conjectures and the equivariant leading term conjecture of Burns, Compositio Math. **150** (2014) 1809-1835.
[29] T. Sano, A new conjecture for Rubin-Stark elements and its applications, this volume.
[30] W. Sinnott, On the Stickelberger ideal and the circular units of a cyclotomic field, Ann. of Math. **108** (1978) 107-134.
[31] J. Tate, *Les conjectures de Stark sur les fonctions L d'Artin en $s = 0$* (notes par D.Bernardi et N. Schappacher), Progress in Math. **47**, Birkhäuser, Boston, 1984.
[32] K. Ventullo, On the rank one abelian Gross-Stark conjecture, preprint 2013, arXiv: 1308.2261.