Discrete mean formulas and the Landau-Siegel zero

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The Landau-Siegel zero

Let χ be a real primitive character to the modulus D. It is known that the Dirichlet L-function $L(s, \chi)$ has at most one real and simple zero $\tilde{\rho}$ satisfying

$$1 - \tilde{\rho} \ll (\log D)^{-1}.$$

Such a zero is called the Landau-Siegel zero. The typical methods to determine zero-free regions for Dirichlet L-functions are unable to eliminate the Landau-Siegel zero for an intrinsic reason. On the other hand, many consequences can be obtained if the Landau-Siegel zero really exists, some of which could be too strong!

Heuristic arguments

Relations between the non-existence of the Landau-Siegel zero and other questions in number theory were investigated by several authors. For example, Goldfield discussed the relation between the non-existence of the Landau-Siegel zero and the order of zeros of the function $L_E(s)L_E(s, \chi)$ at the central point, where E is an elliptic curve; Iwaniec and Sarnak discussed the relation between the non-existence of the Landau-Siegel zero and the non-vanishing of central values of a family of automorphic L-functions. Motivated by their work, we discuss the relation between the non-existence of the Landau-Siegel zero and the distribution of zeros of the function $L(s, \psi)L(s, \psi\chi)$, with ψ belonging to a large set Ψ of primitive characters, in a region Ω.

Heuristic arguments

Let c_1, c_2 and c_3 be constants > 1. Write

$$\mathcal{L} = \log D$$

and

$$P = \exp\{\mathcal{L}^{c_1}\}.$$

Let Ψ denote the set of all primitive characters $\psi(\bmod\, p)$ with $P < p < 2P$, where p is a prime. Let Ω denote the region

$$\{s = \sigma + it : \ 0 < \sigma < 1, \ |t - \mathcal{L}^{c_2}| < \mathcal{L}^{c_3}\}. \qquad (c_3 < c_2).$$

For $\psi \in \Psi$, the average gap between consecutive zeros of the function $L(s, \psi)L(s, \psi\chi)$ in Ω is

$$\sim \frac{2\pi}{\log(P^2 D)} \sim \frac{\pi}{\log P} = \alpha, \qquad \text{say}.$$

Heuristic arguments

In what follows assume that the Landau-Siegel zero $\tilde{\rho}$ of $L(s, \chi)$ exists and very close to 1. It can be shown that there is a subset Ψ_1 of Ψ such that the size of the complement of Ψ_1 in Ψ is relatively small, and such that for $\psi \in \Psi_1$ the following hold.

(i). All the zeros of $L(s, \psi)L(s, \psi\chi)$ in Ω lie on the critical line $\sigma = 1/2$.

(ii). All the zeros of $L(s, \psi)L(s, \psi\chi)$ in Ω are simple.

(iii). The gap between any pair of consecutive zeros of $L(s, \psi)L(s, \psi\chi)$ in Ω is $\sim \alpha$.

This result is not surprising and analogous results were observed by several authors. Our aim is now to derive a contradiction from the gap assertion (iii).

Reduced to evaluating discrete means

The functional equation for $L(s, \psi)$ is written as

$$L(s, \psi) = Z(s, \psi) L(1 - s, \bar{\psi}).$$

On the upper-half plane $t > 0$, the factor $Z(s, \psi)$ is analytic and non-vanishing. Thus we can introduce

$$M(s, \psi) := Z(s, \psi)^{-1/2} L(s, \psi)$$

which is analytic on the upper-half plane. Note that for each ψ, there are two choices of $M(s, \psi)$ up to $\pm$, but the expressions

$$M(s_1, \psi) M(s_2, \psi), \qquad \frac{M(s_1, \psi)}{M(s_2, \psi)}$$

are independent of these choices.

Reduced to evaluating discrete means

Note that both $M(s,\psi)$ and $iM'(s,\psi)$ are real on the critical line. Assume that $\psi \in \Psi_1$. The gap assertion (iii) implies that, for an zero ρ of $L(s,\psi)$ in Ω,

$$\mathcal{C}(\rho,\psi) := \frac{M(\rho+\beta_1,\psi)M(\rho+\beta_2,\psi)M(\rho+\beta_3,\psi)}{iM'(\rho,\psi)} > 0,$$

where β_1, β_2 and β_3 are purely imaginary parameters satisfying

$$\beta_j \sim ij\alpha, \qquad 1 \le j \le 3.$$

Let $\mathcal{Z}(\psi)$ be the set of zeros of $L(s,\psi)$ in Ω. If one can find functions $H(s,\psi)$ and show that

$$\sum_{\psi\in\Psi_1}\sum_{\rho\in\mathcal{Z}(\psi)} \mathcal{C}(\rho,\psi)|H(\rho,\psi)|^2\omega(\rho) < 0 \qquad (*)$$

where ω is a positive smooth weight, a contradiction will follow.

Discussion on the sum in (*)

It is natural to consider the form

$$H(s,\psi) = \sum_{n<X} \frac{a(n)\psi(n)}{n^s} \qquad (**)$$

with X slightly smaller than P in the logarithmic scale. Although we are unable to prove (*) directly, it is interesting that the sum in (*) is ~ 0 for certain choices of $H(s,\psi)$. Precisely this means that, when the sum is evaluated, the coefficient of the main term in the result is equal to 0. In fact, since $L(\rho+\beta_j,\psi)L(\rho+\beta_j,\psi\chi) \sim 0$ by the gap assertion (iii), the sum in (*) should be ~ 0 if $H(s,\psi) = L(\rho+\beta_j,\psi\chi)$. We may not be able to prove it directly due to the length of the sum. However, some linear combinations of the $L(\rho+\beta_j,\psi\chi)$ can be well approximated by the $H(s,\psi)$ of length slightly smaller than P.

A variant of the method could be helpful

We have investigated various types of the coefficients $a(n)$ in (**), and found a number of $H(s, \psi)$ for which the sum in (*) is ~ 0; even we do not have interpretation of some of them. However, it seems that the inequality (*) can not be achieved with $H(s, \psi)$ given by (**).
A variant of the method could be helpful.

Thank you!