On the exceptional set of the Generalized Ramanujan Conjecture

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In 1916, Ramanujan considered the following function

$$\Delta(z) = e^{2\pi iz} \prod_{n=1}^{\infty}(1 - e^{2\pi inz})^{24} = \sum_{n=1}^{\infty} \tau(n) e^{2\pi inz},$$

where $z \in \mathbb{H} = \{z = x + iy : x \in \mathbb{R},\ y > 0\}$ and $\tau(n)$ is called Ramanujan's tau function. He conjectured that

- $\tau(m)\tau(n) = \tau(mn)$ if $(m, n) = 1$;
- $\tau(p^{r+1}) = \tau(p)\tau(p^r) - p^{11}\tau(p^{r-1})$ for p prime and $r > 0$;
- $p^{-11/2}|\tau(p)| \leq 2$ for all primes p.

The first two were proved by Mordell in 1917. The third one is called the Ramanujan Conjecture and was proved by Deligne in 1974.

In fact, the function $\Delta(z)$ is a holomorphic cusp form of weight 12 with respect to

$$\Gamma = SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z},\ ab - cd = 1 \right\}.$$

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Holomorphic modular forms

Let $k > 0$ be an even integer and $f(z) \neq 0$ be a holomorphic complex function on $\mathbb{H}$ and at ∞. We call $f(z)$ a holomorphic modular form of weight k with respect to $SL_2(\mathbb{Z})$ if it satisfies the transformation rule

$$f(\gamma z) = (cz + d)^k f(z) \text{ for all } z \in \mathbb{H} \text{ and } \gamma \in SL_2(\mathbb{Z}),$$

where

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } \gamma z = \frac{az + b}{cz + d} \text{ for all } z \in \mathbb{H}.$$

The Fourier Expansion

Let $f(z)$ be a holomorphic modular form of weight k and

$$\gamma_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Clearly, $\gamma_0 \in SL_2(\mathbb{Z})$, $\gamma_0 z = z + 1$ and $f(z+1) = f(\gamma_0 z) = f(z)$. Hence, we have Fourier expansion

$$f(z) = \sum_{n=0}^{\infty} a_f(n) e(nz).$$

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Holomorphic cusp forms

A holomorphic modular form $f(z)$ of weight k is called a holomorphic cusp form of weight k if $a_f(0) = 0$.

Denote $\mathcal{S}_k$ the linear vector space consisting of all the holomorphic cusp forms of weight k.

Hecke Operators

Let $f(z) \in \mathcal{S}_k$ and $n \in \mathbb{N}$. The Hecke operators T_n are defined on $\mathcal{S}_k$ by

$$(T_n f)(z) = n^{-\frac{k+1}{2}} \sum_{ad=n} a^k \sum_{0 \le b < d} f\left(\frac{az+b}{d}\right).$$

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Holomorphic Hecke eigenforms

There exists a basis $\mathcal{F}_k$ in $\mathcal{S}_k$ such that $\mathcal{F}_k$ consists of common eigenfunctions of all the Hecke operators T_n. The elements of $\mathcal{F}_k$ are called holomorphic Hecke eigenforms.

For any $f \in \mathcal{F}_k$, let $\lambda_f(n)$ denote the n-th Hecke eigenvalue of f.

One can prove that $\Delta(z)$ is a holomorphic Hecke eigenform of weight 12 and its n-th Hecke eigenvalue

$$\lambda_\Delta(n) = n^{-11/2}\tau(n).$$

Then the Ramanujan Conjecture can be rewritten as

$$|\lambda_\Delta(p)| \leq 2 \quad \text{for all primes } p.$$

In 1930s, Petersson generalized the Ramanujan Conjecture to holomorphic Hecke eigenforms of weight k.

The Generalized Ramanujan Conjecture (GRC)

Let $f(z) \in \mathcal{F}_k$ be a holomorphic Hecke eigenform and $\lambda_f(n)$ be its n-th Hecke eigenvalue. Then for any prime p,

$$|\lambda_f(p)| \leq 2.$$

This case was proved by Deligne in 1974.

Maass cusp forms

A smooth function f is called a Maass cusp form for Γ if it satisfies the following properties:

1. f is an eigenfunction of the hyperbolic Laplace operate
$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$
2. $f(\gamma z) = f(z)$ for all $z \in \mathbb{H}$ and all $\gamma \in \Gamma$.
3. $f(x+iy) = O(y^N)$ as $y \to \infty$.
4. For all $z \in \mathbb{H}$, $\int_0^1 f(z+x)dx = 0$.

We denote the subspace consisting of all the Maass cusp forms by $\mathcal{C}(\Gamma \setminus \mathbb{H})$.

Hecke Operators and Complete Orthogonal Basis

The Hecke operators T_n are defined on $\mathcal{C}(\Gamma \setminus \mathbb{H})$ by

$$(T_n f)(z) := \frac{1}{\sqrt{n}} \sum_{ad=n} \sum_{0 \leq b < d} f\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} z \right).$$

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There exists a complete orthonormal basis $\{u_j\}_{j=0}^{\infty}$ in $\mathcal{C}(\Gamma \setminus \mathbb{H})$ such that it consists of eigenfunctions of Δ and T_n, $n = 1, 2, \ldots$, with

$$\Delta u_j = (1/4 + t_j^2) u_j, \quad T_n u_j = \lambda_j(n) u_j$$

where u_0 is a constant function, $0 < t_1 \le t_2 \le \cdots$, and $\lambda_j(n) \in \mathbb{R}$ are Hecke eigenvalues of u_j. We call u_j Hecke-Maass cusp forms. Moreover, we have Weyl's law

$$r(T) := \#\{u_j : 0 < t_j \le T\} = \frac{1}{12} T^2 + O(T \log T).$$

The Generalized Ramanujan Conjecture (GRC)

We have the Generalized Ramanujan Conjecture for Hecke-Maass cusp forms which predicts that

$$|\lambda_j(p)| \leq 2 \text{ for all } u_j \text{ and all primes } p.$$

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Let $\alpha_{u_j,1}(p), \alpha_{u_j,2}(p)$ be the Satake parameters of u_j at p. We have

$$\lambda_j(p) = \alpha_{u_j,1}(p) + \alpha_{u_j,2}(p) \quad \text{and} \quad \alpha_{u_j,1}(p)\alpha_{u_j,2}(p) = 1.$$

Then

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$$|\lambda_j(p)| \leq 2 \iff |\alpha_{u_j,1}(p)| = |\alpha_{u_j,2}(p)| = 1.$$

2003, Kim and Sarnak

$$|\alpha_{u_j,\ell}(p)| \leq p^{7/64} \text{ for } \ell = 1, 2.$$

Question 1.

Given a fixed Hecke-Maass cusp form, could we estimate the number of primes at which GRC fails?

The case of $GL(2)$

Denote S_j the set of primes p with $|\lambda_j(p)| > 2$.

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$$\limsup_{s\to 1^+} \frac{\sum_{p\in S_j} p^{-s}}{\sum_p p^{-s}} \leq \frac{1}{10}.$$

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$$\limsup_{s\to 1^+} \frac{\sum_{p\in S_j} p^{-s}}{\sum_p p^{-s}} \leq \frac{1}{35}.$$

2019, Luo and Zhou (natrual density)

$$\limsup_{x\to\infty} \frac{\#\{p \leq x : |\lambda_j(p)| > 2\}}{\pi(x)} \leq \frac{1}{35}.$$

One key observation, Ramakrishnan, 1997

We can write

$$\alpha_{u_j,1}(p) = e^{i\theta_j(p)} \qquad \text{and} \qquad \alpha_{u_j,2}(p) = e^{-i\theta_j(p)},$$

with $\theta_j(p) \in [0, \pi] \cup i\mathbb{R} \cup \pi + i\mathbb{R}$. It is well-known that for $m \geq 1$,

$$\lambda_j(p^m) = \frac{e^{i(m+1)\theta_j(p)} - e^{-i(m+1)\theta_j(p)}}{e^{i\theta_j(p)} - e^{-i\theta_j(p)}}.$$

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Suppose GRC fails at p. If $\theta_j(p) \in i\mathbb{R}$, put $\theta_j(p) = i\vartheta_j(p)$. If $\theta_j(p) \in \pi + i\mathbb{R}$, put $\theta_j(p) = \pi + i\vartheta_j(p)$. If m is even, we have

$$\lambda_j(p^m) = \frac{e^{(m+1)\vartheta_j(p)} - e^{-(m+1)\vartheta_j(p)}}{e^{\vartheta_j(p)} - e^{-\vartheta_j(p)}} \geq m + 1.$$

Symmetric power L-functions

The symmetric m-th power L-function of u_j is defined by ($\Re s > 1$)

$$L(s, \mathrm{sym}^m u_j) = \prod_p \prod_{k=0}^{m} (1 - \alpha_{u_j}(p)^{m-2k} p^{-s})^{-1} = \sum_{n=1}^{\infty} \lambda_{\mathrm{sym}^m u_j}(n) n^{-s}.$$

It is well-known that

$$\lambda_{\mathrm{sym}^m u_j}(p) = \lambda_j(p^m).$$

Moreover, $L(s, \mathrm{sym}^m u_j)$ are automorphic for $m = 1, 2, 3, 4$ and

$$\lim_{x\to\infty} \frac{\sum_{p\le x} \lambda_{\mathrm{sym}^m u_j}(p)}{\pi(x)} = 0 \text{ and } \lim_{x\to\infty} \frac{\sum_{p\le x} \lambda_{\mathrm{sym}^m u_j}(p)^2}{\pi(x)} = 1.$$

The idea of Luo-Zhou

If GRC fails at p, then$(1 + 3\lambda_j(p^2) + 5\lambda_j(p^4))^2 \geq 35^2$.

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If GRC fails at p, then$(1+3\lambda_j(p^2)+5\lambda_j(p^4))^2 \geq 35^2$.

$$\frac{\#\{p \leq x : |\lambda_j(p)| > 2\}}{\pi(x)} \leq \frac{1}{\pi(x)} \sum_{p \leq x} \frac{(1+3\lambda_j(p^2)+5\lambda_j(p^4))^2}{35^2}$$

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$$= \frac{1}{\pi(x)} \sum_{p \leq x} \frac{-20 + 15\lambda_j(p^2) + 19\lambda_j(p^4) + 30\lambda_j(p^3)^2 + 25\lambda_j(p^4)^2}{35^2}$$

by the Hecke relation $\lambda_j(m)\lambda_j(n) = \sum_{d|(m,n)} \lambda_j\left(\frac{mn}{d^2}\right)$.

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Noting that $\lambda_{\mathrm{sym}^m u_j}(p) = \lambda_j(p^m)$, we have

$$\limsup_{x\to\infty} \frac{\#\{p \leq x : |\lambda_j(p)| > 2\}}{\pi(x)} \leq \frac{1}{35}.$$

Main tool: symmetric power L-functions

The symmetric m-th power L-function of u_j is defined by ($\Re s > 1$)

$$L(s, \mathrm{sym}^m u_j) = \prod_p \prod_{k=0}^{m} (1 - \alpha_{u_j}(p)^{m-2k} p^{-s})^{-1} = \sum_{n=1}^{\infty} \lambda_{\mathrm{sym}^m u_j}(n) n^{-s}.$$

It is well-known that

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Moreover, $L(s, \mathrm{sym}^m u_j)$ are automorphic for $m = 1, 2, 3, 4$ and

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This implies that (note that $\lambda_{\mathrm{sym}^m u_j}(p) = \lambda_j(p^m)$)

$$\limsup_{x \to \infty} \frac{\#\{p \leq x : |\lambda_j(p)| > 2\}}{\pi(x)} \leq \frac{1}{35}.$$

Recall that

$$\Gamma = SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z},\ ab - cd = 1 \right\}.$$

Let

$$G = GL_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R},\ ab - cd \neq 0 \right\},$$

$$K = O_2(\mathbb{R}) = \left\{ \begin{pmatrix} \pm\cos\theta & -\sin\theta \\ \pm\sin\theta & \cos\theta \end{pmatrix} : 0 \leq \theta \leq 2\pi \right\}.$$

Then $\mathbb{H} \cong G/(K \cdot \mathbb{R}^\times)$. Since Maass cusp forms are invariant under the action of Γ, we can view Maass cusp forms on $\Gamma \backslash G/(K \cdot \mathbb{R}^\times)$.

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Generalize to the case:

$$G = GL_n(\mathbb{R}), \quad K = O_n(\mathbb{R}), \quad \Gamma = SL_n(\mathbb{Z}).$$

Let $\mathcal{H} = \{\phi_j\}$ be the set of Hecke-Maass cusp forms for $SL_n(\mathbb{Z})$ with $n \geq 3$. For $T > 100$, define

$$\mathcal{H}_T = \{\phi \in \mathcal{H} : \mu_\phi \in i\mathbb{R}^n, \|\mu_\phi\|_2 \leq T\}$$

where μ_ϕ is the Langlands parameters and $\|\cdot\|_2$ is the Euclidean norm.

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Weyl's Law

$$\#\mathcal{H}_T \sim T^d,$$

where $d = n(n+1)/2 - 1$.

The Satake parameters

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Let p be a fixed prime and $\phi \in \mathcal{H}_T$. Denote

$$\alpha_{\phi,1}(p), \alpha_{\phi,2}(p), \ldots, \alpha_{\phi,n}(p)$$

the corresponding Satake parameters. It is well-known that

$$\alpha_{\phi,1}(p)\alpha_{\phi,2}(p) \cdots \alpha_{\phi,n}(p) = 1$$

and

$$\alpha_{\phi,1}(p) + \alpha_{\phi,2}(p) + \cdots + \alpha_{\phi,n}(p) = A_\phi(p, 1, \ldots, 1),$$

where $A_\phi(p, 1, \ldots, 1)$ is the p-th Hecke eigenvalue of ϕ.

The Generalized Ramanujan Conjecture (GRC)

Similar to the case of $SL_2(\mathbb{Z})$, we also have the Generalized Ramanujan Conjecture which asserts that

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Please note that for $n \geq 3$, GRC is NOT equivalent to

$$|A_\phi(p, 1, \ldots, 1)| = |\alpha_{\phi,1}(p) + \alpha_{\phi,2}(p) + \ldots + \alpha_{\phi,n}(p)| \leq n.$$

The case of $GL(n)$

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2004, Ramakrishnan, n=3

There are infinitely many primes at which GRC holds.

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In what follows, assume ϕ is non-self dual. For any $\alpha > 1$, define

$$S(\alpha) := \left\{ p \text{ primes} : \max_{1 \le i \le n} |\alpha_{\phi,i}(p)| > \alpha \right\}.$$

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2014, Walji, n=3, (upper Dirichlet density)

$$\limsup_{s \to 1^+} \frac{\sum_{p \in S(\alpha)} p^{-s}}{\sum_p p^{-s}} \le \frac{1}{(\alpha + \alpha^{-1} - 1)^2}.$$

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$$\limsup_{s \to 1^+} \frac{\sum_{p \in S(\alpha)} p^{-s}}{\sum_p p^{-s}} \le \frac{1}{(\alpha + \alpha^{-1} - 1)^2}.$$

2014, Walji, n=4, (upper Dirichlet density)

$$\limsup_{s \to 1^+} \frac{\sum_{p \in S(\alpha)} p^{-s}}{\sum_p p^{-s}} \le \frac{1}{(\alpha + \alpha^{-1} - 2)^2} + \frac{1}{4(\alpha + \alpha^{-1} - 1)^2}.$$

This is nontrivial for $\alpha > 2.65509610049736074 5\ldots$

Theorem 1. (Lau, Ng and W., 2021)

Let ϕ be a non-self dual Hecke-Maass cusp form for $SL_3(\mathbb{Z})$. We have

$$\limsup_{x\to\infty} \frac{1}{\pi(x)} \# \left\{ p \leq x : \max_{1\leq i\leq 3} |\alpha_{\phi,i}(p)| > 1 \right\} \leq \frac{14}{25}$$

under the assumption

$$\sum_{p\leq x} |\lambda_{\mathrm{sym}^2\phi}(p)|^2 \sim \pi(x).$$

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under the assumption

$$\sum_{p\le x} |\lambda_{\mathrm{sym}^2\phi}(p)|^2 \sim \pi(x).$$

If we further assume

$$\sum_{p\le x} \lambda_{\phi\times\tilde{\phi}}(p)^2 \sim 2\pi(x),$$

$$\limsup_{x\to\infty} \frac{1}{\pi(x)} \# \left\{ p \le x : \max_{1\le i\le 3} |\alpha_{\phi,i}(p)| > 1 \right\} \le \frac{12}{25}.$$

Sketch of the proof of Theorem 1

Main tool: the Rankin-Selberg L-function

$$L(s,\phi\times\tilde{\phi}) = \prod_p \prod_{i=1}^{3}\prod_{j=1}^{3}(1-\alpha_{\phi,i}(p)\alpha_{\phi,j}(p)^{-1}p^{-s})^{-1} =: \sum_{n\geq 1}\lambda_{\phi\times\tilde{\phi}}(n)n^{-s}.$$

and the symmetric square L-function

$$L(s,\mathrm{sym}^2\phi) = \prod_p \prod_{1\leq i\leq j\leq 3}(1-\alpha_{\phi,i}(p)\alpha_{\phi,j}(p)p^{-s})^{-1} =: \sum_{n\geq 1}\lambda_{\mathrm{sym}^2\phi}(n)n^{-s}.$$

It is known that

$$\lambda_{\phi\times\tilde{\phi}}(p) = |A_\phi(p,1)|^2 \geq 0 \quad \text{and} \quad \lambda_{\mathrm{sym}^2\phi}(p) = A_\phi(p^2,1).$$

However, the automorphy of these L-functions are not known.

Recall the case of $GL(2)$

$$(1 + 3\lambda_j(p^2) + 5\lambda_j(p^4))^2 \geq 35^2 \text{ if GRC fails at } p$$

and

$$\frac{\#\{p \leq x : |\lambda_j(p)| > 2\}}{\pi(x)} \leq \frac{1}{\pi(x)} \sum_{p \leq x} \frac{(1 + 3\lambda_j(p^2) + 5\lambda_j(p^4))^2}{35^2}$$

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$$\leq \frac{1}{\pi(x)} \sum_{p \leq x} \frac{-20+15\lambda_j(p^2)+19\lambda_j(p^4)+30\lambda_j(p^3)^2+25\lambda_j(p^4)^2}{35^2}.$$

Sketch of the proof of Theorem 1

Aim: find a polynomial in $\alpha_{\phi,\ell}(p)$ similar to

$$(1 + 3\lambda_j(p^2) + 5\lambda_j(p^4))^2.$$

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We consider

$$S(x_1, x_2, x_3) = \frac{1}{16} \sum_{1 \le i < j \le 3} (x_i + x_j)^2 (x_i^{-1} + x_j^{-1})^2 + \frac{1}{32} \prod_{1 \le i < j \le 3} (x_i + x_j)(x_i^{-1} + x_j^{-1}).$$

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Then

$$S(\alpha_{\phi,1}(p), \alpha_{\phi,2}(p), \alpha_{\phi,3}(p))$$
$$= \frac{1}{32}(7 + 10\lambda_{\phi \times \tilde{\phi}}(p) + 4|\lambda_{\mathrm{sym}^2\phi}(p)|^2 - \lambda_{\phi \times \tilde{\phi}}(p)^2).$$

Sketch of the proof of Theorem 1

Next, we can prove that for all primes p,

$$S(\alpha_{\phi,1}(p), \alpha_{\phi,2}(p), \alpha_{\phi,3}(p)) \geq \frac{7}{32}.$$

Furthermore, if GRC for ϕ fails at p, we can prove

$$S(\alpha_{\phi,1}(p), \alpha_{\phi,2}(p), \alpha_{\phi,3}(p)) \geq 1.$$

Sketch of the proof of Theorem 1

$$\frac{\#\{p \le x : \max_{1\le i\le 3} |\alpha_{\phi,i}(p)| > 1\}}{\pi(x)} + \frac{7}{32}\left(\frac{\#\{p \le x : \max_{1\le i\le 3} |\alpha_{\phi,i}(p)| \le 1\}}{\pi(x)}\right)$$

$$\le \frac{1}{\pi(x)}\sum_{p\le x} S(\alpha_{\phi,1}(p), \alpha_{\phi,2}(p), \alpha_{\phi,3}(p))$$

$$= \frac{1}{32\pi(x)}\sum_{p\le x}(7 + 10\lambda_{\phi\times\tilde{\phi}}(p) + 4|\lambda_{\mathrm{sym}^2\phi}(p)|^2 - \lambda_{\phi\times\tilde{\phi}}(p)^2).$$

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If we assume

$$\sum_{p\le x} |\lambda_{\mathrm{sym}^2\phi}(p)|^2 \sim \pi(x),$$

$$\limsup_{x\to\infty} \frac{\#\{p \le x : \max_{1\le i\le 3} |\alpha_{\phi,i}(p)| > 1\}}{\pi(x)} \le \frac{14}{25}.$$

Question 2.

Could we estimate the number of Hecke-Maass cusp forms on $GL(n)$ whose Satake parameters at any given prime p fail GRC?

The case of $GL(2)$

1987, Sarnak

$$\frac{1}{r(T)}\#\left\{1 \le j \le r(T) : |\lambda_j(p)| \ge \alpha > 2\right\} \ll T^{-\frac{2\log(\alpha/2)}{\log p}}.$$

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2014, Blomer, Buttcane and Raulf

$$\frac{1}{r(T)}\#\{1 \leq j \leq r(T) : |\lambda_j(p)| \geq \alpha > 2\} \ll_{\varepsilon} T^{-\frac{8\log(\alpha/2)}{\log p}+\varepsilon}.$$

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$$\frac{1}{r(T)} \# \{1 \le j \le r(T) : |\lambda_j(p)| > 2\} \ll \left(\frac{\log p}{\log T}\right)^2.$$

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2017, W. and Xiao

$$\frac{1}{r(T)} \# \{1 \le j \le r(T) : |\lambda_j(p)| > 2\} \ll \left(\frac{\log p}{\log T}\right)^3.$$

The case of $GL(n)$

2014, $n = 3$, Blomer, Buttcane and Raulf

$$\frac{1}{\#\mathcal{H}_T} \# \left\{ \phi \in \mathcal{H}_T : \max_{1 \le \ell \le 3} |\alpha_{\phi,\ell}(p)| > p^{\theta} \right\} \ll T^{-\eta},$$

where η depends on θ and p.

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where η depends on θ and p.

2021, Matz and Templier

$$\frac{1}{\#\mathcal{H}_T}\#\left\{\phi \in \mathcal{H}_T : \max_{1\le\ell\le n} |\alpha_{\phi,\ell}(p)| > p^{\theta}\right\} \ll p^{2\theta}T^{-c\theta}.$$

Theorem 2. (Lau, Ng and W.)

Let p be a fixed prime. We have

$$\frac{1}{\#\mathcal{H}_T}\#\left\{\phi\in\mathcal{H}_T:\max_{1\le\ell\le n}|\alpha_{\phi,\ell}(p)|>1\right\}\ll\left(\frac{\log p}{\log T}\right)^3,$$

where the implied constant depends on n.

Write

$$\alpha_{\phi,1}(p) = e^{i\theta_{\phi,1}(p)}, \cdots, \alpha_{\phi,n}(p) = e^{i\theta_{\phi,n}(p)},$$

where $\theta_{\phi,j}(p) \in \{a + bi : a \in [0, 2\pi), b \in \mathbb{R}\}$ for $j = 1, \cdots, n$.

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Denote $\overline{\theta}_\phi(p) = (\theta_{\phi,1}(p), \ldots, \theta_{\phi,n}(p))$.

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Since the order of $\overline{\theta}_{\phi}(p)$'s entries plays no role in GRC, we shall view $\overline{\theta}_{\phi}(p)$ in $\mathbb{C}^n/\mathfrak{S}_n$ where $\mathfrak{S}_n$ is the symmetric group of degree n.

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GRC is equivalent to

$$\overline{\theta}_\phi(p) \in [0, 2\pi)^n/\mathfrak{S}_n.$$

The Sato-Tate conjecture

Given any $I' = \prod_{j=1}^{n} [a_j, b_j] \subset [0, 2\pi)^n$. We denote by I the image of I' under the canonical map $\rho : [0, 2\pi)^n \to [0, 2\pi)^n / \mathfrak{S}_n$.

Define

$$N_I(\phi; x) := \# \left\{ p \leq x : \ \overline{\theta}_\phi(p) \in I \right\}.$$

The Sato-Tate conjecture can be formulated as

$$\lim_{x \to \infty} \frac{N_I(\phi; x)}{\pi(x)} = \int_I d\mu_{\mathrm{ST}}$$

where $\pi(x)$ counts the number of primes up to x and

$$d\mu_{\mathrm{ST}} = \frac{1}{n!(2\pi)^{n-1}} \prod_{1 \leq \ell < m \leq n} |e^{i\theta_\ell} - e^{i\theta_m}|^2 d\theta_1 \cdots d\theta_{n-1}.$$

Theorem 3.

Suppose that $T = T(x)$ satisfies $\log T/\log x \to \infty$ as $x \to \infty$. We have

$$\frac{1}{\#\mathcal{H}_T} \sum_{\phi \in \mathcal{H}_T} \frac{N_I(\phi; x)}{\pi(x)} = \mu_{ST}(I) + O\left(\frac{\log x}{\log T} + \frac{\log\log x}{\pi(x)}\right),$$

where $\mu_{ST}(I) = \int_I d\mu_{\mathrm{ST}}$ and the implied constant only depends on n.

Theorem 4.

Let $T = T(x)$ be a function satisfying $\frac{\log T}{\sqrt{x}\log\log x} \to \infty$ as $x \to \infty$. Then for any bounded continuous, real-valued function h on $\mathbb{R}$, we have

$$\lim_{x\to\infty} \frac{1}{\#\mathcal{H}_T} \sum_{\phi\in\mathcal{H}_T} h\left(\frac{N_I(\phi;x) - \pi(x)\mu_{\mathrm{ST}}(I)}{\sqrt{\pi(x)(\mu_{\mathrm{ST}}(I) - \mu_{\mathrm{ST}}(I)^2)}}\right)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t)e^{-\frac{t^2}{2}}\,dt.$$

Sketch of the proof of Theorem 2

The idea is similar to the proof of Theorem 1.

Find a polynomial $F(\alpha_{\phi,1}(p),\ldots,\alpha_{\phi,n}(p))$ satisfying

- $F(\alpha_{\phi,1}(p),\ldots,\alpha_{\phi,n}(p))$ are always nonnegative.
- $F(\alpha_{\phi,1}(p),\ldots,\alpha_{\phi,n}(p)) \geq 1$ if ϕ fails GRC at p.

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We also use the following trace formula instead of the properties of L-functions.

Trace formula (Matz-Templier)

Let $n \geq 3$ and p be fixed. Put $\underline{\alpha}_\phi = (\alpha_{\phi,1}(p), \ldots, \alpha_{\phi,n}(p))$. Given any $g \in \mathbb{C}[x_1^{\pm}, \cdots, x_n^{\pm}]^{\mathfrak{S}_n}$. We have

$$\frac{1}{\#\mathcal{H}_T} \sum_{\phi \in \mathcal{H}_T} g(\underline{\alpha}_\phi) = \int_{S^{1^n}/\mathfrak{S}_n} g d\mu_p + O\left(T^{-1/2} p^{A \deg'(g)}\right).$$

Sketch of the proof of Theorem 2

Define for any $x \neq y \in \mathbb{C}$,

$$U_L(x, y) := \frac{1}{L+1} \frac{x^{L+1} - y^{L+1}}{x - y}.$$

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For $x_1, \cdots, x_n \in \mathbb{C}$ with $x_i \neq x_j$ for $i \neq j$, we consider the unordered $n(n-1)/2$ tuple

$$\mathrm{U}_L(x_1, \cdots, x_n) = \{U_L(x_\ell, x_m)\}_{1 \leq \ell < m \leq n}.$$

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Let $s = n(n-1)/2$ and $y_1, \ldots, y_s \in \mathbb{C}$. For $m \leq s$, define

$$f_m(y_1, \ldots, y_s) := \frac{2^m}{m!(s-m)!}\sum_{\sigma \in \mathfrak{S}_s} y_{\sigma(1)}^2 \cdots y_{\sigma(m)}^2.$$

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Define

$$F_L(x_1, \dots, x_n) := \sum_{i=1}^{s} f_i(\mathrm{U}_L(x_1, \cdots, x_n)) f_i(\mathrm{U}_L(x_1^{-1}, \cdots, x_n^{-1})).$$

Sketch of the proof of Theorem 2

Clearly, $F_L(x_1, \ldots, x_n) \in \mathbb{C}[x_1^{\pm}, \cdots, x_n^{\pm}]^{\mathfrak{S}_n}$.

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Firstly, we can prove that for all $\phi \in \mathcal{H}_T$,

$$F_L(\alpha_{\phi,1}(p), \ldots, \alpha_{\phi,n}(p)) \geq 0.$$

Sketch of the proof of Theorem 2

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$$F_L(\alpha_{\phi,1}(p), \ldots, \alpha_{\phi,n}(p)) \geq 0.$$

Secondly, if GRC for ϕ fails at p, we can prove

$$F_L(\alpha_{\phi,1}(p), \ldots, \alpha_{\phi,n}(p)) \geq 1.$$

Sketch of the proof of Theorem 2

$$\frac{\#\left\{\phi \in \mathcal{H}_T : \max_{1\le \ell \le n} |\alpha_{\phi,\ell}(p)| > 1\right\}}{\#\mathcal{H}_T}$$
$$\ll \quad \frac{1}{\#\mathcal{H}_T} \sum_{\phi \in \mathcal{H}_T} F_L(\alpha_{\phi,1}(p), \ldots, \alpha_{\phi,n}(p))$$

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$$\ll \frac{1}{L^3} + p^{LA} T^{-1/2} \qquad \text{(by the trace formula)}$$

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$$\ll \left(\frac{\log p}{\log T}\right)^3$$

by taking

$$L = \left[\frac{\log T}{4A \log p}\right].$$

Thank you!