Fourier coefficients of double Eisenstein series and their analytic continuation

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Contents:

- Basics on (twisted) double Eisenstein series
- Computation of Fourier coefficients
- Analytic continuation
- Rationality of Periods of a twisted Cohen kernel

(I) Double Eisenstein Series

- $\mathcal{S}_k$, $\mathcal{B}_k$: cusp forms, normalized eigenforms for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$
- $f = \sum_{n=1}^{\infty} a(n) q^n \in \mathcal{S}_k$, $q = e^{2\pi i z}$
- $L(f,s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$
- $L^*(f,s)$: its completed L-function

$$L^*(f,s) = \frac{\Gamma(s)}{(2\pi)^s} \sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \int_0^{\infty} f(iy) y^s \frac{dy}{y}$$

- Ana. cont and functional equation $L^*(f,k-s) = (-1)^{k/2} L^*(f,s)$.

Manin's Periods Theorem

Periods of f in the critical strip:

$$L^*(f,1),\dots,L^*(f,k-1)$$

Theorem (Manin, 1973)

For $f\in\mathcal{B}_k$, there exist nonzero real numbers $\omega_\pm(f)$, such that for odd n and even m in $[1,k-1]$

$$\frac{L^*(f,n)}{\omega_+(f)},\ \frac{L^*(f,m)}{\omega_-(f)}\in K_f.$$

- $K_f=\mathbb{Q}(a(n):n=1,2,\dots)$, a number field

The common multiplier $\omega_{\pm}(f)$

Theorem (Rankin 1952 ($n=0$), Zagier 1977)

For even integers $k_1, k_2 \geq 4$, $k = k_1 + k_2 + 2n$ and $n \geq 0$,

$$\langle [E_{k_1}, E_{k_2}]_n, f \rangle = (-1)^{\frac{k_1}{2}} (2\pi i)^n 2^{3-k} \binom{k-2}{n} \frac{k_1 k_2}{B_{k_1} B_{k_2}} \times L^*(f, n+1) L^*(f, n+k_2).$$

- B_k: the Bernoulli number
- $[f, g]_n$: Rankin-Cohen bracket of index n

Theorem (Kohnen-Zagier 1984)

$\omega_{\pm}(f)$ can be chosen to satiesfy $\omega_{+}(f)\omega_{-}(f) = \langle f, f \rangle$.

- $f \mapsto L^*(f, n+1)$ is linear in f, hence gives a unique (*Cohen's kernel*) $R_n \in \mathcal{S}_k$ s.t. $\langle f, R_n \rangle = L^*(f, n+1)$

Question: periods of R_n?

Theorem (Kohnen-Zagier 1984)

For integers $0 < m, n < k-2$, *opposite parity,*

$$\begin{aligned} &2^{2-k}(k-2)!\langle R_m, R_n \rangle \\ =&\rho(m-\tilde{n}+1)m!n! + \rho(-m+\tilde{n}+1)\tilde{m}!\tilde{n}! \\ +&(-1)^{\frac{k}{2}}\rho(m-n+1)m!\tilde{n}! + (-1)^{\frac{k}{2}}\rho(-m+n+1)\tilde{m}!n! \end{aligned}$$

In particular, $\langle R_m, R_n \rangle \in \mathbb{Q}$.

- $\rho(2n) = (-1)^{n+1}\frac{B_{2n}}{(2n)!}$ if $n \geq 0$ and 0 otherwise
- $\tilde{m} = k - 2 - m$
- Kohnen-Zagier's formula includes 0 and $k-2$

Goal

A twisted version

Cohen's kernel with $s \in \mathbb{C}$

Diamantis-O'Sullivan (2010) defined for
$1 < \mathrm{Re}(s) < k-1$

$$\mathcal{C}_k(z,s) = \frac{1}{2}\sum_{\gamma\in\Gamma}(\gamma z)^{-s} j(\gamma,z)^{-k}$$

and proved that $\mathcal{C}_k(z,s) \in \mathcal{S}_k$.

- If set $\mathcal{C}_k(z,s) = 2^{2-k}(-1)^{k/2}\pi e^{-\pi i s/2}\frac{\Gamma(k-1)}{\Gamma(s)\Gamma(k-s)}\mathcal{D}_k(z,s)$, then
$$\langle \mathcal{D}_k(z,s), f\rangle = L^*(\bar{f},s).$$
- $\mathcal{C}_k(z,s)$ has analytic continuation to $s \in \mathbb{C}$.

Product of L-values?

- (Zagier) The “kernel” for

$$f \mapsto L^*(f, n+1)L^*(f, n+k_2), \quad f \in \mathcal{B}_k,$$

  is roughly $[E_{k_1}, E_{k_2}]_n$

- How about that of

$$f \mapsto L^*(f, s)L^*(f, w), \quad f \in \mathcal{B}_k, s, w \in \mathbb{C}?$$

This gives Diamantis-O’Sullivan’s double Eisenstein series.

Definition. (Diamantis-O'Sullivan 2013)

The completed double Eisenstein series:

$$E^*_{s,k-s}(z,w) = \frac{\Gamma(s)\Gamma(k-s)\zeta(1-w+s)\zeta(1-w+k-s)}{e^{-si\pi/2}2^{3-w}\pi^{k+1-w}\Gamma(k-1)}$$

$$\times \sum_{\gamma,\delta\in\Gamma_\infty\backslash\Gamma, c_{\gamma\delta^{-1}}>0} (c_{\gamma\delta^{-1}})^{w-1}\left(\frac{j(\gamma,z)}{j(\delta,z)}\right)^{-s} j(\delta,z)^{-k}$$

They proved

- its absolute and uniform convergence on compact subset of $\mathcal{D}$:

$$2 < \mathrm{Re}(s) < k-2,$$
$$\mathrm{Re}(w)+1 < \min\{\mathrm{Re}(s), k-\mathrm{Re}(s)\}$$

- $E^*_{s,k-s}(\cdot, w)$ is the kernel function for

  "$f \mapsto L^*(f,s)L^*(f,w)$ ":

  $$E^*_{s,k-s}(z,w) = \sum_{f \in \mathcal{B}_k} \frac{L^*(f,s)L^*(f,w)}{\langle f,f \rangle} f(z)$$

- Analy. cont. to $s, w \in \mathbb{C}$, $E^*_{s,k-s}(\cdot, w) \in \mathcal{S}_k$
- 8 Functional Equations ($s \longleftrightarrow w$, $s \longleftrightarrow k-s$)
- Relation: $E^*_{s,k-s}(z,w) \overset{\Gamma,\pi,\zeta}{\approx} \sum_{n=1}^{\infty} n^{w-k} T_n(\mathcal{C}_k(z,s))$

With double Eisenstein series, Diamantis-O'Sullivan (2013)

- recovered Manin's periods theorem
- extended it to general $s, w \in \mathbb{C}$, and to
- twisted periods $L^*(f, s; p/q)$, $p/q \in \mathbb{Q}$

Theorem (Diamantis-O'Sullivan 2013)

With $\omega_\pm(f)$ above and $s \in \mathbb{C}$,

$$L^*(f, s; p/q)/\omega_+(f) \in K(E^*_{k-s,s}(z, 1; p/q))K_f$$
$$L^*(f, s; p/q)/\omega_-(f) \in K(E^*_{k-s,s}(z, 2; p/q))K_f.$$

Their Twisted Setting (Diamantis-O'Sullivan):

- The twisted Cohen kernel: for $p/q \in \mathbb{Q}$,

$$\mathcal{C}_k(z,s;p/q) := \frac{1}{2}\sum_{\gamma\in\Gamma}(\gamma z + p/q)^{-s} j(\gamma,z)^{-k}$$

- The twisted double Eisenstein series

$$E^*_{s,k-s}(z,w;p/q) \overset{\Gamma,\zeta,\pi}{\approx} \sum_{\substack{a,b,c,d\in\mathbb{Z}\\ ad-bc>0}} (ad-bc)^{w-1}\left(\frac{az+b}{cz+d}+\frac{p}{q}\right)^{-s}(cz+d)^{-k}$$

Still holds:

- $E^*_{s,k-s}(z,w;p/q) \overset{\Gamma,\zeta,\pi}{\approx} \sum_{n=1}^{\infty} n^{w-k} T_n(\mathcal{C}_k(z,s;p/q))$
- $\mathcal{C}_k(z,s;p/q)$ is roughly the kernel of

$$f \mapsto L^*(f,k-s;p/q)$$

- $E^*_{s,k-s}(z,w;p/q)$ is the kernel of

$$f \mapsto L^*(f,k-s;p/q)L^*(f,k-w)$$

Naturally, Fourier coefficients of $E^*_{s,k-s}(z,w;p/q)$ tell information on the product of L-values!

For example, its first Fourier coefficient is equal to

$$\sum_{f\in\mathcal{B}_k}\frac{L^*(f,k-s;p/q)L^*(f,k-w)}{\langle f,f\rangle}$$

Choie-Kohnen-Zhang (2020)

For $E^*_{s,k-s}(z,w)$:

- Computed and continued the Fourier coefficients
- And obtained

simultaneous non-vanishing

$$\sum_{f \in \mathcal{B}_k} \frac{L^*(f,s)L^*(f,w)}{\langle f, f\rangle}$$

when $k \gg 0$ on $\rightarrow$

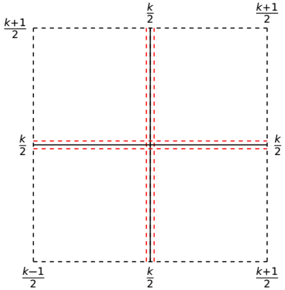

Figure: On the Real Part for Nonvanishing

(II) Fourier Expansion of $E^*_{s,k-s}(z,w;1/2)$

Theorem (You-Zhang 2021)

For $(s,w) \in \mathcal{D}$, $E^*_{s,k-s}(z,w;1/2) \overset{\Gamma,\pi}{\approx} \sum_{m=1}^{\infty} c(m)q^m$, *with* $c(1) = c_1(s,w) =$

$$\frac{(2\pi)^{k-w}\Gamma(s+w-k)}{2^{s-1}\Gamma(s)}\cos(\pi(s+w-k)/2)\zeta(s+w-k,1/2)$$

$$+(-1)^{\frac{k}{2}}\frac{(2\pi)^{k-w}\Gamma(w-s)}{2^{k-s-1}\Gamma(k-s)}\cos(\pi(w-s)/2)\zeta(w-s,1/2)$$

$$+\frac{(-1)^{\frac{k}{2}}(2\pi)^k}{2^s\Gamma(s)\Gamma(k-s)}\sum_{a+c/2>0,c>0,(a,c)=1} c^{s-k}(a+c/2)^{-s}\sum_{n>0} n^{w-1}$$

$$\times\left(e^{\pi is/2}e^{2\pi i\frac{na'}{c}}\,{}_1f_1\left(s,k;-\frac{2\pi in}{c(a+c/2)}\right)+e^{-\pi is/2}e^{-2\pi i\frac{na'}{c}}\,{}_1f_1\left(s,k;\frac{2\pi in}{c(a+c/2)}\right)\right).$$

- General $c(m)$ involves more terms; omitted

- $a' = a^{-1} \pmod{c}$
- ${}_1F_1(\alpha, \beta; z)$: Kummer's degenerate hypergeometric function
- ${}_1f_1(\alpha, \beta; z) = \frac{\Gamma(\alpha)\Gamma(\beta-\alpha)}{\Gamma(\beta)} {}_1F_1(\alpha, \beta; z)$;
- When $\mathrm{Re}(\beta) > \mathrm{Re}(\alpha) > 0$,

$$ {}_1f_1(\alpha, \beta; z) = \int_0^1 e^{zu} u^{\alpha-1} (1-u)^{\beta-\alpha-1} du. $$

Sketch of Proof

Follow the lines of Choie-Kohnen-Zhang (2020):

- Recall that

$$
\begin{aligned}
E^*_{s,k-s}(z,w;\frac{1}{2}) &= \sum_{f\in\mathcal{B}_k} \frac{L^*(f,k-s;\frac{1}{2})L^*(f,k-w)}{\langle f,f\rangle} f(z) \\
&\overset{\Gamma,\pi}{\approx} \sum_{n=1}^{\infty} n^{w-1} \sum_{\gamma\in\mathcal{M}_n} \left(\gamma z+\frac{1}{2}\right)^{-s} j(\gamma,z)^{-k}
\end{aligned}
$$

where $\mathcal{M}_n$: integral matrices of $\det = n$

Split into four cases on $\gamma \in \mathcal{M}_n$:

- $c = 0$: Apply Lipschitz's formula and obtain the contribution

$$\frac{(2\pi)^{k-w}\Gamma(s+w-k)}{2^{s-1}\Gamma(s)} \cos(\pi(s+w-k)/2)\zeta(s+w-k, 1/2)$$

- $a + c/2 = 0$: Similarly the contribution

$$(-1)^{k/2}\frac{(2\pi)^{k-w}\Gamma(w-s)}{2^{k-s-1}\Gamma(k-s)} \cos(\pi(w-s)/2)\zeta(w-s, 1/2)$$

- $(a+c/2)c > 0$: for each (a,c) fix a pair b_0, d_0 with $ad_0 - b_0c = r = \gcd(a,c)$
  - Split the set via r

    $$\left\{\begin{pmatrix} ar & nb_0/r + (t+r\ell)a \\ cr & nd_0/r + (t+r\ell)c \end{pmatrix} : r \mid n, (a,c) = 1, t \bmod r, \ell \in \mathbb{Z}\right\}$$

  - After long calculation, obtain one half of the series involving ${}_1f_1$
- $(a+c/2)c < 0$: almost identical and contribute the second half

(III) Analytic Continuation

- Need Fourier expansion in the critical strip, e.g. in

$$\mathcal{F} = \{(s,w) \in \mathbb{C}^2 : 3/2 < \mathrm{Re}(s), \mathrm{Re}(w) < k-2\}.$$

- As in Choie-Kohnen-Zhang (2020), the continuation is tricky.

- It involves several regions

$$\begin{aligned} \mathcal{D}: \quad & 2 < \mathrm{Re}(s) < k-2, \\ & \quad \mathrm{Re}(w) + 1 < \min\{\mathrm{Re}(s), k - \mathrm{Re}(s)\} \\ \mathcal{D}_1: \quad & 2 < \mathrm{Re}(s) < k-2, \quad \mathrm{Re}(w) < 0 \\ \mathcal{F}: \quad & 3/2 < \mathrm{Re}(s), \mathrm{Re}(w) < k-2 \end{aligned}$$

Recall $c(1) =$

$$\frac{(2\pi)^{k-w}\Gamma(s+w-k)}{2^{s-1}\Gamma(s)}\cos(\pi(s+w-k)/2)\zeta(s+w-k,1/2)$$
$$+(-1)^{\frac{k}{2}}\frac{(2\pi)^{k-w}\Gamma(w-s)}{2^{k-s-1}\Gamma(k-s)}\cos(\pi(w-s)/2)\zeta(w-s,1/2)$$
$$+\frac{(-1)^{\frac{k}{2}}(2\pi)^{k}}{2^{s}\Gamma(s)\Gamma(k-s)}\sum_{a+c/2>0,c>0,(a,c)=1}c^{s-k}(a+c/2)^{-s}\sum_{n>0}n^{w-1}$$
$$\times\left(e^{\pi is/2}e^{2\pi i\frac{na'}{c}}\,{}_1f_1\left(s,k;-\frac{2\pi in}{c(a+c/2)}\right)+e^{-\pi is/2}e^{-2\pi i\frac{na'}{c}}\,{}_1f_1\left(s,k;\frac{2\pi in}{c(a+c/2)}\right)\right).$$

Roughly:

- On the smaller $\mathcal{D}_1$, consider inner sum $\sum_{n>0}$:
  - Open ${}_1f_1$ with its integral expressions for the inner sums
  - Sum n and switch to Hurwitz zeta function
  - This part is more complicated than usual and involves ${}_2F_1$
- The resulting expressions also hold on $\mathcal{D}$
- With the new inner sum, the total expression is now absolutely convergent on $\mathcal{F}$
- $\mathcal{F} \cap \mathcal{D} \neq \emptyset$

We obtained: On $\mathcal{F}$,

$$\frac{2^{2-s-w}\pi^{k+1-w}\Gamma(k-1)}{\Gamma(s)\Gamma(k-s)\Gamma(k-w)}\sum_{f\in\mathcal{B}_k}\frac{L^*(f,k-s;1/2)L^*(f,k-w)}{\langle f,f\rangle}$$

note that $\sum_f = c(1)$

$$=\frac{(2\pi)^{k-w}\Gamma(s+w-k)}{2^{s-1}\Gamma(s)}\cos(\pi(s+w-k)/2)\zeta(s+w-k,1/2)$$

$$+\ \text{(three more similar terms)}$$

$$+(-1)^{\frac{k}{2}}\frac{(2\pi)^{k-w}\Gamma(w)\Gamma(1-w)}{2^{k-s-1}\Gamma(s)\Gamma(k-s-w+1)}\cos(\pi(s-w)/2)\,{}_2F_1\left[\begin{matrix}1-s, & k-s\\ & k-s-w+1\end{matrix}\middle|\frac{1}{2}\right]$$

$$+(-1)^{\frac{k}{2}}\frac{(2\pi)^{k-w}\Gamma(w)\Gamma(1-w)}{2^{s-1}\Gamma(k-s)\Gamma(1+s-w)}\cos(\pi(s+w)/2)\,{}_2F_1\left[\begin{matrix}s+1-k, & s\\ & 1+s-w\end{matrix}\middle|\frac{1}{2}\right]$$

$$+R(s,w)$$

- $R(s,w) = \cos(\pi(s+w)/2) \times (*) + \cos(\pi(s-w)/2) \times (*)$ with $(*)$'s being absolutely convergent series on $\mathcal{F}$

  Key point: when s, w are integers in $\mathcal{F}$ of opposite parity, $R(s,w) = 0$

- ${}_2F_1$ is Gauss' hypergeometric function: When $\mathrm{Re}(\gamma) > \mathrm{Re}(\beta) > 0, |z| < 1$,

$$ {}_2F_1\left[\begin{matrix}\alpha, & \beta \\ & \gamma\end{matrix}\middle| z\right] = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 \frac{t^{\beta-1}(1-t)^{\gamma-\beta-1}}{(1-zt)^{\alpha}} dt. $$

  Important: ${}_2F_1$ is computable on integer arguments when $z = \frac{1}{2}$

(IV) Rationality

Explicit formula of ${}_2F_1$

For integers a, b, n,

$$
{}_2F_1\left[\begin{matrix} a, & b \\ & \frac{1}{2}(a+b+n+1) \end{matrix}\middle| \frac{1}{2}\right]
= \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}(a+b+n+1))}{\Gamma(\frac{b}{2})\Gamma(\frac{1}{2}(b+1))} \frac{\Gamma(\frac{1}{2}(a-b-|n|+1))}{\Gamma(\frac{1}{2}(a-b+n+1))}
\times \sum_{r=0}^{|n|} \binom{|n|}{r} \frac{\delta(n,r)\Gamma(\frac{1}{2}(b+r))}{\Gamma(\frac{1}{2}(a-|n|+r+1))}
$$

- $\delta(n, r) = (-1)^r$ if $n \geq 0$ and 1 otherwise

Evaluating at integers s, w of opposite parity with $(s, w) \in \mathcal{F}$ and applying above formula,

$$\begin{aligned}
&2(2\pi)^{w-k-1}c(1) \\
&= \frac{2^{s+w-k}-1}{2^s\Gamma(s)}\rho(k-w-s+1) + (-1)^{\frac{k}{2}}\frac{2^{w-s}-1}{2^{k-s}\Gamma(k-s)}\rho(s-w+1) \\
&\quad + \frac{(2^{k-s-w}-1)\Gamma(w)}{2^{k-s}\Gamma(k-s)\Gamma(k-w)}\rho(s+w-k+1) + (-1)^{\frac{k}{2}}\frac{(2^{s-w}-1)\Gamma(w)}{2^s\Gamma(s)\Gamma(k-w)}\rho(w-s+1) \\
&\quad + \frac{(-1)^{(|k-2w|+w-s-1)/2}\Gamma(w)}{2\Gamma(s)\Gamma(k-s)\Gamma((|k-2w|+k)/2)} \\
&\qquad \times \sum_{r=0}^{|k-2w|}\delta(k-2w,r)\binom{|k-2w|}{r}\prod_{j=1}^{(k+|k-2w|-2)/2}(j+(-s-|k-2w|+r)/2) \\
&\quad + \frac{(-1)^{(|k-2w|+w+s-1)/2}\Gamma(w)}{2\Gamma(s)\Gamma(k-s)\Gamma((|k-2w|+k)/2)} \\
&\qquad \times \sum_{r=0}^{|k-2w|}\delta(k-2w,r)\binom{|k-2w|}{r}\prod_{j=1}^{(k+|k-2w|-2)/2}(j+(s-k-|k-2w|+r)/2)
\end{aligned}$$

- With the explicit relation

$$\begin{aligned}&\langle \mathcal{C}_k(z,s;1/2),\mathcal{C}_k(z,\overline{w})\rangle\\ &=\frac{2^{2(2-k)}\pi^2 e^{\pi i(w-s)/2}\Gamma(k-1)^2}{\Gamma(s)\Gamma(k-s)\Gamma(w)\Gamma(k-w)}\sum_{f\in\mathcal{B}_k}\frac{L^*(f,k-s;1/2)L^*(f,k-w)}{\langle f,f\rangle}\\ &=\frac{2^{2(2-k)}\pi^2 e^{\pi i(w-s)/2}\Gamma(k-1)^2}{\Gamma(s)\Gamma(k-s)\Gamma(w)\Gamma(k-w)}\cdot\frac{\Gamma(s)\Gamma(k-s)\Gamma(k-w)}{2^{2-s-w}\pi^{k+1-w}\Gamma(k-1)}c(1).\end{aligned}$$

- and employ the relation of $\mathcal{D}_k$ and $\mathcal{C}_k$

We obtain the rationality of the periods of $\mathcal{D}_k(z,m;1/2)$:

Theorem (You-Z. 2021)

For integers $2 \le m, n \le k-2$ of opposite parity,

$$\langle \mathcal{D}_k(z, m; 1/2), \mathcal{D}_k(z, n) \rangle \in \mathbb{Q}.$$

- Extended Kohnen-Zagier 1984: $\langle R_n, R_m \rangle \in \mathbb{Q}$ (Recall that $R_n = \mathcal{D}_k(z, n+1)$)
- The pairs with m or $n = k-2$ do not belong to $\mathcal{F}$: Done with functional equation

Remarks and Questions

- The values $1, k-1$ are not included, because of the absolute convergence for $R(s, w)$. Refine the treatment?

- The non-vanishing result should hold as well

- General rational twists $L^*(f, s; p/q)$? And then general character twists $L^*(f, s; \chi)$?
  Lack formulas for ${}_2F_1$

- Higher Fourier coefficients can also be computed and continued. Applications?

Thank You for Your Attention!