Arithmetic Exponent Pairs: Individual & Averaged

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Related Works

- J. Wu & P. Xi, Arithmetic exponent pairs for algebraic trace functions and applications, with an appendix by Will Sawin, to appear in *Algebra Number Theory*.

- J. Wu & P. Xi, Quadratic polynomials at prime arguments, *Math. Z.* **285** (2017), 631–646.
- P. Xi, Ternary divisor functions in arithmetic progressions to smooth moduli, *Mathematika* **64** (2018), 701–729.
- P. Xi, Counting fundamental solutions to the Pell equation with prescribed size, *Compositio Math.* **154** (2018), 2379–2402.

Outline

- Algebraic exponential sums: background
- Developing the method of arithmetic exponent pairs
- Several applications
- Arithmetic exponent pairs on average

Exponential sums

- $\mathrm{e}(z) := \exp(2\pi i z)$
- $U = \{u_n\} \subseteq \mathbf{R} \rightsquigarrow \mathbf{R}/\mathbf{Z}$
- I an interval

Exponential Sum

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- (Kloosterman) $U = \{\overline{n}/p\}_{n\leqslant x,\ p\nmid n} : p$ is a large prime
- (Hooley) $U = \{ha/n\}_{f(a)\equiv 0 \pmod{n}, n\leqslant x} : h \in \mathbf{Z} \setminus \{0\}$

Exponential sums

GOAL

$$\sum_{n\in I} \mathrm{e}(u_n) \ll |I|\Delta(I), \quad \Delta(I) \searrow 0$$

- We expect $\{u_n\}$ is randomly distributed on $\mathbf{R}/\mathbf{Z}$.
- The problem seems more difficult is I is rather short or u_n is highly oscillating
- Weyl, van der Corput, Vinogradov, Bombieri–Iwaniec, Huxley, Bourgain, Wooley, et al

Algebraic exponential sums

- $q \geqslant 2$ an integer
- V a suitable algebraic variety over $\mathbf{Z}$ or $\mathbf{Z}/q\mathbf{Z}$
- f a rational function over V

Algebraic Exponential Sum

$$\sum_{\mathbf{x} \in V(\mathbf{Z}/q\mathbf{Z})} \mathrm{e}\Big(\frac{f(\mathbf{x})}{q}\Big)$$

Typical examples

- Gauss sum

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$$\tau(a,\chi) = \sum_{x \in \mathbf{Z}/q\mathbf{Z}} \chi(x) \mathrm{e}\Big(\frac{ax}{q}\Big)$$

- Kloosterman sum

$$\mathrm{Kl}(a,q) = \frac{1}{\sqrt{q}} \sum_{x \in (\mathbf{Z}/q\mathbf{Z})^*} \mathrm{e}\Big(\frac{ax+\bar{x}}{q}\Big)$$

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- hyper-Kloosterman sum $(k\geqslant 2)$

$$\mathrm{Kl}_k(a,q)=q^{\frac{1-k}{2}}\sum_{\substack{x_1,x_2,\cdots,x_k\in(\mathbf{Z}/q\mathbf{Z})^*\\ x_1x_2\cdots x_k=a}}\mathrm{e}\Big(\frac{x_1+x_2+\cdots+x_k}{q}\Big)$$

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- Jacobi sum

$$\mathcal{J}(\chi,\eta)=\sum_{x\in(\mathbf{Z}/q\mathbf{Z})^*}\chi(x)\eta(1-x)$$

Incomplete sums

In many applications of harmonic analysis to analytic number theory, we usually encounter *incomplete* sums that do not have nice structures as given by suitable varieties.

For instance, we may consider

$$\sum_{n\in I} a_n F_q(n),$$

where

- F_q is defined over $\mathbf{Z}/q\mathbf{Z}$
- a_n carries the arithmetic structure of n
- I is an interval

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- sums of products of Kloosterman sums

$$\sum_{n\in I}\prod_{1\leqslant j\leqslant r}\mathrm{Kl}(n+h_j,q)$$

Fourier analysis: from incomplete to complete

A basic approach to treat incomplete sums is to apply Fourier analysis, dating back to Pólya–Vinogradov.

Theorem (Pólya–Vinogradov, 1918)

For all non-trivial Dirichlet characters $\chi \pmod q$,

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- A non-trivial bound for shorter I is highly desired in many applications.
- Subconvexity for Dirichlet L-functions

$$L(\frac{1}{2},\chi) \ll q^{\frac{1}{4}-\delta} \quad \leftrightarrow \quad \sum_{n\leqslant\sqrt{q}}\chi(n) \ll q^{\frac{1}{2}-\delta'}.$$

Fourier analysis: from incomplete to complete

The method of Pólya–Vinogradov can be generalized extensively. Consider

$$S := \sum_{n \in I} F_q(n),$$

where I is a certain interval and $F_q : \mathbf{Z}/q\mathbf{Z} \to \mathbf{C}$.

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By Plancherel, we have

$$S = \sum_{n\in\mathbf{Z}} F_q(n) I(n) = \frac{1}{\sqrt{q}} \sum_{h\in\mathbf{Z}} \widehat{F}_q(h) \overline{\widehat{I}\Big(\frac{h}{q}\Big)},$$

where $I(\cdot)$ denotes the characteristic function of I, and the normalized Fourier transform $\widehat{F}_q$ of F_q is given by

$$\widehat{F}_q(h) = \frac{1}{\sqrt{q}} \sum_{x\in\mathbf{Z}/q\mathbf{Z}} F_q(x) \mathrm{e}\Big(\frac{-hx}{q}\Big),$$

$$\widehat{I}(y) = \int_{\mathbf{R}} I(x)\mathrm{e}(-yx)\mathrm{d}x = \int_I \mathrm{e}(-yx)\mathrm{d}x \ll \min\{|I|,\ |y|^{-1}, y^{-2}\}.$$

Fourier analysis: from incomplete to complete

$$\begin{aligned}\Big|S-\frac{1}{\sqrt{q}}\widehat{F}_q(0)\widehat{I}(0)\Big| &\leqslant \frac{1}{\sqrt{q}}\sum_{|h|\geqslant 1}|\widehat{F}_q(h)|\Big|\widehat{I}\Big(\frac{h}{q}\Big)\Big|\\ &\leqslant \frac{1}{\sqrt{q}}\|\widehat{F}_q\|_\infty^*\sum_{|h|\geqslant 1}\Big|\widehat{I}\Big(\frac{h}{q}\Big)\Big|,\end{aligned}$$

where $*$ means the norms are taken over non-zero elements.

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Ideally, one has

$$S=\widehat{F}_q(0)\frac{|I|}{\sqrt{q}}+O(q^{\frac{1}{2}+\varepsilon}) \ll (|I|q^{-1}+1)q^{\frac{1}{2}+\varepsilon}.$$

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- $F_q(n)=\chi(n)$, $\widehat{F}_q=$ normalized Gauss sum

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- $F_q(n) = \chi(n)$, $\widehat{F}_q$ = normalized Gauss sum
- $F_q(n) = \mathrm{e}(\overline{n}/q)$, $\widehat{F}_q$ = Kloosterman sum

Developments and improvements

- (Burgess, 1960-70's)

$$\sum_{M<n\leqslant M+N}\chi(n)\ll N^{1-\frac{1}{r}}q^{\frac{r+1}{4r^2}+\varepsilon},\quad r=1,2,3.$$

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- (Heath-Brown, 1978)

$$\sum_{M<n\leqslant M+N}\chi(n)\ll q^{\frac{1}{6}}N^{\frac{1}{2}+\varepsilon}+q^{-1}N^{1+\varepsilon},\quad q=q_1q_2,q_1\sim q^{\frac{1}{3}},q_2\sim q^{\frac{2}{3}}.$$

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- Nontrivial for $N>q^{\frac{1}{3}+\varepsilon}$, but
  - Burgess bound: $L(\frac{1}{2},\chi)\ll q^{\frac{1}{4}-\frac{1}{16}+\varepsilon}$
  - Heath-Brown bound: $L(\frac{1}{2},\chi)\ll q^{\frac{1}{4}-\frac{1}{12}+\varepsilon}$ for special q
- Burgess' method is specially designed for periodic and completely multiplicative functions; Heath-Brown's method is flexible but only works for special q

How to generalize Heath-Brown's method to other Ψ?

Philosophy in Heath-Brown's method

Heath-Brown did not invoke the Pólya–Vinogradov method directly, before which he introduced a difference process motivated by the method of van der Corput designed for

$$\sum_{n\in I} \mathrm{e}(f(n)),$$

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Heath-Brown named this method as q-analogue of the van der Corput method.

Developments and improvements

Verschärfung der Abschätzung beim Teilerproblem.

Von

J. G. van der Corput in Freiburg (Schweiz).

Es bezeichne $T(n)$ die Anzahl der Teiler der positiven ganzen Zahl n, $\tau(x)$ die summatorische Funktion

$$\tau(x) = \sum_{n \leq x} T(n) = \sum_{n \leq x} \left[\frac{x}{n}\right] \qquad (x \geq 0),$$

C die Eulersche Konstante, $R(x)$ die Funktion

$$R(x) = \tau(x) - x \log x - (2C - 1)x \qquad (x > 0).$$

Über Dirichlets Ergebnis

$$R(x) = O(\sqrt{x})$$

war erst Voronoï[1]) 1903 hinausgekommen, indem er

$$R(x) = O(\sqrt[3]{x} \log x)$$

bewies. Bis jetzt hat man $|R(x)|$ nicht schärfer nach oben abschätzen können, so daß die Abschätzung

$$R(x) = O(x^M) \qquad \left(M < \frac{33}{100}, \text{ unabhängig von } x\right),$$

welche ich in dieser Note beweisen werde, neu ist.

Aus der (mit elementarsten Mitteln beweisbaren) Relation[2])

$$R(x) = -2 \sum_{n \leq \sqrt{x}} \left(\frac{x}{n} - \left[\frac{x}{n}\right] - \frac{1}{2}\right) + O(1)$$

[1]) G. Voronoï, *Sur un problème du calcul des fonctions asymptotiques* [Journal für die reine und angewandte Mathematik 126 (1903), S. 241—282].

[2]) Vgl. z. B. E. Landau, *Über Dirichlets Teilerproblem* [Nachrichten der K. Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-physikalische Klasse, 1920, S. 13—32] S. 15—16.

Inventiones math. 47, 149 – 170 (1978)

Inventiones mathematicae

Hybrid Bounds for Dirichlet L-Functions

D.R. Heath-Brown

Department of Pure Mathematics and Mathematical Statistics, 16, Mill Lane, Cambridge CB2 1SB, England

1. Introduction

Let χ be a character (mod q) and let $L(s,\chi)$ be the corresponding Dirichlet L-function. In this paper we consider the order of magnitude of $L(s,\chi)$ along the critical line $\mathrm{Re}(s)=\frac{1}{2}$. The trivial bound in this context is

$$L(\tfrac{1}{2}+it,\chi) \ll (qT)^{1/4}, \qquad (1)$$

where, as later, $T=|t|+1$. The estimate (1) follows, for example, from Lemma 1. Burgess [2] has given bounds for $L(s,\chi)$ that are sharper than (1) with respect to q; although he does not give the dependence on T explicitly, it is clear that his method yields

$$L(\tfrac{1}{2}+it,\chi) \ll q^{3/16+\varepsilon} T, \qquad (2)$$

for any $\varepsilon > 0$. This estimate has had many applications, for example to sharpenings of the Brun-Titchmarsh theorem on primes in arithmetic progressions.

Burgess' bound (2) is weaker than the trivial bound (1) for $q \leq T^{12}$. However there is an alternative method which improves upon (1) for sufficiently large T; one treats the q-dependence trivially and applies van der Corput's method to sums of the type

$$\sum_n (nq+r)^{-s}.$$

J. G. van der Corput, Verschärfung der Abschäätzung beim Teilerproblem, *Math. Ann.*, **87** (1922), 39–65.

D. R. Heath-Brown, Hybrid bounds for Dirichlet L-functions, *Invent. Math.* **47** (1978), 149–170.

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[1]) G. Voronoï, Sur un problème du calcul des
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van der Corput in arithmetic situations

$$S(\Psi; I) = \sum_{n \in I} \Psi(n).$$

Lemma (*A*- and *B*-processes, Heath-Brown / Irving)

- (*A*-**process**) *Assume* $q = q_1 q_2$ *with* $(q_1, q_2) = 1$ *and* $\Psi_i : \mathbf{Z}/q_i\mathbf{Z} \to \mathbf{C}$. *Define* $\Psi = \Psi_1 \Psi_2$, *then we have*

$$|S(\Psi; I)|^2 \leqslant \|\Psi_2\|_\infty^2 q_2 \Big(|I| + \sum_{0 < |\ell| \leqslant |I|/q_2} \Big| \sum_{n, n+\ell q_2 \in I} \Psi_1(n) \overline{\Psi_1(n + \ell q_2)} \Big| \Big).$$

- (*B*-**process**) *For* $\Psi : \mathbf{Z}/q\mathbf{Z} \to \mathbf{C}$, *we have*

$$S(\Psi; I) \ll \frac{|I|}{\sqrt{q}} \Big(|\widehat{\Psi}(0)| + (\log q) \Big| \sum_{h \in \mathcal{I}} \widehat{\Psi}(h) \mathrm{e}\Big(\frac{ha}{q}\Big) \Big| \Big)$$

for certain $a \in \mathbf{Z}$ *and some interval* $\mathcal{I}$ *not containing* 0 *with* $|\mathcal{I}| \leqslant q/|I|$, *where* $\widehat{\Psi}$ *denotes the (normalized) Fourier transform of* Ψ.

Trace functions as desired

To apply the A- and B-processes iteratively, one shoud expect both of

$$n \mapsto \Psi_1(n)\overline{\Psi_1(n+\ell q_2)}, \quad n \mapsto \widehat{\Psi}(n)$$

are good, in the sense that they still reveal certain oscillations.

Examples –

- $\Psi(n) = \chi(n)$ (subconvexity of Dirichlet L-functions)
- $\Psi(n) = \mathrm{e}(\overline{n}/q)$ (divisor functions in arithmetic progressions, prime gaps)

Trace functions as desired

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All above can be well interpreted in the language of trace functions following the spirit of Deligne–Katz–Fouvry–Kowalski–Michel.

Trace functions as desired

Let $p \neq \ell$ be two primes, and fix an isomorphism $\iota : \overline{\mathbf{Q}}_\ell \to \mathbf{C}$.

- Let $\mathcal{F}$ be an ℓ-adic middle-extension sheaf pure of weight zero, which is lisse on an open set U. The trace function associated to $\mathcal{F}$ is defined by

$$K : x \in \mathbf{F}_p \mapsto \iota(\mathrm{tr}(\mathrm{Frob}_x \mid V_{\mathcal{F}})),$$

  where Frob_x denotes the geometric Frobenius at $x \in \mathbf{F}_p$, and $V_{\mathcal{F}}$ is a finite dimensional $\overline{\mathbf{Q}}_\ell$-vector space, corresponding to a continuous finite-dimensional Galois representation, is unramified at every closed point x of U.

- Define the (analytic) conductor of $\mathcal{F}$ to be

$$\mathfrak{c}(\mathcal{F}) = \mathrm{rank}(\mathcal{F}) + \sum_{x \in S(\mathcal{F})} (1 + \mathrm{Swan}_x(\mathcal{F})),$$

  where $S(\mathcal{F}) \subset \mathbf{P}^1(\overline{\mathbf{F}}_p)$ denotes the set of singularities of $\mathcal{F}$, and $\mathrm{Swan}_x(\mathcal{F}) \geqslant 0$ denotes the Swan conductor of $\mathcal{F}$ at x.

Quasi-orthogonality

P. Deligne, La conjecture de Weil, II, *Publ. Math. IHES* **52** (1980), 137–252.

Proposition (Deligne)

Suppose $\mathcal{F}_1, \mathcal{F}_2$ *are two "admissible sheaves" on* $\mathbf{P}^1_{\mathbf{F}_p}$*, and* K_1, K_2 *are the associated trace functions, respectively. If* $\mathcal{F}_1, \mathcal{F}_2$ *have no common geometrically irreducible components, then there exists an absolute constant* $C > 0$ *such that*

$$\Big|\sum_{x\in\mathbf{F}_p} K_1(x)\overline{K_2(x)}\Big| \leqslant C\cdot\mathfrak{c}(\mathcal{F}_1)^4\mathfrak{c}(\mathcal{F}_2)^4\sqrt{p}.$$

- squareroot cancellation
- mild assumptions on $\mathcal{F}_1, \mathcal{F}_2$

Composite trace functions

Let q be a squarefree number. We consider the *composite* trace function K modulo q, given by the product

$$K(n) = \prod_{p \mid q} K_p(n),$$

where K_p is a trace function associated to some ℓ-adic middle-extension sheaf on $\mathbf{A}^1_{\mathbf{F}_p}$. The value of $K_p(n)$ may depend on the complementary divisor q/p. Moreover, we write

$$K(n) = \prod_{1 \leqslant j \leqslant \mathcal{J}} K(n, q_j),$$

where $q = q_1 q_2 \cdots q_{\mathcal{J}}$ for some $\mathcal{J} \geqslant 1$, q_j's are not necessarily primes but they are pairwise coprime. For each $p \mid q$, we assume $\mathfrak{c}(\mathcal{F}_p) \leqslant \mathfrak{c}$ for some uniform $\mathfrak{c} > 0$.

Convention: $K(n) \equiv 1$ if $q = 1$.

Trace functions via van der Corput

- We concern **admissible** sheaves, which are middle-extension on $\mathbf{A}^1_{\mathbf{F}_p}$, pointwise pure of weight 0 (in the sense of Deligne) and of Fourier type.
- A composite trace function $K \pmod q$ is called to be **admissible**, if the reduction K_p is admissible for each $p \mid q$.

Definition (Amiable sheaf)

- An admissible sheaf $\mathcal{F}_p$ over $\mathbf{F}_p$ is said to be d-amiable if it is geometrically isotypic and no geometrically irreducible component is geometrically isomorphic to an Artin–Schreier sheaf of the form $\mathcal{L}_{\psi(P)}$, where $P \in \mathbf{F}_p[X]$ is of degree $\leqslant d$. In such case, we also say the associated trace function K_p is d-amiable.
- A composite $K \pmod q$ is said to be compositely d-amiable if for each $p \mid q$, K_p can be decomposed into a sum of d-amiable trace functions, in which case we also say the sheaf $\mathcal{F} := (\mathcal{F}_p)_{p \mid q}$ is compositely d-amiable.
- A sheaf (or its associated trace function) is said to be (compositely) ∞-amiable if it is (compositely) amiable for any fixed $d \geqslant 1$.

Trace functions via van der Corput

In A-process, we expect $n \mapsto K(n+a)\overline{K(n)}$ is "amiable" for each $a \in \mathbf{F}_p^\times$.

Lemma (Polymath 8, 2014)

Let d be a positive integer and $p > d$. Suppose $\mathcal{F}$ is a d-amiable admissible sheaf over $\mathbf{F}_p$ with $\mathfrak{c}(\mathcal{F}) \leqslant p$. Then, for each $a \in \mathbf{F}_p^\times$, the sheaf $[+a]^*\mathcal{F} \otimes \check{\mathcal{F}}$ is compositely $(d-1)$-amiable with

$$\mathfrak{c}([+a]^*\mathcal{F} \otimes \check{\mathcal{F}}) \leqslant 5\mathfrak{c}(\mathcal{F})^4.$$

More precisely, the trace function of $[+a]^*\mathcal{F} \otimes \check{\mathcal{F}}$ can be decomposed into the sum of $\leqslant 5\mathfrak{c}(\mathcal{F})^4$ of trace functions, each of which is $(d-1)$-amiable and has a conductor at most $5\mathfrak{c}(\mathcal{F})^4$.

Remark. For any ∞-amiable sheaf $\mathcal{F}$ and large prime p, $[+a]^*\mathcal{F} \otimes \check{\mathcal{F}}$ is also compositely ∞-amiable for each $a \in \mathbf{F}_p^\times$.

Trace functions via van der Corput

In B-process, we expect $n \mapsto \widehat{K}(n)$ is also "amiable".

Lemma (Laumon, Brylinski, Katz, Fouvry–Kowalski–Michel)

Let ψ be a non-trivial additive character of $\mathbf{F}_p$ and $\mathcal{F}$ a Fourier sheaf on $\mathbf{A}^1_{\mathbf{F}_p}$. Then there exists an ℓ-adic sheaf $\mathrm{FT}_\psi(\mathcal{F})$ called the Fourier transform of $\mathcal{F}$, which is also an ℓ-adic Fourier sheaf, with the property that

$$K_{\mathrm{FT}_\psi(\mathcal{F})}(y) = \mathrm{FT}_\psi(K_{\mathcal{F}})(y) := \frac{-1}{\sqrt{p}} \sum_{x \in \mathbf{F}_p} K_{\mathcal{F}}(x)\psi(yx).$$

Furthermore, we have

- $\mathrm{FT}_\psi(\mathcal{F})$ *is pointwise of weight* 0 *on an open set, if and only if* $\mathcal{F}$ *is;*
- $\mathrm{FT}_\psi(\mathcal{F})$ *is geometrically irreducible, or geometrically isotypic, if and only if* $\mathcal{F}$ *is;*
- *We have*

$$\mathfrak{c}(\mathrm{FT}_\psi(\mathcal{F})) \leqslant 10\mathfrak{c}(\mathcal{F})^2.$$

Trace functions via van der Corput

After several steps of A-processes, one may apply the B-process if the resultant sheaves/trace functions are sufficiently "amiable". In addition, we would also like to check the amiability after applying the B-process.

Lemma (J. Wu - X.)

Suppose $r \geqslant 1, d \geqslant 2$ *and* $a \in \mathbf{F}_p^\times$. *If* $\mathcal{F}$ *is a compositely* d*-amiable sheaf on* $\mathbf{A}^1_{\mathbf{F}_p}$ *of rank* r *with* $\mathfrak{c}(\mathcal{F}) \leqslant p$. *Denote by* $\mathcal{G}$ *the Fourier transform of* $[+a]^*\mathcal{F} \otimes \check{\mathcal{F}}$.

- *If* $r = 1$, *then* $\mathcal{G}$ *is compositely* 1*-amiable when* $d = 2$, *and is compositely* ∞*-amiable when* $d \geqslant 3$.
- *If* $r \geqslant 2$, *then* $\mathcal{G}$ *is compositely* 2*-amiable. Moreover, for a given* $a \in \mathbf{F}_p^\times$, *if* $[+a]^*\mathcal{F} \otimes \check{\mathcal{F}}$ *is geometrically irreducible, then* $\mathcal{G}$ *is geometrically irreducible and* r^2*-amiable.*

Trace functions via van der Corput

Examples of amiable trace functions –

- $\psi(f_1(n)\overline{f_2(n)})$, where ψ is a primitive additive character, $f_1, f_2 \in \mathbf{F}_p[X]$, $\deg(f_1) < \deg(f_2) < p$ and $\deg(f_2) \geqslant 1$;
- $\chi(f(n))\psi(g(n))$, where χ is a primitive multiplicative character mod p, ψ is not necessarily primitive, f, g are rational functions and f is not a d-th power of another rational function with d being the order of χ;
- $\mathrm{Kl}_k(n, p)$ as a normalized hyper-Kloosterman sum of rank $k \geqslant 2$;
- The Fourier transforms of the above examples;
- The resultant functions by applying (partially)

$$BA^3BA^2BAB$$

or

$$ABA^3BA^2BAB$$

to the above examples.

- $q \geqslant 3$ is a squarefree number with $P^+(q) < q^\eta$ for any small $\eta > 0$
- K = a compositely amiable trace function mod q
- $\delta \geqslant 1$ with $(\delta, q) = 1$
- $W_\delta : \mathbf{Z}/\delta\mathbf{Z} \to \mathbf{C}$ (*deformation factor*)

Consider

$$\mathfrak{S}(K, W; I) := \sum_{n \in I} K(n) W_\delta(n),$$

where $|I| = N$. We assume $N < q\delta$, i.e., we work on incomplete sums.

- We expect the following bound holds for some (κ, λ, ν) :

$$\mathfrak{S}(K, W; I) \ll_{\eta,\varepsilon,\mathfrak{c}} N^\varepsilon (q/N)^\kappa N^\lambda \delta^\nu \|W_\delta\|_\infty. \qquad (\boldsymbol{\Omega})$$

Arithmetic exponent pairs

Proposition (Initial choices)

If K is compositely 1*-amiable, then* $(\boldsymbol{\Omega})$ *holds for*

$$(\kappa, \lambda, \nu) = (0, 1, 0), \quad (\tfrac{1}{2}, \tfrac{1}{2}, 1).$$

Classical vdC starts from the trivial exponent pair $(0, 1)$; this corresponds to the q-analogue

$$\mathfrak{S}(K, W; I) \ll N^{\varepsilon}(q/N)^{0} N^{1} \delta^{0} \|W_{\delta}\|_{\infty}.$$

However, we can always employ B-process (Poisson summation), so that our initial exponent pairs are in fact related to $(\frac{1}{2}, \frac{1}{2}) = B \cdot (0, 1)$ in the classical case.

Arithmetic exponent pairs

Let $\mathcal{J}, L \geqslant 1$. Denote by $\mathfrak{A}_q(\mathcal{J}, L)$ the set of trace functions $K \pmod q$ such that

- K is compositely $\mathcal{J}$-amiable
- $\widehat{K}$ is compositely L-amiable

Definition (Exponent pairs)

Let $0 \leqslant \kappa \leqslant \frac{1}{2} \leqslant \lambda \leqslant 1$ and $0 \leqslant \nu \leqslant 1$.

We say (κ, λ, ν) is an exponent pair of width $(\mathcal{J}; L)$, if $(\boldsymbol{\Omega})$ holds

- for all $K \in \mathfrak{A}_q(\mathcal{J}, L)$,
- for all $W_\delta : \mathbf{Z}/\delta\mathbf{Z} \to \mathbf{C}$.

An exponent pair of width $(\infty; L)$ with some $L \geqslant 1$ is called an arithmetic exponent pair.

Arithmetic exponent pairs

Theorem (A-process)

Let $\mathcal{J} \geqslant 1$. *If* (κ, λ, ν) *is an exponent pair of width* $(\mathcal{J}; 1)$, *then*

$$A \cdot (\kappa, \lambda, \nu) = \left(\frac{\kappa}{2(\kappa+1)}, \ \frac{\kappa+\lambda+1}{2(\kappa+1)}, \ \frac{1}{2} \right).$$

is an exponent pair of width $(\mathcal{J}+1; 1)$.

Theorem (B-process)

If (κ, λ, ν) *is an exponent pair of width* $(1; 1)$, *then so is*

$$B \cdot (\kappa, \lambda, \nu) = \left(\lambda - \frac{1}{2}, \ \kappa + \frac{1}{2}, \ \nu + \lambda - \kappa \right).$$

List of arithmetic exponent pairs

The following tables give the first several exponent pairs produced by different combinations of A- and B-processes to $(\frac{1}{2}, \frac{1}{2}, 1)$.
Note that ν is omitted in the list since it is not essential in many applications.

Processes	A	A^2	A^3	BA^2
(κ, λ)	$(\frac{1}{6}, \frac{2}{3})$	$(\frac{1}{14}, \frac{11}{14})$	$(\frac{1}{30}, \frac{26}{30})$	$(\frac{2}{7}, \frac{4}{7})$

Processes	BA^3	ABA^2	A^2BA^2	$BABA^2$
(κ, λ)	$(\frac{11}{30}, \frac{16}{30})$	$(\frac{2}{18}, \frac{13}{18})$	$(\frac{2}{40}, \frac{33}{40})$	$(\frac{4}{18}, \frac{11}{18})$

An application to a special case of Schinzel Hypothesis

Conjecture (Schinzel Hypothesis)

Suppose $f_1, f_2, \cdots, f_k$ *are arbitrarily irreducible polynomials with integral coefficients such that* $f_1 f_2 \cdots f_k$ *has no fixed prime factors. Then there should exist infinitely many* n*, such that* $f_1(n), f_2(n), \cdots, f_k(n)$ *take prime values simultaneously. In particular,*

- $n^2 + 1$ *is prime infinitely often;*
- $p + 2$ *is prime infinitely often;*
- $p^2 + 2$ *is prime infinitely often.*

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The existing approaches approximating Schinzel Hypothesis include

- almost prime values: P_r
- greatest prime factors: $P^+(n)$
- prime gaps
- ......

An application to a special case of Schinzel Hypothesis

Theorem (Richert, 1969)

Suppose f *is an irreducible polynomial of degree* k *with integral coefficients, and* $nf(n)$ *has no fixed prime factors. Then there exist infinitely many primes* p*, such that* $f(p) = P_{2k+1}$*. In particular,* $p^2 + 2 = P_5$ *infinitely often.*

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Theorem (J. Wu–X., 2017)

(i) *There are infinitely many primes p such that $p^2 + 2 = P_4$.*
(ii) *There are infinitely many primes p such that $P^+(p^2 + 2) > p^{0.847}$.*

An application to a special case of Schinzel Hypothesis

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(i) There are infinitely many primes p such that $p^2 + 2 = P_4$.
(ii) There are infinitely many primes p such that $P^+(p^2 + 2) > p^{0.847}$.

- Irving (2015) improved Richert's result for $k \geqslant 3$.
- Dartyge (1996) proved $P^+(p^2 + 2) > p^{0.78}$ infinitely often.

Related works – Shifted convolution

- $\lambda_1(1,n)$ = Fourier coefficient of an $SL(3,\mathbf{Z})$ Hecke–Maass cusp form
- $\lambda_2(n)$ = Fourier coefficient of an $SL(2,\mathbf{Z})$ Hecke–Maass cusp form

$$\mathscr{D}_h(X)=\sum_{n\leqslant X}\lambda_1(1,n)\lambda_2(n+h)$$

Munshi (Duke, 2013) proved that

$$\mathscr{D}_h(X)\ll X^{1-\delta},\quad \delta<\frac{1}{26}.$$

Theorem (X., 2018)

Uniformly for $0<|h|\leqslant X$, *we have*

$$\mathscr{D}_h(X)\ll X^{1-\delta},\quad \delta<\frac{1}{22}.$$

- Circle method (Jutila) + Voronoi summation + arithmetic exponent pairs

$$h\mapsto\frac{1}{\sqrt{q}}\sum_{x\ (\mathrm{mod}\ q)}^{*}\mathrm{Kl}(\overline{x}+a,q)\mathrm{e}\Big(\frac{-hx}{q}\Big).$$

Related works – Pell equation

Denote by $\varepsilon_D = t_0 + u_0\sqrt{D}$ the fundamental solution to the Pell equation

$$t^2 - Du^2 = 1.$$

For $\alpha > 0$ and $x \geqslant 2$, define

$$S^{\mathrm{f}}(x,\alpha) := \#\{2 \leqslant D \leqslant x : D \neq \square, \ \varepsilon_D \leqslant D^{\frac{1}{2}+\alpha}\}.$$

Hooley (1984) proved that for $\alpha \in]0, \frac{1}{2}]$,

$$S^{\mathrm{f}}(x,\alpha) \sim \frac{4\alpha^2}{\pi^2}\sqrt{x}\log^2 x.$$

Conjecture (Hooley, Crelle (1984))

Uniformly for $\alpha \in]\frac{1}{2}, 1]$, *we have*

$$S^{\mathrm{f}}(x,\alpha) \sim \frac{1}{\pi^2}(4\alpha - 1)\sqrt{x}\log^2 x.$$

Related works – Pell equation

- Fouvry (Crelle, 2016): For $\alpha \in]\frac{1}{2}, 1]$,

$$S^{\mathrm{f}}(x,\alpha) \geqslant \frac{1}{\pi^2}\Big(4\alpha - 1 - 4\Big(\alpha - \frac{1}{2}\Big)^2 - o(1)\Big)\sqrt{x}\log^2 x.$$

- Bourgain (IMRN, 2015): As $\alpha \to \frac{1}{2}+$,

$$S^{\mathrm{f}}(x,\alpha) \geqslant \frac{1}{\pi^2}\Big(4\alpha - 1 + O\Big(\Big(\alpha - \frac{1}{2}\Big)^{2+c}\Big) - o(1)\Big)\sqrt{x}\log^2 x.$$

Theorem (X., 2018)

Uniformly for $\alpha \in]\frac{1}{2}, \frac{35}{69}]$, we have

$$S^{\mathrm{f}}(x,\alpha) \geqslant \frac{1}{\pi^2}\Big(4\alpha - 1 - \frac{11}{5}\Big(\alpha - \frac{1}{2}\Big)^2 + \frac{6}{5}\Big(\alpha - \frac{1}{2}\Big)^3\Big)\sqrt{x}\log^2 x.$$

Related works – Pell equation

In fact, we have the more general theorem.

Theorem (X., 2018)

For any fixed $\theta \in]0, \frac{1}{2}[$, *we have*

$$S^{\mathrm{f}}(x,\alpha) \geqslant \frac{1}{\pi^2}\Big(4\alpha - 1 - 4\Big(\alpha - \frac{1}{2}\Big)^2 + \frac{1}{6}\rho\Big(\frac{1}{\theta}\Big)F_\theta(\alpha) - o(1)\Big)\sqrt{x}\log^2 x$$

uniformly in $\alpha \in [\frac{1}{2}, 1]$, *where* ρ *is the Dickman function and*

$$F_\theta(\alpha) = \begin{bmatrix} 24\alpha - 4(5+2\theta)\alpha^2 - (7-2\theta), & \alpha \in [\frac{1}{2}, \frac{6}{11+2\theta}], \\ 864(11+2\theta)^{-2} - (7-2\theta), & \alpha \in \,]\frac{6}{11+2\theta}, 1]. \end{bmatrix}$$

The basic idea is transform the problem to the estimate for triple exponential sums

$$\sum_{h\leqslant H}\ \sum_{\substack{m\sim M\\ (2,mn)=(m,n)=1\\ m \text{ is } M^{\theta}\text{-smooth}}}\sum_{n\sim N} \mathrm{e}\Big(\frac{h\overline{n}^2}{m^2}\Big).$$

Here the moduli is a perfect square, but the ideas of van der Corput method also applies.

We use *BAB*-process.

Other applications

- Bounding Dirichlet L-functions $L(\frac{1}{2}, \chi)$ to smooth moduli
- Divisor functions in arithmetic progressions to smooth moduli

$$\sum_{\substack{n \leqslant X \\ n \equiv a (\operatorname{mod} q)}} \tau_\kappa(n) \quad (\kappa = 2, 3).$$

- Distribution of roots of reducible polynomials [Dartyge–Martin, 2019]

$$\sum_{n \leqslant x} \sum_{\substack{a \, (\operatorname{mod} n) \\ f(a) \equiv 0 \, (\operatorname{mod} n)}} \mathrm{e}\Big(\frac{ha}{n}\Big) \ \ (h \neq 0),$$

where one may take $f(x) = x(x+1),\ x(x^2+1),\ x(x+1)(2x+1)$.

In the spirit of q-van der Corput method

$$\sum_{n\in I} F_q(n)$$

- Heath-Brown (2001) proved that $P^+(n^3+2) > n^{1+10^{-303}}$ for infinitely many n, as an approximation to presenting primes by cubic polynomials.
- Pierce (2006) obtained the first non-trivial bound for the size of 3-part of the class group of $\mathbf{Q}(\sqrt{-D})$ with the help of q-vdC.
- Zhang (2014) employed similar ideas to primes in APs to smooth moduli, going beyond $\frac{1}{2}$ in the classical Bombieri–Vinogradov theorem, which allows him to prove the existence of bounded gaps between infinitely many consecutive primes. Polymath8 (2014) pushed the ideas further.
- Irving (2015, 2016) picked up the original ideas of Heath-Brown. He obtained a sub-Weyl bound for $L(\frac{1}{2},\chi)$ to smooth moduli, and can also go beyond the Selberg–Hooley barrier on τ in APs (smooth moduli).
- Blomer & Milićević (2014): moments of modular L-functions ......

Arithmetic exponent pairs on average

- $x \geqslant 3$ large enough, $\eta > 0$ sufficiently small
- $S(Q,\eta) := \{Q < q \leqslant 2Q : P^+(q) \leqslant Q^\eta,\ \mu^2(q) = 1\}$
- K_q is a composite trace function $\pmod q$
- $I = I_q$ is an interval (might depend mildly on q) with $|I| \asymp N$

$$\mathfrak{S}(Q,\eta) = \frac{1}{|S(Q,\eta)|} \sum_{q \in S(Q,\eta)} \sum_{n \in I_q} K_q(n).$$

Arithmetic exponent pairs on average

We may defined the (arithmetic) exponent pair (κ, λ) such that

$$\mathfrak{S}(Q, \eta) \ll Q^{\varepsilon}(Q/|I|)^{\kappa}|I|^{\lambda} \tag{1}$$

holds for all K_q in a suitable family.

Theorem (X., 202+)

If (κ, λ) is an exponent pair of suitable width, then so is

$$C \cdot (\kappa, \lambda) = \Big(\frac{1 + 2\kappa}{2(5 + 4\kappa - 2\lambda)}, \frac{3 + 3\kappa - \lambda}{5 + 4\kappa - 2\lambda}\Big).$$

We have $C \cdot (\frac{1}{6}, \frac{2}{3}) = (\frac{2}{13}, \frac{17}{26})$, which gives a non-trivial bound for $\mathfrak{S}(Q, \eta)$ for $|I| > Q^{\frac{4}{13}+\varepsilon}$, which beats the previous range $|I| > Q^{\frac{1}{3}+\varepsilon}$.

Arithmetic exponent pairs: averaged

By elaborating A- and B-processes more effectively, we will encounter the transform

$$h \mapsto \sum_{a \in \mathbf{F}_p} \overline{\widehat{K}_p(a)} \widehat{K}_p(a+v) \psi(-ha).$$

In particular, we have to characterize the geometric features of K under the above transform. More basically, we need to study the sheaf

$$\mathrm{FT}_\psi(\mathrm{FT}_\psi(\mathcal{F})^\vee \otimes ([+v]^* \mathrm{FT}_\psi(\mathcal{F}))),$$

and hope some essential properties can be kept under such transformations!

Concluding remarks

- With the input from algebraic geometry, Kloosterman sums, as well as many other algebraic exponential sums, may provide powerful tools in modern analytic number theory.
- Methods in analytic number theory can also be employed to demonstrate certain objects in arithmetic geometry, for instance, counting rational points on projective varieties.
- More interfaces to be done!

Thank you for your attention !