On the first derivative of the cyclotomic Katz p-adic L-functions

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Plan

- Trivial zeros of Dirichlet L-functions and p-adic Kubota-Leopoldt p-adic L-functions.
- The existence of the trivial zero of the Katz p-adic L-functions.
- The p-adic regulators and conjectures on the higher derivatives of the Katz p-adic L-functions at the trivial zero.
- Results for general CM fields under the Leopoldt conjectures.

Trivial zeros of complex L-functions

Let $\chi : (\mathbf{Z}/N\mathbf{Z})^\times \to \mathbf{C}^\times$ be a primitive Dirichlet character modulo N and let $L_\infty(s,\chi)$ be the Dirichlet L-function associated with χ. If χ is even, i.e. $\chi(-1) = 1$, then the functional equation reads

$$\pi^{\frac{s}{2}} \cdot \Gamma(\frac{s}{2}) L_\infty(s,\chi) = \mathfrak{g}(\chi) N^{-s} \cdot \pi^{\frac{1-s}{2}} \Gamma(\frac{1-s}{2}) \cdot L(1-s,\chi^{-1}).$$

Here $\mathfrak{g}(\chi)$ is the Gauss sum of χ. The fact that $\Gamma(s/2)$ has a simple pole at $s = 0$ implies that

$$\mathrm{ord}_{s=0} L_\infty(0,\chi) = 1.$$

These zeros caused by the poles of archimedean L-factors are referred to trivial zeros.

Derivatives of complex L-functions at trivial zeros

Let $\zeta_N = e^{\frac{2\pi i}{N}}$. A classical formula: if $\chi \neq 1$ is even, then

$$L'_\infty(0,\chi) = -2^{-1}\sum_{a=1}^{N}\chi(a)\log(1-\zeta_N^a).$$

Let $F = \mathbf{Q}(\zeta_N)^+ = \mathbf{Q}(\cos\frac{2\pi}{N}) \hookrightarrow \mathbf{R}$ and regard χ as a character of $\mathrm{Gal}(F/\mathbf{Q})$ via the Artin map. Then

$$L'_\infty(0,\chi) = \frac{-1}{2}\sum_{\sigma\in\mathrm{Gal}(F/\mathbf{Q})}\chi(\sigma)\log\mathbf{c}_N^\sigma,$$

where $\mathbf{c}_N := (1-\zeta_N)(1-\zeta_N^{-1}) \in F^\times$ is a totally positive element.

Trivial zeros of p-adic L-functions

Let $p > 2$ be a prime and fix an isomorphism $\iota_p : \mathbf{C} \simeq \mathbf{C}_p$ once and for all. Denote by $\omega : (\mathbf{Z}/p\mathbf{Z})^\times \to \mathbf{C}^\times$ the Teichmüller character induced by ι_p. To each $\chi : (\mathbf{Z}/N\mathbf{Z})^\times \to \mathbf{C}^\times$ a primitive odd Dirichlet character modulo N, we can associate the Kubota-Leopodlt p-adic L-function $L_p(s, \chi\omega) : \mathbf{Z}_p \to \mathbf{C}_p$, which enjoys the interpolation formula

$$L_p(1-k, \chi\omega) = (1 - \chi\omega^{1-k}(p)p^{k-1})L_\infty(1-k, \chi\omega^{1-k}) \text{ for } k \geq 1.$$

In particular, $L_p(0, \chi\omega) = (1 - \chi(p))L_\infty(0, \chi)$, and hence

$$L_p(0, \chi\omega) = 0 \text{ if } \chi(p) = 1.$$

In this case, $s = 0$ is called the trivial zero for $s = 0$ is the pole of the local L-factor $(1 - \chi(p)p^{-s})^{-1}$.

$\mathscr{L}$-invariant

Suppose that $\chi(p) = 1$. Let H be the abelian extension of $\mathbf{Q}$ cut out by χ. Let $\mathcal{O}_{H,p}^{\times}$ be the group of p-units. The χ-isotypic part of $\mathcal{O}_{H,p}^{\times}$ is defined by

$$\mathcal{O}_{H,p}^{\times}[\chi] = \left\{ u \in \mathcal{O}_{H,p}^{\times} \otimes_{\mathbf{Z}} \mathbf{C}_p \mid (\sigma \otimes 1)u = (1 \otimes \chi(\sigma))u,\ \sigma \in \mathrm{Gal}(H/\mathbf{Q}) \right\}$$

It is known that $\dim_{\mathbf{C}_p} \mathcal{O}_{H,p}^{\times}[\chi] = 1$.
Let $\log_p : \mathbf{C}_p^{\times} \to \mathbf{C}_p$ be the Iwasawa logarithm. Let $\mathfrak{P}$ be the prime of $\mathcal{O}_H$ induced by ι_p and let

$$\begin{aligned} \log_{\mathfrak{P}} &: H^{\times} \otimes \mathbf{C}_p \to \mathbf{C}_p, \quad \log_{\mathfrak{P}}(\alpha \otimes x) = \log_p(\iota_p(\alpha))x, \\ \mathrm{ord}_{\mathfrak{P}} &: H^{\times} \otimes \mathbf{C}_p \to \mathbf{C}_p, \quad \mathrm{ord}_{\mathfrak{P}}(\alpha \otimes x) = \mathrm{ord}_{\mathfrak{P}}(\alpha)x. \end{aligned}$$

Define the $\mathscr{L}$-invariant by

$$\mathscr{L}(\chi) := -\frac{\log_{\mathfrak{P}}(u)}{\mathrm{ord}_{\mathfrak{P}}(u)}, \quad u \neq 0 \in \mathcal{O}_{H,p}^{\times}[\chi].$$

Derivatives of p-adic L-functions at trivial zeros

Theorem (Gross)

We have

$$L_p'(0, \chi\omega) = \mathscr{L}(\chi) L_\infty(0, \chi).$$

As a consequence, $\mathrm{ord}_{s=0} L_p(s, \chi\omega) = 1$.

This was proved by combining the Gross-Koblitz formula and the theorem of Ferrero-Greenberg. The non-vanishing of $\mathscr{L}(\chi)$ follows from the Brumer-Baker Theorem. The generalization of this derivative formula to totally real fields was solved in a series work of Darmon-Dasgupta-Pollack (2011), Ventullo (2015), and Dasgupta-Kakade-Ventullo (2018) by using the Eisenstein congruence of Hilbert modular forms

Today we will present an analogue of this result for the Katz p-adic L-functions over CM fields by adapting their ideas to the CM congruence of Hilbert modular forms

Let $K \hookrightarrow \mathbf{C}$ be an imaginary quadratic field with the fundamental discriminant $d_K > 0$. Let χ be a primitive Hecke character modulo $\mathfrak{f}$. The classical Hecke L-function of χ is defined by

$$L_\infty(s, \chi) := \sum_{(\mathfrak{a}, \mathfrak{f})=1} \chi(\mathfrak{a}) \mathrm{N}\mathfrak{a}^{-s},$$

where $\mathfrak{a}$ runs over ideals of $\mathcal{O}_K$ and N is the norm from K to $\mathbf{Q}$.

If χ has the infinity type (a, b), i.e. $\chi(\alpha\mathcal{O}_K) = \alpha^{-a}\overline{\alpha}^{-b}$ with $\alpha \equiv 1 \pmod{\mathfrak{f}}$, then the complete L-function is defined by

$$L(s, \chi) := 2(2\pi)^{-s+\max\{a,b\}} \Gamma(s + \max\{a, b\}) L_\infty(s, \chi).$$

There is a functional equation relating $L(s, \chi)$ and $L(1 - s, \chi^{-1})$.

Derivatives of L-functions for imaginary quadratic fields at the trivial zero

When χ has the infinity type $(0,0)$, i.e. χ is a ray class character of conductor $\mathfrak{f}$, $L_\infty(s,\chi)$ has a trivial zero at $s=0$ by the functional equation. Let $K(\mathfrak{f})$ be the ray class field of K of conductor $\mathfrak{f}$. Regarding χ as a character of $\mathrm{Gal}(K(\mathfrak{f})/K)$,we have $\mathrm{ord}_{s=0}L_\infty(s,\chi)=1$ and a classical formula:

$$L'_\infty(0,\chi)=\frac{-1}{12fw_{\mathfrak{f}}}\sum_{\sigma\in\mathrm{Gal}(K(\mathfrak{f})/K)}\chi(\sigma)\log\left|\varphi_{\mathfrak{f}}^{\sigma}\right|_{\mathbf{C}},$$

where

- $w_{\mathfrak{f}}$ is the number of roots of unity in $K^\times$ congruent to 1 modulo $\mathfrak{f}$,
- f is the smallest positive integer in $\mathfrak{f}$;
- $\varphi_{\mathfrak{f}}\in K(\mathfrak{f})^\times$ is the Robert's invariant (elliptic units).

More notation

Before going into the p-adic world, we need to introduce some notation:

- K: CM field and F, a totally imaginary quadratic extension of a totally real subfield of F.
- Let $\mathbf{c}$ be the complex conjugation.
- $\mathcal{W}$: a finite extension of the Witt ring $W(\overline{\mathbb{F}}_p)$.
- $\chi : \mathrm{Gal}(K(\mathfrak{f})/K) \to \mathcal{W}^\times$: a ray class character of conductor $\mathfrak{f}$.
- K_∞: the cyclotomic $\mathbf{Z}_p$-extension of K,
- $\varepsilon_{\mathrm{cyc}} : \mathrm{Gal}(K_\infty/K) \overset{\sim}{\to} 1 + p\mathbf{Z}_p$ is the p-adic cyclotomic character,
- $K(p^\infty) := \cup_{n=1}^\infty K(p^n)$ the ray class field of conductor p^∞.

The Katz p-adic L-functions over CM fields

Let Σ be a CM type of K, i.e. Σ is a subset of $\mathrm{Hom}(K, \mathbf{C})$ such that

$$\Sigma \cap \overline{\Sigma} = \emptyset; \quad \Sigma \cup \overline{\Sigma} = \mathrm{Hom}(K, \mathbf{C}), \quad (\overline{\Sigma} := \Sigma \circ \mathbf{c}).$$

Let $S_p(K)$ be the set of p-adic places of K above p. Let $\Sigma_p \subset S_p(K)$ be the set of p-adic places of K induced by $\iota_p \circ \Sigma$. We say Σ is a *p-ordinary CM type* if

$$\Sigma_p \cap \overline{\Sigma}_p = \emptyset; \quad \Sigma_p \cup \overline{\Sigma}_p = S_p(K).$$

We assume the following p-ordinary condition:

$$\text{Every prime factor of } p \text{ in } F \text{ splits in } K. \qquad \text{(ord)}$$

The assumption (ord) assures the existence of p-ordinary CM types. Fixing a p-ordinary CM type Σ, let $\mathcal{L}_\Sigma(\chi) \in \mathcal{W}[\![\mathrm{Gal}(K_\infty/K)]\!]$ be the Katz p-adic L-function constructed by Katz (when $\mathfrak{f} = (1)$) and Hida-Tilouine (when $\mathfrak{f} \neq (1)$).

Interpolation formula

The p-adic L-function $\mathcal{L}_\Sigma(\chi)$ is characterized by the following interpolation property: there exists a complex period $\Omega = (\Omega_\sigma)_{\sigma\in\Sigma} \in (\mathbf{C}^\times)^\Sigma$ and a p-adic period $\Omega_p = (\Omega_{p,\sigma})_{\sigma\in\Sigma} \in (\mathcal{W}^\times)^\Sigma$ such that for every crystalline characters $\nu : \mathrm{Gal}(K_\infty/K) \to \mathbf{C}_p^\times$ of weight $k\Sigma + j(1-\mathbf{c})$, we have

$$\frac{\nu(\mathcal{L}_\Sigma(\chi))}{\Omega_p^{k\Sigma+2j}} = \frac{[\mathcal{O}_K^\times : \mathcal{O}_F^\times]}{2^d\sqrt{\Delta_F}} \cdot \frac{1}{(\sqrt{-1})^{k\Sigma+j}} \cdot \frac{L(0,\chi\nu_\infty)}{\Omega^{k\Sigma+2j}} \times \prod_{\mathfrak{P}\in\Sigma_p} (1-\chi\nu_\infty(\overline{\mathfrak{P}}))(1-\chi\nu_\infty(\mathfrak{P}^{-1})\mathrm{N}\mathfrak{P}^{-1})$$

($k > 0$ and $j \in \mathbf{Z}_{\geq 0}[\Sigma]$ or $k \leq 1$ and $k\Sigma + j \in \mathbf{Z}_{>0}[\Sigma]$). Here

- ν_∞ is the grossencharacter of type A_0 associated with ϕ such that $\nu_\infty(\mathfrak{Q})$ is the value of ν at the geometric Frobenius $\mathrm{Frob}_\mathfrak{Q}$ at primes $\mathfrak{Q} \nmid p$,
- $L(s,\chi\nu_\infty)$ denotes the complete Hecke L-function for $\chi\nu_\infty$, including the Gamma functions at the archimedean places.

The conjecture on the vanishing order of p-adic L-functions at the trivial zero

We consider the cyclotomic Katz p-adic L-function $L_\Sigma(s,\chi) : \mathbf{Z}_p \to \mathbf{C}_p$ defined by

$$L_\Sigma(s,\chi) := \varepsilon^s_{\mathrm{cyc}}(\mathcal{L}_\Sigma(\chi)).$$

Put

$$r_\Sigma(\chi) := \#\left\{\mathfrak{P} \in \Sigma_p \mid \chi(\overline{\mathfrak{P}}) = 1\right\}.$$

The following is an analogue of Gross' conjecture for the vanishing order of the Katz p-adic L-functions at the trivial zero:

Conjecture

Suppose that $\chi \neq 1$. Then we have

$$\mathrm{ord}_{s=0} L_\Sigma(s,\chi) = r_\Sigma(\chi).$$

Since $s = 0$ is not a critical value for χ and is outside the range of interpolation, it is not clear that $L_\Sigma(0, \chi) = 0$ if $r_\Sigma(\chi) > 0$. In the case of imaginary quadratic fields, it follows from the p-adic Kronecker limit formula that $L_\Sigma(0, \chi) = 0$ if $r_\Sigma(\chi) > 0$.

Our first main result asserts that under certain Leopoldt hypothesis, the trivial zeros occur precisely when $r_\Sigma(\chi) > 0$.

The Σ-Leopoldt conjecture for χ

Let $H = \overline{\mathbf{Q}}^{\operatorname{Ker}\chi} \subset K(\mathfrak{f})$ be the extension cut out by χ. We introduce the following

Conjecture (Σ-Leopoldt conjecture for χ)

The Σ_p-logarithm map

$$\log_{\Sigma_p} : \mathcal{O}_H^{\times}[\chi] \longrightarrow (\mathbf{C}_p)^{\Sigma}, \quad x \mapsto \log_{\Sigma_p}(x) := (\log_p(\iota_p(\sigma(x)))_{\sigma \in \Sigma}$$

is injective.

Remark

- The Σ-Leopoldt conjecture for χ is a standard consequence of the Schanuel's conjecture in transcendental number theory.
- When F is a real quadratic field over $\mathbf{Q}$ and p splits in F and $\chi \neq \chi^{\mathbf{c}}$, then there exist at least two p-ordinary CM types Σ such that the Σ-Leopoldt conjecture for χ holds in view of D. Roy's strong six exponential theorem.

The existence of the trivial zero

Let Δ_F be the absolute discriminant of F. Let $\mathfrak{d}_{K/F}$ be the relative discriminant of K/F and $h_{K/F} = h_K/h_F$ be the relative class number of K/F.

Theorem A (Adel Betina and H.)

Suppose that

(i) $p \nmid 6\Delta_F h_{K/F}$;

(ii) χ *is of prime-to-p order and is unramified at primes dividing* $p\mathfrak{d}_{K/F}$.

(iii) *The Leopoldt conjecture for* F *and the* Σ*-Leopoldt conjecture for* χ *hold.*

Then we have

$$\mathrm{ord}_{s=0} L_\Sigma(s,\chi) > 0 \text{ if and only if } r_\Sigma(\chi) > 0.$$

A two-variable p-adic L-function

Let K_{Σ_p} be the maximal $\mathbf{Z}_p$-extension of K unramified outside Σ_p. Let $\varepsilon_\Sigma : \mathrm{Gal}(K_{\Sigma_p}/K) \to \overline{\mathbf{Z}}_p^\times$ be a character such that $\varepsilon_\Sigma \circ \mathscr{V}$ is the p-adic cyclclotomic character, where $\mathscr{V} : G_F^{\mathrm{ab}} \to G_K^{\mathrm{ab}}$ is the transfer map and let $\varepsilon_{\overline{\Sigma}} := \varepsilon_\Sigma^{\mathbf{c}}$ be the complex conjugation of ε_Σ. Define the two-variable Katz p-adic L-function $\mathscr{L}_\Sigma(-,-,\chi) : \mathbf{Z}_p^2 \to \mathcal{W}$ by

$$\mathcal{L}_\Sigma(s,t,\chi) := \varepsilon_\Sigma^s \varepsilon_{\overline{\Sigma}}^t (\mathcal{L}_\Sigma(\chi)) .$$

Remark

Let $h = h_K$. We have

$$\mathcal{L}_\Sigma(hs, hs, \chi) = L_\Sigma(hs, \chi).$$

An one-variable improved p-adic L-function

Proposition

There exists an one-variable (improved) p-adic L-function $\mathcal{L}^*_\Sigma(s,\chi) : \mathbf{Z}_p \to \mathcal{W}$ *such that*

$$\mathcal{L}_\Sigma(s,0,\chi) = \mathcal{L}^*_\Sigma(s,\chi) \prod_{\mathfrak{P}\in\Sigma_p} (1 - \chi\varepsilon^s_\Sigma(\overline{\mathfrak{P}})).$$

This p-adic function $\mathcal{L}^*_\Sigma(s,\chi)$ can be viewed as the p-adic L-function for χ along the $\mathbf{Z}_p$-extension K_{Σ_p}/K.

We introduce the improved Katz p-adic Eisenstein measure and construct $\mathcal{L}^*_\Sigma(s,\chi)$ by the evaluation of this Eisenstein measure at CM points.

The non-vanishing of $\mathcal{L}^*_\Sigma(0,\chi)$

The proof of Theorem A boils down to proving the non-vanishing of the special value $\mathcal{L}^*_\Sigma(0,\chi)$.

If K is an imaginary quadratic field, we have

$$\mathcal{L}^*_\Sigma(0,\chi) = -\frac{1}{12 f w_{\mathfrak{f}}} \sum_{\sigma \in \mathrm{Gal}(K(\mathfrak{f})/K)} \chi(\sigma^{-1}) \log_p(\sigma(\varphi_{\mathfrak{f}}))$$

by the p-adic Kronecker limit formula, so we get $\mathcal{L}^*_\Sigma(0,\chi) \neq 0$ by the Brumer-Baker theorem.

For general CM fields, we deduce the non-vanishing of $\mathcal{L}^*_\Sigma(0,\chi)$ from our previous work on the one-sided divisibility of the Iwasawa main conjecture for CM fields and the Σ-Leopoldt conjecture.

The p-adic regulator $\mathscr{R}_\chi$

Let $\log_\Sigma$ be the complex logarithm given by

$$\log_\Sigma : \mathcal{O}_H^\times[\chi] \longrightarrow \mathbf{C}^\Sigma \simeq \mathbf{C}_p^\Sigma, \quad x \mapsto (\log(\sigma(x))_{\sigma\in\Sigma}.$$

Define the p-adic regulator $\mathscr{R}_\chi$ of $\mathcal{O}_H^\times[\chi]$ by

$$\mathscr{R}_\chi := \det\left(\log_{\Sigma_p} \circ \log_\Sigma^{-1}\right).$$

Let $\mathcal{O}_{H,\overline{\Sigma}_p}^\times$ be the group of $\overline{\Sigma}_p$-units. Then

$$\dim_{\mathbf{C}_p} \mathcal{O}_{H,\overline{\Sigma}_p}^\times[\chi] = [F : \mathbf{Q}] + r_\Sigma(\chi).$$

Put

$$V_\chi := \ker\left\{\log_{\Sigma_p} : \mathcal{O}_{H,\overline{\Sigma}_p}^\times[\chi] \longrightarrow \mathbf{C}_p^\Sigma\right\}.$$

With the Σ-Leopoldt conjecture for χ ($\iff \mathscr{R}_\chi \neq 0$), we have

$$\mathcal{O}_{H,\overline{\Sigma}_p}^\times[\chi] = \mathcal{O}_H^\times[\chi] \oplus V_\chi; \quad \dim V_\chi = r_\Sigma(\chi).$$

Let $\Sigma_p^0 = \{\mathfrak{P} \in \Sigma_p \mid \chi(\overline{\mathfrak{P}}) = 1\}$. Put

$$\begin{aligned} \mathrm{ord}_{\Sigma_p^0} : V_\chi &\longrightarrow \mathbf{C}_p^{\Sigma_p^0}, \quad \mathrm{ord}_{\Sigma_p^0}(x) = (\mathrm{ord}_{\mathfrak{P}}(x))_{\mathfrak{P}\in\Sigma_p^0} \\ \ell^{\mathrm{cyc}}_{\Sigma_p^0} : V_\chi &\longrightarrow \mathbf{C}_p^{\Sigma_p^0}, \quad \ell^{\mathrm{cyc}}_{\Sigma_p^0}(x) = (\log_p(\mathrm{N}_{K_{\mathfrak{P}}/\mathbf{Q}_p}(x))_{\mathfrak{P}\in\Sigma_p^0}. \end{aligned}$$

Define the cyclotomic $\mathscr{L}$-invariant of V_χ by

$$\mathscr{L}_\chi := \det\left(\ell^{\mathrm{cyc}}_{\Sigma_p^0} \circ \mathrm{ord}^{-1}_{\Sigma_p^0}\right).$$

Higher derivatives of the Katz p-aduc L-functions at the trivial zero

Conjecture

Assume the Σ-Leopoldt conjecture holds for χ. We have $\mathrm{ord}_{s=0} L_p(s,\chi) \geq e := r_\Sigma(\chi)$, *and*

$$\left.\frac{L_p(s,\chi)}{s^e}\right|_{s=0} = \mathscr{L}_\chi \cdot (-1)^e \mathscr{R}_\chi \times \prod_{\mathfrak{P}\in\Sigma_p}(1-\chi(\mathfrak{P}^{-1})\mathrm{N}\mathfrak{P}^{-1}) \prod_{\mathfrak{P}\in\Sigma_p\setminus\Sigma_p^0}(1-\chi(\overline{\mathfrak{P}}))\cdot L^*(0,\chi).$$

Here $L^*(0,\chi)$ is the leading coefficient of $L(s,\chi)$.

Buyukboduk and Sakamoto in 2019 proved that this formula is a consequence of several open conjectures, including the existence of the conjectural Rubin-Stark units and the Iwasawa main conjecture for CM fields.

A conjectural formula for the value $\mathcal{L}_\Sigma^*(0,\chi)$

Inspired by the work of Buyukboduk-Sakamoto, we expect that

Conjecture

$$\mathcal{L}_\Sigma^*(0,\chi) = (-1)^{r_\Sigma(\chi)} \mathscr{R}_\chi \prod_{\mathfrak{P}\in\Sigma_p} (1-\chi(\mathfrak{P}^{-1})\mathrm{N}\mathfrak{P}^{-1}) \cdot L^*(0,\chi).$$

The first derivative formula of $L_\Sigma(s,\chi)$

Define the complex conjugation $\chi^{\mathbf{c}}$ of χ by $\chi^{\mathbf{c}}(\sigma)=\chi(\mathbf{c}\sigma\mathbf{c})$. Recall that we say χ is anticyclotomic if $\chi^{\mathbf{c}}=\chi^{-1}$

Theorem B (B.-H.)

Suppose that χ is anticyclotomic. With the same hypotheses in Theorem A, we have

$$\left.\frac{L_\Sigma(s,\chi)}{s}\right|_{s=0}=\mathscr{L}_\chi\cdot\mathscr{L}_\Sigma^*(0,\chi)\prod_{\mathfrak{P}\in\Sigma_p\setminus\Sigma_p^0}(1-\chi(\overline{\mathfrak{P}})).$$

In particular, this shows that

$$\mathrm{ord}_{s=0}L_\Sigma(s,\chi)=1 \textit{ if and only if } r_\Sigma(\chi)=1 \textit{ and } \mathscr{L}_\chi\neq 0.$$

Remark

If $r_\Sigma(\chi)=1$, the non-vanishing of $\mathscr{L}_\chi$ is a consequence of the Four Exponential conjecture in transcendental number theory.

The proof of Theorem B uses the two-variable p-adic L-function:

- Use the Proposition to compute the first derivative $\frac{d}{ds}\mathcal{L}_\Sigma(s,0,\chi)|_{s=0}$.
- Adapt the method of Dasgupta, Kakde and Ventullo [DKV] to the CM congruence to compute the higher derivatives of the anticylotomic p-adic L-function $\mathcal{L}_\Sigma(s,-s,\chi)$.
- Theorem B follows from the equation

$$\left.\frac{L_\Sigma(s,\chi)}{s}\right|_{s=0} = \left.\frac{2\mathcal{L}_\Sigma(s,0,\chi)-\mathcal{L}_\Sigma(s,-s,\chi)}{s}\right|_{s=0}.$$

Choose $u_{\mathfrak{P}} \in \mathcal{O}_{H,\mathfrak{P}}^{\times}[\chi]$ with $\mathrm{ord}_{\mathfrak{P}}(u_{\mathfrak{P}}) = 1$ and put $\mathscr{L}_{\mathfrak{P}} := \log_p \left(\mathrm{N}_{K_{\mathfrak{P}}/\mathbf{Q}_p}(u_{\mathfrak{P}}) \right)$. Define the anticyclotomic logarithm $\ell_{\mathfrak{P}}^{\mathrm{ac}} := 2\mathscr{L}_{\mathfrak{P}} \cdot \mathrm{ord}_{\mathfrak{P}} - \log_p \circ \mathrm{N}_{K_{\mathfrak{P}}/\mathbf{Q}_p}$ for $\mathfrak{P} \in \Sigma_p^0$.

Theorem C (B.-H.)

With the same hypotheses in Theorem A, we have

- $\mathrm{ord}_{s=0} \mathcal{L}_{\Sigma}(s, -s, \chi) \geq r_{\Sigma}(\chi)$;
- $$\left. \frac{\mathcal{L}_{\Sigma}(s, -s, \chi)}{s^{r_{\Sigma}(\chi)}} \right|_{s=0} = \mathscr{L}_{\chi}^{\mathrm{ac}} \cdot \mathcal{L}_{\Sigma}^{*}(0, \chi) \prod_{\mathfrak{P} \in \Sigma_p \setminus \Sigma_p^0} (1 - \chi(\overline{\mathfrak{P}})).$$

Here $\mathscr{L}_{\chi}^{\mathrm{ac}}$ is the anticyclotomic p-adic regulator with $\log \circ \mathrm{N}_{K_{\mathfrak{P}}/\mathbf{Q}_p}$ *replaced by $\ell_{\mathfrak{P}}^{\mathrm{ac}}$ in the definition.*

The proof is divided into the two steps:

- Choose a nice ray class character ϕ such that $\chi = \phi^{\mathbf{1-c}}$. Note that $\phi \neq \phi^{\mathbf{c}}$ as $\chi \neq 1$ and construct an explicit congruence between the Hida family of CM forms $\boldsymbol{\theta}_{\phi^{\mathbf{c}}}$ and non-CM forms.
- Apply Ribet's method and the techniques of [DKV] to get the higher derivative formula.

Now we briefly explain the first step in the case of $F = \mathbf{Q}$.

Let $\mathscr{A}$ be the ring of rigid analytic functions on the closed unit disk $\mathbf{D} := \left\{ s \in \mathbf{C}_p \mid |s|_p \leq 1 \right\}$. Let N be a prime-to-p integer and ξ be a Dirichlet character. Let $\mathbf{S}(N, \xi)$ be the space of ordinary $\mathscr{A}$-adic cusp forms, consisting of

$$F(s) = \sum_{n>0} a(n, F)(s) q^n \in \mathscr{A}[\![q]\!] \quad (s \in \mathbf{D})$$

such that for any positive integer $k \in \mathbf{D}$ with $k \equiv 0 \pmod{p-1}$,

$$F(k) = \sum_{n>0} a(n, F)(k) q^n$$

is the q-expansion of some p-ordinary elliptic modular form of weight $k+1$, level Np and nebentypus ξ.

Then $\mathbf{S}(N, \xi)$ is a $\mathscr{A}$-module with the usual action of Hecke operators. A classical result in Hida theory asserts that $\mathbf{S}(N, \xi)$ is a finite free $\mathscr{A}$-module.

If $\phi : \mathrm{Gal}(K(\mathfrak{c})/K) \to F^\times$ is a primitive ray class character with $(\mathfrak{c}, pd_K) = 1$, we put

$$\boldsymbol{\theta}_\phi(s) := \sum_{(\mathfrak{a},\mathfrak{p}\mathfrak{c})=1} \phi(\mathfrak{a})(\varepsilon_\Sigma(\mathrm{Frob}_\mathfrak{a}))^{-s} q^{\mathrm{N}\mathfrak{a}} \in \mathscr{A}[\![q]\!] \quad (s \in \mathbf{D}).$$

Then $\boldsymbol{\theta}_\phi \in \mathbf{S} := \mathbf{S}(\Delta_K \mathrm{N}\mathfrak{c}, \phi_+\tau_{K/F})$ is a Hecke eigenform, where $\phi_+ : (\mathbf{Z}/\mathrm{N}\mathfrak{c}\mathbf{Z})^\times \to \mathbf{C}^\times$ is the character $\phi_+(a) := \phi(a\mathcal{O}_K)$ for $a \in \mathbf{Z}$ prime to $\mathrm{N}\mathfrak{c}$. We call $\boldsymbol{\theta}_\phi$ the $\mathscr{A}$-adic CM forms associated with ϕ.

Let $\mathscr{A}^\dagger$ be the stalk of $\mathscr{A}$ at $s = 0$, i.e.

$$\mathscr{A}^\dagger = \left\{ \sum_{n \geq 0} a_n s^n,\ a_n \in \mathbf{C}_p \mid \lim_{n \to \infty} |a_n|_p\, r^n = 0 \text{ for some } r > 0 \right\}.$$

Put $\mathbf{S}^\dagger = \mathbf{S} \otimes_{\mathscr{A}} \mathscr{A}^\dagger$. There is a spectral decomposition of Hecke modules

$$\mathbf{S}^\dagger \otimes_{\mathscr{A}^\dagger} \mathscr{K} = \mathscr{K} \cdot \boldsymbol{\theta}_\phi \bigoplus \mathscr{K} \cdot \boldsymbol{\theta}_{\phi^{\mathbf{c}}} \bigoplus \mathbf{S}^\perp.$$

Here $\mathscr{K}$ is the fractional field of $\mathscr{A}^\dagger$ and $\mathbf{S}^\perp$ is $\mathscr{A}^\dagger$-submodule of $\mathbf{S}^\dagger$ generated by forms orthogonal to $\boldsymbol{\theta}_\phi$ and $\boldsymbol{\theta}_{\phi^{\mathbf{c}}}$.

Let $\mathbf{T}^\perp \subset \mathrm{End}_{\mathscr{A}^\dagger} \mathbf{S}^\perp$ be the $\mathscr{A}^\dagger$-algebra acting on $\mathbf{S}^\perp$ generated by Hecke operators $\{T_\ell\}_{\ell \nmid pN}$ and $\{U_q\}_{q \mid pN}$.

In view of the works of Darmon,-Dasgupta-Pollack and Dasgupta-Kakde-Ventullo, to compute the $\mathscr{L}$-invariant, we will need to construct a non-zero $\mathscr{A}^{\dagger}$-adic form $\mathscr{H} \in \mathbf{S}^{\perp}$ such that

- $\mathscr{H}$ is a generalized Hecke eigenform modulo s^{r+1}, where $r = \mathrm{ord}_{s=0}\mathcal{L}_{\Sigma}(s, -s, \chi)$.
- $\mathscr{H}$ (mod s^{r}) shares the same Hecke eigenvalues with $\boldsymbol{\theta}_{\phi^{\mathsf{c}}}$ (mod s^{r}).

The method of [DKV] tells us that $\mathscr{L}_{\chi}^{\mathrm{ac}}$ can be computed from the Hecke eigenvalues of $\mathscr{H}$ (mod s^{r+1}).

The construction of $\mathscr{H}$ is achieved by the explicit p-adic Rankin-Selberg method.

The p-adic Rankin-Selberg method

Let $\zeta_{F,p}(s) = L_p(s, \mathbf{1})$ be the p-adic Riemann zeta function. For any integer $C \mid N$, define

$$G_C(s) := 1 + \frac{2}{\zeta_p(1-s)} \sum_{n>0} \left(\sum_{d|n} \langle d \rangle^{s-1} \right) q^{Cn}.$$

Here $\langle d \rangle = d\omega(d)^{-1}$. Then $G_C = G_C(s)$ is a p-adic family of Eisenstein series and since $\zeta_p(1-s)$ has a pole at $s = 1$,

$$G_C(0) = 1.$$

Let

$$\theta_\phi := \sum_{(\mathfrak{a},\mathfrak{c})=1} \phi(\mathfrak{a}) q^{\mathrm{N}\mathfrak{a}}$$

be the weight one CM form associated with ϕ. Consider

$$e_{\mathrm{ord}}(G_C \theta_\phi) \in \mathbf{S}(N, \phi_+ \tau_{K/F}) \otimes_{\mathscr{A}} \hat{\mathscr{A}}_0.$$

Here $e_{\mathrm{ord}} = \lim_{n\to\infty} U_p^{n!}$ is Hida's ordinary projector.

By the spectral decomposition, there exist unique A and B in $\mathscr{K} = \mathrm{Frac}\,\mathscr{A}$ such that

$$\mathscr{H} := e_{\mathrm{ord}}(G_C(s)\theta_\phi) - \frac{1}{A}\boldsymbol{\theta}_\phi - \frac{1}{B}\boldsymbol{\theta}_{\phi^{\mathbf{c}}} \in \mathbf{S}^\perp.$$

Proposition (The explicit p-adic Rankin-Selberg convolution)

With a good choice of ϕ and C, the meromorphic function $B = B(s)$ is given by

$$B(s) = \frac{L(0,\tau_{K/F})\mathcal{L}_\Sigma(s,-s,\chi)}{2\mathcal{L}^*_\Sigma(s,0,\chi)}\langle\Delta_K\rangle^s \cdot \frac{\zeta_{F,p}(1-s)}{\mathcal{L}^*_\Sigma(s,0,1)}.$$

Let $r_B := \mathrm{ord}_{s=0}(B)$ and $r_A = \mathrm{ord}_{s=0}(A)$. We put

$$\mathscr{F} := B\cdot\mathscr{H} \text{ if } r_B \geq r_A; \quad \mathscr{F} = A\cdot\mathscr{H} \text{ if } r_A > r_B.$$

We thus get a non-trivial congruence modulo s^{r_B} (resp. s^{r_A} between non-CM form $\mathscr{F}$ in $\mathbf{S}^\perp$ and CM forms $\boldsymbol{\theta}_{\phi^{\mathbf{c}}}$ (resp. $\boldsymbol{\theta}_\phi$).

A special case of the p-adic Kronecker limit formula

To assure we obtain sufficiently many congruence, we need to know $r_B = \mathrm{ord}_{s=0}\mathcal{L}_\Sigma(s,-s,\chi)$. This is done by proving the following special case of the p-adic Kronecker limit formula:

Proposition

Suppose that the Leopoldt conjecture for F holds. For each character χ of the ideal class group of K, we have

$$\left.\frac{2^d \mathcal{L}^*_\Sigma(s,0,\chi)}{\zeta_{F,p}(1-s)}\right|_{s=0} = \begin{cases} h_{K/F} & \text{if } \chi = \mathbf{1}, \\ 0 & \text{if } \chi \neq \mathbf{1}. \end{cases}$$

Thank you very much for the attension!