On canonical systems related to roots of polynomials

Masatoshi Suzuki

Tokyo Institute of Technology

PANT–Kyoto 2021

Research Institute for Mathematical Sciences, Kyoto University
via Zoom

December 10, 2021, 10:45–11:45

The subject is the relationship between the following two matters:

- Distribution of roots of polynomials

$$\text{(half-planes)} \overset{\text{LFT}}{\rightsquigarrow} \text{(the unit circle)}$$

- Inverse problem of (quasi) canonical systems

$\uparrow$ 1st order systems of ODEs

For polynomial

$$f(x) = a_d x^d + a_{d-1} x^{d-1} + \cdots + a_1 x + a_0 \ \in \ \mathbb{C}[x],$$

we define $D_0(f) := 1$,

$$D_1(f) := \det \left[\begin{array}{c|c} a_d & \overline{a_0} \\ \hline a_0 & \overline{a_d} \end{array}\right], \quad D_2(f) := \det \left[\begin{array}{cc|cc} a_d & & \overline{a_0} & \\ a_{d-1} & a_d & \overline{a_1} & \overline{a_0} \\ \hline a_0 & a_1 & \overline{a_d} & \overline{a_{d-1}} \\ & a_0 & & \overline{a_d} \end{array}\right],$$

$$D_3(f) := \det \left[\begin{array}{ccc|ccc} a_d & & & \overline{a_0} & & \\ a_{d-1} & a_d & & \overline{a_1} & \overline{a_0} & \\ a_{d-2} & a_{d-1} & a_d & \overline{a_2} & \overline{a_1} & \overline{a_0} \\ \hline a_0 & a_1 & a_2 & \overline{a_d} & \overline{a_{d-1}} & \overline{a_{d-2}} \\ & a_0 & a_1 & & \overline{a_d} & \overline{a_{d-1}} \\ & & a_0 & & & \overline{a_d} \end{array}\right], \cdots, D_d(f).$$

The sign change of $D_n(f)$ is related to the distribution of roots of f.

The Schur–Cohn Test

Schur (1917, 1918), Cohn (1922)

Let $f(x) \in \mathbb{C}[x]$, $\deg f = d$.

Suppose that $D_n(f) \neq 0$ for all $1 \leq n \leq d$ and let

$$q := \sharp \text{ of sign changes in } (D_0(f), D_1(f), \ldots, D_d(f)).$$

Then

- $f(x)$ has no roots on $\mathbb{T} = \{|x| = 1\}$,
- $f(x)$ has exactly $d - q$ roots inside $\mathbb{T}$ counting multiplicity.

In particular, all roots of f lie inside $\mathbb{T}$ iff $D_n(f) > 0$ for all $1 \leq n \leq d$.

For

$$f(x) = (2x - i)(x-2)(x-3)$$
$$= 2x^3 - (10+i)x^2 + (12+5i)x - 6i,$$

we have $D_0(f) = 1$, $D_1(f) = \det \begin{bmatrix} 2 & 6i \\ -6i & 2 \end{bmatrix} = -32$,

$$D_2(f) = \det \left[\begin{array}{cc|cc} 2 & & 6i & \\ -10-i & 2 & 12-5i & 6i \\ \hline -6i & 12+5i & 2 & -10+i \\ & -6i & & 2 \end{array}\right] = -1800,$$

$$D_3(f) = \det \left[\begin{array}{ccc|ccc} 2 & & & 6i & & \\ -10-i & 2 & & 12-5i & 6i & \\ 12+5i & -10-i & 2 & -10+i & 12-5i & 6i \\ \hline -6i & 12+5i & -10-i & 2 & -10+i & 12-5i \\ & -6i & 12+5i & & 2 & -10+i \\ & & -6i & & & 2 \end{array}\right] = 187200.$$

$(D_0, D_1, D_2, D_3) = (1, -32, -1800, 187200)$ has 2 sign changes.

We confirm that $d - q = 3 - 2 = 1$ is the number of roots inside $\mathbb{T}$.

The most simple case

The theory of **canonical systems** generalizes the exponential function from the perspective of Fourier analysis (Paley–Wiener).

- $I = [0, a)$, $0 < a < \infty$, $u(t,z) : I \times \mathbb{C} \to \mathbb{C}^{2\times 1}$,

$$\frac{\partial}{\partial t} u(t,z) + z \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u(t,z) = 0, \quad \lim_{t \to a} u(t,z) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$$\left(\Leftrightarrow \begin{cases} A'(t,z) - zB(t,z) = 0 \\ B'(t,z) + zA(t,z) = 0 \end{cases} \Leftrightarrow \begin{cases} A''(t,z) + zA(t,z) = 0 \\ B''(t,z) + zB(t,z) = 0 \end{cases} \right)$$

The unique solutions is

$$u(t,z) = \begin{bmatrix} \cos((a-t)z) \\ \sin((a-t)z) \end{bmatrix}$$

and

$$\cos((a-t)z) - i\sin((a-t)z) = \exp(-i(a-t)z).$$

$$H(t): I=[t_0,t_1) \;\to\; \mathrm{Sym}_2(\mathbb{R})=\left\{\begin{bmatrix}\alpha & \beta\\ \beta & \gamma\end{bmatrix} : \alpha,\beta,\gamma\in\mathbb{R}\right\}, \text{ measurable}$$

- The first order system

$$\frac{\partial}{\partial t}u(t,z)+z\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}H(t)\,u(t,z)=0,\quad z\in\mathbb{C}$$

for unknown function $u(t,z): I\times\mathbb{C} \;\to\; \mathbb{C}^{2\times 1}$ is called a quasi canonical system on I (QCS/QCS(H) for short). $H(t)$ is called Hamiltonian.

- QCS is called a canonical system (CS/CS(H)) if

1. $H(t)\geq 0$ for a.e $t\in I$,
2. $H(t)\not\equiv 0$ on $^{\forall}J\subset I$, $|J|>0$,
3. $H(t)=(h_{ij}(t))$, $h_{ij}(t)\in L^1_{\mathrm{loc}}(I)$.

Several classical ODEs are reduced to canonical systems.

Schrödinger equation

$$-y''(x,z) + q(x)y(x,z) = zy(x,z), \quad q(x) \in L^1(I)$$

$\alpha(x)$, $\beta(x)$: the solutions for $z = 0$ with $W = \alpha'\beta - \alpha\beta' = -1$. Define

$$u(x,z) = \begin{bmatrix} \alpha(x) & \beta(x) \\ \alpha'(x) & \beta'(x) \end{bmatrix}^{-1} \begin{bmatrix} y(x,z) \\ y'(x,z) \end{bmatrix}.$$

Then $u(x,z)$ satisfies CS(H) with

$$H(x) = \begin{bmatrix} \alpha(x)^2 & \alpha(x)\beta(x) \\ \alpha(x)\beta(x) & \beta(x)^2 \end{bmatrix}.$$

Other examples: Dirac equation, string equation, etc.

The solution of a canonical system

Suppose that $u(t,z) = {}^t(A(t,z), B(t,z))$ solves a CS on $I = [t_0, t_1)$ with

$$\lim_{t \to t_1} u(t,z) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Then $E(t,z) = A(t,z) - iB(t,z)$ belongs to the Hermite–Biehler class $\mathbb{HB}$ for $\forall t \in I$. $E \in \mathbb{HB}$ is a generalization of the exponential function which is defined to be an entire function satisfying

$$|\overline{E(\bar{z})}| < |E(z)| \quad \text{for} \quad \Im(z) > 0$$

and $E(z) \neq 0$, $\forall z \in \mathbb{R}$.

- $H \rightsquigarrow CS(H) \overset{\text{solve}}{\rightsquigarrow} E \in \mathbb{HB}$ (direct problem)

Inverse problem for canonical systems

- $H \rightsquigarrow CS(H) \overset{\text{solve}}{\rightsquigarrow} E \in \mathbb{HB}$ (direct problem)

Q: Does any $E \in \mathbb{HB}$ come from H (CS(H)) ? (inverse problem)

A: Yes! L. de Branges (1960 $\pm\varepsilon$)

But, in general, it is difficult to determine H explicitly from a given E

Associating a polynomial to a canonical system

For $f(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_0 \in \mathbb{C}[x]$, we consider

$$E_f(z) := e^{irdz/2} f(e^{-irz}), \quad r = \begin{cases} 1, & d = \deg f : \text{even}, \\ 2, & d = \deg f : \text{odd}. \end{cases}$$

(the definition of r is for technical reason)

$$E_f(z) = \begin{cases} a_1 e^{-iz} + a_0 e^{iz}, & \deg f = 1, \\ a_2 e^{-iz} + a_1 + a_0 e^{iz}, & \deg f = 2, \\ a_3 e^{-3iz} + a_2 e^{-iz} + a_1 e^{iz} + a_0 e^{3iz}, & \deg f = 3, \\ a_4 e^{-2iz} + a_3 e^{-iz} + a_2 + a_1 e^{iz} + a_0 e^{2iz}, & \deg f = 4, \ldots \end{cases}$$

- All roots of f are inside $\mathbb{T}$ $\overset{\text{iff}}{\Longleftrightarrow}$ $E_f(z) \in \mathbb{HB}$ (elementary).

$\rightsquigarrow$ $^{\exists}H_f(t)$ (Hamiltonian) by de Branges' result

A few natural questions

- All roots of f are inside $\mathbb{T}$ $\overset{\text{iff}}{\Longleftrightarrow}$ $E_f(z) \in \mathbb{HB}$ (elementary).

$$\rightsquigarrow\ {}^{\exists}H_f(t) \text{ (Hamiltonian) by de Branges' result}$$

Q1. What is the explicit formula of $H_f(t)$?

Q2. Even if f has a root outside the unit circle,
can we get $E_f(z)$ from the solution of a QCS(H) ?

Q3. If the answer to Q2 is yes, what is the explicit formula of H ?

Theorem 1 (inverse problem) Let $f(x) \in \mathbb{C}[x]$, $d = \deg f$.

Suppose that $f(0) \neq 0$, $D_d(f) \neq 0$.Then $^{\exists}\widetilde{H_{f,n}} \in \mathrm{Sym}_2^+(\mathbb{R})$ $(1 \leq n \leq d)$ s.t.

$$E_f(z) = A(0,z) - iB(0,z)$$

for the unique solution ${}^t(A(t,z), B(t,z))$ of QCS for the Hamiltonian

$$H_f(t) = \frac{1}{D_{n-1}(f)D_n(f)}\widetilde{H_{f,n}}, \quad \frac{r(n-1)}{2} \leq t < \frac{rn}{2}, \quad 1 \leq n \leq d$$

on $[0, rd/2)$ ($r = 1$ if d is even, $r = 2$ if d is odd) and

$$\lim_{t \to rd/2} \begin{bmatrix} A(t,z) \\ B(t,z) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(E_f(0) + \overline{E_f(0)}) \\ \frac{i}{2}(E_f(0) - \overline{E_f(0)}) \end{bmatrix}.$$

- $H_f(t) > 0$ on $[0, rd/2)$ if all roots of f lie inside $\mathbb{T}$. (i.e This H_f is H in de Branges inverse theorem)

For $f(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0 \ \in \ \mathbb{C}[x]$, we define

$$\underset{n\times n}{M_n(f)} := \begin{bmatrix} a_d & a_{d-1} & \cdots & a_{d-n+1} \\ & a_d & \cdots & a_{d-n+2} \\ & & \ddots & \vdots \\ & & & a_d \end{bmatrix}, \quad \underset{n\times n}{N_n(f)} := \begin{bmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ & a_0 & \cdots & a_{n-2} \\ & & \ddots & \vdots \\ & & & a_0 \end{bmatrix}.$$

$$\left(\text{Then} \quad D_n(f) = \det \underset{2n\times 2n}{\begin{bmatrix} {}^{\mathrm{t}}M_n(f) & \pm{}^{\mathrm{t}}\overline{N_n(f)} \\ \pm N_n(f) & \overline{M_n(f)} \end{bmatrix}}. \right)$$

For every $1 \leq n \leq d$, using the solutions of linear equations

$$\underbrace{\begin{bmatrix} {}^{\mathrm{t}}M_n(f) & \pm{}^{\mathrm{t}}\overline{N_n(f)} \\ \pm N_n(f) & \overline{M_n(f)} \end{bmatrix}}_{2n\times 2n} \begin{bmatrix} z_n^{\pm}(1) \\ z_n^{\pm}(2) \\ \vdots \\ z_n^{\pm}(n) \\ \overline{z_n^{\pm}(n)} \\ \overline{z_n^{\pm}(n-1)} \\ \vdots \\ \overline{z_n^{\pm}(1)} \end{bmatrix} = \mp \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 2\overline{a_0} \\ 2a_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

for unknowns $z_n^{\pm}(1) \ldots, z_n^{\pm}(n)$, we define the matrix $H_{f,n}$ by

$$\begin{bmatrix} \Re(z_n^+(1)) & \Im(z_n^+(1)) \\ -\Im(z_n^-(1)) & \Re(z_n^-(1)) \end{bmatrix} \cdots \begin{bmatrix} \Re(z_1^+(1)) & \Im(z_1^+(1)) \\ -\Im(z_1^-(1)) & \Re(z_1^-(1)) \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} H_{f,n}$$

$$= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \Re(z_n^+(1)) & \Im(z_n^+(1)) \\ -\Im(z_n^-(1)) & \Re(z_n^-(1)) \end{bmatrix} \cdots \begin{bmatrix} \Re(z_1^+(1)) & \Im(z_1^+(1)) \\ -\Im(z_1^-(1)) & \Re(z_1^-(1)) \end{bmatrix}.$$

Then $\boxed{D_{n-1}(f)D_n(f)H_{f,n} = \widetilde{H_{f,n}}}$. □

For $f(x) = (2x - i)(x - 2)(x - 3) = 2x^3 - (10 + i)x^2 + (12 + 5i)x - 6i$,

we have $H_f(t) = \begin{cases} H_{f,1} & \text{if } 0 \le t < 1, \\ H_{f,2} & \text{if } 1 \le t < 2, \\ H_{f,3} & \text{if } 2 \le t < 3, \end{cases}$ with

$$H_{f,1} = \frac{1}{D_0(f)D_1(f)}\widetilde{H_{f,1}} = \frac{1}{1 \cdot (-32)} \begin{bmatrix} 40 & -24 \\ -24 & 40 \end{bmatrix} < 0,$$

$$H_{f,2} = \frac{1}{D_1(f)D_2(f)}\widetilde{H_{f,2}} = \frac{1}{(-32) \cdot (-1800)} \begin{bmatrix} 40,256 & 35,648 \\ 35,648 & 113,984 \end{bmatrix} > 0,$$

$$H_{f,3} = \frac{1}{D_2(f)D_3(f)}\widetilde{H_{f,3}}$$
$$= \frac{1}{(-1800) \cdot 187200} \begin{bmatrix} 898,617,600 & 988,300,800 \\ 988,300,800 & 121,328,6400 \end{bmatrix} < 0.$$

Result for the direct problem

- $H_f(t)$: locally const., taking values in $\mathrm{SL}_2(\mathbb{R}) \cap \mathrm{Sym}_2(\mathbb{R})$

Theorem 2 (direct problem)

Let $d \in \mathbb{Z}_{>0}$, $H_1, \ldots, H_d \in \mathrm{SL}_2(\mathbb{R}) \cap \mathrm{Sym}_2(\mathbb{R})$, $(A, B) \neq (0, 0)$.
Let $u(t, z) = {}^t(A(t, z), B(t, z))$ be the solution of QCS(H) with

$$H(t) := H_n \quad \text{for} \quad \frac{r(n-1)}{2} \leq t < \frac{rn}{2}, \quad 1 \leq n \leq d,$$

$$\lim_{t \to rd/2} u(t, z) = {}^t(A, B).$$

Then $u(t, z)$ is well-defined and $f(e^{-irz}) := e^{-irdz/2}(A(0, z) - iB(0, z))$ defines a polynomial.

Result for the direct problem

- If $\deg f = d$ and $f(0) \neq 0$, f has $d - q$ roots inside $\mathbb{T}$, where $q = \sharp$ of sign changes in $(H_1, H_2, \dots, H_d)$.

- $\deg f = d$ and $f(0) \neq 0$ if and only if

$$(I - iJH_1)(I - iJH_2)\cdots(I - iJH_d)\begin{bmatrix} A \\ B \end{bmatrix} \neq 0, \ \notin \mathbb{C}\begin{bmatrix} 1 \\ \pm i \end{bmatrix},$$

where $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. □

Correspondence

$$\left\{ f \in \mathbb{C}[x] \;\middle|\; \begin{array}{l} \cdot\ \deg f = d \\ \cdot\ f(0) \neq 0 \\ \cdot\ D_d(f) \neq 0 \\ \cdot\ n \text{ roots in } \mathbb{T} \\ \cdot\ f(1) = 1 \end{array} \right\}$$

inverse problem $\downarrow \uparrow$ direct problem

$$\left\{ (H_1, \ldots, H_d) \;\middle|\; \begin{array}{l} \cdot\ H_1, \ldots, H_d \in \mathrm{SL}_2(\mathbb{R}) \cap \mathrm{Sym}_2(\mathbb{R}) \\ \cdot\ \sharp \text{ of sign changes in } (H_1, \ldots, H_d) \text{ is } d - n \\ \cdot\ (I - iJH_1)(I - iJH_2) \cdots (I - iJH_d) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \neq 0, \ \notin \mathbb{C} \begin{bmatrix} 1 \\ \pm i \end{bmatrix} \end{array} \right\}$$

Re:Theorem 1 Let $f(x) \in \mathbb{C}[x]$, $d = \deg f$.

Suppose that $f(0) \neq 0$, $D_d(f) \neq 0$.Then $^{\exists}\widetilde{H_{f,n}} \in \mathrm{Sym}_2^+(\mathbb{R})$ $(1 \leq n \leq d)$ s.t.

$$E_f(z)\,(= e^{irdz/2} f(e^{-irz})) = A(0,z) - iB(0,z)$$

for the unique solution ${}^t(A(t,z), B(t,z))$ of QCS for the Hamiltonian

$$H_f(t) = \frac{1}{D_{n-1}(f)D_n(f)}\,\widetilde{H_{f,n}}, \quad \frac{r(n-1)}{2} \leq t < \frac{rn}{2}, \quad 1 \leq n \leq d$$

on $[0, rd/2)$ ($r = 1$ if d is even, $r = 2$ if d is odd) and ...

Independent of the construction of $(\widetilde{H_{f,1}}, \ldots, \widetilde{H_{f,d}})$ by using solutions of linear equations, it can also be constructed inductively.
In the rest, we explain the inductive construction.

Define $A_0(z) := \frac{1}{2}(E_f(z) + \overline{E_f(\bar{z})})$ and $B_0(z) := \frac{i}{2}(E_f(z) - \overline{E_f(\bar{z})})$.

If d is even, we write $\boxed{L = d/2}$ and

$$A_0(z) = \sum_{j=0}^{d/2} a_0(L-j)\cos((L-j)z) + \sum_{j=0}^{d/2-1} c_0(L-j)\sin((L-j)z),$$

$$B_0(z) = \sum_{j=0}^{d/2-1} b_0(L-j)\sin((L-j)z) + \sum_{j=0}^{d/2} d_0(L-j)\cos((L-j)z).$$

If d is odd, we write $\boxed{L = d}$ and

$$A_0(z) = \sum_{j=0}^{(d-1)/2} a_0(L-2j)\cos((L-2j)z) + \sum_{j=0}^{(d-1)/2-1} c_0(L-2j)\sin((L-2j)z),$$

$$B_0(z) = \sum_{j=0}^{(d-1)/2-1} b_0(L-2j)\sin((L-2j)z) + \sum_{j=0}^{(d-1)/2} d_0(L-2j)\cos((L-2j)z).$$

For $1 \leq n \leq d$, we define

$$A_n(t,z) = \sum_{j=0}^{d-n} a_n(L-rj)\cos((L-rj-t)z) + \sum_{j=0}^{d-n} c_n(L-rj)\sin((L-rj-t)z),$$

$$B_n(t,z) = \sum_{j=0}^{d-n} b_n(L-rj)\sin((L-rj-t)z) + \sum_{j=0}^{d-n} d_n(L-rj)\cos((L-rj-t)z).$$

($r = 1$ if d is even, $r = 2$ if d is odd)

For each $1 \leq n \leq d$, there are $4(d-n+1)$ indeterminates.

- Equations

$$\begin{aligned} A_n(r(n-1)/2, z) &= A_{n-1}(r(n-1)/2, z), \\ B_n(r(n-1)/2, z) &= B_{n-1}(r(n-1)/2, z) \end{aligned} \quad (2 \leq n \leq d)$$

  or

$$A_1(0, z) = A_0(z), \quad B_1(0, z) = B_0(z) \quad (n = 1)$$

  provide $4(d-n+1) - 2(d-n)$ linear equations.

- Differential equations

$$\begin{aligned} \frac{d}{dt} A_n(t, z) &= z(\beta_n A_n(t, z) + \gamma_n B_n(t, z)), \\ \frac{d}{dt} B_n(t, z) &= -z(\alpha_n A_n(t, z) + \beta_n B_n(t, z)) \end{aligned}$$

  provide $2(d-n)$ linear equations for $a_n(L-rj)$, $b_n(L-rj)$, $c_n(L-rj)$, $d_n(L-rj)$ $(1 \leq j \leq d-n)$ including α_n, β_n, and γ_n as constants.

The resulting linear system for $4(d-n+1)$ indeterminates has unique solution as long as $\alpha_n\gamma_n - \beta_n^2 \neq 0$. The solution includes α_n, β_n and γ_n as constants, but they do not appear in $a_n(L)$, $b_n(L)$, $c_n(L)$, and $d_n(L)$. Therefore, the above differential equations determine α_n, β_n and γ_n as

$$\alpha_n = \frac{b_n(L)^2 + d_n(L)^2}{a_n(L)b_n(L) - c_n(L)d_n(L)}, \quad \beta_n = -\frac{a_n(L)d_n(L) + b_n(L)c_n(L)}{a_n(L)b_n(L) - c_n(L)d_n(L)},$$
$$\gamma_n = \frac{a_n(L)^2 + c_k(L)^2}{a_n(L)b_n(L) - c_n(L)d_n(L)}.$$

Further, we easily find that

$$a_n(L) = a_{n-1}(L) + a_{n-1}(R), \quad b_n(L) = b_{n-1}(L) - b_{n-1}(R),$$
$$c_n(L) = c_{n-1}(L) - c_{n-1}(R), \quad d_n(L) = d_{n-1}(L) + d_{n-1}(R),$$

where $R := L - r(d-n)$. Hence $a_n(k)$, $b_n(k)$, $c_n(k)$, $d_n(k)$ are written in $a_{n-1}(k)$, $b_{n-1}(k)$, $c_{n-1}(k)$, $d_{n-1}(k)$ for $2 \leq n \leq d$, and $a_1(k)$, $b_1(k)$, $c_1(k)$, $d_1(k)$ are written in coefficients of $A_0(z)$ and $B_0(z)$.

As a result, we have

$$\widetilde{H_{f,n}} = D_{n-1}(f)D_n(f)\begin{bmatrix}\alpha_n & \beta_n \\ \beta_n & \gamma_n\end{bmatrix} \quad \text{for} \quad 1 \leq n \leq d.$$

See arxiv:2106.04061 for details.

See also JFA, 281 (2021), arXiv:2012.11121 for the cases of ζ and L.

Thank you for your kind attention !