On arithmetic Dijkgraaf-Witten theory

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Contents

1. What is arithmetic DW theory ?
2. DW TQFT and CFT
3. Arithmetic topology
4. Arithmetic analogies – Our results

1. What is arithmetic Dijkgraaf-Witten theory ?

What is arithmetic Dijkgraaf-Witten theory ?

$\Longleftrightarrow$

- What is Dijkgraaf-Witten theory ?
- What does "arithmetic" mean ?

1. What is arithmetic Dijkgraaf-Witten theory ?

Dijkgraaf-Witten theory

= a (2+1)-dim. **topological quantum field theory (TQFT)**

defined by **Chern-Simons action** with finite gauge group.

TQFT is a framework which provides topological invariants of manifolds, knots and links.

1. What is arithmetic Dijkgraaf-Witten theory ?

What doest "arithmetic" mean ?

arithmetic = arithmetic analogue,

based on the analogies in **arithmetic topology**:

3-dim. topology	**number theory**
3-manifold	number ring
knot	prime

1. What is arithmetic Dijkgraaf-Witten theory ?

Minhyong Kim (2015):

initiated to study an arithmetic analogue of Chern-Simons gauge theory,

arithmetic Chern-Simons theory.

Arithmetic DW theory is an arithmetic analogue of DW theory, based on Kim's **arithmetic CS theory**.

2. DW TQFT and CFT

Dijkgraaf-Witten theory = a (2+1)-dim. **TQFT** (**Atiyah**, 1989)

(2+1)-dim. **TQFT** = a functor

$$\text{cobordism cat. of 2-manifolds} \to \text{cat. of } \mathbb{C}\text{-vector spaces},$$

which satisfies several axioms.

2. DW TQFT and CFT

oriented closed surface $\Sigma \rightsquigarrow$ quantum Hilbert space H_Σ

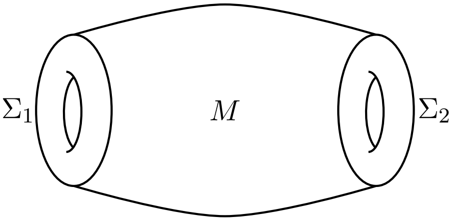
oriented compact 3-manifold $M \rightsquigarrow$ partition function $Z_M \in H_{\partial M}$
(If $\partial M = \emptyset$, $Z(M) := Z_M \in \mathbb{C}$)

s.t.

- **multiplicativity** $\&$ **involutority**:

$$H_{\Sigma_1 \sqcup \Sigma_2} = H_{\Sigma_1} \otimes H_{\Sigma_2}, \quad H_{\Sigma^*} = H_\Sigma^*$$

2. DW TQFT and CFT



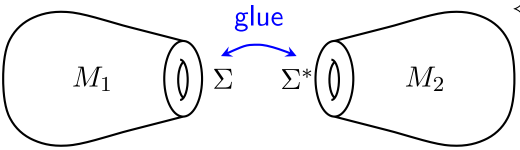
$$\partial M = \Sigma_1^* \sqcup \Sigma_2$$

$$Z_M \in H_{\partial M} = H^*_{\Sigma_1} \otimes H_{\Sigma_2} = \mathrm{Hom}\,(H_{\Sigma_1}, H_{\Sigma_2})$$

★The partition function Z_M is given by a path integral in physical contexts when a gauge group is a Lie group. e.g. **Jones polynomial** (Witten).
In DW theory, Z_M is given by a finite sum.

2. DW TQFT and CFT

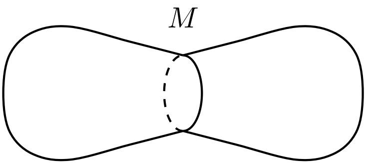
- **gluing property**:



$\Longleftrightarrow$ pairing

$$H_\Sigma \otimes H_\Sigma^* \xrightarrow{<,>} \mathbb{C}$$

$\Downarrow$ $\Downarrow$



$$Z_M = \langle Z_{M_1}, Z_{M_2} \rangle$$

DW TQFT

G : finite group,
$c \in H^3(G, \mathbb{R}/\mathbb{Z})$.

For an oriented compact manifold X with a fixed finite triangulation,
$\mathcal{F}_X$: the space of gauge fields,
$\mathcal{G}_X = \mathrm{Map}(X, G)$: the gauge group,
$\mathcal{F}_X / \mathcal{G}_X = \mathrm{Hom}(\pi_1(X), G)/G$.

Key ingredient - transgression hom. (**Kiyonori Gomi**)

$$\mathrm{tg}_X : C^3(G, \mathbb{R}/\mathbb{Z}) \to C^{3-d}(\mathcal{G}_X, \mathrm{Map}(\mathcal{F}_X, \mathbb{R}/\mathbb{Z})) \ \ (d = \dim(X))$$

$$\begin{cases} \lambda_\Sigma := \mathrm{tg}_\Sigma(c) \text{ for a surface } \Sigma - \textbf{Chern-Simons 1-cocycle} \\ CS_M := \mathrm{tg}_M(c) \text{ for a 3-manifold } M - \textbf{Chern-Simons action}. \end{cases}$$

2. DW TQFT and CFT

The construction of TQFT $<$ **Classical theory** / **Quantum theory**

2. DW TQFT and CFT

Classical theory

oriented closed surface $\Sigma \rightsquigarrow \lambda_\Sigma \in Z^1(\mathcal{G}_\Sigma, \mathrm{Map}(\mathcal{F}_\Sigma, \mathbb{R}/\mathbb{Z}))$,
oriented compact 3-manifold $M \rightsquigarrow CS_M \in C^0(\mathcal{G}_M, \mathrm{Map}(\mathcal{F}_M, \mathbb{R}/\mathbb{Z}))$

s.t. $$dCS_M = \mathrm{res}^* \lambda_{\partial M},$$

$\mathrm{res} : \mathcal{F}_M (\mathrm{resp} : \mathcal{G}_M) \to \mathcal{F}_{\partial M} (\mathrm{resp} : \mathcal{G}_{\partial M})$: restriction map

CS 1-cocycle $\lambda_\Sigma \rightsquigarrow \mathcal{G}_\Sigma$-equiv. complex line bundle on $\mathcal{F}_\Sigma$,
i.e, **prequantization bundle** L_Σ on $\mathcal{F}_\Sigma / \mathcal{G}_\Sigma$.

$$e^{2\pi\sqrt{-1} CS_M} \in \Gamma(\mathcal{F}_M / \mathcal{G}_M, \mathrm{res}^* L_{\partial M}).$$

Quantum theory (Geometric quantization)

oriented closed surface $\Sigma \rightsquigarrow$ quantum Hilbert space H_Σ
oriented compact 3-manifold $M \rightsquigarrow$ partition function $Z_M \in \mathcal{H}_{\partial M}$

quantum Hilbert space:

$$\begin{aligned} H_\Sigma &:= \Gamma(\mathcal{F}_\Sigma/\mathcal{G}_\Sigma, L_\Sigma) \\ &= \{\theta : \mathcal{F}_\Sigma \to \mathbb{C} \,|\, \theta(g.\rho) = e^{2\pi\sqrt{-1}\lambda_\Sigma(\rho)}\theta(\rho),\ (g \in \mathcal{G}_\Sigma, \rho \in \mathcal{F}_\Sigma)\} \end{aligned}$$

DW invariant(partition function):

$$Z_M(\rho) := \frac{1}{\#G} \sum_{\substack{\tilde{\rho} \in \mathcal{F}_M \\ \mathrm{res}(\tilde{\rho}) = \rho}} e^{2\pi\sqrt{-1}CS_M(\tilde{\rho})}$$

⋆ "non-abelian Gauss sum (non-abelian finite theta function)"

DW TQFT and CFT

Ex 1. If M is closed and c is trivial, we have

$$Z(M) = \frac{\#\mathrm{Hom}(\pi_1(M), G)}{\#G}.$$

Ex 2. Let $M \to S^3$ be the double cover ramified over a link $\mathcal{L} = \mathcal{K}_1 \sqcup \cdots \sqcup \mathcal{K}_r$ $(r \geq 2)$.
$G = \mathbb{Z}/2\mathbb{Z}$, $H^3(G, \mathbb{R}/\mathbb{Z}) = \langle c \rangle = \mathbb{Z}/2\mathbb{Z}$.
$D_{\mathcal{L}}$:= the mod 2 linking diagram of $\mathcal{L}$.

Theorem (Hikaru Hirano, Riku Kurimaru, Deng Yuqi).

$$Z(M) = \begin{cases} 2^{r-2} & \text{any connected component of } D_{\mathcal{L}} \text{ is an Euler graph} \\ 0 & \text{otherwise,} \end{cases}$$

Euler graph = one stroke writing circuit.

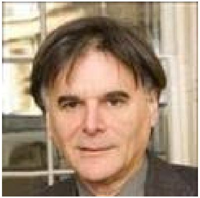
2. DW TQFT and CFT

Relation with 2-dim. conformal field theory (CFT) (Witten)

2-dim. **CFT** $=$ the **modular functor** (**Segal**, 1987)

$$\text{cat. of 2-manifolds} \to \text{cat. of } \mathbb{C}\text{-vector spaces}$$

oriented compact 2-manifold $\Sigma \rightsquigarrow$ the space of **conformal blocks** E_Σ, which satisfies several axioms.
For a closed surface Σ, $E_\Sigma \simeq H_\Sigma$ (Beauville-Laszlo).



2. DW TQFT and CFT

For the case that G is connected Lie group, the construction of E_Σ uses representations of the extension $\widehat{C^\infty(\partial\Sigma, G)}$ of the loop group $C^\infty(\partial\Sigma, G)$ $(\partial\Sigma = S^1 \sqcup \cdots \sqcup S^1)$:

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow \widehat{C^\infty(\partial\Sigma, G)} \longrightarrow C^\infty(\partial\Sigma, G) \longrightarrow 1.$$

Key - Segal-Witten reciprocity law for loop groups:
"This central extension splits over $Hol(\Sigma, G)$".

For $G = \mathbb{C}^\times$,
SW reciprocity law $\Longrightarrow$ the reciprocity law for tame symbols

$$\prod_{p \in \overline{\Sigma}} \{f, g\}_p = 1.$$

2. DW TQFT and CFT

For the case that G is finite, the construction of E_Σ is due to **Brylinski-McLaughlin**, by using the analogy:

G: finite	G: connected
groupoid $\mathcal{C}(S^1, G)$ of pr. G-bundles	loop group $LG := C^\infty(S^1, G)$

2. DW TQFT and CFT

Key ingredients - transgression hom,

$$\mathrm{tg} := \int_{S^1} \circ \, \mathrm{ev}^* : H^3(BG, \mathbb{C}^\times) \to H^2(LBG, \mathbb{C}^\times),$$

where $\mathrm{ev} : LBG \times S^1 \to BG$

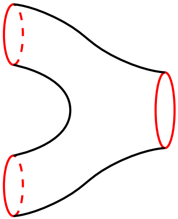
- $H^2(LBG, \mathbb{C}^\times)$ classifies central extensions of $\mathcal{C}(S^1, G)$ by $\mathbb{C}^\times$.
- Brylinski-McLaughlin reciprocity law: The extension of $\mathcal{C}(\partial\Sigma, G)$ obtained by elements of the image of tg splits over $\mathcal{C}(\Sigma, G)$.

BM also give a **group theoretic construction** of E_Σ, using $H^2(LBG, \mathbb{C}^\times) = \oplus_i H^2(Z_G(g_i), \mathbb{C})$ (g_i: representatives of conjugacy classes of G).

Fusion algebra

For $T^2 = S^1 \times S^1$, $H_{T^2} = V_{T^2}$ is equipped with an algebra structure by TQFT and CFT.

- **TQFT** (Wakui). Let Σ be a pair of pants



and consider $M := \Sigma \times S^1$. By DW TQFT, we have the **product**

$$Z_M : H_{T^2} \otimes H_{T^2} \to H_{T^2}.$$

2. DW TQFT and CFT

- **CFT** (Brylinski-McLaughlin). Use the projective representations of $Z_G(g_i)$ and the reciprocity law.

- **Operator algebras** (Dijkgraaf-Vafa-E. Verlinde-H. Verlinde). Physical (orbifold model) approach. Verlinde's formula for the structure constants.

- (**Group theoretic**) (Bannai, Munemasa). Fusion algebra = center of quantum double of group Hopf algebra of G = character algebra.

Analogies

3-dim. topology	**number theory**
3-manifold	number ring
knot	prime

Historical background

- **Gauss** ($1777 \sim 1855$)
  - quadratic residues
  - linking numbers in electro-magnetic theory

3. Arithmetic topology

Historical background

- **Gauss** (1777 $\sim$ 1855)
  - quadratic residues $\rightarrow$ class field theory
  - linking numbers in electro-magnetic theory $\rightarrow$ abelian CS theory

3. Arithmetic topology

- **Grothendieck, Tate, Artin and Verdier** etc (middle of 20th century $\sim$)
  - $S^1 = B\mathbb{Z} = K(\mathbb{Z}, 1), \mathrm{Spec}(\mathbb{F}_q) = B\hat{\mathbb{Z}} = K(\hat{\mathbb{Z}}, 1)$.
  - $\mathrm{Spec}(\mathcal{O}_k)$ enjoys an arithmetic analog of 3-dim. Poincaré duality.

$\Longrightarrow$

$K : S^1 \hookrightarrow M$(3-manifold)	$\mathrm{Spec}(\mathcal{O}_k/\mathfrak{p}) \hookrightarrow \mathrm{Spec}(\mathcal{O}_k)$

$\rightsquigarrow$ **Arithmetic Topology** (Mazur, Kapranov, Reznikov, M.)

3. Arithmetic topology

3-dim. Topology	**Number Theory**
oriented, connected, closed 3-manifold M	compactified number ring $X_k = \mathrm{Spec}(\mathcal{O}_k) \sqcup \{\text{infinite primes}\}$
knot $\mathcal{K} : S^1 \hookrightarrow M$	prime $\{\mathfrak{p}\} = \mathrm{Spec}(\mathcal{O}_k/\mathfrak{p}) \hookrightarrow X_k$
tubular n.b.d of a knot $V_{\mathcal{K}}$	$\mathfrak{p}$-adic integer ring $V_{\mathfrak{p}} = \mathrm{Spec}(\mathcal{O}_{\mathfrak{p}})$
boundary torus $\Sigma_{\mathcal{K}}$	$\mathfrak{p}$-adic field $\Sigma_{\mathfrak{p}} = \mathrm{Spec}(k_{\mathfrak{p}})$
peripheral group $\pi_1(\partial V_{\mathcal{K}})$	local absolute Galois group $\Pi_{\mathfrak{p}} = \mathrm{Gal}(\overline{k}_{\mathfrak{p}}/k_{\mathfrak{p}})$
1st homology $H_1(M)$	ideal class group H_k
2nd homology group $H_2(M)$	unit group $\mathcal{O}_k^{\times}$

3. Arithmetic topology

link $\mathcal{L} = \mathcal{K}_1 \sqcup \cdots \sqcup \mathcal{K}_r$	finite set of maximal ideals $S = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$
tubular n.b.d of a link $V_{\mathcal{L}} = V_{\mathcal{K}_1} \sqcup \cdots \sqcup V_{\mathcal{K}_r}$ boundary tori $\Sigma_{\mathcal{L}} = \Sigma_{\mathcal{K}_1} \sqcup \cdots \sqcup \Sigma_{\mathcal{K}_r}$	union of $\mathfrak{p}_i$-adic integer rings $V_S = \mathrm{Spec}(\mathcal{O}_{\mathfrak{p}_1}) \sqcup \cdots \sqcup \mathrm{Spec}(\mathcal{O}_{\mathfrak{p}_r})$ union of $\mathfrak{p}_i$-adic fields $\Sigma_S = \mathrm{Spec}(k_{\mathfrak{p}_1}) \sqcup \cdots \sqcup \mathrm{Spec}(k_{\mathfrak{p}_r})$
link complement $X_{\mathcal{L}} = \overline{M} \setminus \mathrm{Int}(V_{\mathcal{L}})$ link group $\Pi_{\mathcal{L}} = \pi_1(X_{\mathcal{L}})$	complement of a finite set of primes $X_S = X_k \setminus S$ maximal Galois group with given ramification $\Pi_S = \mathrm{Gal}(k_S/k)$

Dijkgraaf-Witten theory	?

3. Arithmetic topology

link $\mathcal{L} = \mathcal{K}_1 \sqcup \cdots \sqcup \mathcal{K}_r$	finite set of maximal ideals $S = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$
tubular n.b.d of a link $V_{\mathcal{L}} = V_{\mathcal{K}_1} \sqcup \cdots \sqcup V_{\mathcal{K}_r}$	union of $\mathfrak{p}_i$-adic integer rings $V_S = \mathrm{Spec}(\mathcal{O}_{\mathfrak{p}_1}) \sqcup \cdots \sqcup \mathrm{Spec}(\mathcal{O}_{\mathfrak{p}_r})$
boundary tori $\Sigma_{\mathcal{L}} = \Sigma_{\mathcal{K}_1} \sqcup \cdots \sqcup \Sigma_{\mathcal{K}_r}$	union of $\mathfrak{p}_i$-adic fields $\Sigma_S = \mathrm{Spec}(k_{\mathfrak{p}_1}) \sqcup \cdots \sqcup \mathrm{Spec}(k_{\mathfrak{p}_r})$
link complement $X_{\mathcal{L}} = \overline{M} \setminus \mathrm{Int}(V_{\mathcal{L}})$	complement of a finite set of primes $X_S = \overline{X}_k \setminus S$
link group $\Pi_{\mathcal{L}} = \pi_1(X_{\mathcal{L}})$	maximal Galois group with given ramification $\Pi_S = \mathrm{Gal}(k_S/k)$

Dijkgraaf-Witten theory	arithmetic Dijkgraaf-Witten theory

Arithmetic topology

⋆ For an arithmetic analog of DW TQFT, we consider $\Sigma_{\mathfrak{p}} := \mathrm{Spec}(k_{\mathfrak{p}})$ as an arithmetic analog of an oriented, connected surface.

- $\Sigma_{\mathfrak{p}}$ enjoys "2-dim. Poincaré duality" (Tate):

$$\mathrm{inv}_{\mathfrak{p}} : H^2(\mathrm{Spec}(k_{\mathfrak{p}}), \mu_n) = \mathbb{Z}/n\mathbb{Z},$$
$$H^i(\mathrm{Spec}(k_{\mathfrak{p}}), \mu_n) \simeq H_{2-i}(\mathrm{Spec}(k_{\mathfrak{p}}), \mathbb{Z}/n\mathbb{Z}) \ (0 \leq i \leq 2).$$

$\Longrightarrow \Sigma_{\mathfrak{p}} \sim$ "orientable, connected, closed surface".

- $\Sigma_{\mathfrak{p}} \sim$ "boundary torus of a tubular n.b.d of a knot".

knot $\mathcal{K}$	finite prime $\mathrm{Spec}(\mathbb{F}_{\mathfrak{p}})$
tubular n.b.d $V_{\mathcal{K}}$	$\mathfrak{p}$-adic integers $\mathrm{Spec}(\mathcal{O}_{\mathfrak{p}})$
boundary torus $\partial V_{\mathcal{K}} \simeq V_{\mathcal{K}} \setminus \mathcal{K}$	$\Sigma_{\mathfrak{p}} = \mathrm{Spec}(\mathcal{O}_{\mathfrak{p}}) \setminus \mathrm{Spec}(\mathbb{F}_{\mathfrak{p}})$
torus group $\pi_1(\partial V_{\mathcal{K}}) = \langle m, l \mid [m, l] = 1 \rangle$	tame Galois group $\Pi_{\mathfrak{p}}^{\mathrm{tame}} = \langle \tau, \sigma \mid \tau^{\mathrm{N}\mathfrak{p}-1}[\tau, \sigma] = 1 \rangle$

Arithmetic topology

- $\Sigma_{\mathfrak{p}} \sim$ "closed surface of higher genus"

The case that $k_{\mathfrak{p}}$ contains a primitive p-th root of unity and $[k_{\mathfrak{p}} : \mathbb{Q}_p] = d$, $p > 2$.

surface group $\pi_1(\Sigma_r)$	pro-p Galois group $\Pi_{\mathfrak{p}}(p)$
$= \langle \alpha_1, \beta_1, \ldots, \beta_r \vert \prod_{i=1}^{r} [\alpha_i, \beta_i] = 1 \rangle$	$= \langle \tau_1, \ldots, \tau_{d+2} \vert \tau_1^{p^s} [\tau_1, \tau_2] \cdots [\tau_{d+1}, \tau_{d+2}] = 1 \rangle$

- $\Sigma_{\mathfrak{p}} \sim$ "punctured sphere"

The case that $k_{\mathfrak{p}}$ does not contain a primitive p-th root of unity and $[k_{\mathfrak{p}} : \mathbb{Q}_p] = d$.

punctured sphere group $\pi_1(S^2 \setminus r + 2\text{points})$	free pro-p Galois group $\Pi_{\mathfrak{p}}(p)$
$= \langle \alpha_1, \alpha_{r+2} \vert \alpha_1 \cdots \alpha_{r+2} = 1 \rangle$	$= \langle \tau_1, \ldots, \tau_{d+2} \vert \alpha_1 \cdots \alpha_{d+2} = 1 \rangle$

4. Arithmetic analogies

k: a number field
$\mathcal{O}_k$: the ring of alg. integers in k

For a prime ideal $\mathfrak{p}$ of $\mathcal{O}_k$,
$k_\mathfrak{p}$: the $\mathfrak{p}$-adic field,
$\mathcal{O}_\mathfrak{p}$: the ring of $\mathfrak{p}$-adic integers.

$X_k := \mathrm{Spec}(\mathcal{O}_k) \sqcup \{\text{infinite primes}\}$

For a finite set of prime ideals $S = \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$,
$\Sigma_S := \mathrm{Spec}(k_{\mathfrak{p}_1}) \sqcup \cdots \sqcup \mathrm{Spec}(k_{\mathfrak{p}_r})$.
$X_S := X_k \setminus S$.

$\Pi_\mathfrak{p} := \pi_1(\mathrm{Spec}(k_\mathfrak{p})) = \mathrm{Gal}(\overline{k}_\mathfrak{p}/k_\mathfrak{p})$,
$\Pi_S := \pi_1(X_S) = \mathrm{Gal}(k_S/k)$.
(k_S: max. Galois ext. of k unramified outside S)

⋆ We see X_S like a 3-manifold with boundary surface Σ_S.

n : a fixed integer ≥ 2,
G : finite group,
$c \in Z^3(G, \mathbb{Z}/n\mathbb{Z})$.

Assume k contains a primitive n-th root of unity ζ_n.

Arithmetic gauge fields

$\mathcal{F}_S := \prod_{i=1}^{r} \mathrm{Hom}_{\mathrm{cont}}(\Pi_{\mathfrak{p}_i}, G) \curvearrowleft \mathcal{G}_S := G$: conjugate action,
$\mathcal{F}_{X_S} := \mathrm{Hom}_{\mathrm{cont}}(\Pi_S, G) \curvearrowleft \mathcal{G}_{X_S} := G$: conjugate action,
$\mathrm{res} : \mathcal{F}_{X_S} \to \mathcal{F}_S$: restriction map.

4. Arithmetic analogies

Arithmetic CS theory (Minhyong Kim).

CS functional $CS : \mathcal{F}_M \to \mathbb{R}/\mathbb{Z}$	Arithmetic CS functional $CS : \mathcal{F}_{X_S} \to \mathbb{Z}/n\mathbb{Z}$
prequantization bundle L_Σ on $\mathcal{F}_\Sigma/\mathcal{G}_\Sigma$	Arithmetic prequantization bundle L_S on $\mathcal{F}_S/\mathcal{G}_S$

Key ingredient (M. Kim)

- $\cdot$ conjugate action on group cochain c,
- $\cdot$ $H^2(\Pi_{\mathfrak{p}}, \mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$-torsor $\mathcal{L}(\rho_{\mathfrak{p}}) := d^{-1}(c \circ \rho_{\mathfrak{p}})/B^2(\Pi_{\mathfrak{p}}, \mathbb{Z}/n\mathbb{Z})$ for $\rho_{\mathfrak{p}} \in \mathcal{F}_{\mathfrak{p}}$.

($\Rightarrow$ CS 1-cocycle implicit)

Arithmetic analogies

As in the topological case, we consider the "transgression"

$$\mathrm{tg}_\sigma : C^3(G, \mathbb{Z}/n\mathbb{Z}) \to C^2(G, \mathbb{Z}/n\mathbb{Z}) \ \ (\sigma \in G),$$

defined explicitly by

$$\mathrm{tg}_\sigma(c)(g_1, g_2) := c(\sigma, \sigma^{-1}g_1\sigma, \sigma^{-1}g_2\sigma) - c(g_1, \sigma, \sigma^{-1}g_2\sigma) + c(g_1, g_2, \sigma),$$

which is interpreted by Brylinski-McLaughlin's transgression

$$H^3(BG, \mathbb{Z}/n\mathbb{Z}) \to H^2(LBG, \mathbb{Z}/n\mathbb{Z}) = \oplus_i H^2(Z_{g_i}, \mathbb{Z}/n\mathbb{Z})$$

if $\sigma = g_i$.
Using tg and Kim's torsor, we can define the **arithmetic CS 1-cocycle**

$$\lambda_S \in Z^1(\mathcal{G}_S, \mathrm{Map}(\mathcal{F}_S, \mathbb{Z}/n\mathbb{Z})),$$

from which we can construct arithmetic analogs of objects in DW TQFT.

Arithmetic analogies

Arithmetic DW TQFT (H. Hirano, J. Kim, M.).

$\Sigma \rightsquigarrow \lambda_\Sigma$	$\Sigma_S \rightsquigarrow \lambda_S$
$M \rightsquigarrow CS_M$	$X_S \rightsquigarrow CS_{X_S}$
$dCS_M = \mathrm{res}^* \lambda_{\partial M}$	$dCS_{X_S} = \mathrm{res}^* \lambda_S$
prequantization bundle L_Σ $e^{2\pi\sqrt{-1}CS_M} \in \Gamma(\mathcal{F}_M/\mathcal{G}_M, \mathrm{res}^* L_{\partial M})$	arith. prequantization bundle L_S $\zeta_n^{CS_{X_S}} \in \Gamma(\mathcal{F}_{X_S}/\mathcal{G}_{X_S}, \mathrm{res}^* L_S)$
quantum Hilbert space $\Sigma \rightsquigarrow H_\Sigma$ $H_\Sigma := \Gamma(\mathcal{F}_\Sigma/\mathcal{G}_\Sigma, L_\Sigma)$	arith. quantum space $\Sigma_S \rightsquigarrow H_S$ $H_S := \Gamma(\mathcal{F}_S/\mathcal{G}_S, L_S)$
DW invariant $M \rightsquigarrow Z_M \in H_{\partial M}$ $Z_M(\rho) := \frac{1}{\#G} \sum_{\substack{\tilde{\rho} \in \mathcal{F}_M \\ \mathrm{res}(\tilde{\rho}) = \rho}} e^{2\pi\sqrt{-1}CS_M(\tilde{\rho})}$	arith. DW invariant $X_S \rightsquigarrow Z_{X_S} \in H_S$ $Z_{X_S}(\rho) := \frac{1}{\#G} \sum_{\substack{\tilde{\rho} \in \mathcal{F}_{X_S} \\ \mathrm{res}(\tilde{\rho}) = \rho}} \zeta_n^{CS_{X_S}(\tilde{\rho})}$

Arithmetic analogies

Basic properties

• **multiplicativity**:
For disjoint S_1 and S_2, we have

$$H_{S_1 \sqcup S_2} = H_{S_1} \otimes H_{S_2}.$$

• **involutority**:
For $S^* = S$ with opposite orientation (so that $\lambda_{S^*} = -\lambda_S$), we have

$$H_{S^*} = H_S^*.$$

So, for $S = S_1 \sqcup S_2$, we have the pairing

$$\langle \cdot, \cdot \rangle : H_S \times H_{S_2^*} \to H_{S_1}.$$

Arithmetic analogies

• For a "closed" X_k, we can define the arithmetic CS functional CS_{X_k} and the arithmetic DW invariant $Z(X_k)$ (M. Kim, Lee-Park, Hirano):

$$CS_{X_k} : \mathcal{F}_{X_k} := \mathrm{Hom_c}(\pi_1(X_k), G) \to \mathbb{Z}/n\mathbb{Z},$$
$$Z(X_k) := \frac{1}{\#G} \sum_{\rho \in \mathcal{F}_{X_k}} \zeta_n^{CS_{X_k}(\rho)}.$$

• For $V_S = \mathrm{Spec}(\mathcal{O}_{\mathfrak{p}_1}) \sqcup \cdots \sqcup \mathrm{Spec}(\mathcal{O}_{\mathfrak{p}_r})$, we can also define the arithmetic CS functional CS_{V_S} and the arithmetic DW invariant Z_{V_S}:

$$CS_{V_S} : \mathcal{F}_{V_S} \to \mathbb{Z}/n\mathbb{Z},\ \zeta_n^{CS_{V_S}} \in \Gamma(\mathcal{F}_S, \tilde{\mathrm{res}}^* L_S),$$
$$Z_{V_S} \in H_S,$$

where $\tilde{\mathrm{res}} : \mathcal{F}_{V_S} \to \mathcal{F}_S$ is the restriction map.

Arithmetic analogies

Ex 1. If c is trivial, we have

$$Z(X_k) = \frac{\#\mathrm{Hom}(\pi_1(X_k), G)}{\#G}.$$

Ex 2. Let $k = \mathbb{Q}(\sqrt{p_1 \cdots p_r})$, $p_i \equiv 1$ mod 4, $S = \{p_1, \ldots, p_r\}$ $(r \geq 2)$.
$G = \mathbb{Z}/2\mathbb{Z}$, $H^3(G, \mathbb{R}/\mathbb{Z}) = \langle c \rangle = \mathbb{Z}/2\mathbb{Z}$.
$D_S =$ the mod 2 linking diagram of S, $(-1)^{\mathrm{lk}_2(p_i,p_j)} = \left(\frac{p_i}{p_j}\right)$.

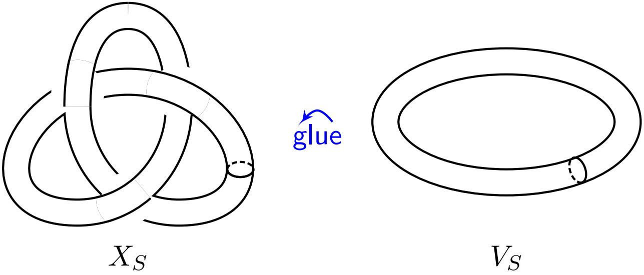
Theorem (Hikaru Hirano, Riku Kurimaru, Deng Yuqi).

$$Z(X_k) = \begin{cases} 2^{r-2} & \text{any connected component of } D_S \text{ is an Euler graph,} \\ 0 & \text{otherwise.} \end{cases}$$

Gluing formula for arithmetic DW invariants (J. Kim, H. Hirano, M.).

Let $\langle \cdot, \cdot \rangle : H_S \times H_S^* \to \mathbb{C}$ be the pairing of arithmetic quantum spaces. Then we have

$$\langle Z_{X_S}, Z_{V_S^*} \rangle = Z(X_k)$$



4. Arithmetic analogies

Arithmetic analogues of CFT and fusion algebras

• We can construct Brylinski type space of "conformal block" E_S for the case that G is a p-group and $\Pi_{\mathfrak{p}_i}(p)$ ($\mathfrak{p}_i \in S$) is the punctured sphere type free pro-p group.

• We may construct H_S as an Bannai-Munemasa's fusion algebra for the case that $\#G | (\mathrm{N}\mathfrak{p}_i - 1)$ and hence $\mathrm{Hom}_{\mathrm{c}}(\Pi_{\mathfrak{p}_i}, G) = \mathrm{Hom}(\pi_1(T^2), G)$.

4. Arithmetic analogies

Questions.

Can we develop arithmetic analogies of the theory of CFT and fusion algebras ?

・ Relation between Segal-Witten-Brylinsky type reciprocity law and Kubota's metaplectic theory for reciprocity law.
Can we develop Kubota's theory to obtain a non-commutative reciprocity law of Segal-Witten-Brylinsky type in arithmetic ?
(Cf. Z_{X_S} is a sort of non-commutative Gauss sum, H_S consists of non-commutative (finite) theta functions.)

・ arithmetic Verlinde type formula, dim H_S etc ..