On the Brumer Stark Conjecture

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Stickelberger's theorem

Let $H = \mathbf{Q}(\mu_m)$ be the mth cyclotomic field. Then

$$(\mathbf{Z}/m\mathbf{Z})^* \xrightarrow{\cong} G = \mathrm{Gal}(H/\mathbf{Q}) \qquad c \mapsto \sigma_c \ (\sigma_c(\zeta_m) = \zeta_m^c)$$

Stickelberger element

$$\Theta(H/\mathbf{Q}) = \sum_{c \in (\mathbf{Z}/m\mathbf{Z})^*} \left(\frac{1}{2} - \left\{ \frac{c}{m} \right\} \right) \sigma_c^{-1} \in \mathbf{Q}[G].$$

Here $\{x\}$ = fractional part of x. The ideal class group $Cl(H)$ of H is a $\mathbf{Z}[G]$-module.

Stickelberger's theorem (1890)

$\Theta(H/\mathbf{Q})\mathbf{Q}[G] \cap \mathbf{Z}[G]$ is contained in the annihilator ideal $\mathrm{Ann}_{\mathbf{Z}[G]}(Cl(H))$ of $Cl(H)$.

Relation with L-values

The element $\Theta(H/\mathbf{Q})$ satisfies: for any $\chi \in \hat{G}$ and $S = \{\text{prime divisors of } m\} \cup \{\infty\}$

$$\chi(\Theta(H/\mathbf{Q})) = L_S(\chi^{-1}, 0).$$

Let F be a totally real field and H be a finite abelian CM extension of F. Put $G = \mathrm{Gal}(H/F)$. Let S be a finite set of primes of F containing all those that ramify in H and all archimedean primes. Let T be a finite set of primes of F, disjoint from S and containing at least two primes of distinct residue characteristic.

$$L_{S,T}(\chi, s) = \prod_{v \notin S} (1 - \chi(v)\mathrm{N}v^{-s})^{-1} \prod_{v \in T} (1 - \chi(v)\mathrm{N}v^{1-s}).$$

Brumer–Stark conjecture

Stickelberger element

$\Theta_{S,T}(H/F) \in \mathbf{C}[G]$ is such that for any $\chi \in \hat{G}$

$$\chi(\Theta_{S,T}(H/F)) = L_{S,T}(\chi^{-1}, 0).$$

Theorem (Siegel, Klingen, Shintani, Cassou-Noguès, Deligne–Ribet)

$\Theta_{S,T}(H/F) \in \mathbf{Z}[G]$.

Let ideal class group $Cl(H)$ is a $\mathbf{Z}[G]$-module.

Brumer–Stark–Tate conjecture

$\Theta_{S,T}(H/F)$ is contained in the annihilator ideal $\mathrm{Ann}_{\mathbf{Z}[G]}(Cl(H))$ of $Cl(H)$.

Strong Brumer–Stark conjecture

For any $\mathbf{Z}[G]$-module M, we put $M^- = (\mathbf{Z}\left[\frac{1}{2}\right][G] \otimes M)/(1+c)$, where c is the complex conjugation. We also put $M^\vee$ for the Pontryagin dual of M. We have an involution $\# : \mathbf{Z}[G] \to \mathbf{Z}[G]$ induced by $g \mapsto g^{-1}$.

Theorem (Dasgupta–K), The strong Brumer–Stark conjecture

$\Theta_{S,T}^{\#}(H/F) \in \mathrm{Fitt}_{\mathbf{Z}[G]^-}(Cl(H)^{-,\vee})$.

Theorem (Dasgupta-K)

$\Theta_{S,T}(H/F) \in \mathrm{Ann}_{\mathbf{Z}[G]^-}(Cl(H)^-)$.

Note that

(i) Fitting ideal is contained in annihilator ideal.

(ii) If M is finite, then $\mathrm{Ann}_{\mathbf{Z}[G]}(M^\vee) = \mathrm{Ann}_{\mathbf{Z}[G]}(M)^{\#}$.

(iii) $\Theta_{S,T}(H/F)$ annihilates M if, and only if, annihilates M^-.

A conjecture of Kurihara

Let S_{ram} be the set of finite places of F that ramify in H and S_∞ be all archimedean places.

Sinnott–Kurihara ideal

$$\mathrm{SKu}^T(H/F) = (\Theta^{\#}_{S_\infty,T}(H/F)) \prod_{v \in S_{\text{ram}}} (\mathrm{N}I_v, 1 - \sigma_v e_v)$$

where I_v is the inertia subgroup, $e_v = \frac{1}{\# I_v}\mathrm{N}I_v = \frac{1}{\# I_v}\sum_{\sigma \in I_v} \sigma \in \mathbf{Q}[G]$.

Kurihara shows that $\mathrm{SKu}^T(H/F) \subset \mathbf{Z}[G]$.

Theorem (Dasgupta–K), Conjecture of Kurihara

$\mathrm{Fitt}_{\mathbf{Z}[G]^-}(Cl^T(H)^{-,\vee}) = \mathrm{SKu}^T(H/F)^-$.

$Cl^T(H)$ is the quotient of fractional ideal of H modulo principal fraction ideals (α) with $\alpha \equiv 1 \pmod{T}$.

Main technical result

Fix an odd prime p. Modify sets S and T.

$$\Sigma = \{v \mid p : v \text{ is ramified in } H\} \cup S_\infty$$

$$\Sigma' = \{v \nmid p : v \text{ is ramified in } H\} \cup T.$$

We have the Ritter–Weiss module $\nabla_{\Sigma,\Sigma'}(H/F)$:

$$0 \to Cl_{\Sigma,\Sigma'}(H)^- \to \nabla_{\Sigma,\Sigma'}(H/F)^- \to X_{H,\Sigma}^- \to 0,$$

where $X_{H,\Sigma}$ is the free abelian group generated by primes above Σ and $Cl_{\Sigma,\Sigma'}(H)$ is the class group corresponding to the extension of H that is unramified outside Σ', completely split at primes above Σ and at most tamely ramified at primes above Σ'.

Theorem (Dasgupta–K), Conjecture of Burns–Kurihara–Sano

$$\mathrm{Fitt}_{\mathbf{Z}[G]^-}(\nabla_{\Sigma,\Sigma'}(H/F)^-) = (\Theta_{\Sigma,\Sigma'}(H/F)).$$

First reduction

$$0 \to Cl_{\Sigma,\Sigma'}(H)^- \to \nabla_{\Sigma,\Sigma'}(H/F)^- \to X_{H,\Sigma}^- \to 0,$$

Theorem (Dasgupta–K)

$\mathrm{Fitt}_{\mathbf{Z}[G]^-}(\nabla_{\Sigma,\Sigma'}(H)^-) = (\Theta_{\Sigma,\Sigma'}(H/F))$.

All the above results are deduced from this one. Firstly it is enough to show

$$\mathrm{Fitt}_{\mathbf{Z}[G]^-}(\nabla_{\Sigma,\Sigma'}(H)^-) \subset (\Theta_{\Sigma,\Sigma'}(H/F))$$

This uses analytic class number formula.
To prove the inclusion we use Hilbert modular forms. Using Ribet–Wiles method we can construct global Galois cohomology classes.

Galois cohomology description of $\nabla_{\Sigma,\Sigma'}(H)^-$

To get a surjective homomorphism from $\nabla_{\Sigma,\Sigma'}(H)^-$ to a $\mathbf{Z}_p[G]^-$-module to B, we need to

(1) A cohomology class κ in $H^1(G_F, B)$ that is unramified outside Σ', tamely ramified at Σ' and locally trivial at Σ.

(2) If B_0 is the image of restriction κ under $H^1(G_F, B) \to H^1(G_H, B) = Hom(G_H, B)$, then $X_{H,\Sigma}^-$ surjects on B/B_0.

(3) Certain compatibility in the extension group $Ext^1_{\mathbf{Z}_p[G]^-}(X_{H,\Sigma}^-, B_0)$.

From now on denote $\Theta_{\Sigma,\Sigma'}(H/F)$ by Θ.

Construction of a cusp form

Assume that H/F is unramified. We work with a quotient R of $\mathbf{Z}_p[G]^-$ so that $\Theta^\# = \Theta^\#_{S_\infty,T}$ is a non-zerodivisor. This is sufficient. Let $\psi : G_F \to R^*$ be the tautological character. Then there is a cusp form with coefficients in R.

$$\mathcal{F}_k(\psi) = e(xW_1(\psi,1)V_{k-1} - W_k(\psi,1) - x\Theta^\# H_k(\psi)).$$

(i) e is Hida's idempotent.

(ii) V_{k-1} is a level 1 form congruent to 1 modulo p^k (Hida).

(iii) $H_k(\psi)$ has constant terms so that $\mathcal{F}_k(\psi)$ is cuspidal (Silliman).

(iv) $W_k(\psi,1)$ has q-expansion with constant coefficient $\Theta^\#_{S_\infty,T}(1-k)$.

(v) $x = \Theta_{S_\infty,T}(1-k)/\Theta$ (non-zerodivisor, takes care of "trivial zeros").

Hecke operators

Important congruence

$$\mathcal{F}_k(\psi) \equiv xW_1(\psi, 1) - W_k(\psi, 1_p) \pmod{x\Theta^{\#}}$$

This can be used to compute action of Hecke operators on $\mathcal{F}_k(\psi)$ modulo $x\Theta^{\#}$.

Let $\mathbf{T}$ be the Hecke operator over R generated by $T_{\mathfrak{l}}$ for all $\mathfrak{l} \nmid p$ and diamond operators $S(\mathfrak{m})$. Let $\tilde{\mathbf{T}}$ be generated by $\mathbf{T}$ and $U_{\mathfrak{p}}$ for all $\mathfrak{p} \mid p$.

Theorem

There is a $R/(x\Theta^{\#})$ algebra W and a surjective R homomorphism $\varphi : \tilde{\mathbf{T}} \to W$ such that the structure map $R/(x\Theta^{\#}) \to W$ is injective.

$$\varphi(S(\mathfrak{m})) = \psi(\mathfrak{m}).$$

$$\varphi(T_{\mathfrak{l}}) = \epsilon_{\text{cyc}}^{k-1} + \psi(\mathfrak{l}).$$

If $\varphi(\prod_{\mathfrak{p}}(U_{\mathfrak{p}} - \psi(\mathfrak{p})))y = 0$ in W for any $y \in R$, then $y \in (\Theta^{\#})$.

Galois representation

Let $\mathfrak{m}$ be the maximal ideal of containing kernel of φ. Put K for the total ring of fraction of the $\mathfrak{m}$-adic completion of $\mathbf{T}_{\mathfrak{m}}$. Then there exists a Galois representation (Hida–Wiles)

$$\rho : G_F \to GL_2(K)$$

such that

(i) ρ is unramified outside p and

$$\mathrm{char}(\rho(\mathrm{Frob}_{\mathfrak{l}}))(x) = x^2 - T_{\mathfrak{l}}x + \boldsymbol{\psi}(\mathfrak{l})\epsilon_{\mathrm{cyc}}^{k-1}.$$

(ii) (Ordinarity at p: for every $\mathfrak{p} \mid p$,

$$\rho|_{G_{\mathfrak{p}}} \sim \begin{pmatrix} \boldsymbol{\psi}\eta_{\mathfrak{p}}^{-1}\epsilon_{\mathrm{cyc}}^{k-1} & * \\ 0 & \eta_{\mathfrak{p}} \end{pmatrix}$$

with $\eta_{\mathfrak{p}}(\varpi^{-1}) = U_{\mathfrak{p}}$.

Ribet's wrench

We can carefully choose a basis (global basis of eigenvectors for $\rho(\tau)$, for some τ is different than local basis) and write

$$\rho(\sigma) = \begin{pmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{pmatrix}.$$

I denote the kernel of φ. Then

$$a(\sigma) \equiv \epsilon_{\mathrm{cyc}}^{k-1}(\sigma) \pmod{I} \qquad d(\sigma) \equiv \boldsymbol{\psi}(\sigma) \pmod{I}.$$

Let $M_{\mathfrak{p}} = \begin{pmatrix} A_{\mathfrak{p}} & * \\ C_{\mathfrak{p}} & * \end{pmatrix}$ be such that

$$\begin{pmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{pmatrix} M_{\mathfrak{p}} = M_{\mathfrak{p}} \begin{pmatrix} \boldsymbol{\psi}\eta_{\mathfrak{p}}^{-1}\epsilon_{\mathrm{cyc}}^{k-1} & * \\ 0 & \eta_{\mathfrak{p}} \end{pmatrix}$$

$\implies b(\sigma) = \frac{A_{\mathfrak{p}}}{C_{\mathfrak{p}}}(\boldsymbol{\psi}\eta_{\mathfrak{p}}^{-1}\epsilon_{\mathrm{cyc}}^{k-1}(\sigma) - a(\sigma))$ for all $\sigma \in G_{\mathfrak{p}}$. Note: $A_{\mathfrak{p}}, C_{\mathfrak{p}} \in K^*$.

Let B be the $\hat{\mathbf{T}}_{\mathfrak{m}}$-submodule of K generated by $b(\sigma)$ for all $\sigma \in G_F$ (In general we need to include $\frac{A_{\mathfrak{p}}}{C_{\mathfrak{p}}}$, for all $\mathfrak{p} \mid p$ that ramify in H).
Put $\overline{B} = B/(I, p^k)B$. As ρ is a representation

$$b(\sigma\sigma') = a(\sigma)b(\sigma') + b(\sigma)d(\sigma'), \qquad \text{for all } \sigma, \sigma' \in G_F.$$

Therefore $\kappa(\sigma) = b(\sigma)\psi^{-1}(\sigma) \in H^1(G_F, \overline{B}(\psi^{-1}))$.
Then we can show:

(1) This class has all the required properties to get a surjection $\nabla_{S_\infty,T}(H/F)^- \to \overline{B}$.
(2) $\overline{B}$ is big enough so that $\mathrm{Fitt}_R(\overline{B}) \subset (\Theta_{S_\infty,T}(H/F), p^k)$. Thus $\mathrm{Fitt}_R(\nabla_{S_\infty,T}(H/F)^-) \subset (\Theta_{S_\infty,T}(H/F))$.