Class numbers of large degree nonabelian number fields

Kwang-Seob Kim (Chosun University)

(joint work with John Miller)

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Previous result

Theorem(2014, K-S Kim)

Let K be the real quadratic field $\mathbb{Q}(\sqrt{653})$. Then under the GRH,

$$\mathrm{Gal}(K_{ur}^{f}/K) \simeq A_5$$

where K_{ur}^{f} is the maximal extension of K which is unramified over all *finite* places.

Idea of proof

Construction of an A_5-unramified extension of K

Let L be the splitting field of

$$x^5 + 3x^3 + 6x^2 + 2x + 1 \tag{0.1}$$

a polynomial with complex roots. Then L is an A_5-extension of $\mathbb{Q}$ and 653 is the only finite prime ramified in this field with ramification index 2. Then LK/K is unramified at all finite places.

Abhyankar's lemma

Let F be a local field. Let E_1 and E_2 be finite extensions of F with ramification indices e_1 and e_2 respectively. Suppose E_2 is tamely ramified and $e_2|e_1$. Then E_1E_2 is an unramified extension of E_1.

Bound of $[K_{ur}^f : LK]$

If we assume GRH, then Odlyzko's bounds imply that $[K_{ur}^f : LK] < 3$. Therefore $\mathrm{Gal}(K_{ur}^f/LK)$ is isomorphic to C_2 or the trivial group. If $\mathrm{Cl}^+(LK)$ is trivial, we are done and so we only need to deal with the case $\mathrm{Cl}(LK) \simeq C_2$.

Finally, we found a contradiction when $\mathrm{Cl}(LK) \simeq C_2$. Thus $\mathrm{Cl}(LK)$ is a trivial group and $\mathrm{Gal}(K_{ur}^f/K) \simeq A_5$ under the assumption of GRH.

Question

Question

Is it possible to show that

$$\mathrm{Gal}(K_{ur}^{f}/K) \simeq A_5$$

without the assumption of GRH?

Remark

- To do this, the first step is to prove that KL has *class number one*.
- If a number field has a large degree and discriminant, the computation of the class number becomes quite difficult.

Main Result

Main Theorem(2016, KS Kim and John Miller)

KL has class number one without assuming GRH.

Remark

In fact, KL has degree 120, and it is the largest degree number field proven unconditionally to have class number 1. Previously, among such fields, the one with largest degree, the real cyclotomic field of conductor 151, has degree 75 proven by John Miller.

Remark on the maximal unramified extension of $\mathbb{Q}(\sqrt{p})$

Main theorem of Kim-2014

Let K be a real quadratic field $\mathbb{Q}(\sqrt{p})$ with narrow class number 1, where p is a prime congruent 1 mod 4 and $p \neq 5$. Suppose that there exists a totally imaginary A_5-extension L over $\mathbb{Q}$ and p is the only prime ramified in this field with ramification index 2. If $\sqrt{p} < B(1920, 0, 960)$, then the class number of KL is one and $K_{ur}^{f} = KL$.

($B(n, r_1, r_2)$ is defined as an infimum of $|d_F|^{1/n_F}$ over all number fields F satisfying $n_F \geq n$ and $\frac{r_1(F)}{n_F} = \frac{r_1}{n}$ (resp. $\frac{r_2(F)}{n_F} = \frac{r_2}{n}$) where $r_1(F)$ (resp. $r_2(F)$) is the number of real (resp. complex) places of a number field F.)

In $B(1920, 0, 960)$, the number 1920 represents

$$16 \times \text{ the degree of } KL.$$

Corollary

With the notations and conditions being the same as above, if the class number of KL is smaller than 16, then the class number of KL is exactly one.

Upper bounds on class numbers of totally complex fields

Theorem

Let K be a totally complex Galois number field of degree n, and let

$$F(x) = \frac{e^{-(x/c)^2}}{\cosh \frac{x}{2}}$$

for some positive constant c. Suppose S is a subset of the prime integers which are unramified in K and factor into principal prime ideals of K of degree f_p.

Let

$$B = \gamma + \log 8\pi - \log \mathrm{rd}(K) - \int_0^\infty \frac{1 - F(x)}{2 \sinh \frac{x}{2}}\, dx$$

$$+2 \sum_{p \in S} \sum_{m=1}^{\infty} \frac{\log p}{p^{f_p m/2}} F(f_p m \log p),$$

where γ is Euler's constant. If $B > 0$ then we have an upper bound for the class number h of K,

$$h < \frac{2c\sqrt{\pi}}{nB}.$$

Proof

We apply Poitou's version of Weil's “explicit formula" for the Dedekind zeta function of the Hilbert class field $H(K)$ of K:

$$\log d(H(K)) = hr_1 \frac{\pi}{2} + hn(\gamma + \log 8\pi) - hn \int_0^\infty \frac{1 - F(x)}{2 \sinh \frac{x}{2}}\, dx$$

$$-hr_1 \int_0^\infty \frac{1 - F(x)}{2 \cosh \frac{x}{2}}\, dx - 4 \int_0^\infty F(x) \cosh \frac{x}{2}\, dx$$

$$+ \sum_\rho \Phi(\rho) + 2 \sum_{\mathfrak{P}} \sum_{m=1}^\infty \frac{\log N\mathfrak{P}}{N\mathfrak{P}^{m/2}} F(m \log N\mathfrak{P})$$

where γ is Euler's constant and $r_1 = 0$ since K is totally complex. The first sum is over the nontrivial zeros of the Dedekind zeta function of $H(K)$, the second sum is over the prime ideals of $H(K)$, and Φ is defined by

$$\Phi(s) = \int_{-\infty}^\infty F(x) e^{(s-1/2)x}\, dx.$$

By our choice of F, the real part of $\Phi(s)$ is nonnegative everywhere in the critical strip. Indeed, on the boundary of the critical strip, the real part

$$\operatorname{Re}\Phi(s) = \operatorname{Re}\int_{-\infty}^{\infty} \frac{e^{-(x/c)^2}}{\cosh\frac{x}{2}} e^{(s-1/2)x}\,dx$$

$$= \int_{-\infty}^{\infty} e^{-(x/c)^2}\cos(x\operatorname{Im}s)\,dx = c\sqrt{\pi}e^{-(c\operatorname{Im}s/2)^2} > 0$$

is positive, and $\operatorname{Re}\Phi(s) \to 0$ as $|\operatorname{Im}s| \to \infty$, so by the maximum modulus principle for harmonic functions, $\operatorname{Re}\Phi(s)$ can not be negative anywhere in the critical strip.

Since the root discriminant $\mathrm{rd}(K)$ of K equals the root discriminant of $H(K)$, we have

$$\log d(H(K)) = hn \log \mathrm{rd}(H(K)) = hn \log \mathrm{rd}(K),$$

and also

$$4 \int_0^\infty F(x) \cosh \frac{x}{2}\, dx = 2c\sqrt{\pi}.$$

We therefore we get the expression

$$hn \log \mathrm{rd}(K) = hn(\gamma + \log 8\pi) - hn \int_0^\infty \frac{1 - F(x)}{2 \sinh \frac{x}{2}}\, dx$$

$$- 2c\sqrt{\pi} + \sum_\rho \Phi(\rho) + 2 \sum_{\mathfrak{P}} \sum_{m=1}^\infty \frac{\log N\mathfrak{P}}{N\mathfrak{P}^{m/2}} F(m \log N\mathfrak{P}).$$

We rearrange this to get the identity

$$h = \frac{2c\sqrt{\pi}}{Q}, \tag{0.2}$$

where

$$Q = n\Big[\gamma + \log 8\pi - \mathcal{G}(F) - \log \mathrm{rd}(K) + \frac{1}{hn}\sum_{\rho}\Phi(\rho) + \frac{2}{hn}\sum_{\mathfrak{P}}\sum_{m=1}^{\infty}\frac{\log N\mathfrak{P}}{N\mathfrak{P}^{m/2}}F(m\log N\mathfrak{P})\Big].$$

Here

$$\mathcal{G}(F) = \int_0^{\infty}\frac{1-F(x)}{2\sinh\frac{x}{2}}\,dx.$$

To get an upper bound for the class number h, we need to bound from below the sum over the zeros and the sum over the primes. The sum $\sum_\rho \Phi(\rho)$ over the critical zeros is nonnegative since the real part of $\Phi(s)$ is nonnegative on the critical strip. Thus we know that

$$Q > nB.$$

In conclusion,

$$h = \frac{2c\sqrt{\pi}}{Q} < \frac{2c\sqrt{\pi}}{nB}, \tag{0.3}$$

We note that principal ideals in K totally split in the Hilbert class field of K.

To find a nontrivial lower bound for the sum over prime ideals of the Hilbert class field, we consider the contribution of the hn/f_p prime ideals $\mathfrak{P}$ of degree f_p that lie over some unramified rational prime p:

$$\frac{2}{hn}\sum_{\mathfrak{P}|p}\sum_{m=1}^{\infty}\frac{\log N\mathfrak{P}}{N\mathfrak{P}^{m/2}}F(m\log N\mathfrak{P}) = 2\sum_{m=1}^{\infty}\frac{\log p}{p^{f_p m/2}}F(f_p m\log p).$$

Summing this contribution over an arbitrary set S of unramified primes gives a lower bound for the sum over the prime ideals, proving the theorem.

An integral basis for KL

In order to apply the previous Theorem, we must find sufficiently many integral elements of small prime power norm. To do this, we first must compute a basis of the ring of integers $\mathcal{O}_{KL}$ of KL. In general, it is difficult to compute an integral basis for a number field with such large degree and takes an unfeasibly long time using the commonly implemented algorithms. Fortunately, Jordi Guàrdia, Jesús Montes and Enric Nart studied and recently implemented an algorithm that allows for fast computation of an integral basis. In addition,we obtained better basis

$$\mathcal{C} = (c_1, c_2, \dots, c_{120}), \quad c_i \in \mathbb{R}^{120}$$

by LLL-algorithm.

Finding elements of $\mathcal{O}_{KL}$ with small multiplicative norm

Table: Generators of some small degree 1 primes in $\mathcal{O}_{KL}$

Element	Norm
$c_2 + c_{17} + c_{46}$	3571
$c_3 - c_9 + c_{48}$	5477
$c_{14} - c_{41}$	7499
$c_1 - c_{48} + c_{64}$	8867
$c_{11} + c_{75}$	15679
$c_2 - c_{54} + c_{70}$	17203
$c_3 + c_{12} - c_{100}$	20047
$c_{23} + c_{104}$	25343
$c_{25} - c_{74}$	31477
$c_{10} - c_{21} + c_{97}$	34613
$c_{49} - c_{80}$	35537

Element	Norm
$c_{71} + c_{74}$	43787
$c_{30} + c_{62}$	44879
$c_1 - c_{17} + c_{77}$	45361
c_{95}	46271
$c_2 - c_{28} - c_{62}$	48341
$c_{23} - c_{53} + c_{85}$	54311
$c_2 + c_{31} + c_{76}$	95327
$c_{36} + c_{49}$	111611
$c_3 + c_{22} + c_{66}$	113081
$c_{23} - c_{62}$	137927
$c_7 - c_{89}$	139999

Table: Generators of some small degree 2 and degree 3 primes in $\mathcal{O}_{KL}$

Element	Norm
$c_9 + c_{62}$	13^2
c_{71}	83^2
c_{94}	89^2
$c_4 + c_{17} - c_{76}$	137^2
$c_8 - c_{62}$	227^2

Element	Norm
c_{69}	229^2
$c_{22} + c_{25}$	251^2
$c_{24} - c_{75}$	383^2
$c_7 + c_{42}$	433^2
$c_1 - c_7 + c_{68}$	11^3

Table: Generators of some composite ideals in $\mathcal{O}_{KL}$

Element	Norm
$c_2 + c_{65}$	$13^2 * 19^2$
$c_9 - c_{89}$	$13^2 * 7^3$
$c_{24} - c_{70}$	$13^2 * 6361$
$c_{35} + c_{91}$	$13^2 * 10753$
$c_{68} + c_{93}$	$13^2 * 11681$
$c_{18} + c_{91}$	$13^2 * 109^2$
$c_3 + c_{39} - c_{74}$	$13^2 * 23^3$
$c_1 + c_{68} + c_{78}$	$19^2 * 12619$
$c_2 - c_{21} + c_{44}$	$13^2 * 16561$

Table 3 list some integral elements and their norms. Consider, for example, the element $c_2 + c_{65}$, which has norm $13^2 \cdot 19^2$. Since we also know that $c_9 + c_{62}$ generates a degree 2 prime ideal of norm 13^2, we can divide $c_2 + c_{65}$ by the appropriate Galois conjugate of $c_9 + c_{62}$ to find an integral element of norm 19^2. Since 19 does not totally split in KL, we know that 19 factors into degree 2 principal prime ideals. By this way, we can show that the following primes totally split in KL into degree 1 principal prime ideals:

$$6361, 10753, 11681, 12619, 16561, 19963, 23431,$$

$$23531, 32309, 33403, 41621, 48179, 56359, 58601.$$

Proof of Main Theorem

By searching for elements of small norm, and taking quotients where necessary, we find a number of primes that can be included in the set S of unramified primes that factor into principal prime ideals. In particular, the following 36 primes totally split into degree 1 principal primes in KL:

$$3571, 5477, 6361, 7499, 8867, 10753, 11681, 12619, 15679,$$

$$16561, 17203, 19963, 20047, 23431, 23531, 25343, 31477,$$

$$32309, 33403, 34613, 35537, 41621, 43787, 44879, 45361,$$

$$46271, 48179, 48341, 54311, 56359, 58601, 95327, 111611,$$

$$113081, 137927, 139999.$$

Also, the following 11 primes factor into degree 2 principal primes in KL:

$$13, 19, 83, 89, 109, 137, 227, 229, 251, 383, 433.$$

Finally, the following three primes factor into degree 3 principal primes in KL:

$$7, 11, 23.$$

If we include these primes in our set S and set $c = 24.5$, we find that

$$2 \sum_{p \in S} \sum_{m=1}^{\infty} \frac{\log p}{p^{f_p m/2}} F(f_p m \log p) > 0.18797.$$

We can numerical calculate the integral and find that

$$\int_0^\infty \frac{1 - F(x)}{2\sinh\frac{x}{2}}\,dx < 0.70010.$$

Since the root discriminant of KL is $\sqrt{653}$, we have

$$B = \gamma + \log 8\pi - \log \mathrm{rd}(KL) - \int_0^\infty \frac{1 - F(x)}{2\sinh\frac{x}{2}}\,dx$$

$$+2\sum_{p\in S}\sum_{m=1}^{\infty} \frac{\log p}{p^{f_p m/2}} F(f_p m \log p)$$

$$> 0.57721 + 3.22417 - 3.24079 - 0.70010 + 0.18797 = 0.04846.$$

Therefore, we get an upper bound for the class number of KL:

$$h_{KL} < \frac{2c\sqrt{\pi}}{nB} < \frac{2 \times 24.5\sqrt{\pi}}{120 \times 0.04846} < 14.94.$$

Since the class number is an integer, we deteremine that

$$h_{KL} \leq 14.$$

Thus we conclude that the class number of KL is 1.

Thank You For Coming.