On the Equivariant Polylogarithm and the Special Values of Hecke L-functions for totally real fields

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Background

Case for $F = \mathbb{Q}$: Lerch Zeta Functions

The *universal generating function* for special values of *Lerch zeta functions*

Lerch Zeta Function

For ξ: root of unity in $\mathbb{C}$

$$\mathcal{L}(\xi, s) := \sum_{n=1}^{\infty} \frac{\xi^n}{n^s}, \qquad \mathrm{Re}(s) > 1$$

- For $\xi = 1$, coincides with Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$
- Analytic continuation to $s \in \mathbb{C}$, holomorphic if $\xi \neq 1$

Lerch zeta functions related to Dirichlet L-functions

Case for $F = \mathbb{Q}$: Relation to Dirichlet L-functions

Dirichlet L-Function

$N > 0$: integer, $\chi : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}^\times$: primitive Dirichlet character

$$L(\chi, s) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

Fix ξ: primitive N-th root of unity. For $c_\chi(\xi) := N^{-1} \sum_{m=1}^{N} \chi(m)\xi^{-m}$, we have the finite Fourier expansion

$$\chi(n) = \sum_{m=1}^{N} c_\chi(\xi^m)\xi^{mn}$$

for any $n \in \mathbb{Z}$, hence

$$L(\chi, s) = \sum_{m=1}^{N} c_\chi(m)\mathcal{L}(\xi^m, s)$$

Case for $F = \mathbb{Q}$: Universal Generating Function

Let

$$\mathcal{G}(t) := \frac{t}{1-t}$$

Theorem (Classical)

For any $N > 1$ and non-trivial N-th root of unity $\xi \in \mathbb{C}^\times$, we have

$$\left(t\frac{d}{dt}\right)^k G(t)\Big|_{t=\xi} \mathcal{L}(\xi, -k), \qquad k \in \mathbb{N}$$

In other words, $\mathcal{G}(t)$ knows all the Lerch zeta values for ALL non-trivial roots of unity ξ at ANY non-positive integer

- $\mathcal{G}(t)$: rational function on $\mathbb{G}_m$
- $\left(t\frac{d}{dt}\right)$: algebraic differential
- Roots of unity ξ are the torsion points of $\mathbb{G}_m$

Universal Generating Function for More General Number Fields

- $F = \mathbb{Q}$ (Classical)

  Lerch Zeta Value $\quad \mathbb{G}_m \quad \mathcal{G}(t) = \dfrac{t}{1-t}$

- F: Imaginary Quadratic Field (Robert 1973, Coates-Wiles 1977, B– Kobayashi 2010)

  Eisenstein-Kronecker Number $\quad E \times E \quad \dfrac{\theta(s \oplus t)}{\theta(s)\theta(t)}$

- F: Totally Real Field (Today)

  Generalized Lerch Zeta Function $\quad \mathbb{T} \quad$ Shintani Generating Class

- F: CM Field and its Extension(Kings-Sprang, arXiv:1912.03657)

  Generalized Eisenstein-Kronecker Number $\quad A \times A^\vee \quad$ Eisenstein-Kronecker Class

Totally Real Field

- Eisenstein Series
  Siegel-Klingen,
  Deligne-Ribet (1980)
- Cone Zeta Function and its Generating Function
  Shintani (1976)
  Barsky (1978), Cassou-Noguès (1979)
- Eisenstein Cocycle
  Sczech (1993), Solomon (1998,1999), Hu–Solomon (2001), Hill (2007), Speiss (2014), Charollois-Dasgupta (2014), Charollois-Dasgupta-Greenberg (2015),. . .
- Topological Polylogarithm
  Blottière (2008),
  Beilinson-Kings-Levin (2018)

Outline

Part I: Generalized Algebraic Torus

Part II: Equivariant Polylogarithm Class
- Case for $F = \mathbb{Q}$
- Case for F: totally real

Part III: Relation to Shintani Generating Class
- Case for $F = \mathbb{Q}$
- Main Theorem

Conjectures

Part I: Generalized Algebraic Torus

Algebraic Torus

- F: Totally Real Field, $\mathcal{O}_F$: Ring of Integers, $g := [F : \mathbb{Q}]$
- $\mathfrak{a}$: Fractional Ideal of F.

Definition (Algebraic Torus)

$$\mathbb{T}^{\mathfrak{a}} := \mathrm{Hom}_{\mathbb{Z}}(\mathfrak{a}, \mathbb{G}_m)$$

- Affine algebraic group over $\mathbb{Z}$, $\forall\mathbb{Z}$-algebra R $\mathbb{T}^{\mathfrak{a}}(R) = \mathrm{Hom}_{\mathbb{Z}}(\mathfrak{a}, R^\times)$

$$\xi : \mathfrak{a} \longrightarrow R^\times \qquad \xi(\alpha + \alpha') = \xi(\alpha)\xi(\alpha') \qquad \forall\alpha, \alpha' \in \mathfrak{a}$$

  Parameterizes additive characters
- Used by N. Katz in "Another Look at p-Adic L-Functions for Totally Real Fields" Mathematische Annalen (1981)

Affine Scheme

Explicit Description

$$\mathbb{T}^{\mathfrak{a}} = \operatorname{Spec} \mathbb{Z}[t^{\alpha} \mid \alpha \in \mathfrak{a}]$$

$$t^{\alpha} t^{\alpha'} = t^{\alpha+\alpha'} \quad \forall \alpha, \alpha' \in \mathfrak{a}$$

$\alpha_1, \ldots, \alpha_g$: $\mathbb{Z}$-basis of $\mathfrak{a}$ $\quad \Rightarrow \quad \mathbb{Z}[t^{\alpha} \mid \alpha \in \mathfrak{a}] = \mathbb{Z}[t^{\pm\alpha_1}, \ldots, t^{\pm\alpha_g}]$

$$\mathbb{T}^{\mathfrak{a}} \overset{\text{non-canonical}}{\cong} \mathbb{G}_m \times \cdots \times \mathbb{G}_m$$

Case $F = \mathbb{Q}$ and $\mathfrak{a} = \mathbb{Z}$

$$\mathbb{T}^{\mathbb{Z}} := \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{G}_m) = \mathbb{G}_m$$

Uniformization

- Let $\mathfrak{a}^* = \mathfrak{a}^{-1}\mathfrak{d}^{-1}$, where $\mathfrak{d}$: different of F

Uniformization

$$(F \otimes_{\mathbb{Q}} \mathbb{C})/\mathfrak{a}^* \xrightarrow{\cong} \mathbb{T}^{\mathfrak{a}}(\mathbb{C}) = \mathrm{Hom}_{\mathbb{Z}}(\mathfrak{a}, \mathbb{C}^\times)$$

$$u \mapsto \xi_u(\alpha) = e^{2\pi i \,\mathrm{Tr}(u\alpha)}$$

- $\mathrm{Tr}(u\alpha) := \sum_{\tau \in I} u_\tau \alpha^\tau \quad I = \mathrm{Hom}(F, \mathbb{R}) \quad \alpha^\tau := \tau(\alpha) \quad u = (u_\tau) \in F \otimes \mathbb{C} \cong \prod_{\tau \in I} \mathbb{C}$

⋆ **Similarity to CM Elliptic Curve Case**

$$\mathbb{C}/\mathfrak{a}^* \xrightarrow{\cong} E(\mathbb{C})$$

Case $F = \mathbb{Q}$

Uniformization

$$(F \otimes_{\mathbb{Q}} \mathbb{C})/\mathfrak{a}^* \xrightarrow{\cong} \mathbb{T}^{\mathfrak{a}}(\mathbb{C}) = \mathrm{Hom}_{\mathbb{Z}}(\mathfrak{a}, \mathbb{C}^{\times})$$

$$u \mapsto \xi_u(\alpha) = e^{2\pi i \,\mathrm{Tr}(u\alpha)}$$

for the case $F = \mathbb{Q}$ and $\mathfrak{a} = \mathbb{Z}$ given for $\mathbb{T}^{\mathbb{Z}} = \mathbb{G}_m$ as

$$\mathbb{C}/\mathbb{Z} \xrightarrow{\cong} \mathbb{G}_m(\mathbb{C}) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{C}^{\times})$$

$$u \mapsto \xi_u(\alpha) = e^{2\pi i u\alpha}$$

Equivariance

- $F_+ := \{x \in F \mid \tau(x) > 0 \ \forall \tau \in I\}$
- $\Delta = \mathcal{O}_{F_+}^\times \cong \mathbb{Z}^{g-1}$: group of totally positive units ($\Delta = \{1\}$ if $F = \mathbb{Q}$)

$\forall \varepsilon \in \Delta$

$$\langle \varepsilon \rangle \colon \mathbb{T}^{\mathfrak{a}} \longrightarrow \mathbb{T}^{\mathfrak{a}}$$

map induced by $t^\alpha \mapsto t^{\varepsilon\alpha}$ gives action on $\mathbb{T}^{\mathfrak{a}} = \operatorname{Spec} \mathbb{Z}[t^\alpha \mid \alpha \in \mathfrak{a}\}$

(equivalent to action given by multiplication by ε on $\mathfrak{a}$ in $\mathbb{T}^{\mathfrak{a}} = \operatorname{Hom}_{\mathbb{Z}}(\mathfrak{a}, \mathbb{C})$)

$$\langle \varepsilon \rangle \colon \mathbb{T}^{\mathfrak{a}}(\mathbb{C}) \longrightarrow \mathbb{T}^{\mathfrak{a}}(\mathbb{C})$$

$$\xi(\alpha) \qquad \mapsto \qquad \xi^{\varepsilon}(\alpha) := \xi(\varepsilon\alpha) \qquad \forall \alpha \in \mathfrak{a}$$

Equivariant action of Δ on $\mathbb{T}^{\mathfrak{a}}$

Equivariance: Generalized

- $F_+^\times$: group of totally positive elements in F

$\forall x \in F_+^\times$

$$\langle x \rangle : \mathbb{T}^{x\mathfrak{a}} \longrightarrow \mathbb{T}^{\mathfrak{a}}$$

map induced by $t^\alpha \mapsto t^{x\alpha}$

$$\langle x \rangle : \mathbb{T}^{x\mathfrak{a}}(\mathbb{C}) \longrightarrow \mathbb{T}^{\mathfrak{a}}(\mathbb{C})$$

$$\xi(\alpha) \qquad \mapsto \qquad \xi^x(\alpha) := \xi(x\alpha)$$

Map from $\mathbb{T}^{x\mathfrak{a}}$ to $\mathbb{T}^{\mathfrak{a}}$

Idea: Take All Choices

Generalized Algebraic Torus

Definition (Generalized Algebraic Torus)

$\mathfrak{I}$: group of all non-zero fractional ideals of F

$$\mathbb{T} := \coprod_{\mathfrak{a} \in \mathfrak{I}} \mathbb{T}^{\mathfrak{a}}$$

The map $\langle x \rangle : \mathbb{T}^{x\mathfrak{a}} \to \mathbb{T}^{\mathfrak{a}}$ for all $x \in F_+^\times$ gives action

$$\langle x \rangle : \mathbb{T} \to \mathbb{T}$$

Equivariant action of $F_+^\times$ on $\mathbb{T}$

Quotient Stack

We will consider the Equivariant Polylogarithm on $\mathbb{T}$, which may be regarded as the Polylogarithm on the quotient stack $\mathcal{T} := \mathbb{T}/F_+^\times$

$\mathfrak{C}$: fractional ideals representing narrow class group $\mathrm{Cl}_F^+(1)$

$$\mathcal{T} := \mathbb{T}/F_+^\times = \Big(\coprod_{\mathfrak{a}\in\mathfrak{I}} \mathbb{T}^{\mathfrak{a}}\Big)/F_+^\times \cong \coprod_{\mathfrak{a}\in\mathfrak{C}} (\mathbb{T}^{\mathfrak{a}}/\Delta)$$

Isomorphic to finite sum of quotient stacks of form $\mathbb{T}^{\mathfrak{a}}/\Delta$

- $\mathrm{Cl}_F^+(1) := \mathfrak{I}/P_+$
- $P_+ := \{(x) \mid x \in F_+^\times\}$

Torsion Points

For an *integral* ideal $\mathfrak{g} \subset \mathcal{O}_F$, we define the group of $\mathfrak{g}$-torsion points by

$$\mathbb{T}^{\mathfrak{a}}[\mathfrak{g}] := \mathrm{Hom}_{\mathbb{Z}}(\mathfrak{a}/\mathfrak{g}\mathfrak{a}, \mathbb{G}_m) \hookrightarrow \mathbb{T}^{\mathfrak{a}} = \mathrm{Hom}_{\mathbb{Z}}(\mathfrak{a}, \mathbb{G}_m)$$

We let

$$\mathbb{T}[\mathfrak{g}] := \coprod_{\mathfrak{a} \in \mathfrak{I}} \mathbb{T}^{\mathfrak{a}}[\mathfrak{g}], \qquad \mathscr{T}[\mathfrak{g}] \coloneqq \mathbb{T}[\mathfrak{g}]/F_+^\times$$

For any *integral* ideal $\mathfrak{b} \subset \mathcal{O}_F$, the inclusion $\mathfrak{a}\mathfrak{b} \subset \mathfrak{a}$ induces $\mathbb{T}^{\mathfrak{a}} \longrightarrow \mathbb{T}^{\mathfrak{a}\mathfrak{b}}$, which induces maps

$$\rho(\mathfrak{b})\colon \mathbb{T} \longrightarrow \mathbb{T}, \qquad \rho(\mathfrak{b})\colon \mathbb{T}[\mathfrak{g}] \longrightarrow \mathbb{T}[\mathfrak{g}],$$

Lemma (B–, Hagihara, Yamada, Yamamoto)

ρ gives a transitive action of $\mathrm{Cl}_F^+(\mathfrak{g})$ on $\mathscr{T}[\mathfrak{g}]$

- $\mathrm{Cl}_F^+(\mathfrak{g}) := \mathfrak{I}/P_+(\mathfrak{g}), \qquad P_+(\mathfrak{g}) := \{(\beta) \mid \beta \equiv 1 \bmod^\times \mathfrak{g}\}$
- Note: Class Field Theory $\mathrm{Cl}_F^+(\mathfrak{g}) \cong \mathrm{Gal}(F(\mathfrak{g})/F)$

Question

⋆ **Similarity to CM Elliptic Curve Case (Rough)**

- K: imaginary quadratic field
- $\mathfrak{a} \in \mathfrak{I}$, $E^{\mathfrak{a}}$: CM elliptic curve defined over $K(1)$, CM in $\mathfrak{a}$

$$\mathcal{E} = \Big(\coprod_{\mathfrak{a}\in\mathfrak{I}} E^{\mathfrak{a}}\Big)/K^{\times}, \qquad \mathcal{E}[\mathfrak{g}] = \Big(\coprod_{\mathfrak{a}\in\mathfrak{I}} E^{\mathfrak{a}}[\mathfrak{g}]\Big)/K^{\times}$$

Construction similar to that of ρ gives the action of Hecke character on $\mathcal{E}$ and $\mathcal{E}[\mathfrak{g}]$

Theory of Complex Multiplication

Action of ρ $\Leftrightarrow$ Action of $\mathrm{Gal}(K^{ab}/K)$ on torsion points

Question: Is there some way to equip $\mathcal{T}$ with a F-structure so that $\mathcal{T}[\mathfrak{g}]$ has a natural action of $\mathrm{Gal}(F(\mathfrak{g})/F)$, which is compatible with the action of ρ?

Part II: Equivariant Polylogarithm Class

Work in Progress

Case for $F = \mathbb{Q}$

The cohomology

$$\mathscr{H} := H^1(\mathbb{G}_m(\mathbb{C}), \mathbb{R})^\vee = \mathbb{R}(1)$$

has a *Hodge structure* of pure weight 2.

Definition (The Logarithm Sheaf)

The Logarithm Sheaf is a certain admissible unipotent pro-variation of mixed $\mathbb{R}$-Hodge structures $\mathbb{L}\mathrm{og}$ on $\mathbb{G}_m(\mathbb{C})$ such that

$$\mathrm{Gr}^W_\bullet \mathbb{L}\mathrm{og} \cong \prod_{k \geq 0} \mathrm{Sym}^k \mathscr{H} \cong \prod_{k \geq 0} \mathbb{R}(k)$$

For any torsion point $\xi \in \mathbb{G}_m(\mathbb{C})$, the Logarithm sheaf satisfies the splitting principle

$$i_\xi^* \mathbb{L}\mathrm{og} \cong \prod_{k \geq 0} \mathbb{R}(k)$$

Case for $F = \mathbb{Q}$: The Polylogarithm Class

Let $U^{\mathbb{Z}} := \mathbb{G}_m \setminus \{1\} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$. The residue at 1 gives a canonical isomorphism

$$H^1_{\mathscr{D}}(U^{\mathbb{Z}}, \mathbb{L}\mathrm{og}) \cong \mathbb{R},$$

where $H^1_{\mathscr{D}}(U^{\mathbb{Z}}, \mathbb{L}\mathrm{og})$ is the *Deligne-Beilinson cohomology* of $U^{\mathbb{Z}}$ with coefficients in $\mathbb{L}\mathrm{og}$, given as

$$H^1_{\mathscr{D}}(U^{\mathbb{Z}}, \mathbb{L}\mathrm{og}) = \mathrm{Ext}^1_{\mathsf{VMHS}_{\mathbb{R}}(U^{\mathbb{Z}})}(\mathbb{R}(0), \mathbb{L}\mathrm{og})$$

Definition (Beilinson-Deligne, Huber-Wildeshaus)

The *polylogarithm class* is the element

$$\mathrm{pol} \in H^1_{\mathscr{D}}(U^{\mathbb{Z}}, \mathbb{L}\mathrm{og})$$

which maps to 1 through the isomorphism $H^1_{\mathscr{D}}(U^{\mathbb{Z}}, \mathbb{L}\mathrm{og}) \cong \mathbb{R}$.

Case for $F = \mathbb{Q}$: Construction is Motivic

Let $U^{\mathbb{Z}} := \mathbb{G}_m \setminus \{1\} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$. By the works of Beilinson and Deligne, there exists a motivic meaning to the sheaf $\mathbb{L}\mathrm{og}$, and the residue at 1 gives a canonical isomorphism

$$H^1_{\mathrm{mot}}(U^{\mathbb{Z}}, \mathbb{L}\mathrm{og}) \cong \mathbb{Q},$$

where $H^1_{\mathrm{mot}}(U^{\mathbb{Z}}, \mathbb{L}\mathrm{og})$ is the *motivic cohomology* of $U^{\mathbb{Z}}$ with coefficients in $\mathbb{L}\mathrm{og}$. We may define the *motivic polylogarithm* $\mathrm{pol} \in H^1_{\mathrm{mot}}(U^{\mathbb{Z}}, \mathbb{L}\mathrm{og})$ similarly. We have a commutative diagram

$$\begin{array}{ccccc} \mathrm{pol} & \in & H^1_{\mathrm{mot}}(U^{\mathbb{Z}}, \mathbb{L}\mathrm{og}) & \xrightarrow{\cong} & \mathbb{Q} \\ \big\downarrow & & \big\downarrow{\scriptstyle r_{\mathscr{D}}} & & \big\downarrow \\ \mathrm{pol} & \in & H^1_{\mathscr{D}}(U^{\mathbb{Z}}, \mathbb{L}\mathrm{og}) & \xrightarrow{\cong} & \mathbb{R}, \end{array}$$

where $r_{\mathscr{D}}$ is the regulator map.

Case for $F = \mathbb{Q}$: Specialization to Torsion Points

Theorem (Beilinson-Deligne, Huber-Wildeshaus)

For any torsion point $\xi \in U^{\mathbb{Z}}$, the specialization

$$i_\xi^* \mathrm{pol} \in H^1_{\mathscr{D}}(\xi, i_\xi^* \mathbb{L}\mathrm{og}) \cong \prod_{k \geq 0} H^1_{\mathscr{D}}(\xi, \mathbb{R}(k)) \cong \prod_{k>0} \mathbb{R}$$

satisfies

$$i_\xi^* \mathrm{pol} = (\mathrm{Li}_k(\xi))_{k>0}$$

Here,

$$\mathrm{Li}_k(t) = \sum_{n=1}^{\infty} \frac{t^n}{n^k}$$

is the *polylogarithm function*

Case for $F = \mathbb{Q}$: Implications

$$\mathsf{pol} \longmapsto (c_k(\xi))$$

$$\begin{array}{ccc} H^1_{\mathrm{mot}}(U^{\mathbb{Z}}, \mathbb{L}\mathrm{og}) & \xrightarrow{i_\xi^*} & \prod_{k>0} H^1_{\mathrm{mot}}(\xi, \mathbb{R}(k)) \\ \downarrow r_{\mathscr{D}} & & \downarrow r_{\mathscr{D}} \\ H^1_{\mathscr{D}}(U^{\mathbb{Z}}, \mathbb{L}\mathrm{og}) & \xrightarrow{i_\xi^*} & \prod_{k>0} H^1_{\mathscr{D}}(\xi, \mathbb{R}(k)) \end{array}$$

$$\mathsf{pol} \longmapsto (\mathrm{Li}_k(\xi))$$

Commutativity: The polylogarithm values are the image by $r_{\mathscr{D}}$ of motivic objects $c_k(\xi)$

$\Rightarrow$ Beilinson conjecture for Dirichlet L-functions

See for example Neukirch 1988 [4]

Motivation and Results

The construction of the polylogarithm extended from $\mathbb{G}_m$ to general algebraic groups (Huber and Kings, 2018 [2]).

Question

By considering the equivariant polylogarithm for $\mathbb{T} = \coprod_{\mathfrak{a} \in \mathfrak{I}} \mathbb{T}^{\mathfrak{a}}$, can the same method be used to attack the Beilinson conjecture for Hecke L-functions of totally real fields?

Not yet clear. We give some observations.

Our Results

- Construction of the Polylogarithm in Equivariant Deligne-Beilinson Cohomology
- Relation to Shintani Generating Class

Logarithm Sheaf

Let $\mathbb{T} = \coprod_{\mathfrak{a} \in \mathfrak{I}} \mathbb{T}^{\mathfrak{a}}$ with action of $F_+^\times$, and let $U = \coprod_{\mathfrak{a} \in \mathfrak{I}} U^{\mathfrak{a}}$ for $U^{\mathfrak{a}} = \mathbb{T}^{\mathfrak{a}} \setminus \{1\}$. Let

$$\mathscr{H}_{\mathfrak{a}} := H^1(\mathbb{T}^{\mathfrak{a}}, \mathbb{R})^\vee = \bigoplus_{j=1}^{g} \mathbb{R}(1),$$

and let $\mathscr{H}$ be the sheaf on $\mathbb{T}$ given by $\mathscr{H}_{\mathfrak{a}}$ on $\mathbb{T}^{\mathfrak{a}}$

Definition (The Logarithm Sheaf)

The Logarithm Sheaf is a certain $F_+^\times$-equivariant admissible unipotent pro-variation of mixed $\mathbb{R}$-Hodge structures $\mathbb{L}\mathrm{og}$ on $\mathbb{T}(\mathbb{C})$ such that

$$\mathrm{Gr}_\bullet^W \mathbb{L}\mathrm{og} \cong \prod_{k \geq 0} \mathrm{Sym}^k \mathscr{H}$$

This Logarithm sheaf also satisfies the splitting principle

Equivariant Variation of mixed $\mathbb{R}$-Hodge structures

- We define a variation of mixed $\mathbb{R}$-Hodge structures $\mathbb{V}$ on $\mathbb{T}$, to be a family of variation of mixed $\mathbb{R}$-Hodge structures $\mathbb{V} = (\mathbb{V}_{\mathfrak{a}})_{\mathfrak{a}\in\mathscr{I}}$ on $\mathbb{T}^{\mathfrak{a}}(\mathbb{C})$
- It is $F_+^\times$-equivariant, if we fix isomorphisms $\iota_{x,\mathfrak{a}}$ for $x \in F_+^\times$ and $\mathfrak{a} \in \mathfrak{I}$

$$\iota_{x,\mathfrak{a}} \colon \langle x\rangle^*\mathbb{V}_{\mathfrak{a}} \xrightarrow{\cong} \mathbb{V}_{x\mathfrak{a}}$$

  satisfying standard compatibility with respect to composition
- Equivariant cohomology $H^m(\mathbb{T}/F_+^\times, \mathbb{V})$ is equipped with mixed $\mathbb{R}$-Hodge structure
- There exists spectral sequence

$$E_2^{p,q} = H^p(F_+^\times, H^q(\mathbb{T}, \mathbb{V})) \Rightarrow H^{p+q}(\mathbb{T}/F_+^\times, \mathbb{V})$$

Cohomology of $\mathbb{L}\mathrm{og}_{\mathfrak{a}}$ on $\mathbb{T}^{\mathfrak{a}}$

Let $g = [F : \mathbb{Q}]$

Theorem (cf. Huber-Kings)

We have

$$H^m(\mathbb{T}^{\mathfrak{a}}, \mathbb{L}\mathrm{og}_{\mathfrak{a}}) = \begin{cases} \mathbb{R}(-g) & m = g \\ \{0\} & m \neq g \end{cases}$$

Let $U^{\mathfrak{a}} := \mathbb{T} \setminus \{1\}$. We may calculate the cohomology on U via the localizing sequence

$$\cdots \to H^m(\mathbb{T}^{\mathfrak{a}}, \mathbb{L}\mathrm{og}_{\mathfrak{a}}) \to H^m(U, \mathbb{L}\mathrm{og}_{\mathfrak{a}}) \to H^{m+1}_{\{1\}}(\mathbb{T}^{\mathfrak{a}}, \mathbb{L}\mathrm{og}_{\mathfrak{a}}) \to \cdots$$

noting that

$$H^m_{\{1\}}(\mathbb{T}^{\mathfrak{a}}, \mathbb{L}\mathrm{og}_{\mathfrak{a}}) = \begin{cases} \left(\prod_{k=0}^{\infty} \mathrm{Sym}^k \mathscr{H}_{\mathfrak{a}}\right)(-g) & m = 2g \\ 0 & m \neq 2g \end{cases}$$

Cohomology of $\mathbb{L}\mathrm{og}_{\mathfrak{a}}$ on $U^{\mathfrak{a}}$

Let $U^{\mathfrak{a}} := \mathbb{T}^{\mathfrak{a}} \setminus \{1\}$

Theorem (cf. Huber-Kings)

If $g > 1$, we have

$$H^m(U^{\mathfrak{a}}, \mathbb{L}\mathrm{og}_{\mathfrak{a}}) = \begin{cases} \mathbb{R}(-g) & m = g \\ \left(\prod_{k=0}^{\infty} \mathrm{Sym}^k \mathscr{H}_{\mathfrak{a}}\right)(-g) & m = 2g-1 \\ \{0\} & \textit{otherwise} \end{cases}$$

If $g = 1$, we have

$$H^m(U^{\mathfrak{a}}, \mathbb{L}\mathrm{og}_{\mathfrak{a}}) = \begin{cases} \mathbb{R}(-1) \oplus \prod_{k=0}^{\infty} \mathbb{R}(k-1) & m = 1 \\ \{0\} & m \neq 1 \end{cases}$$

Equivariant Cohomology of $\mathbb{L}\mathrm{og}$ on U

Calculate equivariant cohomology of $\mathbb{L}\mathrm{og}_{\mathfrak{a}}$ on $U^{\mathfrak{a}}$

$$E_2^{p,q} = H^p(\Delta, H^q(U^{\mathfrak{a}}, \mathbb{L}\mathrm{og}_{\mathfrak{a}})) \Rightarrow H^{p+q}(U^{\mathfrak{a}}/\Delta, \mathbb{L}\mathrm{og}_{\mathfrak{a}}),$$

noting that $H^{g-1}(\Delta, \mathbb{R}(-g)) = \mathbb{R}(-g)$ and $H^0(\Delta, \mathrm{Sym}^k \mathscr{H}_{\mathfrak{a}}) = \begin{cases} \mathbb{R}(k) & g|k \\ \{0\} & \text{otherwise} \end{cases}$

Theorem (B–, Bekki, Hagihara, Ohshita, Yamada, Yamamoto)

For any $g \geq 1$, we have an exact sequence

$$0 \to \mathbb{R}(-g) \to H^{2g-1}(U^{\mathfrak{a}}/\Delta, \mathbb{L}\mathrm{og}_{\mathfrak{a}}) \to \prod_{n=0}^{\infty} \mathbb{R}((n-1)g) \to 0$$

Cohomology $H^m(U^{\mathfrak{a}}/\Delta, \mathbb{L}\mathrm{og}_{\mathfrak{a}})$ for $m < 2g-1$ vanish or have weight $2g$

The Polylogarithm Class

For $U = \coprod_{\mathfrak{a}\in\mathfrak{I}} U^{\mathfrak{a}}$

$$H^{2g-1}(U/F_+^\times, \mathbb{L}\mathrm{og}) \cong \bigoplus_{\mathfrak{a}\in\mathfrak{C}} H^{2g-1}(U^{\mathfrak{a}}/\Delta, \mathbb{L}\mathrm{og}_{\mathfrak{a}})$$

Equivariant Deligne-Beilinson cohomology is defined to fit into the spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p_{\mathrm{MHS}_{\mathbb{R}}}(\mathbb{R}(0), H^q(U/F_+^\times, \mathbb{L}\mathrm{og})) \Rightarrow H^{p+q}_{\mathscr{D}}(U/F_+^\times, \mathbb{L}\mathrm{og})$$

Previous theorem gives canonical isomorphism

$$H^{2g-1}_{\mathscr{D}}(U/F_+^\times, \mathbb{L}\mathrm{og}) \cong \bigoplus_{\mathrm{Cl}_F^+(1)} \mathrm{Ext}^0_{\mathrm{MHS}_{\mathbb{R}}}(\mathbb{R}(0), H^{2g-1}(U/F_+^\times, \mathbb{L}\mathrm{og})) \cong \bigoplus_{\mathrm{Cl}_F^+(1)} \mathbb{R},$$

noting that we have

$$\mathrm{Ext}^0_{\mathrm{MHS}_{\mathbb{R}}}(\mathbb{R}(0), \mathbb{R}(n)) \cong \begin{cases} \mathbb{R} & n = 0 \\ \{0\} & n \neq 0 \end{cases} \quad \mathrm{Ext}^1_{\mathrm{MHS}_{\mathbb{R}}}(\mathbb{R}(0), \mathbb{R}(n)) \cong \begin{cases} (2\pi i)^{n-1}\mathbb{R} & n > 0 \\ \{0\} & n \leq 0 \end{cases}$$

The Polylogarithm Class

Our argument shows that we have a canonical isomorphism

$$H^{2g-1}_{\mathscr{D}}(U/F^{\times}_{+}, \mathbb{L}\mathrm{og}) \cong \bigoplus_{\mathrm{Cl}^{+}_{F}(1)} \mathbb{R}$$

Definition (B–, Bekki, Hagihara, Ohshita, Yamada, Yamamoto)

We define the *equivariant polylogarithm class* for the generalized torus to be the element

$$\mathrm{pol} \in H^{2g-1}_{\mathscr{D}}(U/F^{\times}_{+}, \mathbb{L}\mathrm{og})$$

which maps to $(1, \ldots, 1)$ through the isomorphism $H^{2g-1}_{\mathscr{D}}(U/F^{\times}_{+}, \mathbb{L}\mathrm{og}) \cong \bigoplus_{\mathrm{Cl}^{+}_{F}(1)} \mathbb{R}$

Part III: Relation to Shintani Generating Class

Case for $F = \mathbb{Q}$: Lerch Zeta Functions

The *universal generating function* for special values of *Lerch zeta functions*

Lerch Zeta Function

For ξ: root of unity in $\mathbb{C}$

$$\mathcal{L}(\xi, s) := \sum_{n=1}^{\infty} \frac{\xi^n}{n^s}, \qquad \mathrm{Re}(s) > 1$$

- For $\xi = 1$, coincides with Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$
- Analytic continuation to $s \in \mathbb{C}$, holomorphic if $\xi \neq 1$
- Since $\mathbb{G}_m(\mathbb{C}) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{C}^\times)$, ξ may be viewed as a character $\xi(n) = \xi^n$ for $n \in \mathbb{Z}$

Lerch zeta functions related to *Dirichlet L-functions*, and also to the *polylogarithm function*

$$\mathcal{L}(\xi, k) = \mathrm{Li}_k(\xi)$$

Case for $F = \mathbb{Q}$: Relation to Dirichlet L-functions

Dirichlet L-Function

$N > 0$: integer, $\chi : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}^\times$: primitive Dirichlet character

$$L(\chi, s) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

Fix ξ: primitive N-th root of unity. For $c_\chi(\xi) := N^{-1} \sum_{m=1}^{N} \chi(m)\xi^{-m}$, we have the finite Fourier expansion

$$\chi(n) = \sum_{m=1}^{N} c_\chi(\xi^m)\xi^{mn}$$

for any $n \in \mathbb{Z}$, hence

$$L(\chi, s) = \sum_{m=1}^{N} c_\chi(m)\mathcal{L}(\xi^m, s)$$

Case for $F = \mathbb{Q}$: Shintani Generating Function

Let

$$\mathcal{G}(t) := \frac{t}{1-t} \quad \in \quad H^0(U^{\mathbb{Z}}, \mathcal{O}_{U^{\mathbb{Z}}})$$

Theorem (Classical)

For any $N > 1$ and non-trivial N-th root of unity $\xi \in \mathbb{C}^\times$, we have

$$\left(t\frac{d}{dt}\right)^k \mathcal{G}(t)\Big|_{t=\xi} = \mathcal{L}(\xi, -k), \qquad k \in \mathbb{N}$$

$\mathcal{G}(t)$ is the *universal generating function* of non-positive Lerch zeta value

Shintani Generating Class is the generalization to the case of totally real fields of $\mathcal{G}(t)$

Finite Hecke Character

For a finite Hecke character

$$\chi\colon\ \mathrm{Cl}_F^+(\mathfrak{g}) \to \mathbb{C}^\times$$

of conductor $\mathfrak{g}$, we may extend χ by zero to a function on the group of fractional ideals $\mathfrak{I}$.

Hecke L-function

Hecke L-function of χ defined as

$$L(\chi, s) := \sum_{\mathfrak{a} \subset \mathcal{O}_F} \frac{\chi(\mathfrak{a})}{N\mathfrak{a}^s}$$

This function converges for $\mathrm{Re}(s) > 1$, has analytic continuation to $s \in \mathbb{C}$

Case $F = \mathbb{Q}$: finite Hecke character

For $F = \mathbb{Q}$ and $\mathfrak{g} = (N)$ for $N > 0$, let

$$\chi\colon \mathrm{Cl}^{+}_{\mathbb{Q}}(\mathfrak{g}) \to \mathbb{C}^{\times}$$

be a finite Hecke character. Let

$$\chi_{\mathbb{Z}}(n) := \chi((n)), \qquad n\text{: integer} > 0,$$

then this defines a Dirichlet character $\chi_{\mathbb{Z}}\colon (\mathbb{Z}/N\mathbb{Z})^{\times} \to \mathbb{C}^{\times}$. The Hecke L-function for χ coincides with the Dirichlet L-function for $\chi_{\mathbb{Z}}$ in this case.

$$L(\chi, s) = L(\chi_{\mathbb{Z}}, s) = \sum_{m=1}^{N} c_{\chi_{\mathbb{Z}}}(\xi^m)\mathcal{L}(\xi^m, s)$$

What is the generalization of Lerch zeta function for totally real fields?

Lerch Zeta Function: General F

Definition (Our Definition of Lerch Zeta Function)

For any $\mathfrak{a} \in \mathrm{Cl}_F^+(1)$ and finite additive character $\xi \in \mathbb{T}^{\mathfrak{a}}(\mathbb{C}) := \mathrm{Hom}_{\mathbb{Z}}(\mathfrak{a}, \mathbb{C}^\times)$, let

$$\mathcal{L}(\xi\Delta, s) := \sum_{\alpha \in \Delta \backslash \mathfrak{a}_+} \frac{\xi\Delta(\alpha)}{N(\mathfrak{a}^{-1}\alpha)^s}$$

where $\xi\Delta := \sum_{\varepsilon \in \Delta/\Delta_\xi} \xi^\epsilon$, $\quad \Delta_\xi = \{\varepsilon \in \Delta \mid \xi^\varepsilon = \xi\}$

Then for any finite Hecke character $\chi\colon \mathrm{Cl}_F^+(\mathfrak{g}) \to \mathbb{C}^\times$, we have

$$L(\chi, s) = \sum_{\mathfrak{a} \in \mathrm{Cl}_F^+(1)} \sum_{\xi \in \mathbb{T}^{\mathfrak{a}}[\mathfrak{g}]/\Delta} c_\chi(\xi)\mathcal{L}(\xi\Delta, s)$$

for suitable constants $c_\chi(\xi)$. Hecke L-function expressed by Lerch zeta function!

Shintani Generating Class

The *Shintani Generating Class* is a canonical equivariant coherent cohomology class

$$\mathcal{G}(t) \in H^{g-1}(U/F_+^\times, \mathcal{O}_U).$$

Differential given by $\partial(t^\alpha) = N(\alpha)t^\alpha$ induces a differential on $H^{g-1}(U/F_+^\times, \mathscr{O}_\mathbb{T})$.

Theorem (B., Hagihara, Yamada, Yamamoto)

For any integer $k \geq 0$ and any torsion point ξ in $U(\overline{\mathbb{Q}})$, we have

$$\begin{array}{ccc} H^{g-1}(U/F_+^\times, \mathscr{O}_\mathbb{T}) & & \partial^k \mathcal{G}(t) \\ \downarrow{\scriptstyle i_\xi^*} & & \\ H^{g-1}(\xi\Delta/\Delta, \mathscr{O}_{\xi\Delta}) = \mathbb{Q}(\xi) & & \partial^k \mathcal{G}(t)\big|_{t=\xi\Delta} = \mathcal{L}(\xi\Delta, -k), \end{array}$$

where $i_\xi : \xi\Delta \to U$ is equivariant with respect to the action of Δ.

de Rham Shintani Generating Class

There exists a natural homomorphism

$$H^{g-1}(U/F_+^\times, \mathscr{O}_\mathbb{T}) \to H_{\mathrm{dR}}^{2g-1}(U/F_+^\times, \mathscr{L}\mathrm{og})$$

obtained via wedge product with

$$\frac{dt^{\alpha_1}}{t^{\alpha_1}} \wedge \cdots \wedge \frac{dt^{\alpha_g}}{t^{\alpha_g}}$$

on each open set $U^{\mathfrak{a}}_{\alpha_1,\ldots,\alpha_g} := \mathbb{T}^{\mathfrak{a}} \setminus (\{t^{\alpha_1} \neq 1\} \cup \cdots \cup \{t^{\alpha_g} \neq 1\})$ for $\alpha_1, \ldots, \alpha_g \in \mathfrak{a}$

Definition

We define the de Rham Shintani generating class $\mathcal{S}$ to be the image of $\mathcal{G}(t)$ with respect to the above homorphism

Case $g = 1$

$$\mathcal{G}(t) = \frac{t}{1-t} \qquad \mapsto \qquad \mathcal{S} = \frac{dt}{1-t}$$

Main Theorem

There exists a natural injection

$$i : H^{2g-1}_{\mathscr{D}}(U/F_+^\times, \mathbb{L}\mathsf{og}) \xrightarrow{\cong} \mathsf{Hom}_{\mathsf{MHS}_\mathbb{R}}(\mathbb{R}(0), H^{2g-1}(U/F_+^\times, \mathbb{L}\mathsf{og})) \hookrightarrow H^{2g-1}_{\mathsf{dR}}(U/F_+^\times, \mathcal{L}\mathsf{og})$$

Theorem (B–, Bekki, Hagihara, Ohshita, Yamada, Yamamoto)

In $H^{2g-1}_{\mathsf{dR}}(U/F_+^\times, \mathcal{L}\mathsf{og})$, we have

$$i(\mathsf{pol}) = \mathcal{S}$$

In other words, the polylogarithm coincides wth the de Rham Shintani class

Proof: The residue of Shintani generating class is 1 at 1 on each component

※ Shintani generating class is the de Rham realization of the polylogarithm class

Remark

- Beilinson-Kings-Levin ([1] 2018) gives relation between Topological Polylogarithm and Special Values of Hecke L-functions
- Classical Polylogarithm Function

$$\mathrm{Li}_{k+1}(s) = \int_0^s \mathrm{Li}_k(t)\frac{dt}{t}, \qquad \mathrm{Li}_0(t)\frac{dt}{t} = \frac{dt}{1-t} = \mathcal{S}$$

  de Rham Shintani generating class gives the algebraic differential which is the “start" of the iterated integral of polylogarithm function
- We may hope that the “specialization" of the polylogarithm in this case may be related to special values of Hecke L-functions – even in the noncritical case

Appendix: Conjectures

Specialization

For torsion $\xi \in \mathbb{T}$, there exists an equivariant inclusion $i_{\xi\Delta} \colon \xi\Delta \to \mathbb{T}$, which induces the specialization

$$H_{\mathscr{D}}^{2g-1}(U/F_+^\times, \mathbb{L}\mathrm{og}) \xrightarrow{i_{\xi\Delta}^*} H_{\mathscr{D}}^{2g-1}(\xi\Delta/\Delta, i_{\xi\Delta}^*\mathbb{L}\mathrm{og}) \cong \prod_{k>0}^{\infty} H_{\mathscr{D}}^{2g-1}(\xi\Delta/\Delta, \mathbb{R}(gk))$$

PROBLEM: We have

$$H_{\mathscr{D}}^{2g-1}(\xi\Delta/\Delta, \mathbb{R}(gk)) = \mathrm{Ext}^g_{\mathrm{MHS}_{\mathbb{R}}}(\mathbb{R}(0), H^{g-1}(\xi\Delta/\Delta, \mathbb{R}(gk)))$$

which is *zero* for $g > 1$ since Ext^g in the category of mixed $\mathbb{R}$-Hodge structures

Specialization

IDEA: Use the category of mixed *plectic* $\mathbb{R}$-Hodge structures $\mathsf{MHS}^{l}_{\mathbb{R}}$ proposed by Nekovar and Scholl 2016 [3]. Assuming the existence of such theory, plectic Deligne-Beilinson cohomology should fit into the spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p_{\mathsf{MHS}^l_{\mathbb{R}}}(\mathbb{R}(0), H^q(U/F_+^\times, \mathbb{L}\mathrm{og})) \Rightarrow H^{p+q}_{\mathscr{D}^l}(U/F_+^\times, \mathbb{L}\mathrm{og}),$$

where $\mathsf{MHS}^{l}_{\mathbb{R}}$ is the category of mixed plectic $\mathbb{R}$-Hodge structures. Assuming such theory, we may prove

$$H^{p+q}_{\mathscr{D}}(U/F_+^\times, \mathbb{L}\mathrm{og}) \cong H^{p+q}_{\mathscr{D}^l}(U/F_+^\times, \mathbb{L}\mathrm{og})$$

We have the specialization

$$H^{2g-1}_{\mathscr{D}^l}(U/F_+^\times, \mathbb{L}\mathrm{og}) \xrightarrow{i^*_{\xi\Delta}} H^{2g-1}_{\mathscr{D}^l}(\xi\Delta/\Delta, i^*_{\xi\Delta}\mathbb{L}\mathrm{og}) \cong \prod_{k>0}^{\infty} H^{2g-1}_{\mathscr{D}^l}(\xi\Delta/\Delta, \mathbb{R}(gk))$$

Conjecture

We have

$$H^{2g-1}_{\mathscr{D}^I}(\xi\Delta/\Delta, \mathbb{R}(gk)) = \mathrm{Ext}^g_{\mathrm{MHS}^I_{\mathbb{R}}}(\mathbb{R}(0), H^{g-1}(\xi\Delta/\Delta, \mathbb{R}(gk))),$$

and we have $\mathrm{Ext}^g_{\mathrm{MHS}^I_{\mathbb{R}}}(\mathbb{R}(0), H^{g-1}(\xi\Delta/\Delta, \mathbb{R}(gk))) \cong \mathbb{R}$ for $k > 0$.

Conjecture

For any torsion point $\xi \in U^{\mathbb{Z}}$, the specialization

$$i^*_{\xi\Delta}\,\mathrm{pol} \in H^{2g-1}_{\mathscr{D}}(\xi\Delta/\Delta, i^*_{\xi}\mathbb{L}\mathrm{og}) \cong \prod_{k\geq 0} H^{2g-1}_{\mathscr{D}^I}(\xi\Delta/\Delta, \mathbb{R}(gk)) \cong \prod_{k>0} \mathbb{R}$$

satisfies

$$i^*_{\xi\Delta}\,\mathrm{pol} = (\mathcal{L}(\xi\Delta, k))_{k>0}$$

This is a generalization of the result of Beilinson-Deligne for the case $F = \mathbb{Q}$.

Conjecture

$$\mathsf{pol} \longmapsto (c_k(\xi\Delta))$$

$$\begin{array}{ccc}
H^{2g-1}_{\mathrm{mot}}(U/F_+^\times, \mathbb{L}\mathrm{og}) & \xrightarrow{i^*_{\xi\Delta}} & \prod_{k>0} H^{2g-1}_{\mathrm{mot}^I}(\xi\Delta/\Delta, \mathbb{R}(k)) \\
\downarrow r_{\mathscr{D}} & & \downarrow r_{\mathscr{D}} \\
H^{2g-1}_{\mathscr{D}}(U/F_+^\times, \mathbb{L}\mathrm{og}) & \xrightarrow{i^*_{\xi\Delta}} & \prod_{k>0} H^{2g-1}_{\mathscr{D}^I}(\xi\Delta/\Delta, \mathbb{R}(k))
\end{array}$$

$$\mathsf{pol} \overset{?}{\longmapsto} (\mathcal{L}(\xi\Delta, k))$$

Conclusion

There exists an isomorphism

$$H^{2g-1}_{\mathscr{D}^I}(\xi\Delta/\Delta, \mathbb{R}(gk)) \cong \bigwedge^{g} H^1_{\mathscr{D}}(\xi, \mathbb{R}(k)).$$

If we can further prove that:

- The construction of the equivariant polylogarithm is motivic
- There are motivic version of the plectic specialization maps
- Everything is functorial, i.e. the diagrams are commutative

Then for Hecke character χ which is totally non-critical,

Conjecture $\Rightarrow$ Beilinson conjecture for Hecke L-function of χ

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