A RTF Approach to Unitary Friedberg-Jacquet Periods.

Jingwei Xiao

Joint work (in progress) with Wei Zhang

PANT Conference, Dec 9th, 2021

Motivations.

Automorphic Periods.

- F number field, G reductive group over F, $[G] = G(F)\backslash G(\mathbb{A}_F)$.
- H subgroup of G.
- π cuspidal automorphic representation of G.
- Automorphic periods:

$$\mathcal{P}_H(\phi) := \int_{[H]} \phi(h)dh.$$

- π is called H-distinguished if there exists $\phi \in \pi$ such that $\mathcal{P}_H(\phi) \neq 0$. This condition is often related to values of L-functions or functorial image from other groups.

Automorphic Periods.

- Waldspurger formula: K quadratic extension of F. Take $(G, H) = (\mathrm{GL}_{2,F}, K^\times)$, π cuspidal automorphic on G, $\phi \in \pi$:

$$|\mathcal{P}_H(\phi)|^2 \sim L(\mathsf{BC}(\pi), \frac{1}{2}).$$

- GGP: K/F as above. Take $(G, H) = (\mathrm{U}(n) \times \mathrm{U}(n+1), \mathrm{U}(n))$, $\pi = \pi_n \times \pi_{n+1}$ cuspidal automorphic rep on G, $\phi \in \pi$:

$$|\mathcal{P}_H(\phi)|^2 \sim L(\mathsf{BC}(\pi), \frac{1}{2}) := L(\mathsf{BC}(\pi_n) \times \mathsf{BC}(\pi_{n+1}), \frac{1}{2}).$$

Automorphic Periods.

- In this talk, we consider:

$$(G, H) = (\mathrm{U}(2n), \mathrm{U}(n) \times \mathrm{U}(n)).$$

Conjecture (Zhang-X.)

Let π be tempered cuspidal on G and let Π be base change of π to $\mathrm{GL}_{2n,F}$. The following are equivalent:

- *π is H-distinguished.*
- $\mathrm{Hom}_{H(\mathbb{A}_{F_0})}(\pi, \mathbb{C}) \neq 0$, *$L(\Pi, \wedge^2, s)$ has a pole at $s = 1$ and $L(\Pi, \frac{1}{2}) \neq 0$.*

Furthermore ther should be relations:

$$|\mathcal{P}_H(\phi)|^2 \sim L(\Pi, \frac{1}{2}).$$

Today, I will discuss a Relative Trace Formula (RTF) approach to this conjecture.

Special Cycles on Shimura Varieties.

Special cycles on Shimura varieties have arithmetic meaning.

- Gross-Zagier formula: $K/\mathbb{Q}$ imaginary quadratic, Heegner points $x_K \in X_0(N)$, π cuspidal automorphic rep of $\mathrm{GL}_{2,\mathbb{Q}}$ of weight 2, then

$$\langle x_{K,\pi}, x_{K,\pi} \rangle_{\mathrm{NT}} \sim L'(\mathsf{BC}(\pi), \frac{1}{2}).$$

- Arithmetic GGP Conjecture: K as above, consider $H = \mathrm{U}(n)$, $G = \mathrm{U}(n) \times \mathrm{U}(n+1)$ and $\Delta : H \to G$. Over $\mathbb{R}$ we assume they are $\mathrm{U}(1, n-1) \to \mathrm{U}(1, n-1) \times \mathrm{U}(1, n)$. Then $\mathrm{Sh}_H \to \mathrm{Sh}_G$ and $2 \dim(\mathrm{Sh}_H) + 1 = \dim(\mathrm{Sh}_G)$. One conjectures for $\pi = \pi_n \times \pi_{n+1}$ cuspidal automorphic rep of G that appearing in the cohomology $H^*(\mathrm{Sh}_G)$:

$$\langle \mathrm{Sh}_{H,\pi}, \mathrm{Sh}_{H,\pi} \rangle_{\mathsf{BB}} \sim L'(\mathsf{BC}(\pi_n) \times \mathsf{BC}(\pi_{n+1}), \frac{1}{2}).$$

Special Cycles on Shimura Varieties.

- Let K as above and consider $(G, H) = (\mathrm{U}(2n), \mathrm{U}(n) \times \mathrm{U}(n))$. Over $\mathbb{R}$ we assume that they are giving by

$$\mathrm{U}(1, n-1) \times \mathrm{U}(0, n) \to \mathrm{U}(1, 2n-1).$$

We have $2\dim(\mathrm{Sh}_H) + 1 = \dim(\mathrm{Sh}_G)$. For π cuspidal automorphic rep of G that appearing in the cohomology $H^*(\mathrm{Sh}_G)$, we expect:

"Conjecture" (Zhang-X.)

$$\langle \mathrm{Sh}_{H,\pi}, \mathrm{Sh}_{H,\pi} \rangle_{\mathsf{BB}} \sim L'(\mathsf{BC}(\pi), \frac{1}{2}).$$

Twists.

Besides $(\mathrm{U}(2n), \mathrm{U}(n) \times \mathrm{U}(n))$, we can also consider (G, H) that are some inner twists.

- F/F_0 quadratic, B simple central algebra over F with an involution $* : B \to B$ whose restriction on F is non-trivial. Let
$$G = \{b \in B^\times | bb^* = 1\}.$$
- E_0/F_0 quadratic with an embedding $E_0 \to B$ such that $*$ fixes E_0. Let
$$H = \{b \in (B^{E_0})^\times | bb^* = 1\},$$
where B^{E_0} is the centralizer of E_0 in B.

We can formulate similar conjectures on periods and heights for any (G, H) as above. For the height, one requires F/F_0 to be CM and E_0 to be totally real.

Special Cycles on Shimura Varieties.

- Locally, at almost all the places (G_v, H_v) is given by one of the following

$$(\mathrm{GL}_{2n}, \mathrm{GL}_n \times \mathrm{GL}_n), (\mathrm{GL}_{2n}, \mathrm{GL}_{n,M}),$$

$$(\mathrm{U}_{2n}, \mathrm{U}_n \times \mathrm{U}_n), (\mathrm{U}_{2n}, \mathrm{GL}_{n,M}).$$

  $M =$ Unramified quadratic extension. The situation is a hybrid of Fridberg-Jacquet, unitary Fridberg-Jacquet, Jacquet-Guo.
- The value $L'(\mathsf{BC}(\pi), \frac{1}{2})$ has been studied extensively in Kulda's program using arithmetic Theta correspondence. The differences are we allow Shimura varieties that are inner form of $\mathrm{U}(2n)$ and for each Shimura variety, by allowing bi-quadratic, we obtain some new cycles on it.

A Relative Trace Formula Approach.

FJ Periods.

Let $(G,H) = (\mathrm{U}(2n), \mathrm{U}(n) \times \mathrm{U}(n))$. To study (G,H) periods, it is often useful to look at its base change, which is isomorphic to $(\mathrm{GL}_{2n}, \mathrm{GL}_n \times \mathrm{GL}_n)$, these periods are studied by Friedberg-Jacquet.

Theorem (Friedberg-Jacquet, 1993)

Let π be an cuspidal automorphic rep of GL_{2n} with trivial central character. Then the following are equivalent.

- *There exists $\phi \in \pi$ such that $\mathcal{P}_{\mathrm{GL}_n \times \mathrm{GL}_n}(\phi) \neq 0$.*
- *$L(\pi, \wedge^2, s)$ has a pole at 1 and $L(\pi, \frac{1}{2}) \neq 0$.*

FJ Periods.

The above theorem is motivated by the Bump-Friedberg integral:

- Let $\Phi \in C_c^\infty(\mathbb{A}^n)$, define Eisenstein series on GL_n

$$E(h, \Phi, s, \eta) = |h|^s \eta(h) \int_{[F^\times]} \sum_{v \in F^n - \{0\}} \Phi(avh)|a|^{ns}\eta(a)^n da.$$

  This is basically the theta series for GL_n acting on F^n.
- Then Bump-Friedberg integral. Take $\phi \in \pi$

$$\int_{[\mathrm{GL}_n \times \mathrm{GL}_n]} \phi \begin{pmatrix} h_1 & \\ & h_2 \end{pmatrix} |h_1 h_2^{-1}|^{s'} E(h_2, \Phi, s, \eta) dh_1 dh_2$$

  equals $L(\pi, \wedge^2 \otimes \eta, s+1) L(\pi, s' + \frac{1}{2}) \times$ Local terms.

The FJ periods is related to $\eta \equiv 1$, taking $\mathrm{Res}_{s=0}$ and $s' = 0$.

RTF: Unitary Side.

Let $(G, H) = (\mathrm{U}(2n), \mathrm{U}(n) \times \mathrm{U}(n))$. Let $f \in C_c^\infty(G(\mathbb{A}))$, and define the kernel function

$$k_f(g, h) = \sum_{x \in G(F)} f(g^{-1}xh).$$

For any $\phi \in L^2([G])$, we have

$$(\pi(f)\phi)(g) := \int_{G(\mathbb{A})} f(h)\phi(gh)dh = \int_{[G]} k_f(g, h)\phi(h)dh.$$

This shows

$$k_f(g, h) \approx \sum_{\pi} \sum_{\phi \in \mathsf{OB}(\pi)} (\pi(f)\phi)(g)\overline{\phi(h)}.$$

RTF: Unitary Side.

The RTF for (G, H) is obtained by consider the following integral:

$$J(f) = \int_{[H]} \int_{[H]} k_f(h_1, h_2) dh_1 dh_2.$$

It admits two expansions

$$\sum_{\pi} J_\pi(f) = J(f) = \sum_{\gamma \in H \backslash G(F_0)/H} J_\gamma(f),$$

with

$$J_\pi(f) = \sum_{\phi \in \mathsf{OB}(\pi)} \mathcal{P}_H(\pi(f)\phi) \overline{\mathcal{P}_H(\phi)},$$

$$J_\gamma(f) = \text{integral of } f \text{ on } H(\mathbb{A})\gamma H(\mathbb{A}).$$

RTF: Linear Side.

Let $(G', H') = (\mathrm{GL}_{2n,F}, \mathrm{GL}_{n,F} \times \mathrm{GL}_{n,F})$. Let $f' \in C_c^\infty(G'(\mathbb{A}))$, and define the kernel function

$$k_{f'}(g, h) = \sum_{x \in G'(F)} f'(g^{-1}xh).$$

Let $\Phi \in C_c^\infty(\mathbb{A}^n)$ and recall

$$E(g, \Phi, s, \eta) := |g|^s \eta(g) \int_{[F^\times]} \sum_{v \in F^n - \{0\}} \Phi(avg)|a|^{ns}\eta^n(a)da.$$

Consider the following

$$I(f', \Phi, s) = \int_{[H' \times H']} k_{f'}(h_1, h_2) E(h_1^{(2)}, \Phi, s, \eta)|h_1 h_2|^s \eta(h_2) dh_1 dh_2.$$

Here $\eta =$ the quadratic character associate to K/F.

RTF: Linear Side.

$I(f', \Phi, s)$ admits two expansions:

- $I(f', \Phi, s) = \sum I_{\pi'}(f', \Phi, s)$, where

  $$I_{\pi'}(f', \Phi, s) \sim L(\Pi, s + \frac{1}{2}) L(\pi', \wedge^2 \otimes \eta, s+1) \operatorname{Res}_{s=0} L(\pi', \wedge^2, s+1).$$

  Here $\Pi = \mathrm{BC}_K(\pi')$. This uses Bump-Friedberg.
- In general, if a group G acts on a variety X, $\phi \in C_c^\infty(X(\mathbb{A}))$, define theta series $\theta_\phi(g) = \sum_{x \in X(F)} \phi(g^{-1}x)$, then $\int_{[G]} \theta_\phi(g) dg$ can be expanded as orbital integrals of ϕ.
- $I(f', \Phi, s)$ corresponds to $H' \times H'$ acting on $G' \times F^n$, where only the second factor of the first H' acts on F^n. Therefore

  $$I(f', \Phi, s) = \sum_\gamma I_\gamma(f', \Phi, s).$$
- To consider periods, set $s = 0$. To consider derivative, set $\frac{d}{ds}|_{s=0}$.

Comparison.

We compare I and J:

$$\sum_{\pi} J_{\pi}(f) = J(f) = \sum_{\delta} J_{\delta}(f),$$

$$\sum_{\pi'} I_{\pi'}(f', \Phi) = I(f', \Phi) = \sum_{\gamma} I_{\gamma}(f', \Phi).$$

The approach to establish the conjecture will be the following:

- Identify $\delta \longrightarrow \gamma$ and prove given f, there exists (f', Φ) such that $I_{\gamma}(f', \Phi) = J_{\delta}(f)$.
- Show the matching above satisfies the fundamental lemma.

These will imply $J(f) = I(f', \Phi)$ for many choices of test functions.Then separate the desired relation $J_{\pi}(f) = I_{\pi'}(f', \Phi)$.

Two variants.

- There is a variant for the twisted (G,H) where we change the GL side in the following way:

$$I(f', \Phi, s) = \int_{[H']} \int_{[H']} k_f(h_1, h_2) |h_1 h_2|^s E(h_1^{(2)}, \Phi, s, \eta_F) \eta_{F'}(h_2) dh_1 dh_2$$

- There is an arithmetic version for the derivative. We consider on the GL side $I'(f', \Phi, 0)$ and on the unitary side $\langle \pi(f)\mathrm{Sh}_H, \mathrm{Sh}_H \rangle$.

Local Results.

Main Results.

Theorem (Zhang-X.)

Fundamental lemma holds for the comparisons above.

This includes the case $(\mathrm{U}(2n), \mathrm{U}(n) \times \mathrm{U}(n))$, its twisted version and the arithmetic version. The twisted case involves a similar argument and for the arithmetic version, the additoinal input is the local Kulda-Rapoport conjecture proved by Li-Zhang. I will focus on the case $(\mathrm{U}(2n), \mathrm{U}(n) \times \mathrm{U}(n))$ afterwards.

Backgrounds: Hironaka's transform.

- Let $\mathcal{H}_K$ and $\mathcal{H}_F$ be the spherical Hecke algebra of $\mathrm{GL}_{n,K}$ and $\mathrm{GL}_{n,F}$, $\mathcal{H}_F$ is a free $\mathcal{H}_K$-module via base change.
- Let H_n be the space of $n \times n$ non-degenerate Hermitian matrices. Then $\mathrm{GL}_{n,K}$ acts on H_n and the space of sphercial functions $C_c^\infty(\mathsf{H}_n)^{\mathrm{GL}_n(\mathcal{O}_K)}$ is also an $\mathcal{H}_K$-module.

Theorem (Y. Hironaka)

There is a canonical isomorphism

$$\mathsf{Hir} : C_c^\infty(\mathsf{H}_n)^{\mathrm{GL}_n(\mathcal{O}_K)} \to \mathcal{H}_F$$

compatible with the $\mathcal{H}_K$-module structure.

A Local Identity.

- Let $f' \otimes \Phi \in C_c^\infty(\mathrm{GL}_n(F) \times F^n), (\gamma, v) \in \mathrm{GL}_n(F) \times F^n$, define

$$\mathrm{Orb}(f' \otimes \Phi, (\gamma, v)) = \int_{\mathrm{GL}_n(F)} f'(g\gamma g^{-1})\Phi(gv)\eta(g)dg.$$

- Let $f \in C_c^\infty(\mathrm{H}_n), \delta \in \mathrm{H}_n$, define

$$\mathrm{Orb}^{st}(f, \delta) = \sum_{\delta' \sim \delta} \int_{\mathrm{U}_n/T_{\delta'}} f(g.\delta')dg.$$

Theorem (Zhang-X.)

For any $f \in C_c^\infty(\mathrm{H}_n)^K$ and matching orbits $\delta \sim (\gamma, v)$. We have

$$\mathrm{Orb}^{st}(f, \delta) = \Delta(\delta, (\gamma, v))\,\mathrm{Orb}(\mathsf{Hir}(f) \otimes 1_{\mathcal{O}_F^n}, (\gamma, v)).$$

We will show our fundamental lemma reduces to this.

The proof.

Theorem (Zhang-X.)

Fundamental lemma holds for the comparisons above.

This roughly means over any local places, we have $J_\delta(f_0') = I_\gamma(f_0, \Phi_0)$ for certain particular choices of f_0', f_0 and Φ_0.
Reduction:

- Identify $\delta \longrightarrow \gamma$ for regular semisimple orbits.
- γ are orbits for $(G', H') = (\mathrm{GL}_{2n}, \mathrm{GL}_n \times \mathrm{GL}_n)$, $H' \times H'$ acting on $G' \times F^n$. This reduces to the action of GL_n on $\mathrm{GL}_n \times F^n$.
- δ are orbits for $(G, H) = (\mathrm{U}(2n), \mathrm{U}(n) \times \mathrm{U}(n)), H \times H$ acts on G. This reduces to the action of $\mathrm{U}(n)$ on H_n.

The proof.

Theorem (Zhang-X.)

For any $f \in C_c^\infty(\mathrm{H}_n)^K$ and matching orbits $\delta \sim (\gamma, v)$. We have

$$\mathrm{Orb}^{st}(f, \delta) = \Delta(\delta, (\gamma, v))\, \mathrm{Orb}(\mathsf{Hir}(f) \otimes 1_{\mathcal{O}_F^n}, (\gamma, v)).$$

- We first prove this when f is the unit function: $f = \mathrm{H}_n(\mathcal{O}_F)$. This is deduced from my thesis.
- Global method. We globalize the situation and consider the comparison of the following RTFs:
  - $(\mathrm{GL}_{n,K}, \mathrm{U}_n, \mathrm{U}_n)$.
  - $(\mathrm{GL}_{n,F} \times \mathrm{GL}_{n,F}, \Delta(\mathrm{GL}_{n,F}), \Delta(\mathrm{GL}_{n,F}))$ with a weight factor $E(g, \Phi, \eta, s)$ on one of the $\Delta(\mathrm{GL}_{n,F})$.

  The spectral sides are well understood (Feigon-Lapid-Offen, Jacquet-Ye). This gives identities on the spectral side. We use that to deduce identities on the geometric side.

Thank you!