Period Relations for Standard L-functions of Symplectic Type

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Overview

Results of Leibniz and Euler

- Leibniz (1674):

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^{(n \mod 2)}}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \cdots .$$

- Euler (1734, published in 1740):

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots ;$$

$$\frac{\pi^4}{90} = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots ;$$

Results of Leibniz and Euler

- Euler (1734, published in 1740):

$$\frac{\pi^2}{8} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots .$$

$$\frac{\pi^3}{32} = \sum_{n=0}^{\infty} \frac{(-1)^{(n \mod 2)}}{(2n+1)^3} = \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \cdots .$$

$$\frac{\pi^4}{96} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots .$$

$$\frac{5\pi^5}{1536} = \sum_{n=0}^{\infty} \frac{(-1)^{(n \mod 2)}}{(2n+1)^5} = \frac{1}{1^5} - \frac{1}{3^5} + \frac{1}{5^5} - \cdots .$$

Find a suitable 'generating function' and do Fourier expansion.

Example

One can study the Fourier expansion of $\frac{\pi-x}{2}$ and get the Leibniz series.

Riemann Zeta Function and Dirichlet L-Function

B. Riemann (1859):

$$\zeta(s) := \sum_{n=1}^{+\infty} \frac{1}{n^s} = \prod_{\text{prime } p} (1 - p^{-s})^{-1}.$$

P. Dirichlet (1837):

$$L(s, \chi) = \sum_{n=1}^{+\infty} \frac{\chi(n)}{n^s} = \prod_{\text{prime } p} (1 - \chi(p) p^{-s})^{-1},$$

where χ is a Dirichlet character modulo N.

Analytic Property

Absolute convergence for $\mathrm{Re}(s) > 1$; meromorphic continuation to $s \in \mathbb{C}$; functional equation relating s and $1 - s$.

Reformulation of Leibniz's and Euler's Results

Theorem

- $\zeta(2k)$ $(k = 1, 2, 3 \cdots)$ *are rational multiples of* π^{2k}.
- *For the even primitive Dirichlet character* χ *modulo* 4, $L(2k, \chi)$ $(k = 1, 2, 3, \cdots)$ *are rational multiples of* π^{2k}.
- *For the odd primitive Dirichlet character* χ *modulo* 4, $L(2k-1, \chi)$ $(k = 1, 2, 3, \cdots)$ *are rational multiples of* π^{2k-1}.

Special Values of Dirichlet L-function

- $L(1-k,\chi) = -\frac{B_{k,\chi}}{k}$. Here $B_{k,\chi}$ are generalized Bernoulli numbers defined by
$$\sum_{j=1}^{N} \chi(j) \cdot \frac{te^{jt}}{e^{Nt}-1} = \sum_{k=0}^{\infty} B_{k,\chi} \cdot \frac{t^k}{k!}.$$
- Then one has the special values $L(k,\chi)$ via functional equation of $L(s,\chi)$. Guass sum will show up then.

- k: positive integer
- χ : Dirichlet character modulo N, $\chi(-1) = (-1)^k$.
- Modular form of weight k and level N:

$$f(\gamma.z) = \chi(d)(cz+d)^k f(z)$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ runs over the group

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,|\, c \equiv 0 (\mod N) \right\}.$$

Cusp Form and Fourier Expansion

- $f(z) = \sum_{n=-m}^{+\infty} a_n e^{2\pi i n z}$
- holomorphic modular form: $f(z) = \sum_{n=0}^{+\infty} a_n e^{2\pi i n z}$
- cusp form: $f(z) = \sum_{n=1}^{+\infty} a_n e^{2\pi i n z}$.

Examples

- Given a complex number τ with $\mathrm{Im}\tau > 0$, the holomorphic Eisenstein series
$$E_{2k}(z) := \sum_{(m,n)\in\mathbb{Z}^2-\{(0,0)\}} \frac{1}{(m+n\tau)^{2k}}$$
is a holomorphic modular form of weight $2k$ ($k \geq 2$).
- Set $q = e^{2\pi iz}$. The modular discriminant (Ramanujan)
$$\Delta(z) := q\prod_{n=1}^{\infty}(1-q^n)^{24}$$
is a cusp form of weight 12.

G. Shimura's Work on GL_2

Dictionary

- $S_k(N, \chi)$: space of cusp forms of weight k and level N.
- $f(z) \in S_k(N, \chi)$, Fourier expansion $f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi \mathrm{i} n z}$.
- $\sigma \in \mathrm{Aut}(\mathbb{C})$, $f^{\sigma}(z) := \sum_{n=1}^{\infty} a_n^{\sigma} e^{2\pi \mathrm{i} n z}$
- ψ: another primitive Dirichlet character.
- $\mathcal{G}(\psi)$: Gauss sum.
- Set 'the L-function' $L(s, f \otimes \psi) := \sum_{n=1}^{\infty} \psi(n) a_n n^{-s}$. This is the $D(s, f, \psi)$ in Shimura's paper (1976).

G. Shimura's Work on GL_2

We formulate a 'version' of Shimura's theorems on the scenario of GL_2.

Theorem (Shimura(1976,1978))

There exists $\Omega_{f^\sigma} \in \mathbb{C}^\times$ *such that*

$$\sigma\left(\frac{L(j, f \otimes \psi)}{(2\pi \mathrm{i})^j \cdot \mathcal{G}(\psi) \cdot \Omega_f}\right) = \frac{L(j, f^\sigma \otimes \psi^\sigma)}{(2\pi \mathrm{i})^j \cdot \mathcal{G}(\psi^\sigma) \cdot \Omega_{f^\sigma}}$$

for all positive integers $j < k$ *and all* ψ *such that* $\psi_\infty(-1) = (-1)^j$.

A Corollary

Corollary

Set $\mathbb{Q}(f)$ to be the field generated by all Fourier coefficients a_n and $\mathbb{Q}$; $\mathbb{Q}(\psi)$ to be field generated by all $\psi(n)$ and $\mathbb{Q}$. Define $\mathbb{Q}(f,\psi) := \mathbb{Q}(f)\mathbb{Q}(\psi)$. Then exists $\Omega_f \in \mathbb{C}^\times$ such that

$$\frac{L(j, f \otimes \psi)}{(2\pi \mathrm{i})^j \cdot \mathcal{G}(\psi) \cdot \Omega_f} \in \mathbb{Q}(f,\psi)$$

for all positive integers $j < k$ and all ψ such that $\psi_\infty(-1) = (-1)^j$.

What's next?

- Cusp form $f \leftrightarrow$ cuspidal representation Π.
- L-function: Shimura's $L(s, f \otimes \psi) \leftrightarrow$ usual automorphic L-function $L(s - \frac{1}{2}, \Pi \otimes \psi)$.
- Π_∞ is a discrete series, hence cohomological.

Study the higher degree L-function via cohomology method.

Automorphic L-Function for GL_m

- k : number field
- $\mathbb{A}$: ring of adeles
- $\Pi = \hat{\otimes}'_{\nu}\Pi_{\nu}$: irreducible smooth automorphic representation of $\mathrm{GL}_m(\mathbb{A})$.
- r: a finite dimensional representation of $\mathrm{GL}_m(\mathbb{C})$, the dual group of GL_m.
- Complete automorphic L-function

$$L(s, \Pi, r) := \prod_{\nu} L_{\nu}(s, \Pi_{\nu}, r).$$

Analytic Properties of Standard L-Function

Theorem (Tate (1950) for $m = 1$; Godement-Jacquet (1972) for $m > 1$)

Let χ be an automorphic character of $\mathrm{k}^\times \backslash \mathbb{A}^\times$. Then $L(s, \Pi \otimes \chi)$ satisfies the following properties:

- *$L(s, \Pi \otimes \chi)$ has a meromorphic continuation to $s \in \mathbb{C}$ (holomorphic when $m > 1$)*
- *It satisfies the functional equation*

$$L(s, \Pi \otimes \chi) = \epsilon(s, \Pi \otimes \chi) L(1 - s, \Pi^\vee \otimes \chi^{-1}).$$

Langlands Reciprocity Conjecture for GL_n

Langlands Reciprocity Conjecture

There exists a one-to-one correspondence between irreducible motives over k of rank m with coefficients in $\overline{\mathbb{Q}}$ and irreducible algebraic cuspidal automorphic representations of $\mathrm{GL}_m(\mathbb{A})$, which respects L-functions.

Deligne Conjecture

Deligne Conjecture (1979)

Let M be a critical motive over a number field k, with coefficients in a number field E. Let $c^+ \in E \otimes \mathbb{C}$ be the determinant of the period map, which is defined up to multiplication by $E^\times$. Then we have $\frac{L(0,M)}{c^+} \in E$.

In this talk, we aim to attack the analogue of Deligne's Conjecture for automorphic L-function (due to D. Blasius, 1997).

Essentially Self-Dual Representations

It is expected that most motives are essentially self-dual, hence we consider irreducible algebraic cuspidal automorphic representation Π which is essentially self-dual in the sense that

$$\Pi^\vee \simeq \Pi \otimes \eta^{-1}$$

for some automorphic character $\eta : F^\times \backslash \mathbb{A}^\times \to \mathbb{C}^\times$.

$$\Pi^\vee \simeq \Pi \otimes \eta^{-1} \to \begin{cases} L(s, \Pi, \mathrm{Sym}^2 \otimes \eta^{-1}) \text{ has a simple pole at } s = 1 \\ L(s, \Pi, \bigwedge^2 \otimes \eta^{-1}) \text{ has a simple pole at } s = 1 \end{cases}$$

Only focus on the second case.

Twisted Exterior Square L-function

Theorem (Jacquet-Shalika (1988), Asgari-Shahidi (2006,2011), Hundley-Sayag (2009))

$$
\begin{aligned}
&L(s, \Pi, \bigwedge^2 \otimes \eta^{-1}) \textit{ has a simple pole at } s = 1 \\
\Leftrightarrow &\Pi \textit{ has a nonzero } (\eta, \psi)\textit{-Shalika period} \\
\Leftrightarrow &m = 2n \textit{ and } \Pi \textit{ is a functorial transfer from } \mathrm{GSpin}_{2n+1}.
\end{aligned}
$$

If so, the transfer respects local L-parameter at each archimedean local places.

- $G := \mathrm{GL}_{2n}$
- S: Shalika subgroup of G,

$$S := \left\{ \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix} \begin{pmatrix} 1_n & x \\ 0 & 1_n \end{pmatrix} \mid h \in \mathrm{GL}_n, x \in \mathrm{Mat}_n \right\}.$$

- Z_{2n}: center of G
- ψ: non-trivial unitary character of $\mathrm{k}\backslash\mathbb{A}$.
- $\eta \otimes \psi$: character of $S(\mathbb{A})$,

$$(\eta \otimes \psi)(\begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix} \begin{pmatrix} 1_n & x \\ 0 & 1_n \end{pmatrix}) = \eta(\det h)\psi(\mathrm{Tr}(x)).$$

Global Shalika Period

Definition

We say that an irreducible cuspidal automorphic representation Π of $G(\mathbb{A})$ has a nonzero (η, ψ)-Shalika period if its central character equals η^n, and there exists $\varphi \in \Pi$ such that

$$\int_{Z_{2n}(\mathbb{A})S(\mathrm{k})\backslash S(\mathbb{A})} \varphi(g)(\eta \otimes \psi)^{-1}(g)dg \neq 0.$$

Archimedean Local Langlands Correspondence for GL_m

- $\mathbb{K}$: an archimedean local field.
- $\mathcal{E}_{\mathbb{K}}$: the set of continuous field embeddings of $\mathbb{K}$ into $\mathbb{C}$.
- $W_{\mathbb{K}}$: the Weil group of $\mathbb{K}$

$$W_{\mathbb{K}} := \begin{cases} \overline{\mathbb{K}}^{\times} \sqcup \mathrm{j} \cdot \overline{\mathbb{K}}^{\times}, & \text{if } \mathbb{K} \simeq \mathbb{R}; \\ \mathbb{K}^{\times}, & \text{if } \mathbb{K} \simeq \mathbb{C}, \end{cases}$$

Theorem (Special Case of Langlands 1989)

One-to-one correspondence:

$$\begin{aligned} &\{\textit{irreducible Casselman-Wallach representations of } \mathrm{GL}_m(\mathbb{K})\}/\sim \\ \leftrightarrow &\{\textit{completely reducible } m\textit{-dimensional representations of } W_{\mathbb{K}}\}/\sim \end{aligned}$$

Critical Places

Definition

- Let $\Pi_{\mathbb{K}}$ be an algebraic irreducible Casselman-Wallach representation of $\mathrm{GL}_m(\mathbb{K})$. A number in $\frac{m-1}{2}+\mathbb{Z}$ is called a **critical place** for $\Pi_{\mathbb{K}}$ if it is not a pole of the local L-function $L(s,\Pi_{\mathbb{K}})$ or $L(1-s,\Pi_{\mathbb{K}}^{\vee})$.
- Given an algebraic irreducible cuspidal automorphic representation $\Pi=\widehat{\otimes}'_{\nu}\Pi_{\nu}$ of $\mathrm{GL}_m(\mathbb{A})$, a number in $\frac{m-1}{2}+\mathbb{Z}$ is called a critical place for Π if it is a critical place for Π_{ν} for all $\nu \mid \infty$.

Regular Algebraic Representations of $W_{\mathbb{K}}$

- $\mathcal{E}_{\overline{\mathbb{K}}} = \{\iota, \bar{\iota}\}$.
- Every completely reducible finite dimensional representation ρ of $\overline{\mathbb{K}}^{\times}$ has the form

$$\iota^{a_1}\bar{\iota}^{b_1} \oplus \iota^{a_2}\bar{\iota}^{b_2} \oplus \cdots \oplus \iota^{a_m}\bar{\iota}^{b_m}, \quad (m \geq 0,\, a_i, b_i \in \mathbb{C},\, a_i - b_i \in \mathbb{Z}),$$

where $\iota^{a}\bar{\iota}^{b}$ is the character

$$z \mapsto (\iota(z))^{a-b}(\iota(z)\bar{\iota}(z))^{b},$$

of $\overline{\mathbb{K}}^{\times}$, for all $a, b \in \mathbb{C}$ with $a - b \in \mathbb{Z}$.

Regular Algebraic Representations of $W_{\mathbb{K}}$

Definition

- Representation ρ of $\overline{\mathbb{K}}^{\times}$ is **algebraic**: if all a_i's and b_i's are integers.
- Representation ρ of $\overline{\mathbb{K}}^{\times}$ is **regular**: if a_i's are pairwise distinct, and b_i's are also pairwise distinct.
- A completely reducible finite dimensional representation ρ of the Weil group $W_{\mathbb{K}}$ is **algebraic** (or **regular**): if so is $\rho|_{\overline{\mathbb{K}}^{\times}}$.

Regular Algebraic Representations of GL_m

Definition

- An irreducible Casselman-Wallach representation $\Pi_{\mathbb{K}}$ of $\mathrm{GL}_m(\mathbb{K})$ is said to be **algebraic** (or **regular**) if so is the Langlands parameter of $\Pi_{\mathbb{K}} \otimes |\det|_{\mathbb{K}}^{\frac{1-m}{2}}$.
- An irreducible cuspidal automorphic representation $\Pi = \widehat{\otimes}'_\nu \Pi_\nu$ of $\mathrm{GL}_m(\mathbb{A})$ is said to be **algebraic** (or **regular**) if so is Π_ν for all $\nu \mid \infty$.

Theorem of L. Clozel

Theorem (Clozel (1990))

Let $\Pi = \Pi_f \otimes \Pi_\infty$ *be an irreducible regular algebraic cuspidal representation of* GL_m. *Then* Π_∞ *is essentially tempered, and cohomological in the following sense: there is a unique irreducible finite dimensional algebraic representation* F *of* $\mathrm{GL}_m(\mathrm{k} \otimes_{\mathbb{Q}} \mathbb{C})$, *called the* **coefficient system** *of* Π, *such that the total continuous cohomology*

$$\mathrm{H}^*_{\mathrm{ct}}(\mathrm{GL}_m(\mathbb{A}_\infty)^0; \Pi_\infty \otimes F^\vee) \neq 0.$$

Dictionary:

- $\mathbb{K}$: archimedean local field.
- $\Pi_{\mathbb{K}}$: essentially tempered cohomological representation of $G_{\mathbb{K}} := \mathrm{GL}_{2n}(\mathbb{K})$ of symplectic type.
- $F_{\mathbb{K}} = \otimes_{\iota \in \mathcal{E}_{\mathbb{K}}} F_{\iota}$: coefficient system.
- Highest weight of F_{ι}:

$$\nu_{\iota} = (\nu_1^{\iota} \geq \nu_2^{\iota} \geq \cdots \geq \nu_{2n}^{\iota}) \in \mathbb{Z}^{2n}$$

.

Proposition

There exist integers $\{w_{\iota}\}_{\iota \in \mathcal{E}_{\mathbb{K}}}$ such that

$$\nu_1^{\iota} + \nu_{2n}^{\iota} = \nu_2^{\iota} + \nu_{2n-1}^{\iota} = \cdots = \nu_{2n}^{\iota} + \nu_1^{\iota} = w_{\iota} \qquad \text{for all } \iota \in \mathcal{E}_{\mathbb{K}}.$$

Balanced Coefficient System

Definition

For an integer j, we say that the coefficient system $F_{\mathbb{K}} = \otimes_{\iota \in \mathcal{E}_{\mathbb{K}}} F_\iota$ is j-balanced if

$$\mathrm{Hom}_{\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})}(F_\iota^\vee, \det^j \otimes \det^{-j-w_\iota}) \neq 0 \qquad \text{for all } \iota \in \mathcal{E}_{\mathbb{K}}.$$

We say the coefficient system $F_{\mathbb{K}}$ is **balanced** if it is j-balanced for some integer j.

- Balanced coefficient system $\Rightarrow$ existence of critical place.
- In many cases, for example when k has at least one real place, critical place $\Rightarrow$ balanced coefficient system.
- For the coefficient system of the global representation Π, we can define balanceness in the same way.

Notations and Assumptions

- Π: irreducible regular algebraic cuspidal representation of $\mathrm{GL}_{2n}(\mathbb{A})$ (hence cohomological).
- Assume that the coefficient system of Π is balanced.
- $\frac{1}{2} + j \in \frac{1}{2} + \mathbb{Z}$: an arbitrary critical place of Π (existence is guaranteed).
- $\mathbb{Q}(\Pi) = \mathbb{Q}(\Pi_f)$ rationality field of Π in the sense of L.Clozel. Number field.
- η: automorphic character of $\mathrm{k}^\times \backslash \mathbb{A}^\times$ such that $L(s, \Pi, \bigwedge^2 \otimes \eta^{-1})$ has a simple pole at $s = 1$.
- $\mathbb{Q}(\eta)$: rationality field of η. This is a number field, since η is algebraic.

Notations and Assumptions

- sgn: the unique nontrivial quadratic character for a Lie group with two connected components.
- χ: an arbitrary automorphic character $\mathrm{k}^\times \backslash \mathbb{A}^\times$ of finite order such that $\chi_\nu = \mathrm{sgn}^j$ for every real place ν of k.
- $\mathbb{Q}(\chi)$: rationality field of χ. This is a number field.
- $\mathbb{Q}(\Pi, \chi, \eta) := \mathbb{Q}(\Pi)\mathbb{Q}(\eta)\mathbb{Q}(\chi)$. This is a number field.
- $\mathcal{G}(\chi)$: Guass sum.

Statement of the Main Theorem

Theorem (Jiang-Sun-Tian, Preprint (2021))

There exists a nonzero complex number Ω_Π *such that*

$$\frac{L(\frac{1}{2}+j,\Pi\otimes\chi)}{\mathrm{i}^{jn\cdot[\mathrm{k}\,:\,\mathbb{Q}]}\cdot\mathcal{G}(\chi)^n\cdot\Omega_\Pi}\in\mathbb{Q}(\Pi,\eta,\chi),$$

for every critical place $\frac{1}{2}+j$ *and every such* χ.

Reference: arXiv:1909.03476.

Comparison to other people's related work

- For fixed critical place + totally real field, the theorem is proved by Ash-Ginzburg (1994) and Grobner-Raghuram (2014)
- Assuming a key ingredient 'uniform cohomological test vector' + toally real field: F. Januszewski (preprint, 2018).

Ingredients of the Proof

- Friedberg-Jacquet Integral.
- Archimedean period relation.
- Non-archimedean period relation.
- Non-vanishing hypothesis.

Key and breakthrough: Existence of uniform cohomological test vector.

Global Friedberg-Jacquet Integral (1993)

- Π has an (η, ψ)-Shalika model.
- $H := \left\{ \begin{pmatrix} h_1 & \\ & h_2 \end{pmatrix} \mid h_1, h_2 \in \mathrm{GL}_n \right\} \simeq \mathrm{GL}_n \times \mathrm{GL}_n$.
- Global Friedberg Jacquet Integral $Z(\varphi_\Pi, s, \chi, \eta)$:

$$\int_{Z_{2n}(\mathbb{A})H(\mathrm{k})\backslash H(\mathbb{A})} \varphi_\Pi(\begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix})|\frac{\det g_1}{\det g_2}|^{s-\frac{1}{2}}\chi(\frac{\det g_1}{\det g_2})\eta^{-1}(\det g_2)dg_1dg_2.$$

Factorization of Friedberg-Jacquet Integral

- Euler Product:

$$\begin{aligned}
&Z(\varphi_\Pi, s, \chi, \eta)\\
=&\prod_\nu Z_\nu(\mathcal{S}_{\varphi_\nu}, s, \chi_\nu)\\
:=&\prod_\nu \int_{\mathrm{GL}_n(\mathrm{k}_\nu)} \mathcal{S}_{\varphi_\nu}\begin{pmatrix} g & 0 \\ 0 & 1_n \end{pmatrix} \chi_\nu(\det g)|\det g|_\nu^{s-\frac{1}{2}} dg
\end{aligned}$$

Local Friedberg-Jacquet Integral

- Define local Friedberg-Jacquet integral for a local representation Π_ν of $G(\mathrm{k}_\nu)$:

$$Z_\nu(v, s, \chi_\nu) := \int_{\mathrm{GL}_n(\mathrm{k}_\nu)} \langle \lambda_\nu, \Pi_\nu(\begin{pmatrix} g & 0 \\ 0 & 1_n \end{pmatrix})v \rangle \chi_\nu(\det g) |\det g|_\nu^{s-\frac{1}{2}} dg$$

- $\lambda_\nu \in \mathrm{Hom}_{S(\mathrm{k}_\nu)}(\Pi_\nu, \eta_\nu \otimes \psi_\nu)$ is a non-zero Shalika functional. (Uniqueness Theorem by F. Chen and B. Sun (2019))

Analytic Properties of Local Friedberg-Jacquet Integral

Theorem (Friedberg-Jacquet (1993); Aizenbud-Gourevitch-Jacquet (2009))

- $Z_\nu(v, s, \chi_\nu)$ *converges absolutely for* $\mathrm{Re}(s)$ *sufficiently large.*
- *Meromorphic continuation to a function* $s \in \mathbb{C}$ *which is a holomorphic multiple of* $L(s, \Pi_\nu \otimes \chi_\nu)$.

Define normalized local Friedberg-Jacquet integral

$$Z_\nu^\circ(v, s, \chi_\nu) := \frac{1}{L(s, \Pi_\nu \otimes \chi_\nu)} Z_\nu(v, s, \chi_\nu).$$

Existence of Uniform Cohomological Test Vector

Dictionary:

- ν: archimedean local place.
- Π_ν: essentially tempered cohomological representation of $G_\nu := \mathrm{GL}_{2n}(\mathrm{k}_\nu)$ of symplectic type.
- F_ν: coefficient system.
- K_ν: standard maximal compact subgroup of G_ν.
- τ_ν: the unique minimal K_ν-type of Π_ν (by a theorem of D. Vogan (1984)).
- $\xi_{\chi_\nu} := (\chi_\nu \circ \det) \otimes \left((\chi_\nu^{-1} \cdot \eta_\nu^{-1}) \circ \det\right)$, character of H_ν.
- $C_\nu := H_\nu \cap K_\nu$, the maximal compact subgroup of H_ν.

Existence of Uniform Cohomological Test Vector

Theorem (Jiang-Sun-Tian, Preprint)

There exists a vector $v \in \tau_\nu$ *such that* $Z_\nu^\circ(v, s, \chi_\nu) = 1$ *for all* $s \in \mathbb{C}$.

Remark

A weaker version was proved by B. Sun (2019) when ν is real.

Proof

- Construct another integral $\Lambda_\nu(v, s, \chi_\nu)$ at local archimedean places (via orbit method) and find uniform cohomological test vector for this integral (Chen-Jiang-Lin-Tian 2020 real case; Lin-Tian 2020 complex case)
- Identify $\Lambda_\nu(v, s, \chi_\nu)$ with local Friedberg-Jacquet integral (Jiang-Sun-Tian Preprint).

Dictionary:

- $\mathbb{K} = \mathrm{k}_\nu$.
- $\frac{1}{2} + j \in \frac{1}{2} + \mathbb{Z}$: critical place.
- $\xi_{\nu,j} := \otimes_{\iota \in \mathcal{E}_{\mathbb{K}}} (\det^j \otimes \det^{-j-w_\iota})$.
- $\lambda_{F_\nu, j} \in \mathrm{Hom}_{H_\nu}(F_\nu^\vee \otimes \xi_{\nu,j}^\vee, \mathbb{C}) - \{0\}$.
- sgn_ν: non-trivial quadratic character of $\mathbb{K}^\times$ in the real case; trivial character in the complex case.
- $Z_\nu^\circ(\,\cdot\,, \frac{1}{2} + j, \mathrm{sgn}_\nu^j) \in \mathrm{Hom}_{H_\nu}(\Pi_\nu \otimes \xi_{\nu,j}, \mathbb{C})$, normalized local Friedberg-Jacquet integral.
- d_ν: dimension of the modular symbol.

$$d_\nu := \begin{cases} n^2 + n - 1, & \text{if } \mathbb{K} \simeq \mathbb{R}; \\ 2n^2 - 1, & \text{if } \mathbb{K} \simeq \mathbb{C}. \end{cases}.$$

Non-vanishing Hypothesis of Archimedean Modular Symbol

Define archimedean modular symbol:

$$\begin{aligned}
\mathcal{P}_{\nu,j} \quad : \quad & \mathrm{H}^{d_\nu}_{\mathrm{ct}}(\mathbb{R}_+^\times \backslash G_\nu^0; \Pi_\nu \otimes F_\nu^\vee) \\
\to \quad & \mathrm{H}^{d_\nu}_{\mathrm{ct}}(\mathbb{R}_+^\times \backslash H_\nu^0; \Pi_\nu \otimes F_\nu^\vee) \\
= \quad & \mathrm{H}^{d_\nu}_{\mathrm{ct}}(\mathbb{R}_+^\times \backslash H_\nu^0; (\Pi_\nu \otimes \xi_{\nu,j}) \otimes (F_\nu^\vee \otimes \xi_{\nu,j}^\vee)) \\
\to \quad & \mathrm{H}^{d_\nu}_{\mathrm{ct}}(\mathbb{R}_+^\times \backslash H_\nu^0; \mathbb{C}),
\end{aligned}$$

- first arrow: restriction of cohomology.
- last arrow: induced by $Z_\nu^\circ(\,\cdot\,, \frac{1}{2} + j, \mathrm{sgn}_\nu^j) \otimes \lambda_{F_\nu, j}$.

Non-vanishing Hypothesis of Archimedean Modular Symbol

Theorem (Sun (2019); Jiang-Sun-Tian, Preprint (2019))

The archimedean modular symbol

$$\mathcal{P}_{\nu,j} : \mathrm{H}^{d_\nu}_{\mathrm{ct}}(\mathbb{R}^\times_+ \backslash G^0_\nu; \Pi_\nu \otimes F^\vee_\nu) \to \mathrm{H}^{d_\nu}_{\mathrm{ct}}(\mathbb{R}^\times_+ \backslash H^0_\nu; \mathbb{C})$$

is nonzero.

Proof

Existence of uniform cohomological test vector + Sun's technique for non-vanishing hypothesis in the real case.

Remark

As complex places are involved, this is the first case that does not have a 'degree match' condition.

Archimedean Period Relations

Theorem (Jiang-Sun-Tian, Preprint)

Under suitable normalization of $\lambda_{F_\nu,j}$, there exists $\epsilon_\nu \in \{\pm 1\}$ such that the linear map

$$\epsilon_\nu^j \cdot \mathrm{i}^{-jn\cdot[\mathbb{K}:\mathbb{R}]} \cdot \mathcal{P}_{\nu,j} : \mathrm{H}_{\mathrm{ct}}^{d_\nu}(\mathbb{R}_+^\times \backslash G_\nu^0; \Pi_\nu \otimes F_\nu^\vee) \to \mathrm{H}_{\mathrm{ct}}^{d_\nu}(\mathbb{R}_+^\times \backslash H_\nu^0; \mathbb{C})$$

is independent of the critical place $\frac{1}{2} + j$.

Key: existence of uniform cohomological test vectors.

Non-archimedean Period Relations

Now ν is non-archimedean.

Theorem (Jiang-Sun-Tian, Preprint)

There exists a unique $\mathbb{Q}(\Pi_\nu, \eta_\nu)$-rational structure on the G_ν-module Π_ν with the following property: for all $s \in \frac{1}{2} + \mathbb{Z}$, the linear functional

$$\mathcal{G}(\chi_\nu)^n \cdot Z_\nu^\circ(\,\cdot\,, s, \chi_\nu) : \Pi_\nu \otimes \xi_{\chi_\nu, s-\frac{1}{2}} \to \mathbb{C}$$

is non-zero and defined over $\mathbb{Q}(\Pi_\nu, \eta_\nu, \chi_\nu)$.

Non-archimedean Period Relations

By combining all non-archimedean local places, we have a $\mathbb{Q}(\Pi_f, \eta_f)$-rational structure on Π_f. The linear functional

$$\mathcal{G}(\chi_f)^n \cdot Z_f^\circ(\cdot, \frac{1}{2} + j, \chi_f) : \Pi_f \otimes \xi_{\chi_f, j} \to \mathbb{C}$$

is nonzero and defined over $\mathbb{Q}(\Pi_f, \eta_f, \chi_f)$.

Rational Structure of Cohomology Groups

Proposition

There exists a unique $\mathbb{Q}(\Pi, \eta)$-rational structure on

$$\mathrm{H}^{d_\infty}_{\mathrm{ct}}(\mathbb{R}^\times_+ \backslash G^0_\infty; \Pi_\infty \otimes F^\vee)$$

such that the natural isomorphism

$$\mathrm{H}^{d_\infty}_{\mathrm{ct}}(\mathbb{R}^\times_+ \backslash G^0_\infty; \Pi_\infty \otimes F^\vee) \otimes \Pi_f \xrightarrow{\sim} \mathrm{H}^{d_\infty}_{\mathrm{ct}}(\mathbb{R}^\times_+ \backslash G^0_\infty; \Pi \otimes F^\vee)$$

is defined over $\mathbb{Q}(\Pi, \eta)$.

Define

$$\mathcal{X}_G := (G(\mathrm{k})\mathbb{R}_+^\times)\backslash G(\mathbb{A})/K_\infty^0.$$

Then define the modular symbol

$$\begin{aligned}
\mathcal{P}_j :& \mathrm{H}_{\mathrm{ct}}^{d_\infty}(\mathbb{R}_+^\times\backslash G_\infty^0; \Pi \otimes F^\vee) \otimes \mathrm{H}_{\mathrm{ct}}^0(\mathbb{R}_+^\times\backslash H_\infty^0; \xi_{\chi,j} \otimes \xi_{\infty,j}^\vee) \\
\xrightarrow{\iota_\Pi \otimes \iota_j} & \mathrm{H}_{\mathrm{c}}^{d_\infty}(\mathcal{X}_G, F^\vee) \otimes \mathrm{H}^0(\mathcal{X}_H, \xi_{\infty,j}^\vee) \\
\xrightarrow{\iota^* \otimes 1} & \mathrm{H}_{\mathrm{c}}^{d_\infty}(\mathcal{X}_H, F^\vee) \otimes \mathrm{H}^0(\mathcal{X}_H, \xi_{\infty,j}^\vee) \\
\xrightarrow{\lambda_F \otimes 1} & \mathrm{H}_{\mathrm{c}}^{d_\infty}(\mathcal{X}_H, \mathbb{C}) \\
\xrightarrow{\int_{\mathcal{X}_H}} & \mathbb{C}.
\end{aligned}$$

Commutative Diagram

$$\begin{array}{ccc} \mathrm{H}_{\mathrm{ct}}^{d_\infty}(\mathbb{R}_+^\times\backslash G_\infty^0;\Pi_\infty\otimes F^\vee)\otimes(\Pi_f\otimes\xi_{\chi_f,j}) & \xrightarrow{\mathcal{P}_{\infty,j}\otimes Z_f^\circ(\,\cdot\,,\frac{1}{2}+j,\chi_f)} & \mathbb{C} \\ = \Big\downarrow & & \Big\downarrow L(\frac{1}{2}+j,\Pi\otimes\chi) \\ \mathrm{H}_{\mathrm{ct}}^{d_\infty}(\mathbb{R}_+^\times\backslash G_\infty^0;\Pi\otimes F^\vee)\otimes\mathrm{H}_{\mathrm{ct}}^0(\mathbb{R}_+^\times\backslash H_\infty^0,\xi_{\chi,j}\otimes\xi_{\infty,j}^\vee) & \xrightarrow{\mathcal{P}_j} & \mathbb{C} \end{array}$$

Proof of the Main Theorem

$$\frac{L(\frac{1}{2}+j, \Pi \otimes \chi)}{\mathrm{i}^{jn\cdot[\mathrm{k}:\mathbb{Q}]} \cdot \mathcal{G}(\chi)^n \cdot \Omega_\Pi} \in \mathbb{Q}(\Pi, \eta, \chi),$$

Proof: Compare Rational Structures.

- Archimedean period relation + Non-vanishing hypothesis produces a nonzero
$$\Omega_\Pi := \left(\epsilon^j \cdot \mathrm{i}^{-jn\cdot[\mathrm{k}:\mathbb{Q}]} \cdot \mathcal{P}_{\infty,j}([\omega])\right)^{-1},$$
where $[\omega]$ is a rational cohomological class.
- Non-archimedean period relation produces $\mathcal{G}(\chi)^n$.

Recap of Shimura's Result

Recall Shimura's result

$$\sigma\left(\frac{L(j, f \otimes \psi)}{(2\pi \mathrm{i})^j \cdot \mathcal{G}(\psi) \cdot \Omega_f}\right) = \frac{L(j, f^\sigma \otimes \psi^\sigma)}{(2\pi \mathrm{i})^j \cdot \mathcal{G}(\psi^\sigma) \cdot \Omega_{f^\sigma}}.$$

Analogously, we aim to prove

$$\sigma\left(\frac{L(\frac{1}{2} + j, \Pi \otimes \chi)}{\mathrm{i}^{jn \cdot [\mathrm{k} : \mathbb{Q}]} \cdot \mathcal{G}(\chi)^n \cdot \Omega_\Pi}\right) = \frac{L(\frac{1}{2} + j, \Pi^\sigma \otimes \chi^\sigma)}{\mathrm{i}^{jn \cdot [\mathrm{k} : \mathbb{Q}]} \cdot \mathcal{G}(\chi^\sigma)^n \cdot \Omega_{\Pi^\sigma}}.$$

Strategy

- Algebraicity of special values of L-functions come from rational structure of cohomological groups.
- Need a *suitable* way to identify the cohomology of archimedean component of Π^σ when σ varies.

Recent preprint of Li-Liu-Sun shows this kind of result for the Rankin-Selberg integrals for $\mathrm{GL}_n \times \mathrm{GL}_{n-1}$.

The End. Thank you!