Hypergeometric functions over finite fields and a Whipple formula

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- Hypergeometric Functions, How Special Are They? by Frits Beukers. (An article in Notices of the AMS)
- Hypergeometric Motives by Roberts and Rodriguez-Villegas.

Outline

Hypergeometric Series

Hypergeometric Series (HGS): Let $a_i, b_i \in \mathbb{Q}$, $z \in \mathbb{C}$.

$$ {}_nF_{n-1}\left[\begin{matrix} a_1 & a_2 & \cdots & a_n \\ & b_2 & \cdots & b_n \end{matrix} ; \ z\right] := \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_n)_k}{(1)_k (b_2)_k \cdots (b_n)_k} z^k, $$

where $(a)_0 = 1$, $(a)_n = a(a+1)\cdots(a+n-1) = \Gamma(a+n)/\Gamma(a)$.

As a function of z, $F(\alpha, \beta; z)$ satisfies an order-n ordinary differential equation

$$ [\theta(\theta + b_2 - 1)\cdots(\theta + b_n - 1) - z(\theta + a_1)\cdots(\theta + a_n)]\, F = 0, \quad \theta = z\frac{d}{dz}, $$

which is a Fuchsian equation with regular singularities at 0, 1, and ∞.

Some Applications in Number Theory

- For some sepcial a, b, c in $\mathbb{Q}$, the ${}_2F_1$ can be interpreted as modular forms on arithmetic triangle groups.

Example

Let λ be the modular λ-function. Then ${}_2F_1\left[\begin{matrix}\frac{1}{2} & \frac{1}{2} \\ & 1\end{matrix} ; \lambda\right]^2 = \theta_3^4$, where θ_3 is the Jacobi theta function $\theta_3(\tau) = \sum_{k\in\mathbb{Z}} q^{k^2/2}$.

- For some sepcial a, b, c, the ${}_2F_1$ can be viewed as periods of algebraic curves.

Legendre Family

For $\lambda \in \mathbb{Q}$ and $\lambda \neq 0, 1$, let $E_\lambda : y^2 = x(1-x)(1-\lambda x)$ be the elliptic curve in Legendre normal form.

- A period of E_λ is

$$\Omega(E_\lambda) = \int_0^1 \frac{dx}{y} = \int_0^1 \frac{dx}{\sqrt{x(1-x)(1-\lambda x)}}.$$

and

$$\frac{\Omega(E_\lambda)}{\pi} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}+n-1}{n}^2 \lambda^n = {}_2F_1\left[\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ & 1 \end{matrix} ; \lambda\right].$$

For almost all prime p, if $\lambda \neq 0, 1 \mod p$,

$$\#\widetilde{E_\lambda}(\mathbb{F}_p) = p + 1 + \sum_{x \in \mathbb{F}_p} \phi\left(x(1-x)(1-\lambda x)\right),$$

where ϕ is the quadratic character of $\mathbb{F}_p^\times$.

The value

$$a_p(\lambda) = -\sum_{x \in \mathbb{F}_p} \phi\left(x(1-x)(1-\lambda x)\right)$$

- is the trace of Frobenius map;
- is the p-th Fourier coefficient of certain modular form.
- $$a_p(\lambda) \equiv \sum_{n=0}^{\frac{p-1}{2}} \frac{\left(\frac{1}{2}\right)_n^2}{(1)_n^2} (\lambda)^n =: {}_2F_1\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ & 1 \end{bmatrix}; \lambda\Bigg]_{\frac{p-1}{2}} \mod p$$

Examples

$$\mathcal{A}_\lambda : \quad s^2 = xy(x-\lambda y)(y-1)(1-x)$$

Ahlgren: For $p > 2$,

$$ {}_3F_2\left[\begin{matrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ & 1 & 1 \end{matrix}; 1\right]_{p-1} := \sum_{n=0}^{p-1} \frac{\left(\frac{1}{2}\right)_n^3}{(1)_n^3} \equiv a_p(\eta(4\tau)^6) \mod p^2.$$

Osburn-Straub/Li-Long-T.:

$$ {}_3F_2\left[\begin{matrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ & 1 & 1 \end{matrix}; 1\right] = \frac{16}{\pi^2} L(\eta(4\tau)^6, 2).$$

Examples

Kilbourn:

$$ {}_4F_3\left[\begin{matrix}\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\\ & 1 & 1 & 1\end{matrix};\ 1\right]_{p-1} \equiv a_p(\eta(2\tau)^4\eta(8\tau)^4) \mod p^3 $$

Zagier:

$$ {}_4F_3\left[\begin{matrix}\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\\ & 1 & 1 & 1\end{matrix};\ 1\right] = \frac{16}{\pi^2}L(\eta(2\tau)^4\eta(8\tau)^4, 1). $$

Quintic in $\mathbb{P}^4$

- Mirror quintic threefold:

$$\mathcal{V}(\lambda) : y_1 + \cdots + y_5 - x_1 = 0, \quad 5^{-5}\lambda x_1^5 = y_1 \cdots y_5.$$

- The Picard-Fuchs differential operator of $\mathcal{V}(\lambda)$ is

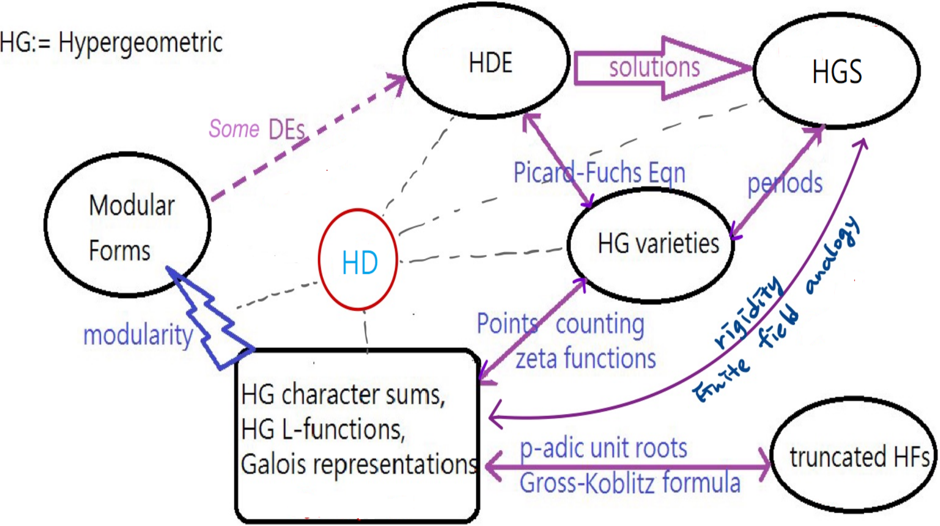
$$\theta^4 - 5\lambda(5\theta+1)(5\theta+2)(5\theta+3)(5\theta+4), \quad \theta = \lambda\frac{d}{d\lambda}.$$

- Schoen (1986): $\mathcal{V}(1)$ is modular.
- McCarthy (2012): (Rodriguez-Villegas' conjecture (2003))

$${}_4F_3\left[\begin{matrix}\frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} \\ & 1 & 1 & 1\end{matrix}; 1\right]_{p-1} \equiv a_p(f) \mod p^3,$$

where $a_p(f)$ is the pth Fourier coefficient of an explicit Heck eigenform f of wight 4 on $\Gamma_0(25)$.

Hypergeometric Functions in Number Theory



A hypergeometric datum consists of two multi-sets $\alpha = \{a_1, \cdots, a_n\}$ and $\beta = \{1, b_2, \cdots, b_n\}$ and an argument z with $a_i, b_j \in \mathbb{Q}$, and $a_i - b_j \notin \mathbb{Z}$ for all i, j.

- A multi-set α is said to be defined over $\mathbb{Q}$ if $\prod_{j=1}^{n}(X - e^{2\pi i a_j}) \in \mathbb{Z}[X]$.
- A hypergeometric datum $\{\alpha, \beta, z\}$ is said to be defined over $\mathbb{Q}$ if both α, β are defined over $\mathbb{Q}$ and $z \in \mathbb{Q}$.
- Beukers-Cohen-Mellit (BCM): When the multi-sets α and β are defined over $\mathbb{Q}$, there is an explicit model associated to the hypergeometric datum. (Magma package: Hypergeometric Motive)

Example

- For the multi-sets

$$\alpha = \{1/3, 2/3, 1/3, 2/3\}, \quad \beta = \{1, 1, 1, 1\},$$

  the corresponding BCM-model is given by

$$W: \quad \sum_{i=1}^{4} x_i = \sum_{i=1}^{4} y_i = 0, \quad (x_1 y_1)^3 = 3^6 (x_2 x_3 x_4 y_1 y_2 y_3),$$

  which is the rigid Calabi-Yau 3-fold labelled as Batyrev-van Straten's $V_{3,3}$.
- The number of rational points on the completion of W over $\mathbb{F}_p$ is

$$|\, \overline{W}(\mathbb{F}_p^\times) \,| = Q(p) - H(\alpha, \beta; 1),$$

  where

$$\mathbb{F}^{BCM}(\alpha, \beta; 1) := H(\alpha, \beta; 1) = \frac{1}{1-p} \sum_{\chi \in \widehat{\mathbb{F}_p^\times}} \left[\frac{g(\chi^3) g(\chi^{-1})^2}{g(\chi)} \chi(3^{-3}) \right]^2.$$

Theorem (Long-T.-Yui-Zudlin)

Let $d_1, d_2 \in \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}\}$ *or*

$$(d_1, d_2) = (1/5, 2/5), (1/8, 3/8), (1/10, 3/10), (1/12, 5/12),$$

then for each prime $p > 5$*, the truncated hypergeometric series*

$$ {}_4F_3 \left[\begin{matrix} d_1 & 1-d_1 & d_2 & 1-d_2 \\ & 1 & 1 & 1 \end{matrix} ; \ 1 \right]_{p-1} \equiv a_p(f_\alpha) \mod p^3, $$

where $a_p(f_\alpha)$ *is the pth coefficient of an explicit weight-4 Hecke eigenform* f_α *associated to the corresponding rigid Calabi–Yau manifold via the modularity theorem for rigid Calabi–Yau manifolds.*

The cases $(1/2, 1/2)$, $(1/5, 2/5)$, and $(1/2, 1/3)$ are verified by Kilbourn, McCarthy, and Fuselier-MacCarty, respectively.

Motivated by Long's work, in the joint work with Li and Long, we consider hypergeometric motives corresponding to this Whipple's formula

$$
{}_7F_6\left[\begin{matrix}\frac{1}{2} & \frac{5}{4} & c & 1-c & \frac{1-p}{2} & f & 1-f \\ & \frac{1}{4} & \frac{3}{2}-c & \frac{1}{2}+c & 1+\frac{p}{2} & \frac{3}{2}-f & \frac{1}{2}+f\end{matrix};\ 1\right]
= C \times \left(p \cdot {}_4F_3\left[\begin{matrix}\frac{1}{2} & \frac{1-p}{2} & f & 1-f \\ & 1-\frac{p}{2} & \frac{3}{2}-c & \frac{1}{2}+c\end{matrix};\ 1\right]\right),
$$

where $C = \frac{\Gamma(\frac{p}{2})\Gamma(\frac{3}{2}-f)\Gamma(\frac{1}{2}+f)\Gamma(\frac{p}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{1-p}{2})\Gamma(1+\frac{p}{2}-f)\Gamma(\frac{p}{2}+f)}$, and investigate the basic questions:

- Galois representation interpretation?
- Modularity?
- HGS-evaluation and periods?

Consider the Whipple formula

$$
{}_7F_6\left[\begin{matrix}\frac{1}{2} & \frac{5}{4} & c & 1-c & \frac{1-p}{2} & f & 1-f \\ & \frac{1}{4} & \frac{3}{2}-c & \frac{1}{2}+c & 1+\frac{p}{2} & \frac{3}{2}-f & \frac{1}{2}+f\end{matrix}; 1\right]
= C \times \left(p \cdot {}_4F_3\left[\begin{matrix}\frac{1}{2} & \frac{1-p}{2} & f & 1-f \\ & 1-\frac{p}{2} & \frac{3}{2}-c & \frac{1}{2}+c\end{matrix}; 1\right]\right),
$$

where $C = \frac{\Gamma(\frac{p}{2})\Gamma(\frac{3}{2}-f)\Gamma(\frac{1}{2}+f)\Gamma(\frac{p}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{1-p}{2})\Gamma(1+\frac{p}{2}-f)\Gamma(\frac{p}{2}+f)}$.

Hypergeometric data with $z = 1$:

$$
\alpha_6 = \left\{\frac{1}{2}, c, 1-c, \frac{1}{2}, f, 1-f\right\}, \beta_6 = \left\{1, \frac{3}{2}-c, \frac{1}{2}+c, 1, \frac{3}{2}-f, \frac{1}{2}+f\right\};
$$

$$
\alpha_4 = \left\{\frac{1}{2}, \frac{1}{2}, f, 1-f\right\}, \beta_4 = \left\{1, 1, \frac{3}{2}-c, \frac{1}{2}+c\right\}.
$$

Main Results-1 (simple verion)

Galois sub-representations: mainly based on the works of Katz and Beukers-Cohen-Mellit.

For any pair (c, f) in the list

$$\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{2},\frac{1}{3}\right),\left(\frac{1}{2},\frac{1}{6}\right),\left(\frac{1}{3},\frac{1}{3}\right),\left(\frac{1}{6},\frac{1}{6}\right),\left(\frac{1}{5},\frac{2}{5}\right),\left(\frac{1}{10},\frac{3}{10}\right),$$

there exists a representation ρ_ℓ^{BCM} associated to the hypergeometric datum $\{\alpha_6, \beta_6; 1\}$ such that the lifted representation ρ_ℓ^{BCM} of $G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ decomposes as

$$\rho_\ell^{BCM} \text{"} \cong \text{"} \sigma_{sym,\ell} \oplus \sigma_{alt,\ell} \oplus \sigma_{1,\ell}$$

of dimensions 2, 2, and 1, respectively.

Main Results-2

(c,f)	$\mathcal{J}$	$\mathcal{J}\cdot \mathrm{Tr}\,\rho_\ell^{BCM}(\mathrm{Frob}_p)$
$(\frac{1}{2},\frac{1}{2})$	1	$a_p(f_{8.6.a.a})+p\cdot a_p(f_{8.4.a.a})+\left(\frac{-1}{p}\right)p^2$
$(\frac{1}{2},\frac{1}{3})$	p	$a_p(f_{4.6.a.a})+p\cdot a_p(f_{12.4.a.a})+\left(\frac{3}{p}\right)p^2$
$(\frac{1}{3},\frac{1}{3})$	p^2	$a_p(f_{6.6.a.a})+p\cdot a_p(f_{18.4.a.a})+\left(\frac{-1}{p}\right)p^2$
$(\frac{1}{2},\frac{1}{6})$	p	$p\cdot a_p(f_{8.4.a.a})+p\cdot a_p(f_{24.4.a.a})+\left(\frac{3}{p}\right)p^2$
$(\frac{1}{5},\frac{2}{5})$	p^2	$p\cdot a_p(f_{10.4.a.a})+p\cdot a_p(f_{50.4.a.d})+\left(\frac{-5}{p}\right)p^2$
$(\frac{1}{6},\frac{1}{6})$	p^2	$p^2\cdot a_p(f_{24.2.a.a})+p^2\cdot a_p(f_{72.2.a.a})+\left(\frac{-1}{p}\right)p^2$
$(\frac{1}{10},\frac{3}{10})$	p^2	$p^2\cdot a_p(f_{40.2.a.a})+p^2\cdot a_p(f_{200.2.a.b})+\left(\frac{-5}{p}\right)p^2$

The Hecke eigenforms are given by their corresponding LMFDB labels.

Evaluations and Periods

Theorem (Li-Long-T.)

$$
{}_6F_5\left[\begin{matrix}\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ & 1 & 1 & 1 & 1 & 1\end{matrix};1\right] = 16\int_{1/2+i/2}^{-1/2+i/2} f_{8.6.a.a}\left(\frac{\tau}{2}\right)\tau^2 d\tau.
$$

$$
{}_7F_6\left[\begin{matrix}\frac{1}{2} & \frac{5}{4} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ & \frac{1}{4} & 1 & 1 & 1 & 1 & 1\end{matrix};1\right] = \frac{32i}{\pi}\int_{1/2+i/2}^{-1/2+i/2} \tau f_{8.4.a.a}(\tau/2) d\tau,
$$

where the path is taken to be the hyperbolic geodesic from $\frac{1+i}{2}$ to $\frac{-1+i}{2}$ going counter clockwise.

Evaluations and Periods

Theorem (Li-Long-T.)

$$ {}_6F_5\left[\begin{matrix}\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ & 1 & 1 & 1 & 1 & 1\end{matrix}\,;\,1\right] = 16\int_{-1/2+i/2}^{1/2+i/2} f_{8.6.a.a}\left(\frac{\tau}{2}\right)\tau^2 d\tau, $$

where the path is taken to be the hyperbolic geodesic from $\frac{1+i}{2}$ to $\frac{-1+i}{2}$ going counter clockwise.

Osburn-Straub-Zudilin For odd prime $p > 5$,

$$ {}_6F_5\left[\begin{matrix}\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ & 1 & 1 & 1 & 1 & 1\end{matrix}\,;\,1\right]_{p-1} \equiv a_p(f_{8.6.a.a}) \mod p^3. $$

Hypergeometric Functions in This Talk

For a given hypergeometric datum, we consider

FF hypergeometric functions | Galois representations

- character sum identities
- hypergeometric evaluations
- modular forms on arithmetic triangle groups

Recent relevant works include: Dembélé-Panchishkin-Voight-Zudilin, Osburn-Straub, and a survey and new conjectures by Dawsey-McCarthy.

Hypergeometric Functions over Finite Fields

In literature, there some different versions of the finite field hypergeometric functions:

- Greene, Fuselier-Long-Ramakrishna-Swisher-T. (FF integral expression)
- Beukers-Cohen-Mellit (BCM), McCarthy, Otsubo (FF $\Gamma(\cdot)$-functions)
- Katz (HG sums/sheaves in Daxin Xu's talk, FF Riemann Scheme)
  - Furusho (ℓ-adic HGS)

Euler's integral representation: When $c > b > 0$,

$$ {}_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix}; \lambda\right] = \frac{1}{B(b, c-b)} \int_0^1 x^{b-1}(1-x)^{c-b-1}(1-\lambda x)^{-a} dx $$

where

$$ B(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1} dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} $$

is the beta function, and $\Gamma(\cdot)$ is the gamma function.

$$ \begin{aligned} {}_2P_1\left[\begin{matrix} a & b \\ & c \end{matrix}; \lambda\right] &= \int_0^1 x^{b-1}(1-x)^{c-b-1}(1-\lambda x)^{-a} dx \\ &= {}_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix}; \lambda\right] B(b, c-b). \end{aligned} $$

Period Functions over Finite Fields

Let p be a prime, and $q = p^s$.

- Let $\widehat{\mathbb{F}_q^\times}$ denote the group of multiplicative characters on $\mathbb{F}_q^\times$.
- Extend $\chi \in \widehat{\mathbb{F}_q^\times}$ to $\mathbb{F}_q$ by setting $\chi(0) = 0$.
- Denote $\overline{\chi}$ the complex conjugation of χ, $\overline{\chi} = \chi^{-1}$.

Definition

Let $\lambda \in \mathbb{F}_q$, and A, B, $C \in \widehat{\mathbb{F}_q^\times}$. Define

$$ {}_2\mathbb{P}_1 \left[\begin{matrix} A & B \\ & C \end{matrix} ; \lambda \right] = \sum_{x \in \mathbb{F}_q} B(x)\overline{B}C(1-x)\overline{A}(1-\lambda x). $$

This is a finite field analogue of

$$ {}_2P_1 \left[\begin{matrix} a & b \\ & c \end{matrix} ; \lambda \right] := \int_0^1 x^{b-1}(1-x)^{c-b-1}(1-\lambda x)^{-a} dx $$

Let $X_\lambda^{[N;i,j,k]}$ be the smooth model of

$$C_\lambda^{[N;i,j,k]}: \; y^N = x^i(1-x)^j(1-\lambda x)^k.$$

We can choose ${}_2P_1\left[\begin{matrix}\frac{k}{N} & \frac{N-i}{N} \\ & \frac{2N-i-j}{N}\end{matrix};\lambda\right]$ as a period of $C_\lambda^{[N;i,j,k]}$.

Let q be an odd prime power, and let i,j,k be natural numbers with $1 \le i,j,k < N$. Further, let $\eta_N \in \widehat{\mathbb{F}_q^\times}$ be a character of order N. Then for $\lambda \in \mathbb{F}_q \setminus \{0,1\}$,

$$\#\widetilde{X}_\lambda^{[N;i,j,k]}(\mathbb{F}_q)\text{"} = \text{"}1+q+\sum_{m=1}^{N-1}\sum_{x\in\mathbb{F}_q}\eta_N^m(x^i(1-x)^j(1-\lambda x)^k).$$

$$\#\widetilde{X}_\lambda^{[N;i,j,k]}(\mathbb{F}_q)\text{"} = \text{"}1+q+\sum_{m=1}^{N-1}{}_2\mathbb{P}_1\left[\begin{matrix}\eta_N{}^{km} & \eta_N{}^{-im} \\ & \eta_N{}^{-(i+j)m}\end{matrix};\lambda\right].$$

For any characters A_0, A_i, B_i, $i = 1, \cdots, n$ in $\widehat{\mathbb{F}_q^\times}$, define

$$
{}_{n+1}\mathbb{P}_n \left[\begin{matrix} A_0 & A_1 & \cdots & A_n \\ & B_1 & \cdots & B_n \end{matrix} ; \lambda \right]
:= \sum_{x \in \mathbb{F}_q} A_n(x) \overline{A}_n B_n(1-x) \cdot {}_{n}\mathbb{P}_{n-1} \left[\begin{matrix} A_0 & A_1 & \cdots & A_{n-1} \\ & B_1 & \cdots & B_{n-1} \end{matrix} ; \lambda x \right].
$$

When $\lambda \neq 0$,

$$
{}_{n+1}\mathbb{P}_n \left[\begin{matrix} A_0 & A_1 & \cdots & A_n \\ & B_1 & \cdots & B_n \end{matrix} ; \lambda \right]
= \frac{(-1)^{n+1}}{q-1} \cdot \left(\prod_{i=1}^{n} A_i B_i(-1) \right) \cdot \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \binom{A_0 \chi}{\chi} \binom{A_1 \chi}{B_1 \chi} \cdots \binom{A_n \chi}{B_n \chi} \chi(\lambda),
$$

where $\binom{A}{B} := -B(-1) J(A, \overline{B}) = -B(-1) \sum_{t \in \mathbb{F}_q} A(t) B(1-t).$

${}_2\mathbb{F}_1$-hypergeometric Functions over Finite Fields

$$\Gamma(a) := \int_0^\infty x^a e^{-x} \frac{dx}{x} \quad \rightarrow \quad g(A) := \sum_{x \in \mathbb{F}_q} A(x) \zeta_p^{tr(x)}$$

$$B(a,b) := \int_0^1 x^{a-1}(1-x)^{b-1} dx \quad \rightarrow \quad J(A,B) := \sum_{x \in \mathbb{F}_q} A(x)B(1-x)$$

Define

$${}_2\mathbb{F}_1\left[\begin{matrix} A & B \\ & C \end{matrix}; \lambda\right] = \frac{1}{J(B, C\overline{B})} {}_2\mathbb{P}_1\left[\begin{matrix} A & B \\ & C \end{matrix}; \lambda\right],$$

which is a finite field analogue of

$${}_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix}; \lambda\right] = \frac{1}{B(b, c-b)} {}_2P_1\left[\begin{matrix} a & b \\ & c \end{matrix}; \lambda\right]$$

$$\begin{array}{rcl} \Gamma(a) & \leftrightarrow & g(A) \\ B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} & \leftrightarrow & J(A,B) = \frac{g(A)g(B)}{g(AB)} \text{ if } AB \neq \varepsilon \\ \Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin a\pi}, \quad a \notin \mathbb{Z} & \leftrightarrow & g(A)g(\overline{A}) = A(-1)q, \quad A \neq \varepsilon \end{array}$$

Gauss' multiplication formula

$$\Gamma(ma)(2\pi)^{(m-1)/2} = m^{ma-\frac{1}{2}}\Gamma(a)\Gamma\left(a+\frac{1}{m}\right)\cdots\Gamma\left(a+\frac{m-1}{m}\right)$$

$$\updownarrow$$

Hasse-Davenport relation

$$-g(A^m)\prod_{\substack{\chi\in\widehat{\mathbb{F}_q^\times}\\ \chi^m=\varepsilon}} g(\chi) = A(m^m)\prod_{\substack{\chi\in\widehat{\mathbb{F}_q^\times}\\ \chi^m=\varepsilon}} g(A\chi)$$

Transformation and Evaluation Formulas

are developed theoretically by Ahlgren, Beukers, Barman , Cohen, Evans, Greene, Katz, McCarthy, Mellit, Papanikolas, Penniston, Tripathi,...

- Fuselier-Long-Ramakrishna-Swisher-T. We give a systematic method to obtain certain type of algebraic transformations of 2P1-functions.
- Hoffman-T. We give a general comparison theorem that states roughly that from a known transformation formula over $\mathbb{C}$ one can deduce a transformation formula over $\mathbb{F}_q$, up to a Galois twist.

Quadratic Formula

Theorem (Fuselier, Long, Ramakrishna, Swisher, and T.)

Let $B, D \in \widehat{\mathbb{F}_q^\times}$, and set $C = D^2$. When $D \neq \phi$, $B \neq D$ and $x \neq \pm 1$, we have

$$\overline{C}(1-x)\ {}_2\mathbb{F}_1\left[\begin{matrix} D & D\phi\overline{B} \\ & C\overline{B} \end{matrix};\ \frac{-4x}{(1-x)^2}\right] = {}_2\mathbb{F}_1\left[\begin{matrix} C & B \\ & C\overline{B} \end{matrix};\ x\right].$$

This is the analogue to the classical result

$$(1-z)^{-c}\,{}_2F_1\left[\begin{matrix} \frac{c}{2} & \frac{1+c}{2}-b \\ & c-b+1 \end{matrix};\ \frac{-4z}{(1-z)^2}\right] = {}_2F_1\left[\begin{matrix} c & b \\ & c-b+1 \end{matrix};\ z\right].$$

Theorem (Fuselier-Long-Ramakrishna-Swisher-T.)

Let q be an odd prime power, $z \neq 1 \in \mathbb{F}_q^\times$, ϕ *be the quadratic character, and* $A \in \widehat{\mathbb{F}_q^\times}$ *of order larger than 2. Then*

$$ {}_2\mathbb{F}_1\left[\begin{matrix} A & \phi A \\ & \phi \end{matrix} ; \ z\right] = $$

$$\begin{cases} 0, & \text{if } z \text{ is not a square,} \\ \overline{A}^2(1+\sqrt{z}) + \overline{A}^2(1-\sqrt{z}), & \text{if } z \text{ is a square.} \end{cases}$$

This is the analogue to the classical result

$$ {}_2F_1\left[\begin{matrix} a & a+\frac{1}{2} \\ & \frac{1}{2} \end{matrix} ; \ z\right] = \frac{1}{2}\left((1+\sqrt{z})^{-2a} + (1-\sqrt{z})^{-2a}\right). $$

Theorem (Fuselier-Long-Ramakrishna-Swisher-T.)

Let q be an odd prime power, $z \neq 1 \in \mathbb{F}_q^\times$, ϕ be the quadratic character, and $A \in \widehat{\mathbb{F}_q^\times}$ of order larger than 2. Then

$$\frac{1}{J(\phi A, \overline{A})} {}_2\mathbb{P}_1 \left[\begin{matrix} A & \phi A \\ & \phi \end{matrix} ; z\right] = {}_2\mathbb{F}_1 \left[\begin{matrix} A & \phi A \\ & \phi \end{matrix} ; z\right] =$$

$$\begin{cases} 0, & \text{if } z \text{ is not a square}, \\ \overline{A}^2(1+\sqrt{z}) + \overline{A}^2(1-\sqrt{z}), & \text{if } z \text{ is a square}. \end{cases}$$

This is the analogue to the classical result

$${}_2F_1 \left[\begin{matrix} a & a+\frac{1}{2} \\ & \frac{1}{2} \end{matrix} ; z\right] = \frac{1}{2}\left((1+\sqrt{z})^{-2a} + (1-\sqrt{z})^{-2a}\right).$$

$$ {}_2\mathbb{P}_1\left[\begin{matrix}\overline{\eta}_4 & \eta_4 \\ & \phi\end{matrix}\ ;\ b^2\right] = J(\eta_4,\eta_4)\,\phi(1+b) + J(\eta_4,\eta_4)\,\phi(1-b). $$

$$ {}_2\mathbb{P}_1\left[\begin{matrix}\eta_4 & \overline{\eta}_4 \\ & \phi\end{matrix}\ ;\ b^2\right] = J(\overline{\eta}_4,\overline{\eta}_4)\,\phi(1+b) + J(\overline{\eta}_4,\overline{\eta}_4)\,\phi(1-b). $$

$$ y^4 = x(x-1)(x-b^2) $$

$$ y^2 = x^3 + (1+b)^2 x \qquad y^2 = x(x-1)(x-b^2) \qquad y^2 = x^3 + (1-b)^2 x $$

For a given hypergeometric datum $\{\alpha, \beta; z\}$,

- let $z = \lambda \in \mathbb{Q}^\times$ and $\mathbb{F}_q$ be a finite field of characteristic $p \nmid lcd(\alpha, \beta; \lambda)$.
- Fix a generator ω of $\widehat{\mathbb{F}_q^\times}$. Let $A_i = \omega^{(q-1)a_i}$ and $B_i = \omega^{(q-1)b_i}$.

$$\mathbb{P}(\alpha, \beta; \lambda) := {}_n\mathbb{P}_{n-1}\left[\begin{matrix} A_1 & A_2 & \cdots & A_n \\ & B_2 & \cdots & B_n \end{matrix} ; \lambda \right]$$

Self-dual: The pair α, β is said to be self-dual if it is congruent to the pair $-\alpha, -\beta$ mod $\mathbb{Z}$. A hypergeometric datum $HD = \{\alpha, \beta; \lambda\}$ is said to be self-dual if the pair α, β is self-dual and λ is totally real; it is *defined over* $\mathbb{Q}$ if the pair α, β is defined over $\mathbb{Q}$ and $\lambda \in \mathbb{Q}^\times$.

Theorem (Katz and Beukers-Cohen-Mellit) Given a datum HD = $\{\alpha, \beta; \lambda\}$ with $\alpha = \{a_1, \cdots, a_n\}$ and $\beta = \{1, b_2, \cdots, b_n\}$, set $M = lcd(\alpha \cup \beta)$, $G(M) := \mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_M)\right)$, and let $\lambda \in \mathbb{Z}[\zeta_M, 1/M] \setminus \{0\}$. Let ℓ be a prime. There exists an ℓ-adic Galois representation $\rho_{HD,\ell} : G(M) \to GL(W_\lambda)$ unramified almost everywhere such that at each prime ideal $\wp$ of $\mathbb{Z}[\zeta_M, 1/(M\ell\lambda)]$ with norm $N(\wp) = |\kappa_\wp|$,

$$\begin{aligned}\mathrm{Tr}\rho_{HD,\ell}(\mathrm{Frob}_\wp) =& (-1)^{n-1}\omega_\wp^{(N(\wp)-1)a_1}(-1)\mathbb{P}(\alpha, \beta; 1/\lambda; \kappa_\wp) \\ =& N(\wp)^* \cdot \chi_{HD}(\wp) \cdot \mathbb{F}^{BCM}(\alpha, \beta; 1/\lambda; \kappa_\wp),\end{aligned}$$

where $\mathrm{Frob}_\wp$ stands for the geometric Frobenius conjugacy class of $G(M)$ at $\wp$, and $\chi_{HD}(\wp)$ is a character of $G(M)$.

Theorem (Katz and Beukers-Cohen-Mellit)-cont'd

- When $\lambda \neq 1$, $dim_{\overline{\mathbb{Q}}_\ell} W_\lambda = n$ and all roots of the characteristic polynomial of $\rho_{DH,\ell}(\mathrm{Frob}_\wp)$ are algebraic numbers and have absolute value $N(\wp)^{(n-1)/2}$.
  - If *HD* is self-dual, then W_λ admits a symmetric (resp. alternating) bilinear pairing if n is odd (resp. even).
- When $\lambda = 1$, $dim_{\overline{\mathbb{Q}}_\ell} W_\lambda = n - 1$.
  - If *HD* is self-dual, then $\rho_{HD,\ell}$ has a subrepresentation $\rho^{prim}_{HD,\ell}$ of dimension $2\lfloor \frac{n-1}{2} \rfloor$ whose representation space admits a symmetric (resp. alternating) bilinear pairing if n is odd (resp. even). All roots of the characteristic polynomial of $\rho^{prim}_{HD,\ell}(\mathrm{Frob}_\wp)$ have absolute value $N(\wp)^{(n-1)/2}$.
- If the *HD* is defined over $\mathbb{Q}$, there exits an ℓ-adic representation $\rho^{BCM}_{HD,\ell}$ of $G_{\mathbb{Q}}$ such that

$$\rho^{BCM}_{HD,\ell}\,|_{G(M)} \simeq \rho_{HD,\ell}.$$

Example

- For the HD

$$\{\alpha = \{1/3, 2/3, 1/3, 2/3\}, \quad \beta = \{1, 1, 1, 1\}; \quad 1\},$$

$$\mathbb{F}^{BCM}(\alpha, \beta; 1) = \frac{1}{1-p} \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \left[\frac{g(\chi^3) g(\overline{\chi})^2}{g(\chi)} \chi(3^{-3}) \right]^2.$$

- **Modularity:**

$$\mathbb{F}^{BCM}(\alpha, \beta; 1) = a_p(f_{27.4.a.a}) + p.$$

FF Whipple

$$ {}_6F_5\left[\begin{matrix}\frac{1}{2} & c & 1-c & \frac{1}{2} & f & 1-f \\ & \frac{3}{2}-c & \frac{1}{2}+c & 1 & \frac{3}{2}-f & \frac{1}{2}+f\end{matrix};\ 1\right] $$

Lemma For a fixed finite field $\mathbb{F}_q$ of characteristic $p > 2$, and C, $F \in \widehat{\mathbb{F}_q^\times}$, we have

$$ {}_6\mathbb{P}_5\left[\begin{matrix}\phi & C & \overline{C} & \phi & F & \overline{F} \\ & \phi\overline{F} & \phi F & \varepsilon & \phi\overline{C} & \phi C\end{matrix};\ 1\right] = \sum_{t\in\mathbb{F}_q} \phi(t)\, {}_3\mathbb{P}_2\left[\begin{matrix}\phi & C & \overline{C} \\ & \phi\overline{F} & \phi F\end{matrix};\ t\right]^2 $$

Example: $(c, f) = (1/2, 1/3)$

For $p \equiv 1 \mod 6$, let η be a primitive character of order 6,

$$
{}_6\mathbb{P}_5\left[\begin{matrix}\phi & \phi & \phi & \phi & \eta^2 & \overline{\eta}^2 \\ & \eta & \overline{\eta} & \varepsilon & \varepsilon & \varepsilon\end{matrix};\ 1\right] = \sum_t \phi(t)\,{}_3\mathbb{P}_2\left[\begin{matrix}\phi & \eta^2 & \overline{\eta}^2 \\ & \varepsilon & \varepsilon\end{matrix};\ t\right]^2
$$

$$
\overset{Clausen}{=} \sum_{t\neq 1}\phi(t)\left({}_2\mathbb{P}_1\left[\begin{matrix}\eta & \eta^2 \\ & \varepsilon\end{matrix};\ t\right]^2 - \phi(1-t)p\right)^2 + {}_3\mathbb{P}_2\left[\begin{matrix}\eta^2 & \overline{\eta}^2 & \phi \\ & \varepsilon & \varepsilon\end{matrix};\ 1\right]^2
$$

$$
\overset{Clausen+ \atop Induct.}{=} \sum_{t\neq 1}\phi(t)\,{}_2\mathbb{P}_1\left[\begin{matrix}\eta & \eta^2 \\ & \varepsilon\end{matrix};\ t\right]^4 - 2p\phi(-1)\cdot{}_4\mathbb{P}_3\left[\begin{matrix}\eta^2 & \overline{\eta}^2 & \phi & \phi \\ & \varepsilon & \varepsilon & \varepsilon\end{matrix};\ 1\right] + p^2
$$

$$
+ {}_3\mathbb{P}_2\left[\begin{matrix}\eta^2 & \overline{\eta}^2 & \phi \\ & \varepsilon & \varepsilon\end{matrix};\ 1\right]^2,
$$

$$
{}_3\mathbb{P}_2\left[\begin{matrix}\eta^2 & \overline{\eta}^2 & \phi \\ & \varepsilon & \varepsilon\end{matrix};\ 1\right] = J(\eta,\eta^2)^2 + J(\overline{\eta},\overline{\eta}^2)^2.
$$

Example - Hecke Eigenforms

For $p \equiv 1 \mod 6$,

$$-{}_4\mathbb{P}_3\left[\begin{matrix}\eta^2 & \overline{\eta}^2 & \phi & \phi \\ & \varepsilon & \varepsilon & \varepsilon\end{matrix};\ 1\right] = a_p(f_4) + \phi(-1)p,$$

$$\begin{aligned}&-\ {}_6\mathbb{P}_5\left[\begin{matrix}\phi & \eta^2 & \overline{\eta}^2 & \phi & \phi & \phi \\ & \eta & \overline{\eta} & \varepsilon & \varepsilon & \varepsilon\end{matrix};\ 1\right]\\ =&\phi(-1)a_p(f_6) - p\phi(-1)\cdot {}_4\mathbb{P}_3\left[\begin{matrix}\eta^2 & \overline{\eta}^2 & \phi & \phi \\ & \varepsilon & \varepsilon & \varepsilon\end{matrix};\ 1\right]\\ =&\phi(-1)a_p(f_6) + p\cdot\phi(-1)a_p(f_4) + p^2,\end{aligned}$$

where f_4 is the weight-4 normalized Hecke eigenform on $\Gamma_0(12)$ (or $\Gamma_0(36)$), and f_6 is the weight-6 normalized Hecke eigenform on $\Gamma_0(4)$.

Example - Hecke Eigenforms

For $p \equiv 5 \mod 6$??

$$
{}_4\mathbb{P}_3\left[\begin{matrix}\eta^2 & \overline{\eta}^2 & \phi & \phi \\ & \varepsilon & \varepsilon & \varepsilon\end{matrix}; 1\right] = \frac{\phi(-1)}{p-1}\sum_{\chi}\frac{g(\phi\chi)^2}{g(\phi)^2}\frac{g(\eta^2\chi)}{g(\eta^2)}\frac{g(\overline{\eta}^2\chi)}{g(\overline{\eta}^2)}g(\overline{\chi})^4
$$

$$
\overset{BCM}{\longleftrightarrow} \frac{\phi(-1)}{p-1}\sum_{\chi}\frac{g(\chi^2)^2 g(\chi^3) g(\overline{\chi})^4}{g(\chi)^3}\chi(2^{-4}3^{-3}).
$$

FF Clausen Formula (Evans-Greene): Assume $E, K \in \widehat{\mathbb{F}_q^\times}$ such that none of $E, K\phi, EK, E\overline{K}$ is trivial. Suppose $EK = S^2$ is a square. When $t \neq 0, 1$, we have

$$
\begin{aligned}
&{}_3\mathbb{P}_2\left[\begin{matrix}\phi & E & \overline{E} \\ & K & \overline{K}\end{matrix}\,;\; t\right] \\
=&\phi(1-t)\left({}_2\mathbb{P}_1\left[\begin{matrix}\phi K\overline{S} & S \\ & K\end{matrix}\,;\; t\right]{}_2\mathbb{P}_1\left[\begin{matrix}\phi\overline{K}S & \overline{S} \\ & \overline{K}\end{matrix}\,;\; t\right] - q\right),
\end{aligned}
$$

When $t = 1$, we have

$$
{}_3\mathbb{P}_2\left[\begin{matrix}\phi & E & \overline{E} \\ & K & \overline{K}\end{matrix}\,;\; 1\right] = \frac{J(EK, \overline{E}K)}{J(\phi, \overline{K})}\left(J(S\overline{K}, \phi\overline{S})^2 + J(\phi S\overline{K}, \overline{S})^2\right).
$$

HD

$$
{}_7F_6\left[\begin{matrix}\frac{1}{2} & \frac{5}{4} & c & 1-c & \frac{1-p}{2} & f & 1-f \\ & \frac{1}{4} & \frac{3}{2}-c & \frac{1}{2}+c & 1+\frac{p}{2} & \frac{3}{2}-f & \frac{1}{2}+f\end{matrix};1\right]
= C\times\left(p\cdot {}_4F_3\left[\begin{matrix}\frac{1}{2} & \frac{1-p}{2} & f & 1-f \\ & 1-\frac{p}{2} & \frac{3}{2}-c & \frac{1}{2}+c\end{matrix};1\right]\right).
$$

Hypergeometric data:

$$
\alpha_6=\left\{\frac{1}{2},c,1-c,\frac{1}{2},f,1-f\right\},\beta_6=\left\{1,\frac{3}{2}-c,\frac{1}{2}+c,1,\frac{3}{2}-f,\frac{1}{2}+f\right\};
$$

$$
\alpha_4=\left\{\frac{1}{2},\frac{1}{2},f,1-f\right\},\beta_4=\left\{1,1,\frac{3}{2}-c,\frac{1}{2}+c\right\}.
$$

$$
\alpha_3=\left\{\frac{1}{2},f,1-f\right\},\beta_3=\left\{1,\frac{3}{2}-c,\frac{1}{2}+c\right\}.
$$

Theorem–ideas

Set $q = N(\wp)$, one has

$$"\mathbb{P}(\alpha_6(c,f), \beta_6(c,f); 1) \longleftrightarrow \sum_{t \in \kappa_\wp} \phi_\wp(t) \cdot \mathrm{Tr}_\rho \left(\mathrm{Frob}_\wp \left(W_t \otimes W_t\right)\right)",$$

$$"q \cdot \mathbb{P}(\alpha_4(c,f), \beta_4(c,f); 1) \longleftrightarrow -\psi_{c,f}(\wp) q^2 + \sum_{t \in \kappa_\wp} \phi_\wp(t) \cdot \mathrm{Tr}\,\mathrm{Frob}_\wp \left(\mathrm{Alt}^2 W_t \right),"$$

where $\mathrm{Alt}^2 W_t$ denotes the alternating square of W_t and $\psi_{c,f}$ is an explicit quadratic character of $G_{\mathbb{Q}(\zeta_N)}$ depending only on c, f.

Theorem (Li-Long-T.) For a fixed pair (c, f) in the list

$$\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{2},\frac{1}{3}\right),\left(\frac{1}{2},\frac{1}{6}\right),\left(\frac{1}{3},\frac{1}{3}\right),\left(\frac{1}{6},\frac{1}{6}\right),\left(\frac{1}{5},\frac{2}{5}\right),\left(\frac{1}{10},\frac{3}{10}\right),$$

set $N := lcd\left(\frac{1+2f-2c}{4}, \frac{3-2f-2c}{4}\right)$ and $M := lcd\,(\alpha_3, \beta_3)$.
Then for any fixed prime ℓ and any $\lambda \in \mathbb{Z}[\zeta_M, 1/M]$, there is an ℓ-adic representation $\rho_{\lambda,\ell} : G(M) \to GL_{\overline{\mathbb{Q}}_\ell}(W_\lambda)$ such that at each prime ideal $\wp$ of $\mathbb{Z}[\zeta_M, 1/(2N\ell)]$ and $\lambda \not\equiv 0 \mod \wp$,

$$\mathrm{Tr}_{\rho_{\lambda,\ell}}(\mathrm{Frob}_\wp) = \phi_\wp(-1)\mathbb{P}\left(\alpha_3, \beta_3; 1/\lambda; \kappa_\wp\right),$$

where $\phi_\wp(\cdot)$ is the quadratic character of $\kappa_\wp$.

Theorem-Cont'd

For any $\lambda \in \mathbb{Z}[\zeta_{M(c,f)}] \setminus \{0,1\}$, at a prime ideal $\wp$ of $\mathbb{Z}[\zeta_N, 1/N\ell]$, we have

$$\mathrm{Tr}\,\mathrm{Alt}^2\rho_{\lambda,\ell}|_{G(N)}(\mathrm{Frob}_\wp) = \phi_\wp(1-1/\lambda)N(\wp)\cdot\mathbb{P}(\alpha_3,\beta_3;1/\lambda;\kappa_\wp)$$

and

$$\begin{aligned}&\mathrm{Tr}\,\mathrm{Sym}^2\rho_{\lambda,\ell}|_{G(N)}(\mathrm{Frob}_\wp)\\ =&\mathbb{P}(\alpha_3,\beta_3;1/\lambda;\kappa_\wp)^2-\phi_\wp(1-1/\lambda)N(\wp)\cdot\mathbb{P}(\alpha_3,\beta_3;1/\lambda;\kappa_\wp).\end{aligned}$$

Further, $\mathrm{Sym}^2\rho_{\lambda,\ell}|_{G(N(c,f))}$ contains a 1-dimensional subrepresentation.

$$HD_1(c,f) := \{\alpha_6(c,f),\ \beta_6(c,f);\ 1\}\,;$$
$$HD_2(c,f) := \{\alpha_4(f),\ \beta_4(c);\ 1\}\,.$$

Theorem (Li-Long-T.) Let $M(c,f) := lcd(HD_2(c,f))$, and $N(c,f) := lcd(\frac{1+2f-2c}{4}, \frac{3-2f-2c}{4})$. Then either $N(c,f) = 2M(c,f)$ or $N(c,f) = M(c,f)$. Given any prime ℓ,

$$\rho_{HD_1(c,f),\ell}|_{G(N(c,f))} \cong (\epsilon_\ell \otimes \rho_{HD_2(c,f),\ell})|_{G(N(c,f))} \oplus \sigma_{sym,\ell},$$

where ϵ_ℓ is the ℓ-adic cyclotomic character, and $\sigma_{sym,\ell}$ is a 2-dimensional representation of $G(N(c,f))$ that can be computed explicitly.

Modularity

Theorem: Given a prime ℓ and a 2-dimensional absolutely irreducible representation ρ of $G_{\mathbb{Q}}$ over $\overline{\mathbb{Q}}_\ell$ that is odd, unramified at almost all primes, and its restriction to a decomposition subgroup D_ℓ at ℓ is crystalline with Hodge-Tate weight $\{0, r\}$ where $1 \le r \le \ell - 2$ and $\ell + 1 \nmid 2r$, then ρ is modular and corresponds to a weight $r + 1$ holomorphic Hecke eigenform.

Weight: determined by the zigzag procedure.

Theorem(Serre):
Suppose the trace of the representation ρ of $G_{\mathbb{Q}}$ is $\mathbb{Z}$-valued. Then the p-exponents of the conductor of ρ are bounded by 8 for $p = 2$, by 5 for $p = 3$, and by 2 for all other bad primes.