On the Rankin–Selberg problem

Bingrong Huang

Shandong University

Dec. 09, 2021

PANT - Kyoto 2021

@ RIMS, Kyoto University

Contents

1. Arithmetic functions
2. The Rankin–Selberg theorem
3. The Friedlander–Iwaniec theorem
4. The Rankin–Selberg problem and main result
5. Analytic twisted sum of GL(3) Fourier coefficients
6. Other results

Arithmetic functions

Let $a : \mathbb{N} \to \mathbb{C}$ $(n \mapsto a(n))$ be an arithmetic function.

We care about

- Magnitude of arithmetic functions (e.g. the size of $|a(n)|$ as $n \to \infty$)
- Averages of arithmetic functions (e.g. estimate of $\sum_{n \le x} a(n)$ as $x \to \infty$)

Generating series:

- $F(z) = \sum_{n \ge 1} a(n) e(nz)$ where $e(z) = e^{2\pi i z}$
- $L(s) = \sum_{n \ge 1} a(n) n^{-s}$

Dirichlet convolution $a = b * c$ i.e. $a(n) = \sum_{\ell m = n} b(\ell) c(m)$. We have

$$\sum_{n \ge 1} \frac{a(n)}{n^s} = \sum_{\ell \ge 1} \frac{b(\ell)}{\ell^s} \cdot \sum_{m \ge 1} \frac{c(m)}{m^s}.$$

Example I: Prime Number Theorem

Let $\mathbb{P} = \{2, 3, 5, 7, 11, \cdots\}$ be the set of all prime numbers. Define

$$\mathbf{1}_{\mathbb{P}}(n) = \begin{cases} 1, & \text{if } n \text{ is prime,} \\ 0, & \text{otherwise.} \end{cases}$$

PNT (Hadamard and de la Vallée Poussin 1896):

$$\sum_{p \le x} 1 = \sum_{n \le x} \mathbf{1}_{\mathbb{P}}(n) = \frac{x}{\log x} + o\left(\frac{x}{\log x}\right).$$

$$\text{Riemann Hypothesis} \Longleftrightarrow \sum_{n \le x} \mathbf{1}_{\mathbb{P}}(n) = \int_2^x \frac{\mathrm{d}t}{\log t} + O(x^{1/2+\varepsilon}).$$

Example II: Dirichlet's divisor problem

The divisor function: $\tau(n) = \sum_{d|n} 1 = (u * u)(n) \ll n^{\varepsilon}$.

Dirichlet's hyperbola method (1849):

$$\sum_{n \leq x} \tau(n) = x \log x + (2\gamma - 1)x + O(x^{1/2}).$$

Define $\Delta(x) := \sum_{n \leq x} \tau(n) - (x \log x + (2\gamma - 1)x)$.

Harmonic analysis (Voronoi 1904): $\Delta(x) = O(x^{1/3} \log x)$.

Exponential pairs (van der Corput 1922): $\Delta(x) = O(x^{1/3-\delta})$.

Conjecture: $\Delta(x) = O(x^{1/4+\varepsilon})$.

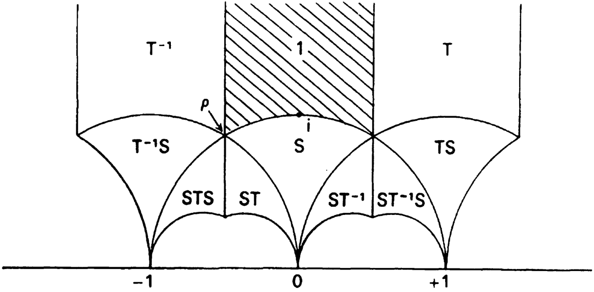
Hardy (1916): $\Delta(x) = \Omega(x^{1/4}(\log x)^{1/4} \log \log x)$.

The upper half plane

$\mathbb{H} = \{z = x + iy : y > 0\}$ the upper half plane.

$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1,\ a, b, c, d \in \mathbb{Z} \right\}$ the modular group.

$\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{H}$ by linear fractional transformations $\gamma z = \frac{az+b}{cz+d}$.



$\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$

Cusp forms

Let $k \geq 2$ be an even integer. A holomorphic function $f : \bar{\mathbb{H}} \to \mathbb{C}$ is a **modular form** of weight k if f satisfies

$$f(\gamma z) = (cz + d)^k f(z), \quad \forall \gamma \in \mathrm{SL}_2(\mathbb{Z}),\ z \in \mathbb{H}.$$

Since $f(z+1) = f(z)$, we have the Fourier expansion

$$f(z) = a_f(0) + \sum_{n \geq 1} a_f(n) n^{\frac{k-1}{2}} e(nz).$$

If $a_f(0) = 0$, then f is called a **cusp form**.

$M_k =$ the space of modular forms of weight k.
$S_k =$ the subspace of cusp forms.
S_k is a Hilbert space with inner product $\langle f, g \rangle = \int_{\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}} f(z) \overline{g(z)} y^k \frac{\mathrm{d}x \mathrm{d}y}{y^2}$.
$\dim S_k = k/12 + O(1)$.

Hecke operators

Let $n \in \mathbb{N}$. The n-th Hecke operator $T(n)$ is defined by $(f \in S_k)$

$$T(n)f(z) = n^{\frac{k-1}{2}} \sum_{ad=n} d^{-k} \sum_{0 \le b < d} f\left(\frac{az+b}{d}\right)$$
$$= \sum_{m \ge 1} \left(\sum_{d|(m,n)} a_f\left(\frac{mn}{d^2}\right) \right) m^{\frac{k-1}{2}} e(mz).$$

Then we have:

- $T(n) : S_k \to S_k$.
- $T(m)T(n) = \sum_{d|(m,n)} T\left(\frac{mn}{d^2}\right)$.
- $\langle T(n)f, g\rangle = \langle f, T(n)g\rangle$.
- $\exists$ an orthonormal basis H_k of S_k which consists of Hecke eigenforms.
- if $f \in H_k$ and $T(n)f = \lambda_f(n)f$, then $a_f(n) = a_f(1)\lambda_f(n)$.
- if $f \in H_k$ then $\lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)} \lambda_f\left(\frac{mn}{d^2}\right)$
  (in particular, $\lambda_f(n)$ is multiplicative).

Second moment

- In the early 20th century, people wanted to know the size of $|\lambda_f(n)|$, since it is related to number of representations of an integer by a quadratic form. Since $\lambda_f(n) \in \mathbb{R}$, we consider $\sum_{n \le x} \lambda_f(n)^2$ instead.

Theorem (Rankin 1939 and Selberg 1940)

Let $f \in H_k$. We have
$$\sum_{n \le x} \lambda_f(n)^2 = c_f x + O(x^{3/5}).$$

- Conjecture: $\sum_{n \le x} \lambda_f(n)^2 = c_f x + O(x^{3/8+\varepsilon})$.
- Y.-K. Lau, G. Lü, and J. Wu (2011):
$$\sum_{n \le x} \lambda_f(n)^2 - c_f x = \Omega(x^{3/8}), \quad \text{for } f \in H_k.$$
- Generalized Riemann Hypothesis (GRH) gives
$$\sum_{n \le x} \lambda_f(n)^2 = c_f x + O(x^{1/2+\varepsilon}).$$

The Rankin–Selberg L-functions

Assume $f \in H_k$.

The Rankin–Selberg L-function: $L(s, f \times f) = \sum_{m \geq 1} \frac{\lambda_{f \times f}(m)}{m^s} = \zeta(2s) \sum_{n \geq 1} \frac{\lambda_f(n)^2}{n^s}$.

The Euler product: $L(s, f \times f) = \prod_p \left(1 - \frac{\alpha_f(p)^2}{p^s}\right)^{-1} \left(1 - \frac{1}{p^s}\right)^{-2} \left(1 - \frac{\beta_f(p)^2}{p^s}\right)^{-1}$.

The Rankin–Selberg method (unfolding method):

$$\Lambda(s, f \times f) = \gamma(s, f \times f) L(s, f \times f) = \Lambda(1 - s, f \times f),$$

where

$$\gamma(s, f \times f) = \pi^{-2s} \Gamma\left(\frac{s+k}{2}\right) \Gamma\left(\frac{s+k-1}{2}\right) \Gamma\left(\frac{s+1}{2}\right) \Gamma\left(\frac{s}{2}\right).$$

$\Lambda(s, f \times f)$ admits a meromorphic continuation to $s \in \mathbb{C}$ of order 1 with at most poles at $s = 0$ and $s = 1$.

Proof of Rankin–Selberg Theorem

By the Rankin–Selberg method we get (e.g. Landau, Friedlander–Iwaniec)

$$\sum_{m\leq x} \lambda_{f\times f}(m) = C_f x + O(x^{3/5+\varepsilon}).$$

Note that $\lambda_f(n)^2 = \sum\sum_{\ell^2 m=n} \mu(\ell)\lambda_{f\times f}(m)$. Hence we have

$$\begin{aligned}\sum_{n\leq x} \lambda_f(n)^2 &= \sum\sum_{\ell^2 m\leq x} \mu(\ell)\lambda_{f\times f}(m)\\ &= \sum_{\ell\leq x^{1/2}} \mu(\ell)\left(C_f\frac{x}{\ell^2} + O(x^{3/5+\varepsilon}\ell^{-6/5})\right)\\ &= \frac{C_f}{\zeta(2)}x + O(x^{3/5+\varepsilon}).\end{aligned}$$

□

L-functions of degree d

More generally, consider an arithmetic function $\lambda_F(n)$ such that its Dirichlet series $L(s,F)$ is an **L-function of degree** d**:**

$$L(s,F)=\sum_{n\geq 1}\frac{\lambda_F(n)}{n^s}=\prod_p\prod_{j=1}^{d}\left(1-\frac{\alpha_j(p)}{p^s}\right)^{-1},\quad \mathrm{Re}(s)>1.$$

Gamma factor: $\gamma(s,F)=\pi^{-ds/2}\prod_{j=1}^{d}\Gamma\left(\frac{s-\kappa_j}{2}\right)$.

The **complete L-function**

$$\Lambda(s,F)=q(F)^{s/2}\gamma(s,F)L(s,F)$$

admits an analytic continuation to a meromorphic function for $s\in\mathbb{C}$ of order 1 with at most poles at $s=0$ and $s=1$.

Functional equation:

$$\Lambda(s,F)=\varepsilon(F)\Lambda(1-s,\bar{F}),$$

where $\bar{F}$ is the dual of F for which $\lambda_{\bar{F}}(n)=\overline{\lambda_F(n)}$, $\gamma(s,\bar{F})=\overline{\gamma(\bar{s},F)}$, $q(\bar{F})=q(F)$, and $\varepsilon(F)$ is the root number of $L(s,F)$ satisfying that $|\varepsilon(F)|=1$.

Friedlander–Iwaniec

Theorem (Friedlander–Iwaniec 2005)

Assume $\lambda_F(n) \ll n^{\varepsilon}$. Then we have

$$\sum_{n \le x} \lambda_F(n) = \mathrm{Res}_{s=1} \frac{L(s,F)x^s}{s} + O_F(x^{\frac{d-1}{d+1}+\varepsilon}).$$

Example: Let $f \in H_k$. Thanks to Deligne 1972, we have $|\lambda_f(n)| \le \tau(n)$. We at least have

$$\sum_{n \le x} \lambda_f(n) = O(x^{1/3+\varepsilon}).$$

(Hecke, Walfisz, ..., Deligne, Hafner and Ivić, Rankin, J. Wu, H. Tang, Z. Xu, L. Yang, ...)

- Compare to Dirichlet divisor problem.

Friedlander–Iwaniec: proof

Proof sketch:

- Perron's formula: $\sum_{n\leq x}\lambda_F(n) = \frac{1}{2\pi i}\int_{1+\varepsilon-iT}^{1+\varepsilon+iT} L(s,F)\frac{x^s}{s}\mathrm{d}s + O(\frac{x^{1+2\varepsilon}}{T})$.
- Shifting the contour: $\sum_{n\leq x}\lambda_F(n) = \mathrm{Res}_{s=1}\frac{L(s,F)x^s}{s} + I(x) + O(\frac{x^{1+2\varepsilon}}{T})$ where $I(x) = \frac{1}{2\pi i}\int_{-\varepsilon-iT}^{-\varepsilon+iT} L(s,F)\frac{x^s}{s}\mathrm{d}s$.
- Changing variable $s \rightsquigarrow 1-s$ and applying functional equation: $I(x) = \frac{1}{2\pi i}\int_{1+\varepsilon-iT}^{1+\varepsilon+iT} L(1-s,F)\frac{x^{1-s}}{1-s}\mathrm{d}s = \frac{\varepsilon(F)}{2\pi i}\int_{1+\varepsilon-iT}^{1+\varepsilon+iT} G(s)L(s,\bar{F})\frac{x^{1-s}}{1-s}\mathrm{d}s$.
- Stationary phase method: $I(x) \rightsquigarrow C_F x^{\frac{d-1}{2d}}\sum_{n\asymp N}\overline{\lambda_F(n)}n^{-\frac{d+1}{2d}}e(\pm T(n/N)^{1/d})$, with $N \asymp T^d/x$.
- Bounding dual sum trivially: $\sum_{n\leq x}\lambda_F(n) = \mathrm{Res}_{s=1}\frac{L(s,F)x^s}{s} + O(T^{\frac{d-1}{2}} + \frac{x^{1+2\varepsilon}}{T})$. Take $T = x^{\frac{2}{d+1}}$. □

Conjecture:

$$\sum_{n\leq x}\lambda_F(n) = \mathrm{Res}_{s=1}\frac{L(s,F)x^s}{s} + O_F(x^{\frac{d-1}{2d}+\varepsilon}).$$

GRH implies:

$$\sum_{n\leq x}\lambda_F(n) = \mathrm{Res}_{s=1}\frac{L(s,F)x^s}{s} + O_F(x^{\frac{1}{2}+\varepsilon}).$$

The Rankin–Selberg problem

The Rankin–Selberg problem

Let $f \in H_k$. Can we unconditionally prove

$$\sum_{n \le x} \lambda_f(n)^2 = c_f x + O(x^{3/5-\delta}),$$

for some $\delta > 0$?

Main result

Theorem 1 [H. 2021]

If $f \in H_k$, then we have

$$\sum_{n \le x} \lambda_f(n)^2 = c_f x + O(x^{3/5-\delta}),$$

for any $\delta < 1/560$.

- The same result holds for a Hecke–Maass cusp form f for $\mathrm{SL}(2, \mathbb{Z})$.
- The mean square of the divisor function: (Ramanujan, Wilson, ..., Ramachandra–Sankaranarayanan, Jia–Sankaranarayanan)

$$\sum_{n \le x} \tau(n)^2 = x P_3(\log x) + O(x^{1/2} (\log x)^5).$$

The symmetric square L-functions

Let $f \in H_k$. The symmetric square lift L-function:

$$L(s, \mathrm{sym}^2 f) = \zeta(2s) \sum_{n \geq 1} \frac{\lambda_f(n^2)}{n^s}$$

$$= \prod_p \left(1 - \frac{\alpha_f(p)^2}{p^s}\right)^{-1} \left(1 - \frac{1}{p^s}\right)^{-1} \left(1 - \frac{\beta_f(p)^2}{p^s}\right)^{-1}.$$

Shimura (1975): The complete L-function $\Lambda(s, \mathrm{sym}^2 f) = \gamma(s, \mathrm{sym}^2 f) L(s, \mathrm{sym}^2 f)$ admits an analytic continuation to an entire function for $s \in \mathbb{C}$ of order 1.

Functional equation:

$$\Lambda(s, \mathrm{sym}^2 f) = \Lambda(1 - s, \mathrm{sym}^2 f).$$

Gelbart and Jacquet (1978): $\mathrm{sym}^2 f$ is an automorphic cuspidal representation for GL(3).

We have $f \times f = 1 \boxplus \mathrm{sym}^2 f$, that is

$$L(s, f \times f) = \zeta(s) L(s, \mathrm{sym}^2 f).$$

The GL(1) ⊞ GL(3) case

By $L(s, f \times f) = \zeta(s)L(s, \mathrm{sym}^2 f)$ we have

$$\lambda_{f\times f}(n) = \lambda_{1\boxplus \mathrm{sym}^2 f}(n) = (u * \lambda_{\mathrm{sym}^2})(n) = \sum_{\ell m = n} \lambda_{\mathrm{sym}^2 f}(m).$$

Let Φ be a Hecke–Maass cusp form for $SL(3, \mathbb{Z})$.
Let $A_\Phi(1, n)$ be the normalized Fourier coefficients of Φ.
The *generalized Ramanujan conjecture* (GRC) for Φ asserts that $A_\Phi(1, n) \ll n^{o(1)}$.

Theorem 2 [H. 2021]

Assuming GRC for Φ, then we have

$$\sum_{n\le X} \lambda_{1\boxplus\Phi}(n) = L(1, \Phi)\, X + O_\Phi(X^{3/5-\delta}),$$

for any $\delta < 1/560$.

Furthermore, if $\Phi = \mathrm{sym}^2 f$, then we don't need to assume GRC for Φ.

Dual sum

The dual sum (e.g. Friedlander–Iwaniec):

$$S = \sum_{n \asymp N} \lambda_{1 \boxplus \Phi}(n) e(T(n/N)^{1/4}),$$

with $T = x^{2/5+\delta}$ and $N \asymp T^4/x = x^{3/5+4\delta}$.

So for some L, M such that $LM \asymp N$,

$$S \rightsquigarrow \sum_{\ell \asymp L} \sum_{m \asymp M} A(1,m) e(T(\ell/L)^{1/4}(m/M)^{1/4}),$$

- If $L \gg T^\eta$ then we use exponential pairs to get nontrivial bounds (Weyl, van der Corput, ..., Bourgain).
- If $L \ll T^\eta$ then $M \gg x^{3/5+4\delta-\eta} \gg T^{3/2-\rho}$.

Analytic twisted sum of GL(3) Fourier coefficients

Let $A(1,m)$ be the Fourier coefficients of a GL(3) automorphic form, e.g. $A(1,m) = \lambda_{\mathrm{sym}^2 f}(m)$. Consider the following sum

$$\sum_{m\geq 1} A(1,m) e\left(T\varphi\left(\frac{m}{M}\right)\right) V\left(\frac{m}{M}\right),$$

where $T \geq 1$ is a large parameter, φ is some fixed real-valued smooth function, and $V \in C_c^\infty(\mathbb{R}_{>0})$ and satisfying that $V^{(j)} \ll_j 1$ for all $j \geq 0$.

- Munshi [JAMS 2015] proved the first nontrivial bound for $\varphi(u) = \log u$ with $M \leq T^{3/2+\varepsilon}$, and then proved the subconvexity bounds of GL(3) L-functions in the T-aspect.
- This was strengthened to the above bound for $\varphi(u) = \log u$ and $M \leq T^{3/2+\varepsilon}$ by Aggarwal.
- For $\varphi(u) = u^\beta$ and $T = \alpha M^\beta$, Kumar–Mallesham–Singh proved nontrivial upper bounds (with bounds depending on α).

Analytic twisted sums: Main result

To bound

$$\sum_{\ell \asymp L} \left| \sum_{m \asymp M} A(1,m) e(T(\ell/L)^{1/4}(m/M)^{1/4}) \right|,$$

we prove the following theorem.

Theorem 3 [H. 2021]

Assume $\varphi(u) = u^{\beta}$ with $\beta \in (0,1)$. Then we have

$$\mathscr{S} := \sum_{m \geq 1} A(1,m) e\left(T\varphi\left(\frac{m}{M}\right)\right) V\left(\frac{m}{M}\right) \ll T^{3/10} M^{3/4+\varepsilon},$$

if $T^{6/5} \leq M \leq T^{8/5-\varepsilon}$.

Trivial bound for $\mathscr{S}$ is $O(M)$. For our application we need $\beta = 1/4$.

Sketch of proof of Theorem 3

The Duke–Friedlander–Iwaniec delta method:

$$\delta(n,0) = \frac{1}{Q}\sum_{1\le q\le Q}\frac{1}{q}\sideset{}{^\star}\sum_{a \bmod q} e\left(\frac{na}{q}\right)\int_{\mathbb{R}} g(q,x)e\left(\frac{nx}{qQ}\right)\mathrm{d}x,$$

for some large Q and certain $g(q,x)$. So (generic case)

$$\mathscr{S} \rightsquigarrow \sum_{m\asymp M}\sum_{n\asymp M} A(1,n)e\left(T\varphi\left(\frac{m}{M}\right)\right) \cdot \frac{1}{Q}\sum_{q\asymp Q}\frac{1}{q}\sideset{}{^\star}\sum_{a \bmod q} e\left(\frac{(m-n)a}{q}\right)\int_{x\asymp 1} e\left(\frac{(m-n)x}{qQ}\right)\mathrm{d}x.$$

Rearranging the sums and integral we get

$$\mathscr{S} \rightsquigarrow \frac{1}{Q}\sum_{q\asymp Q}\frac{1}{q}\sideset{}{^\star}\sum_{a \bmod q}\int_{x\asymp 1}\sum_{m\asymp M} e\left(\frac{ma}{q}\right)e\left(T\varphi\left(\frac{m}{M}\right)+\frac{mx}{qQ}\right) \cdot \sum_{n\asymp M} A(1,n)e\left(\frac{-na}{q}\right)e\left(\frac{-nx}{qQ}\right)\mathrm{d}x.$$

We need to save M plus a little more.

Sketch of proof of Theorem 3, cont

By Poisson summation formula and Voronoi summation formula, we get

$$\mathscr{S} \rightsquigarrow \frac{1}{Q}\sum_{q\asymp Q}\frac{1}{q}\int_{x\asymp 1} M \sum_{|m|\asymp QT/M} \mathcal{V}(m,q,x)$$

$$\cdot q \sum_{n_2\asymp M^2/Q^3} \frac{A(n_2,1)}{n_2} S\left(-\bar{m}, n_2; q\right) \Psi_x^{\pm}\left(\frac{n_2}{q^3}\right) \mathrm{d}x,$$

where $S(a,b;c) := \sum^*{}_{d(c)} e\left(\frac{ad+b\bar{d}}{c}\right)$ is the classical Kloosterman sum,

$\mathcal{V}(m,q,x) \ll T^{-1/2}$, and $\Psi_x^{\pm}\left(\frac{n_2}{q^3}\right) \ll \left(\frac{n_2}{q^3}M\right)^{1/2}$.

We save $\frac{MQ}{M\frac{QT}{M}\frac{1}{T^{1/2}}} = \frac{M}{T^{1/2}}$ from the m-sum and a-sum;

$\frac{M}{q^{3/2}(\frac{n_2}{q^3}M)^{1/2}} \asymp \frac{Q^{3/2}}{M^{1/2}}$ from the n-sum.

We will save $\sqrt{\frac{M}{Q^2}} = \frac{M^{1/2}}{Q}$ from the x-integral.

In total we save $\frac{MQ^{1/2}}{T^{1/2}}$ (for some Q such that $M^{1/3} \le Q \le M^{1/2}$).

Sketch of proof of Theorem 3, cont

By Cauchy:

$$\mathscr{S} \ll \frac{M}{Q^{1/2}} \frac{M}{Q^{3/2}} \mathcal{T}^{1/2},$$

where

$$\mathcal{T} \rightsquigarrow \sum_{n \asymp M^2/Q^3} \frac{1}{n} \cdot \left| \sum_{q \asymp Q} \frac{1}{q} \sum_{m \asymp QT/M} S\left(-\bar{m}, n; q\right) \mathcal{W}(m, n, q) \right|^2.$$

Opening the square and applying the Poisson modulo qq':

$$\mathcal{T} \rightsquigarrow \sum_{q \asymp Q} \frac{1}{q} \sum_{m \asymp QT/M} \sum_{q' \asymp Q} \frac{1}{q'} \sum_{m' \asymp QT/M} \frac{1}{qq'} \sum_{n \in \mathbb{Z}} \mathfrak{C}(n) \mathfrak{I}(n),$$

where the character sum is given by

$$\mathfrak{C}(n) := \sum_{b \bmod qq'} S\left(-\bar{m}, b; q\right) S\left(\bar{m}', -b; q'\right) e\left(\frac{nb}{qq'}\right).$$

Sketch of proof of Theorem 3, cont

Diagonal term ($n = 0$):
The generic terms will be $q = q'$ and $m = m'$. So we save (for $\mathscr{S}$)

$$\left(Q \cdot \frac{QT}{M}\right)^{1/2} = \frac{QT^{1/2}}{M^{1/2}}. \quad \text{Hence in total: } \frac{MQ^{1/2}}{T^{1/2}} \cdot \frac{QT^{1/2}}{M^{1/2}} = M^{1/2}Q^{3/2}.$$

Off-Diagonal terms ($n \neq 0$):
The length of the dual sum is $\frac{Q^2 \frac{M}{Q^2}}{\frac{M^2}{Q^3}} = \frac{Q^3}{M}$. We can save Q from the character sums (square root cancellation) and $\sqrt{\frac{M}{Q^2}}$ from the integral transforms. So we save (for $\mathscr{S}$)

$$\left(Q \cdot \sqrt{\frac{M}{Q^2}} \cdot \frac{1}{\frac{Q^3}{M}}\right)^{1/2} = \frac{M^{3/4}}{Q^{3/2}}. \quad \text{Hence in total: } \frac{MQ^{1/2}}{T^{1/2}} \cdot \frac{M^{3/4}}{Q^{3/2}} = \frac{M^{7/4}}{QT^{1/2}}.$$

The best choice is $Q = \frac{M^{1/2}}{T^{1/5}}$, which proves Theorem 3. □

The GL(1) ⊞ GL(2) case

Let $f \in H_k$. Consider the arithmetic function $\lambda_{1\boxplus f}(n) = \sum_{\ell m = n} \lambda_f(m)$, that is, its Dirichlet series is

$$L(s, 1 \boxplus f) = \sum_{n \geq 1} \frac{\lambda_{1\boxplus f}(n)}{n^s} = \zeta(s) L(s, f).$$

This is a degree three case, and we have (even a trivial application of GRH)

$$\sum_{n \leq x} \lambda_{1\boxplus f}(n) = c_f x + O(x^{1/2+\varepsilon}).$$

Theorem (H., Yongxiao Lin, and Zhiwei Wang 2021)

We have

$$\sum_{n \leq x} \lambda_{1\boxplus f}(n) = c_f x + O(x^{1/2-\delta_3}),$$

for any $\delta_3 < 4/739$.

Here we have used the classical result on analytic twisted sums of GL(2) Fourier coefficients due to Jutila.

The $GL(1) \boxplus (GL(2) \otimes GL(2))$ case

Let $f \in H_k$ and $g \in H_\ell$. Consider the arithmetic function $\lambda_{1\boxplus(f\otimes g)}(n) = \sum_{ab^2c=n} \lambda_f(c)\lambda_g(c)$, that is, its Dirichlet series is

$$L(s, 1 \boxplus (f \otimes g)) = \sum_{n\geq 1} \frac{\lambda_{1\boxplus(f\otimes g)}(n)}{n^s} = \zeta(s)L(s, f \otimes g).$$

Theorem (H., Qingfeng Sun, and Huimin Zhang 2021)

Assume $f \neq g$. Then we have

$$\sum_{n\leq x} \lambda_{1\boxplus(f\otimes g)}(n) = c_{f,g}x + O(x^{2/3-\delta_5}),$$

for any $\delta_5 < 1/356$.

- Yongxiao Lin and Qingfeng Sun improved the exponent $5/7$ for the $GL(3) \otimes GL(2)$ case under GRC.
- Huimin Zhang improves (in progress) $3/4$ for the $GL(1) \boxplus (GL(3) \otimes GL(2))$ case under GRC.

Analytic twisted sums of GL(2) × GL(2) Fourier coefficients

The key to our improvement is the following estimate:

Theorem (H., Qingfeng Sun, and Huimin Zhang 2021)

Let $\varphi(x) = \alpha \log x$ *or* αx^β ($\beta \in (0,1)\backslash\{1/2, 3/4\}$, $\alpha \in \mathbb{R}\backslash\{0\}$). *Let* $V(x) \in \mathcal{C}_c^\infty(1,2)$ *with total variation* $\mathrm{Var(V)} \ll 1$ *and satisfying the condition*

$$V^{(j)}(x) \ll_j \triangle^j$$

for any integer $j \geq 0$ *with* $\triangle \ll t^{1/2-\varepsilon}$ *for any* $\varepsilon > 0$. *Then we have*

$$\sum_{n=1}^{\infty} \lambda_f(n)\lambda_g(n) e\left(t\varphi\left(\frac{n}{X}\right)\right) V\left(\frac{n}{X}\right) \ll_{f,g,\varphi,\varepsilon} t^{2/5} X^{3/4+\varepsilon}$$

for $t^{8/5} < X < t^{12/5}$.

Previously, Acharya, Sharma and Singh proved the upper bound $O(t^{7/16}X^{3/4+\varepsilon})$ for the case $\varphi(x) = \alpha \log x$ and $X < t^{1+\varepsilon}$.

Short intervals and arithmetic progressions

Friedlander–Iwaniec:

$$\sum_{x<n\leq x+y} \lambda_F(n) - \operatorname*{Res}_{s=1} \frac{L(s,F)((x+y)^s - x^s)}{s} = o(y),$$

if $y \geq x^{\frac{d-1}{d+1}+\varepsilon}$.

Arithmetically we have

$$\sum_{\substack{n\leq x \\ n\equiv a \bmod q}} \lambda_F(n) - \frac{1}{\varphi(q)} \sum_{\substack{n\leq x \\ (n,q)=1}} \lambda_F(n) \ll_A \frac{x}{q}(\log x)^{-A}$$

for $q \leq x^{\frac{2}{d+1}-\varepsilon}$.

For degree three case, breaking $1/2$ was done for $\mathrm{GL}(1) \boxplus \mathrm{GL}(1) \boxplus \mathrm{GL}(1)$ case by Friedlander–Iwaniec, Heath-Brown, Fouvry–Kowalski–Michel, Ping Xi; for $\mathrm{GL}(1) \boxplus \mathrm{GL}(2)$ case by Kowalski–Michel–Sawin.

Thank you for your attention!