On the GIT stratification of prehomogeneous vector spaces

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References

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Today's topics

- Previous work
- Main Theorem
- GITstratification
- Finding $\mathfrak{B}$
- When $S_\beta \neq \emptyset$?
- $G_k \backslash S_{\beta\,k}$
- Smaller prehomogeneous vector spaces

Previous work

I was going to talk about the following result last year.
k: non-normal cubic field unramified at $2, 3$ for simplicity
F: quadratic field
Δ_F: discriminant of F
h, R: class number, regulator

Thm:

$$\lim_{X\to\infty} X^{-2} \sum_{[F:\mathbb{Q}]=2, |\Delta_F|<X} \frac{h_{k\cdot F} R_{k\cdot F}}{h_F R_F} = |\Delta_k|^{\frac{1}{2}} h_k R_k \zeta_k(2) \prod_p E_p.$$

Main Theorem

k: fixed field
G: connected reductive $\quad \chi : G \to \mathrm{GL}_1$: character
V: representation of G

Def: (G, V, χ) is a prehomogeneous vector space if
$\exists$ open orbit, $P(x) \in k[V] \setminus k$ s.t. $P(gx) = \chi(g)P(x)$.
$V^{\mathrm{ss}} = \{x \in V \mid P(x) \neq 0\}$.

V^{ss}: semi-stable points
$V \setminus (V^{\mathrm{ss}} \cup \{0\})$: unstable points

Def: $\mathrm{Ex}_n(k) = \mathrm{H}^1(k, \mathfrak{S}_n)$.
If $n = 2, 3$, $\mathrm{Ex}_n(k) \leftrightarrow \{[k : \mathbb{Q}] = n\}$/iso

Main Theorem

(1) $G = \mathrm{GL}_3 \times \mathrm{GL}_3 \times \mathrm{GL}_2$, $V = \mathrm{Aff}^3 \otimes \mathrm{Aff}^3 \otimes \mathrm{Aff}^2$,
(2) $G = \mathrm{GL}_6 \times \mathrm{GL}_2$, $V = \wedge^2 \mathrm{Aff}^6 \otimes \mathrm{Aff}^2$.
(3) $G = \mathrm{GL}_5 \times \mathrm{GL}_4$, $V = \wedge^2 \mathrm{Aff}^5 \otimes \mathrm{Aff}^4$.

These are prehomogeneous vector spaces (the choice of χ turns out to be obvious).

Thm (Wright-Y, 92):
For (1), (2), $G_k \backslash V_k^{\mathrm{ss}} \cong \mathrm{Ex}_3(k)$.
For (3), $G_k \backslash V_k^{\mathrm{ss}} \cong \mathrm{Ex}_5(k)$.

Main Theorem

From now on k is a perfect field.
The notion of GIT stratification was established by Kempf, Ness, Kirwan in 1980's (will be explained). By the GIT stratification,

$$V_k \setminus (V_k^{\mathrm{ss}} \cup \{0\}) = \bigcup_{\beta \in \mathfrak{B}} S_{\beta\, k}$$
$$S_{\beta\, k} = G_k \times_{P_{\beta\, k}} Y_{\beta\, k}^{\mathrm{ss}}$$

Thm (Tajima-Y) We determined the GIT stratification for (1)–(3). (S_β can be empty.)

We give more precise statements.

Main Theorem

SP means “single point”.

Thm (Tajima-Y)
(a) For (1), there are $\mathbf{16}$ non-empty strata S_β. Moreover, except for one stratum S_{β_0}, $G_k \backslash S_{\beta\, k} =$ SP whereas $G_k \backslash S_{\beta_0\, k} \cong \mathrm{Ex}_2(k)$
(b) For (2), there are $\mathbf{13}$ non-empty strata S_β. Moreover, except for one stratum S_{β_0}, $G_k \backslash S_{\beta\, k} =$ SP whereas $G_k \backslash S_{\beta_0\, k} \cong \mathrm{Ex}_2(k)$.

For (3), we need more definitions.

Main Theorem

$\mathrm{Prg}_2(k)$: k-isomorphism classes of k-forms of PGL_2.
$\mathrm{QF}_4(k)$: k-isomorphism classes of algebraic groups of the form $\mathrm{GO}(Q)^\circ$ where $Q \in \mathrm{Sym}^2\mathrm{Aff}^4$.
$\mathrm{IQF}_4(k) \subset \mathrm{QF}_4(k)$: inner forms of $\mathrm{GO}(Q_0)^\circ$ (Q_0 split).

Thm (Tajima-Y) For (3), we have the following.
(a) There are 61 non-empty strata S_β.
(b) Suppose that $\mathrm{ch}\,(k) \neq 2$. If $S_\beta \neq \emptyset$ then $G_k \backslash S_{\beta\, k}$ is (i) SP (ii) $\mathrm{Ex}_2(k)$ (iii) $\mathrm{Ex}_3(k)$ (iv) $\mathrm{Prg}_2(k)$ or (v) $\mathrm{IQF}_4(k)$.

Main Theorem

Moreover the number of S_β's for (i)–(v) are as follows.

Type	Number of S_β's
SP	43
$\mathbf{Ex}_2(k)$	12
$\mathbf{Ex}_3(k)$	3
$\mathrm{Prg}_2(k)$	2
$\mathbf{IQF}_4(k)$	1

The case $k = \mathbb{C}$ is known by Kimura–Muro (79), Kimura–Kasai (85), Ozeki (90).

GIT stratification

G: split (for simplicity) connected reductive
V: finite dimensional representation of G
$G_{\mathrm{st}} \subset G$: split connected reductive
$T_0 \subset Z(G)$: non-trivial split torus

Assume $G_{\mathrm{st}} \cap T_0$ is finite, $G = T_0 G_{\mathrm{st}}$.
Assume T_0 acts on V by scalar multiplication.

$(T_0 \cap G_{\mathrm{st}}) \subset T_{\mathrm{st}} \subset G_{\mathrm{st}}$ maximal split torus
$X_*(T_{\mathrm{st}})$: the group of 1PS's
$X^*(T_{\mathrm{st}})$: the group of rational characters.
$\mathfrak{t} = X_*(T_{\mathrm{st}}) \otimes \mathbb{R}$, $\mathfrak{t}_{\mathbb{Q}} = X_*(T_{\mathrm{st}}) \otimes \mathbb{Q}$,
$\mathfrak{t}^* = X^*(T_{\mathrm{st}}) \otimes \mathbb{R}$, $\mathfrak{t}^*_{\mathbb{Q}} = X^*(T_{\mathrm{st}}) \otimes \mathbb{Q}$.

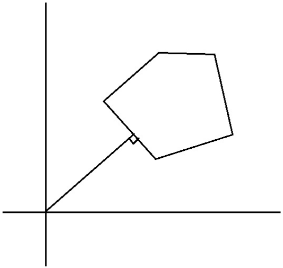
GIT stratification

$\mathbb{W}$: Weyl group of G_{st}
$\mathfrak{t}^*_+ \subset \mathfrak{t}^*$: Weyl chamber
$(\ ,\ )_*$: $\mathbb{W}$-invariant inner product on $\mathfrak{t}^*$,
$\|\ \|_*$ the norm by $(\ ,\ )_*$
$N = \dim V$.
$x = (x_1, \ldots, x_N)$: coordinate on V by which T_{st} acts diagonally.
$\gamma_i \in \mathfrak{t}^*$ the weight of x_i
$\Gamma = \{\gamma_1, \ldots, \gamma_N\}$
For $\mathfrak{I} \subset \Gamma$, Conv $\mathfrak{I}$ is its convex hull β: the closest point of Conv $\mathfrak{I}$ to the origin.
$\mathfrak{B}$: the set of such β which lie in $\mathfrak{g}^*_+$.

GIT stratification

$M_\beta = Z_G(\beta)$

One can define subspaces $Z_\beta, W_\beta \subset V$ and a parabolic subgroup P_β with M_β its Levy part s.t. M_β, P_β act on $Z_\beta, Y_\beta = Z_\beta \oplus W_\beta$ respectively.



GIT stratification

$G_{\beta,\mathrm{st}} \subset M_\beta$: "appropiate scalar direction removed"
$Z_\beta^{\mathrm{ss}} \subset Z_\beta$: semi-stable points with respect to $G_{\beta,\mathrm{st}}$
$Y_\beta^{\mathrm{ss}} = \{(z,w) \mid z \in Z_\beta^{\mathrm{ss}}\}$
$S_\beta^{\mathrm{ss}} = GY_\beta^{\mathrm{ss}}$

Then its is known that

$$V_k \setminus (V_k^{\mathrm{ss}} \cup \{0\}) = \bigcup_{\beta \in \mathfrak{B}} S_{\beta\, k}$$
$$S_{\beta\, k} = G_k \times_{P_{\beta\, k}} Y_{\beta\, k}^{\mathrm{ss}}$$

GIT stratification

If k is a number field, $G_{\mathbb{A}} = KP_{\beta\,\mathbb{A}}$ (K max. compact) and Φ is a K-invariant function,

$$\int_{G_{\mathbb{A}}/G_k} *** \sum_{x \in S_{\beta\, k}} \Phi(gx)dg$$

$$= \int_{P_{\beta\,\mathbb{A}}/P_{\beta\, k}} *** \sum_{x \in Y^{ss}_{\beta\, k}} \Phi(px)dp$$

Finding $\mathfrak{B}$

One can use computer to determine $\mathfrak{B}$
We found $\mathfrak{B}$ for (1)–(3) and

(4) $G = \mathrm{GL}_8, V = \wedge^3 \mathrm{Aff}^8$

The total of CPU time is about 10 minutes

The longest computation: (4) contribution from simplices of dimension 6.

Finding $\mathfrak{B}$

Step 1: X: the set of 7 points of Γ. Find $\mathbb{W}\backslash X$. $\#X$ is about 230 million. $\#\mathbb{W} = 8! = 40320$. $\#\mathbb{W}\backslash X = 7812$.

Step 2: For each element of $\mathbb{W}\backslash X$, make its convex hull, throw away if the dimension is not 6, compute β and move it to $\mathfrak{t}^*_+$.

Step 3: Do the same for $6, \ldots, 1$ points of Γ, remove duplication

When $S_\beta \neq \emptyset$?

In order to show $S_\beta \neq \emptyset$ ($\Leftrightarrow Z_\beta^{\mathrm{ss}} \neq \emptyset$), one has to construct enough (relative) invariant polunomials. Z_β can be reducible even if V is irreducible.

Example: $G_1 = G_2 = \mathrm{GL}_3, G_3 = \mathrm{GL}_2, W_1 = W_2 = \mathrm{Aff}^3$
$G = G_1 \times G_2 \times G_3 \times \mathrm{GL}_1$
$V = \wedge^2 W_2 \oplus W_1 \otimes W_2 \otimes \mathrm{Aff}^2$

W_1, W_2: standard representations of G_1, G_2 respetively.
$t \in \mathrm{GL}_1$ acts on $(x_1, x_2) \in V$ by (tx_1, x_2)
One has to construct two relative invariant polynomials.
One of them is easy.

When $S_\beta \neq \emptyset$?

Elements of $W_1 \otimes W_2 \otimes \mathbf{Aff}^2$ are $x = (A_1, A_2)$ $(A_1, A_2 \in \mathrm{M}_3)$

$v = (v_1, v_2)$ variables
$B(v) = \det(v_1 A_1 + v_2 A_2)$ is a binary cubic form
$P_1(x)$: the discriminant of $B(x)$
The other one is slightly tricky.

When $S_\beta \neq \emptyset$?

$\Phi_1 : W_1 \otimes W_2 \otimes \mathrm{Aff}^2 \to (W_1 \otimes W_2 \otimes \mathrm{Aff}^2)^{6\otimes}$ is

$$\Phi_1(x) = \overbrace{x \otimes \cdots \otimes x}^{6}.$$

$\wedge^3 W_1, \wedge^3 W_2 \cong \mathrm{Aff}^1$, $\wedge^2 \mathrm{Aff}^2 \cong \mathrm{Aff}^1$

$\Phi_2 : (W_1 \otimes W_2 \otimes \mathrm{Aff}^2)^{6\otimes} \to (W_2 \otimes \mathrm{Aff}^2)^{6\otimes}$ is

$$\Phi_2((v_{11} \otimes v_{12} \otimes v_{13}) \otimes \cdots \otimes (v_{61} \otimes v_{62} \otimes v_{63}))$$
$$= (v_{11} \wedge v_{21} \wedge v_{31})(v_{41} \wedge v_{51} \wedge v_{61})(v_{12} \otimes v_{13}) \otimes \cdots \otimes (v_{62} \otimes v_{63}).$$

$\Phi_3 : (W_2 \otimes \mathrm{Aff}^2)^{6\otimes} \to W_2^{6\otimes}$ is

$$\Phi_3((v_{12} \otimes v_{13}) \otimes \cdots \otimes (v_{62} \otimes v_{63}))$$
$$= (v_{13} \wedge v_{43})(v_{23} \wedge v_{53})(v_{33} \wedge v_{63}) v_{12} \otimes \cdots \otimes v_{62}.$$

When $S_\beta \neq \emptyset$?

$\Phi_4 : W_2^{6\otimes} \to W_2^{3\otimes}$ is

$\Phi_4(v_{12} \otimes \cdots \otimes v_{62}) = -(v_{12} \wedge v_{22} \wedge v_{42}) v_{32} \otimes v_{52} \otimes v_{62}$.

$\Phi = \Phi_4 \circ \Phi_3 \circ \Phi_2 \circ \Phi_1$

$$P_2(x_1, x_2) = \frac{1}{6}(x_1 \otimes x_1 \otimes x_1) \wedge \Phi(x_2)$$

for $x_1 \in \wedge^2 W_2, x_2 \in W_1 \otimes W_2 \otimes \mathrm{Aff}^2$
The coefficient $\frac{1}{6}$ can be justified.
Using P_1, P_2, one can show $V^{\mathrm{ss}} \neq \emptyset$ where

$$\chi(g) = (\det g_1)^5 (\det g_2)^8 (\det g_3)^{-12} t^{-15}$$

When $S_\beta \neq \emptyset$?

To show $Z_\beta^{\mathrm{ss}} = \emptyset$, enough to show the following:

1. If $x \in Z_\beta$, $\exists g \in M_\beta$ s.t. some coordinates of $y = gx$ are $\mathbf{0}$.

2. For such y, $\exists$ a 1PS (1 parameter subgroup) $\lambda : \mathrm{GL}_1 \to M_\beta$ s.t. weights of all non-zero coordinates of y are positive and λ^β is trivial.

SL_2 acts on Aff^2. If $x = [x_1, x_2]$, $\exists g \in \mathrm{SL}_2$ s.t.
$y = gx = [y_1, 0]$.
If $\lambda(t) = \mathrm{diag}(t, t^{-1})$, $\lambda(t)y = [ty_1, 0]$.
No semi-stable points

When $S_\beta \neq \emptyset$?

If $P(x) \neq 0$ is an invariant polynomial on Z_β, $P(\lambda(t)gx) \neq 0$. Since weights of non-zero coordinates are positive, $\lambda(t)gx = tz$. $t \to 0$ and $P(\lambda(t)gx) = 0$ Contradiction!

Example: Case (3) coordinates: $x_{ij,k}$
$1 \leq i < j \leq 5, k = 1, \ldots, 4$
$\beta = \frac{1}{90}(-16, -1, -1, 4, 14, -10, -10, 5, 15)$
Z_β coordinates:

$x_{45,1}, x_{25,2}, x_{35,2}, x_{24,3}, x_{34,3}, x_{15,4}, x_{23,4}$
$M_\beta = \mathrm{GL}_1 \times \mathrm{GL}_2 \times \mathrm{GL}_1^2 \times \mathrm{GL}_2 \times \mathrm{GL}_1^2$

When $S_\beta \neq \emptyset$?

The first $\mathbf{GL}_2$ acts on $(x_{25,2}, x_{35,2})$ or $(x_{24,3}, x_{34,3})$.
One can make $x_{25,2} = 0$ or $x_{24,3} = 0$.
$x_{25,2} = 0$ does not work. Assume $x_{24,3} = 0$.
$\lambda(t) =$
$(\mathrm{diag}(t^{-8}, t^{-39}, t^{41}, t^{-8}, t^{14}), \mathrm{diag}(t^{-1}, t^{30}, t^{-28}, t^{-1}))$
Weights

$x_{45,1}$	$x_{25,2}$	$x_{35,2}$	$x_{34,3}$	$x_{15,4}$	$x_{23,4}$
5	5	85	5	5	1

So $Z_\beta^{\mathrm{ss}} = \emptyset$.

$G_k \backslash S_{\beta\, k}$

Suppose $S_\beta \neq \emptyset$.

$$G_k \backslash S_{\beta\, k} \cong P_{\beta\, k} \backslash Y^{ss}_{\beta\, k}$$

It turns out

$$P_{\beta\, k} \backslash Y^{ss}_{\beta\, k} \cong M_{\beta\, k} \backslash Z^{ss}_{\beta\, k}$$

$P_\beta = M_\beta U_\beta$ (U_β unipotent radical)
If $x \in Z^{ss}_{\beta\, k}$, we showed $U_\beta \cap G_x$ connected
If U connected unipotent, k perfect field
$\Rightarrow U$ split
$\Rightarrow \mathrm{H}^1(k, U) = \{1\}$
$\Rightarrow$ Enough to show one can eliminate W_β for one $x \in Z^{ss}_{\beta\, k}$.

$G_k \backslash S_{\beta\, k}$

For $M_{\beta\, k} \backslash Z^{\mathrm{ss}}_{\beta\, k}$, find good $R \in Z^{\mathrm{ss}}_{\beta\, k}$.
Lie alg. computation
$\Rightarrow M_\beta R \subset Z^{\mathrm{ss}}_\beta$ open, $M_{\beta,R}$ smooth reductive.

If
#{relative invariant poly.} = #{irred. factors of} Z_β

$\Rightarrow Z^{\mathrm{ss}}_{\beta\, k^{\mathrm{sep}}} = M_{\beta\, k^{\mathrm{sep}}} R$
$\Rightarrow M_{\beta\, k} \backslash Z^{\mathrm{ss}}_k \cong \mathrm{H}^1(k, M_{\beta,R})$

Smaller representations

(G, V) one of (1)–(3)
(M_β, Z_β) is a prehomogeneous vector space.
Does the GIT stratification of (M_β, Z_β) follow from that of (G, V)?
$\Rightarrow$ Almost "Yes".
If β' is a vector in the parametrizing set of the GIT stratification of (M_β, Z_β),

$$\beta + \beta' = \beta'' = w\beta''' \quad w \in \mathbb{W},\ \beta''' \in \mathfrak{B}$$

$Z_{\beta''}$ similarly defined. $\overline{Z}^{\mathrm{ss}}_{\beta'}$ considered for (M_β, Z_β)

Smaller representations

Condition: $Z_{\beta''} \subset Z_\beta$, $M_{\beta''} \subset M_\beta$,

Prop: Suppose Condition is satisfied.

(1) $Z_{\beta''}^{\mathrm{ss}} = \overline{Z}_{\beta'}^{\mathrm{ss}}$.

(2) $M_{\beta''\,k} \backslash Z_{\beta''\,k}^{\mathrm{ss}} \cong \overline{M}_{\beta'\,k} \backslash \overline{Z}_{\beta'\,k}^{\mathrm{ss}}$

(3) $P_{\beta''\,k} \backslash Y_{\beta''\,k}^{\mathrm{ss}} \cong \overline{P}_{\beta'\,k} \backslash \overline{Y}_{\beta'\,k}^{\mathrm{ss}}$ (some condition on the unipotent part)

(4) $\overline{S}_{\beta'} \neq \emptyset$ if and only if $S_{\beta''} \neq \emptyset$.

Smaller representations

Examples of prehomogeneous vector spaces contained in (1)–(4)

$(\mathrm{GL}_4 \times \mathrm{GL}_2, \wedge^2 \mathrm{Aff}^4 \otimes \mathrm{Aff}^2)$

$(\mathrm{GL}_5 \times \mathrm{GL}_3, \wedge^2 \mathrm{Aff}^5 \otimes \mathrm{Aff}^3)$

$(\mathrm{GL}_4 \times \mathrm{GL}_3, \wedge^2 \mathrm{Aff}^4 \otimes \mathrm{Aff}^3)$

$(\mathrm{GL}_7, \wedge^3 \mathrm{Aff}^7)$

$(\mathrm{GL}_6, \wedge^3 \mathrm{Aff}^6)$

Case (1) is contained in Case (3).

Many reducible prehomogeneous vector spaces are contained in (3).