Stickelberger ideals and Fitting ideals of class groups for abelian number fields

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Abstract

In this paper, we determine completely the initial Fitting ideal of the minus part of the ideal class group of an abelian number field over $\mathbf{Q}$ up to the 2-component. This answers an open question of Mazur and Wiles [11] up to the 2-component, and proves Conjecture 0.1 in [8]. We also study Brumer's conjecture and prove a stronger version for a CM-field, assuming certain conditions, in particular on the Galois group.

0 Introduction

0.1. The ideal class group of a cyclotomic field is one of the most classical and important objects in number theory. For example, the class number has been intensively studied since Kummer. But in many situations, we need the action of the Galois group on the ideal class group. The study of the Galois action also has a tradition going back to Kummer.

Let $K = \mathbf{Q}(\mu_N)$ denote the cyclotomic field obtained by adjoining the N-th roots of unity for some integer $N > 0$, and let Cl_K be the ideal class group. The most fundamental work on the Galois action on Cl_K was done by Stickelberger after the earlier works by Gauss and Kummer. We denote by $\mu(K)$ the group of all roots of unity in K, and by $\mathrm{Ann}(\mu(K))$ the annihilator ideal in $\mathbf{Z}[\mathrm{Gal}(K/\mathbf{Q})]$ as a $\mathbf{Z}[\mathrm{Gal}(K/\mathbf{Q})]$-module. Let θ_K be the Stickelberger element (see §1 for the definition). Then $\mathrm{Ann}(\mu(K))\theta_K$ is in $\mathbf{Z}[\mathrm{Gal}(K/\mathbf{Q})]$ and annihilates Cl_K, namely $\mathrm{Ann}(\mu(K))\theta_K \subset \mathrm{Ann}(\mathrm{Cl}_K)$ (cf. Stickelberger [15]). (This was proved by Kummer when N is a prime.) Since the Stickelberger element is related to the values of L-functions, this can be regarded as a relation between the analytic object and the algebraic object, namely the class group.

Let K be an imaginary abelian field with $[K : \mathbf{Q}] < \infty$. In their celebrated paper [11], in which Mazur and Wiles proved the Iwasawa main conjecture, they asked what the Fitting ideal of Cl_K^- is as a $\mathbf{Z}[\mathrm{Gal}(K/\mathbf{Q})]$-module where Cl_K^- is the cokernel of $\mathrm{Cl}_{K^+} \longrightarrow \mathrm{Cl}_K$. In general, for a commutative ring R and a finitely presented R-module M such that $R^m \stackrel{f}{\longrightarrow} R^n \longrightarrow M \longrightarrow 0$ is exact, the Fitting ideal (the initial Fitting ideal) of M is defined to be the ideal of R generated by all $n \times n$ minors of the matrix A which corresponds to f. We denote it by $\mathrm{Fitt}_R(M)$. By definition, we always have $\mathrm{Fitt}_R(M) \subset \mathrm{Ann}_R(M)$.

In this paper, we neglect the 2-primary components of the modules, and always work over $\mathbf{Z}' = \mathbf{Z}[1/2]$. We put $R_K = \mathbf{Z}'[\mathrm{Gal}(K/\mathbf{Q})]$, and $\mathrm{Cl}'_K = \mathrm{Cl}_K \otimes \mathbf{Z}'$ which we regard as an R_K-module. Since 2 is invertible in R_K, R_K is decomposed into $R_K = R_K^+ \oplus R_K^-$ where $R_K^\pm$ denotes the part on which the complex conjugation acts as ± 1. Any R_K-module M is also decomposed into $M = M^+ \oplus M^-$. We are interested in the R_K^--module $(\mathrm{Cl}'_K)^-$ which is the same as $\mathrm{Coker}(\mathrm{Cl}_{K^+} \longrightarrow \mathrm{Cl}_K) \otimes \mathbf{Z}'$. In [8], we defined the Stickelberger ideal $\Theta_K \subset R_K$ (whose definition we will recall in §1), which is constructed from Stickelberger elements of several abelian fields. In this paper, we prove Conjecture 0.1 in [8], namely,

Theorem 0.1 *For any imaginary abelian field K of finite degree over $\mathbf{Q}$, we have*

$$\mathrm{Fitt}_{R_K^-}((\mathrm{Cl}'_K)^-) = \Theta_K^-.$$

Remark 0.2 (1) Stickelberger's theorem implies that $\mathrm{Ann}_{R_K^-}((\mathrm{Cl}'_K)^-) \supset \Theta_K^-$. The above theorem gives the exact algebraic meaning of Θ_K^- (namely, it is equal to the Fitting ideal which is smaller than the annihilator). In this sense, Theorem 0.1 can be regarded as a refinement of the classical Stickelberger's theorem.

(2) Taking the limit of Theorem 0.1 with respect to the cyclotomic $\mathbf{Z}_p$-extension and taking an odd character component, we recover the usual main conjecture in Iwasawa theory. On the other hand, this theorem contains more information than the usual main conjecture (because the main conjecture gives only information on the character components), so this theorem is a refinement of the Iwasawa main conjecture. Namely, this theorem gives a more refined relationship between the algebraic object (the left hand side) and the analytic object (the right hand side). (We also mention that in the proof of the theorem we use the main conjecture as an important ingredient, so our argument does not give a new proof of the main conjecture.)

(3) In some places in the literature, $\mathrm{Ann}(\mu(K))\theta_K \subset \mathbf{Z}[\mathrm{Gal}(K/\mathbf{Q})]$ is called the Stickelberger ideal for K. But Iwasawa and Sinnott thought this too small and defined a larger Stickelberger ideal $\Theta_K^{IS} \subset \mathbf{Z}[\mathrm{Gal}(K/\mathbf{Q})]$. They proved that the index is expressed by using $h_K^- = \#\,\mathrm{Cl}_K^-$ ([12], [13], [7]).

In particular, if $K = \mathbf{Q}(\mu_N)$, their theorem implies $(R^-_{\mathbf{Q}(\mu_N)} : (\Theta^{IS}_{\mathbf{Q}(\mu_N)} \otimes \mathbf{Z}')^-) = \#(\mathrm{Cl}'_{\mathbf{Q}(\mu_N)})^-$ (cf. [12]). Our Stickelberger ideal $\Theta^-_{\mathbf{Q}(\mu_N)}$ coincides with $(\Theta^{IS}_{\mathbf{Q}(\mu_N)} \otimes \mathbf{Z}')^-$ if $K = \mathbf{Q}(\mu_N)$, but not in general.

(4) For example, for a cyclotomic field $K = \mathbf{Q}(\mu_N)$, we define our Θ_K by $\Theta_K = \Theta'_K \cap R_K$, using some R_K-module Θ'_K in $\mathbf{Q}[\mathrm{Gal}(K/\mathbf{Q})]$ (see §1). In some places in the literature, $\mathrm{Ann}(\mu(K))\Theta'_K \subset \mathbf{Z}[\mathrm{Gal}(K/\mathbf{Q})]$ is used as the Stickelberger ideal (because to treat the product is sometimes easier than to treat the intersection), but this ideal is smaller than our Θ_K, and does not fit well with the Fitting ideal.

(5) We will explain how one can obtain the information on the Galois action on Cl_K in the simplest example. Suppose $K = \mathbf{Q}(\mu_{57})$. Then we can compute $\Theta_{\mathbf{Q}(\mu_{57})} = (9, \sigma_{40} - 7, \sigma_{-1} + 1) \subset R_{\mathbf{Q}(\mu_{57})}$ where σ_i is the element in $\mathrm{Gal}(\mathbf{Q}(\mu_{57})/\mathbf{Q})$ such that $\sigma_i(\zeta) = \zeta^i$ for $\zeta \in \mu_{57}$. There is a unique maximal ideal $\mathfrak{m} = (3, \sigma_{40} - 1, \sigma_{-1} + 1)$ of $R_{\mathbf{Q}(\mu_{57})}$ which contains $\Theta_{\mathbf{Q}(\mu_{57})}$. Since $\sigma_{40} - 7$ is not in $\mathfrak{m}^2$, Theorem 0.1 implies that $\mathrm{Fitt}_{R^-_{\mathbf{Q}(\mu_{57})}}((\mathrm{Cl}'_{\mathbf{Q}(\mu_{57})})^-) \not\subset \mathfrak{m}^2$. This shows that $(\mathrm{Cl}'_{\mathbf{Q}(\mu_{57})})^-$ is generated by one element (because if it was generated by exactly n elements with $n \geq 2$, the Fitting ideal would be generated by the determinants of $n \times n$ matrices with entries in $\mathfrak{m}$, so would be in $\mathfrak{m}^n$). Hence, $(\mathrm{Cl}'_{\mathbf{Q}(\mu_{57})})^-$ is isomorphic to $R^-_{\mathbf{Q}(\mu_{57})}/(9, \sigma_{40} - 7, \sigma_{-1} + 1)$. In this case, the class number of $K = \mathbf{Q}(\mu_{57})$ is 9, so $\mathrm{Cl}_{\mathbf{Q}(\mu_{57})} = (\mathrm{Cl}'_{\mathbf{Q}(\mu_{57})})^-$, and we can determine the Galois action as follows;

$$\mathrm{Cl}_{\mathbf{Q}(\mu_{57})} \simeq \mathbf{Z}[\mathrm{Gal}(\mathbf{Q}(\mu_{57})/\mathbf{Q})]/(9, \sigma_{40} - 7, \sigma_{-1} + 1).$$

Note that the information that $\Theta_{\mathbf{Q}(\mu_{57})} \subset \mathrm{Ann}_{R_{\mathbf{Q}(\mu_{57})}}(\mathrm{Cl}_{\mathbf{Q}(\mu_{57})})$ (Stickelberger) and $(R^-_{\mathbf{Q}(\mu_{57})} : \Theta^-_{\mathbf{Q}(\mu_{57})}) = \#\,\mathrm{Cl}_{\mathbf{Q}(\mu_{57})}$ (Iwasawa-Sinnott) does not even determine the structure of $\mathrm{Cl}_{\mathbf{Q}(\mu_{57})}$ as an abelian group (namely, there are still two possibilities $\mathbf{Z}/3 \oplus \mathbf{Z}/3$ and $\mathbf{Z}/9$). On the other hand, our isomorphism above, of course, implies that $\mathrm{Cl}_{\mathbf{Q}(\mu_{57})}$ is cyclic of order 9 as an abelian group. (This can be also obtained easily from genus theory.)

(6) Theorem 0.1 was already proved in several special cases. For example, for a prime p which does not divide $[K : \mathbf{Q}]$, the p-component of Theorem 0.1 is a theorem of Mazur and Wiles (Chap. I §10 in [11] Theorem 2). If $K = \mathbf{Q}(\mu_{p^n})$ for some prime p, Theorem 0.1 was proved by Greither in [3] (for more, see [8] §0).

(7) Very roughly speaking, a key idea of the proof of Theorem 0.1 is to consider many abelian number fields (subfields of K and abelian fields which contain K) simultaneously.

(8) Theorem 0.1 implies the following beautiful result by D. Solomon (cf. [14]), which is a generalization of the theorem of Mazur and Wiles we referred

to in (6). Let p be an odd prime number, and let $\psi : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \longrightarrow \overline{\mathbf{Q}}_p^\times$ be *any* odd Dirichlet character whose order could be divisible by p. We denote by K the imaginary abelian field corresponding to $\mathrm{Ker}\,\psi$. We will define the ψ-quotient A_K^ψ at the end of §4. We exclude special ψ (see the end of §4), and regard the generalized Bernoulli number $B_{1,\psi^{-1}}$ as an element of $O_\psi = \mathbf{Z}_p[\mathrm{Image}\,\psi]$. Then Theorem 0.1 immediately implies

$$\# A_K^\psi = \# O_\psi/(B_{1,\psi^{-1}})$$

which is a theorem of Solomon (see Corollary 4.3). For more information on A_K^ψ than the order, for example, concerning on the structure of A_K^ψ, see [9] and [10].

0.2. We can apply our method to a slightly more general setting, which we will describe. Let p be an odd prime number, k a totally real number field, and K/k a finite abelian extension such that K is a CM-field. The Stickelberger element $\theta_{K/k}$ is defined by

$$\theta_{K/k} = \sum_{\sigma \in \mathrm{Gal}(K/k)} \zeta(0,\sigma)\sigma^{-1} \in \mathbf{Q}[\mathrm{Gal}(K/k)]$$

where $\zeta(s,\sigma) = \sum_{(\frac{K/k}{\mathfrak{a}})=\sigma} N(\mathfrak{a})^{-s}$ is the partial zeta function. For simplicity, we suppose that a primitive p-th root of unity is not in K. Then by Deligne and Ribet we know $\theta_{K/k} \in \mathbf{Z}_p[\mathrm{Gal}(K/k)]$. We study the p-component of Brumer's conjecture, namely the problem of whether $\theta_{K/k}$ kills the p-component $A_K = \mathrm{Cl}_K \otimes \mathbf{Z}_p$ of the ideal class group of K. More precisely, we study a stronger version that considers whether $\theta_{K/k}$ is in the Fitting ideal of A_K. Generally, the answer is no (see [6]). But for example, Greither proved that the equivariant Tamagawa number conjecture implies this property for the Pontryagin dual of A_K, at least in our setting (see [5]). There are several cases where this stronger version holds for A_K itself. For example, this holds if K/k is "nice" (cf. Greither [3]) and if we assume there is "no trivial zero" in the p-adic L-functions and some extra conditions (see Greither [4] §3 and Theorem 0.4 in [8]). It seems difficult to remove the assumption on the trivial zeros, but in this paper, we give examples for which the stronger version holds even if "trivial zeros" occur. Instead of the non-existence of trivial zeros, we assume a strong condition on $\mathrm{Gal}(K/k)$.

We assume that there is a finite abelian extension K'/k such that K' is a CM-field, $K \subset K'$, K'/K is a p-extension which is unramified everywhere, and K'/k satisfies the condition

$$(\mathrm{A}_p) \qquad \Gamma_{K'} = P_{\mathfrak{l}_1} \times ... \times P_{\mathfrak{l}_r}$$

where $\Gamma_{K'}$ is the Sylow p-subgroup of $\mathrm{Gal}(K'/k)$, $\mathfrak{l}_1$,...,$\mathfrak{l}_r$ are all the ramifying primes of k in K whose ramification indices are divisible by p, and $P_{\mathfrak{l}_i}$ is the Sylow p-subgroup of the inertia group of $\mathfrak{l}_i$ in $\mathrm{Gal}(K'/k)$ $(1 \le i \le r)$.

Theorem 0.3 *We assume that K is contained in K' which satisfies the above conditions, and $\mu_p \not\subset K'$. Moreover, we assume that for every prime $\mathfrak{p}$ of k above p, the ramification index of $\mathfrak{p}$ in $k/\mathbf{Q}$ is odd, the μ-invariant of the cyclotomic $\mathbf{Z}_p$-extension K_∞/K is zero, and all the primes $\mathfrak{p}$ of k above p are unramified in K. Then we have*

$$\theta_{K/k} \in \mathrm{Fitt}_{\mathbf{Z}_p[\mathrm{Gal}(K/k)]}(A_K).$$

In particular, $\theta_{K/k} A_K = 0$.

Note that this theorem can be applied to the case where a prime above p splits completely in K/k (the case where trivial zeros occur) if K is contained in some good extension K'/k satisfying the above conditions. Moreover, we will define the Stickelberger ideal $\Theta_{K'/k}$ in §3, and will prove in §3 that $\mathrm{Fitt}_{\mathbf{Z}_p[\mathrm{Gal}(K'/k)]^-}(A_{K'}^-) = \Theta_{K'/k}^- \otimes \mathbf{Z}_p$ (see Theorem 3.6). The fields which satisfy the condition (A_p) naturally appear in the theory of Euler systems, and Theorem 0.3 has an application to the structure theorem in the style of [9] for the class groups of CM-fields, which we hope to study in our forthcoming paper.

In §4, we study the Fitting ideal for the m-th layer K'_m of the cyclotomic $\mathbf{Z}_p$-extension K'_∞/K' with some $m > 0$, and prove the analogous result (see Theorem 4.1).

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Erratum The first-named author would like to give a correction concerning the paper [8]. He is very grateful to D. Solomon for his question on Corollary 0.7 in [8].
Page 41 Corollary 0.7: The claim should be "Conjecture 0.1 is true for $F = K_n$ (the n-th layer of the cyclotomic $\mathbf{Z}_p$-extension)" instead of "Conjecture 0.1 is true for $F = K(\mu_{p^n})$", in order to apply Theorems 0.5 and 0.6.

Notation
For a positive integer n, μ_n denotes the group of all n-th roots of unity. For a group G and a G-module M, we denote by M^G the G-invariant part of M (the maximal subgroup of M on which G acts trivially), and by M_G the G-coinvariant of M (the maximal quotient of M on which G acts trivially). For a prime number p, $\mathrm{ord}_p : \mathbf{Q}^\times \longrightarrow \mathbf{Z}$ is the normalized additive valuation such that $\mathrm{ord}_p(p) = 1$.

1 Stickelberger ideals

1.1. Notation. Let $K'/\mathbf{Q}$ be an abelian extension, and let K be a subfield of K' such that $[K':K]<\infty$. We denote by

$$c_{K'/K}:\mathbf{Z}[\mathrm{Gal}(K'/\mathbf{Q})]\longrightarrow\mathbf{Z}[\mathrm{Gal}(K/\mathbf{Q})]$$

the ring homomorphism induced by the restriction $\sigma\mapsto\sigma_{|K}$ for any $\sigma\in\mathrm{Gal}(K'/\mathbf{Q})$. The homomorphism of $\mathbf{Z}$-modules

$$\nu_{K'/K}:\mathbf{Z}[\mathrm{Gal}(K/\mathbf{Q})]\longrightarrow\mathbf{Z}[\mathrm{Gal}(K'/\mathbf{Q})]$$

is defined by $\sigma\mapsto\Sigma_{c_{K'/K}(\tau)=\sigma}\tau$ for any $\sigma\in\mathrm{Gal}(K/\mathbf{Q})$. These notations will be used for any group rings such as $\mathbf{Q}[\mathrm{Gal}(K/\mathbf{Q})]$, $\mathbf{Z}_p[\mathrm{Gal}(K/\mathbf{Q})]$, etc.

1.2. Let K be an imaginary abelian field of finite degree over $\mathbf{Q}$.

First of all, we recall the definition of the Stickelberger ideal of $K/\mathbf{Q}$ in [8]. We define the Stickelberger element by

$$\theta_K=\sum_{\sigma\in\mathrm{Gal}(K/\mathbf{Q})}\zeta(0,\sigma)\sigma^{-1}\in\mathbf{Q}[\mathrm{Gal}(K/\mathbf{Q})]$$

where $\zeta(s,\sigma)=\sum_{(\frac{K/\mathbf{Q}}{a})=\sigma}a^{-s}$ is the partial zeta function. Namely, if the conductor of K is N, θ_K is the image of

$$\sum_{\substack{a=1\\(a,N)=1}}^{N}(\frac{1}{2}-\frac{a}{N})\sigma_a^{-1}\in\mathbf{Q}[\mathrm{Gal}(\mathbf{Q}(\mu_N)/\mathbf{Q})]$$

where σ_a is the element of $\mathrm{Gal}(\mathbf{Q}(\mu_N)/\mathbf{Q})$ such that $\sigma_a(\zeta)=\zeta^a$ for any $\zeta\in\mu_N$.

Suppose that N is the conductor of K, and $N=\ell_1^{e_1}\cdot\ldots\cdot\ell_r^{e_r}$ is the prime decomposition of K (we take $e_1,\ldots,e_r>0$). We say K satisfies the assumption (A) if

$$\text{(A)}\qquad \mathrm{Gal}(K/\mathbf{Q})=I_{\ell_1}\times\ldots\times I_{\ell_r}$$

where I_{ℓ_i} is the inertia subgroup of ℓ_i in $\mathrm{Gal}(K/\mathbf{Q})$. A typical example is, of course, $K=\mathbf{Q}(\mu_N)$. We first assume that K satisfies this condition (A).

In this paper, we neglect the 2-primary components of ideal class groups. We put $\mathbf{Z}'=\mathbf{Z}[1/2]$, and $R_K=\mathbf{Z}'[\mathrm{Gal}(K/\mathbf{Q})]$. Let $\mathcal{F}(K)$ be the set of all intermediate fields of $K/\mathbf{Q}$. We define an R_K-module Θ'_K to be the R_K-submodule of $\mathbf{Q}[\mathrm{Gal}(K/\mathbf{Q})]$ generated by all $\nu_{K/F}(\theta_F)$ for $F\in\mathcal{F}(K)$, namely

$$\Theta'_K=\langle\{\nu_{K/F}(\theta_F)\mid F\in\mathcal{F}(K)\}\rangle\subset\mathbf{Q}[\mathrm{Gal}(K/\mathbf{Q})].$$

We need not use all subfields of K. We denote by $\mathcal{D}$ the set of all numbers of the form $d = \ell_1^{n_1} \cdot \ldots \cdot \ell_r^{n_r}$ where $n_i = 0$ or e_i for all i such that $1 \leq i \leq r$. For any $d \in \mathcal{D}$, we define K_d to be the maximal subfield in K which is unramified outside d over $\mathbf{Q}$. If $d = \ell_{i_1}^{e_{i_1}} \cdot \ldots \cdot \ell_{i_s}^{e_{i_s}}$,

$$\mathrm{Gal}(K_d/\mathbf{Q}) = I_{\ell_{i_1}} \times \ldots \times I_{\ell_{i_s}}$$

holds. Hence, K_d is ramified at primes dividing d, and K/K_d is unramified outside N/d and ramified at primes dividing N/d. The fields $K_{\ell_1^{e_1}}, \ldots, K_{\ell_r^{e_r}}$ are linearly disjoint over $\mathbf{Q}$, and their compositum is K. Suppose that $F \in \mathcal{F}(K)$, and $F/\mathbf{Q}$ is unramified outside d and ramified at primes dividing d. By definition, F is in K_d. We know $c_{K_d/F}(\theta_{K_d}) = \theta_F$ by Lemma 2.1 in [8], so

$$\nu_{K/F}(\theta_F) = [K_d : F]\nu_{K/K_d}(\theta_{K_d}).$$

This shows that Θ'_K is generated as an R_K-module by all $\nu_{K/K_d}(\theta_{K_d})$'s;

$$\Theta'_K = \langle \{\nu_{K/K_d}(\theta_{K_d}) \mid d \in \mathcal{D}\} \rangle \subset \mathbf{Q}[\mathrm{Gal}(K/\mathbf{Q})].$$

In the same way, we can easily check that our Θ'_K coincides with $\Theta'_{K/\mathbf{Q}}$ in [8]. We define the Stickelberger ideal Θ_K to be

$$\Theta_K = \Theta'_K \cap R_K.$$

For any imaginary abelian extension $K/\mathbf{Q}$ of finite degree, we can take K' which satisfies the condition (A) and K'/K is unramified everywhere (K' is unique for K; cf. Lemma 2.3 in [8]). We define Θ_K to be

$$\Theta_K = c_{K'/K}(\Theta_{K'}) \subset R_K.$$

(In [8], Θ_K was denoted by $\Theta_{K/\mathbf{Q}}$.)

1.3. Suppose that p is an odd prime number. We put $R_{K,p} = R_K \otimes \mathbf{Z}_p = \mathbf{Z}_p[\mathrm{Gal}(K/\mathbf{Q})]$. In order to prove Theorem 0.1, it is enough to prove

$$\mathrm{Fitt}_{R_{K,p}^-}((\mathrm{Cl}'_K)^- \otimes \mathbf{Z}_p) = \Theta_K^- \otimes \mathbf{Z}_p$$

for all odd primes p. In the following, we fix an odd prime number p, and we will prove the above equality. We write $\mathrm{Gal}(K/\mathbf{Q}) = \Delta_K \times \Gamma_K$ where $\#\Delta_K$ is prime to p and Γ_K is a p-group. Then $R_{K,p}$ is semi-local and is decomposed as follows. The group ring $\mathbf{Z}_p[\Delta_K]$ is semi-local, and isomorphic to a product of discrete valuation rings. Let $\hat{\Delta}_K$ be the group of $\overline{\mathbf{Q}}_p^\times$-valued characters of Δ_K. We say two characters χ_1 and χ_2 are $\mathbf{Q}_p$-conjugate if

$\sigma\chi_1 = \chi_2$ for some $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$. Using this equivalence relation on $\hat{\Delta}_K$, we have $\mathbf{Z}_p[\Delta_K] = \bigoplus_\chi O_\chi$, and

$$R_{K,p} = \mathbf{Z}_p[\mathrm{Gal}(K/\mathbf{Q})] = \bigoplus_\chi O_\chi[\Gamma_K]$$

where the sum is taken over the equivalence classes of $\hat{\Delta}_K$ (we choose a character χ from each equivalence class), and $O_\chi = \mathbf{Z}_p[\mathrm{Image}\,\chi]$ which is the $\mathbf{Z}_p[\Delta_K]$-module such that Δ_K acts via χ (cf. [8] 1.4). Note that the order of χ is prime to p, so O_χ is a discrete valuation ring which is unramified over $\mathbf{Z}_p$.

For any $R_{K,p}$-module M, we denote by M^χ the $O_\chi[\Gamma_K]$-module $M \otimes_{\mathbf{Z}_p[\Delta_K]} O_\chi = M \otimes_{R_{K,p}} O_\chi[\Gamma_K]$. We have decomposition $M = \bigoplus_\chi M^\chi$. In the following, for the ideal class group we use the notation $A_K = \mathrm{Cl}_K \otimes \mathbf{Z}_p$, and

$$A_K^\chi = A_K \otimes_{\mathbf{Z}_p[\Delta_K]} O_\chi = A_K \otimes_{R_{K,p}} O_\chi[\Gamma_K]$$

which is an $O_\chi[\Gamma_K]$-module. We also consider $(\Theta_K \otimes \mathbf{Z}_p)^\chi$. By the above decomposition of $R_{K,p}$, A_K, and $(\Theta_K \otimes \mathbf{Z}_p)$, in order to prove Theorem 0.1, it is enough to prove

Theorem 1.1 *Let p be an odd prime number, and let χ be an odd character of Δ_K. Then we have*

$$\mathrm{Fitt}_{O_\chi[\Gamma_K]}(A_K^\chi) = (\Theta_K \otimes \mathbf{Z}_p)^\chi.$$

1.4. In this subsection, we begin with the following lemma which is more or less well-known.

Lemma 1.2 *Let F, F' be CM-fields such that $F \subset F'$ and F'/F is an abelian extension. Then we have an exact sequence*

$$(\bigoplus I_v(F'/F) \otimes \mathbf{Z}_p)^- \longrightarrow (A_{F'}^-)_{\mathrm{Gal}(F'/F)} \longrightarrow A_F^- \longrightarrow 0$$

where v runs over all finite primes of F, and $I_v(F'/F)$ is the inertia group of v in $\mathrm{Gal}(F'/F)$.

Proof. Let F'' be the intermediate field such that F'/F'' is a p-extension and $[F'' : F]$ is prime to p. Since $\mathrm{Gal}(F'/F) = \mathrm{Gal}(F'/F'') \times \mathrm{Gal}(F''/F)$, for a prime v of F, taking a prime w_0 of F'' which is above v, we have

$$(\bigoplus_{w|v} I_w(F'/F''))_{\mathrm{Gal}(F''/F)} \simeq I_{w_0}(F'/F'') \simeq I_v(F'/F) \otimes \mathbf{Z}_p.$$

This implies $(\bigoplus I_w(F'/F''))^-_{\mathrm{Gal}(F''/F)} = (\bigoplus I_v(F'/F)\otimes\mathbf{Z}_p)^-$ (where w (resp. v) runs over all finite primes of F'' (resp. F)). It is easy to see $(A^-_{F''})_{\mathrm{Gal}(F''/F)} \simeq A^-_F$. Therefore, in order to prove this lemma, we may assume F'/F is a p-extension. Then this lemma follows from Proposition 5.2 in [8].

As in the definition of Θ_K, suppose that K'/K is unramified everywhere. By Lemma 1.2, the norm map induces an isomorphism $(A^-_{K'})_{\mathrm{Gal}(K'/K)} \xrightarrow{\simeq} A^-_K$. Hence, for any odd character $\chi \in \hat{\Delta}_K$, regarding it as a character of $\Delta_{K'}$, we have $c_{K'/K}(\mathrm{Fitt}_{O_\chi[\Gamma_{K'}]}(A^\chi_{K'})) = \mathrm{Fitt}_{O_\chi[\Gamma_K]}(A^\chi_K)$. Therefore, in order to prove Theorem 1.1, we may assume that K satisfies the condition (A) in subsection 1.2.

In the rest of §1 and §2, we always assume that K satisfies the condition (A). Since $L(0,\psi) \neq 0$ for any odd character ψ of $\mathrm{Gal}(K/\mathbf{Q})$, the ψ-component of $\Theta'_K \otimes \mathbf{Q}$ does not vanish, and $(\Theta'_K \otimes \mathbf{Q})^- = (R_K \otimes \mathbf{Q})^- = \mathbf{Q}[\mathrm{Gal}(K/\mathbf{Q})]^-$. Since $\mathbf{Z}_p$ is flat over $\mathbf{Z}'$, we can easily check that

$$(\Theta_K \otimes \mathbf{Z}_p)^- = (\Theta'_K \cap R_K)^- \otimes \mathbf{Z}_p = (\Theta'_K \otimes \mathbf{Z}_p)^- \cap R_{K,p}$$

in $R_{K,p} \otimes \mathbf{Q} = \mathbf{Q}_p[\mathrm{Gal}(K/\mathbf{Q})]$. Therefore, taking the χ-component we also obtain

Lemma 1.3 *We have* $(\Theta_K \otimes \mathbf{Z}_p)^\chi = (\Theta'_K \otimes \mathbf{Z}_p)^\chi \cap O_\chi[\Gamma_K]$.

Suppose that K contains a primitive p-th root of unity, namely $\mu_p \subset K$. Let $\omega : \Delta_K \longrightarrow \mathbf{Z}_p^\times$ be the Teichmüller character, namely the character giving the action on μ_p. We denote the image of θ_K in $(\mathbf{Q}_p[\mathrm{Gal}(K/\mathbf{Q})])^\chi = \mathbf{Q}_p(\mathrm{Image}\,\chi)[\Gamma_K]$ by θ_K^χ. It is well-known that if $\chi \neq \omega$, θ_K^χ is in $R^\chi_{K,p} = O_\chi[\Gamma_K]$. Hence we have $(\Theta_K \otimes \mathbf{Z}_p)^\chi = (\Theta'_K \otimes \mathbf{Z}_p)^\chi$ for $\chi \neq \omega$.

2 The ω-component

In this section, we fix an odd prime number p, and work over $R_{K,p} = \mathbf{Z}_p[\mathrm{Gal}(K/\mathbf{Q})]$. In this section, we will prove Theorem 1.1 for $\chi = \omega$. First of all, we need the following lemma which is well-known (Washington [17], Lemma 6.9 and its proof).

Lemma 2.1 *For* $\mathbf{Q}(\mu_{p^m})$ *with* $m > 0$*, we put*

$$I = \{\alpha \in R_{\mathbf{Q}(\mu_{p^m}),p} \mid \alpha\theta_{\mathbf{Q}(\mu_{p^m})} \in R_{\mathbf{Q}(\mu_{p^m}),p}\}.$$

Then we have $I = \mathrm{Ann}_{R_{\mathbf{Q}(\mu_{p^m}),p}}(\mu_{p^m})$. *(For an* R*-module* M*,* $\mathrm{Ann}_R(M)$ *means the annihilator of* M*.)*

In this section, we assume that K satisfies the condition (A), and we will use the same notation as in the previous section. We fix an odd prime number p, and assume that $\mu_p \subset K$. Hence p is ramified in K, and one of $\ell_1,\ldots,\ell_r$ is p. Therefore, p^m with some $m > 0$ is in $\mathcal{D}$. This also implies that $\mu_{p^m} \subset K$, and $K_{p^m} = \mathbf{Q}(\mu_{p^m})$.

Lemma 2.2 *Suppose that $\beta \in R_{K,p}$ satisfies $\beta\nu_{K/\mathbf{Q}(\mu_{p^m})}(\theta_{\mathbf{Q}(\mu_{p^m})}) \in R_{K,p}$. Then there is an element $\alpha \in \mathrm{Ann}_{R_{\mathbf{Q}(\mu_{p^m}),p}}(\mu_{p^m})$ such that*

$$\beta\nu_{K/\mathbf{Q}(\mu_{p^m})}(\theta_{\mathbf{Q}(\mu_{p^m})}) = \nu_{K/\mathbf{Q}(\mu_{p^m})}(\alpha\theta_{\mathbf{Q}(\mu_{p^m})}).$$

Proof. Put $c = c_{K/\mathbf{Q}(\mu_{p^m})}$ and $\nu = \nu_{K/\mathbf{Q}(\mu_{p^m})}$. Since $\beta\nu(\theta_{\mathbf{Q}(\mu_{p^m})}) = \nu(c(\beta)\theta_{\mathbf{Q}(\mu_{p^m})})$, $c(\beta)\theta_{\mathbf{Q}(\mu_{p^m})}$ is in $R_{\mathbf{Q}(\mu_{p^m}),p}$. Hence Lemma 2.1 implies $c(\beta) \in \mathrm{Ann}_{R_{\mathbf{Q}(\mu_{p^m}),p}}(\mu_{p^m})$, and we can take $\alpha = c(\beta)$.

We put $\mathcal{D}_0 = \{d \in \mathcal{D} \mid p \text{ does not divide } d\}$, and $\mathcal{D}_1 = \{d \in \mathcal{D} \mid p \text{ divides } d\}$. Suppose at first that d is in $\mathcal{D}_1$. Then p^m divides d by definition. We define $d' = d/p^m$, which is prime to p by definition. For any integer a, we write $a = p^m[\frac{a}{p^m}]+r_a$ where $[\frac{a}{p^m}]$, $r_a \in \mathbf{Z}$ are integers satisfying $0 \leq r_a < p^m$.

We have

$$\begin{aligned}
\theta_{K_d} &= c_{\mathbf{Q}(\mu_d)/K_d}(\sum_{\substack{a=1\\(a,d)=1}}^{d}(\frac{1}{2}-\frac{a}{d})\sigma_a^{-1}) = c_{\mathbf{Q}(\mu_d)/K_d}(\sum_{\substack{a=1\\(a,d)=1}}^{d}(\frac{1}{2}-\frac{p^m[\frac{a}{p^m}]+r_a}{d})\sigma_a^{-1}) \\
&= c_{\mathbf{Q}(\mu_d)/K_d}(\nu_{\mathbf{Q}(\mu_d)/\mathbf{Q}}(\frac{1}{2}) - \frac{1}{d'}\sum_{\substack{a=1\\(a,d)=1}}^{d}[\frac{a}{p^m}]\sigma_a^{-1} - \frac{1}{d}\sum_{\substack{a=1\\(a,p)=1}}^{p^m} a\nu_{\mathbf{Q}(\mu_d)/\mathbf{Q}(\mu_{p^m})}(\sigma_a^{-1})) \\
&= \frac{e_d}{2}\nu_{K_d/\mathbf{Q}}(1) - c_{\mathbf{Q}(\mu_d)/K_d}(\frac{1}{d'}\sum_{\substack{a=1\\(a,d)=1}}^{d}[\frac{a}{p^m}]\sigma_a^{-1}) - \frac{e_d}{d}\nu_{K_d/\mathbf{Q}(\mu_{p^m})}(\sum_{\substack{a=1\\(a,p)=1}}^{p^m} a\sigma_a^{-1})
\end{aligned}$$

where $e_d = [\mathbf{Q}(\mu_d) : K_d]$. Therefore, we know

$$\theta_{K_d} - \frac{e_d}{d'}\nu_{K_d/\mathbf{Q}(\mu_{p^m})}(\theta_{\mathbf{Q}(\mu_{p^m})}) \in R_{K_d,p} \ .$$

Put $c_d = e_d/d'$. Since

$$[\mathbf{Q}(\mu_d) : \mathbf{Q}(\mu_{p^m})] = [\mathbf{Q}(\mu_{d'}) : \mathbf{Q}] = d'\prod_{\ell|d'}(1-\frac{1}{\ell}),$$

c_d can be also written as

$$c_d = (\prod_{\ell|d'}(1-\frac{1}{\ell}))[K_d : \mathbf{Q}(\mu_{p^m})]^{-1}.$$

Next, suppose that $d \in \mathcal{D}_0$. Then, since K_d does not contain a primitive p-th root of unity, θ_{K_d} is in $R_{K_d,p}$.

Proposition 2.3 *$\Theta_K \otimes \mathbf{Z}_p \subset R_{K,p}$ is generated by the following elements;*
(i) *$\nu_{K/K_d}(\theta_{K_d})$ for all $d \in \mathcal{D}_0$,*
(ii) *$\nu_{K/K_d}(\theta_{K_d} - c_d\nu_{K_d/\mathbf{Q}(\mu_{p^m})}(\theta_{\mathbf{Q}(\mu_{p^m})}))$ for all $d \in \mathcal{D}_1$ such that $d \neq p^m$,*
(iii) *$\nu_{K/\mathbf{Q}(\mu_{p^m})}(\alpha\theta_{\mathbf{Q}(\mu_{p^m})})$ for $\alpha \in \mathrm{Ann}_{R_{\mathbf{Q}(\mu_{p^m}),p}}(\mu_{p^m})$.*

Proof. We have already seen that the above elements are in $R_{K,p}$. We will show that Θ_K is generated by these elements. Suppose that x is an element of Θ_K. By the definition of Θ'_K, x can be written as $x = \Sigma_{d\in\mathcal{D}}\alpha_d\nu_{K/K_d}(\theta_{K_d})$ with $\alpha_d \in R_{K,p}$. We write

$$\begin{aligned} x &= \sum_{d\in\mathcal{D}_0}\alpha_d\nu_{K/K_d}(\theta_{K_d}) + \sum_{d\in\mathcal{D}_1\setminus\{p^m\}}\alpha_d\nu_{K/K_d}(\theta_{K_d} - c_d\nu_{K_d/\mathbf{Q}(\mu_{p^m})}(\theta_{\mathbf{Q}(\mu_{p^m})})) \\ &\quad +(\alpha_{p^m} + \sum_{d\in\mathcal{D}_1\setminus\{p^m\}}\alpha_d c_d)\nu_{K/\mathbf{Q}(\mu_{p^m})}(\theta_{\mathbf{Q}(\mu_{p^m})}). \end{aligned}$$

Since x is in $R_{K,p}$, we know $(\alpha_{p^m} + \sum_{d\in\mathcal{D}_1\setminus\{p^m\}}\alpha_d c_d)\nu_{K/\mathbf{Q}(\mu_{p^m})}(\theta_{\mathbf{Q}(\mu_{p^m})}) \in R_{K,p}$. It follows from Lemma 2.2 that we can take $\alpha \in \mathrm{Ann}_{R_{\mathbf{Q}(\mu_{p^m}),p}}(\mu_{p^m})$ such that

$$(\alpha_{p^m} + \sum_{d\in\mathcal{D}_1\setminus\{p^m\}}\alpha_d c_d)\nu_{K/\mathbf{Q}(\mu_{p^m})}(\theta_{\mathbf{Q}(\mu_{p^m})}) = \nu_{K/\mathbf{Q}(\mu_{p^m})}(\alpha\theta_{\mathbf{Q}(\mu_{p^m})}).$$

This completes the proof.

Let K_∞/K be the cyclotomic $\mathbf{Z}_p$-extension. In §3 of [8], we defined the Stickelberger ideal Θ_{K_∞}, which is an ideal of the completed group ring $\mathbf{Z}_p[[\mathrm{Gal}(K_\infty/\mathbf{Q})]]$ (Θ_{K_∞} was denoted by $\Theta_{K_\infty/\mathbf{Q}}$ in [8]). The definition is as follows. Let $\mathcal{D}_1$ be as above. For any $d \in \mathcal{D}_1$, we denote by $K_{d,\infty}/K_d$ the cyclotomic $\mathbf{Z}_p$-extension, and by $K_{d,n}$ the n-th layer. The element $\theta_{K_{d,\infty}}$ in the total quotient ring of $\mathbf{Z}_p[[\mathrm{Gal}(K_{d,\infty}/\mathbf{Q})]]$ is defined as the "projective limit" of $\theta_{K_{d,n}}$; more precisely, $\theta_{K_{d,\infty}}$ is the element satisfying the property that $(\sigma - \kappa(\sigma))\theta_{K_{d,\infty}} \in \mathbf{Z}_p[[\mathrm{Gal}(K_{d,\infty}/\mathbf{Q})]]$ is the projective limit of $(\sigma - \kappa(\sigma))\theta_{K_{d,n}} \in \mathbf{Z}_p[\mathrm{Gal}(K_{d,n}/\mathbf{Q})]$ for all $\sigma \in \mathrm{Gal}(K_{d,\infty}/\mathbf{Q})$ where $\kappa : \mathrm{Gal}(K_{d,\infty}/\mathbf{Q}) \longrightarrow \mathbf{Z}_p^\times$ is the cyclotomic character. We define Θ'_{K_∞} to be the $\mathbf{Z}_p[[\mathrm{Gal}(K_\infty/\mathbf{Q})]]$-module in the total quotient ring of $\mathbf{Z}_p[[\mathrm{Gal}(K_\infty/\mathbf{Q})]]$ generated by all $\nu_{K_\infty/K_{d,\infty}}(\theta_{K_{d,\infty}})$ for $d \in \mathcal{D}_1$. The ideal Θ_{K_∞} is defined by $\Theta_{K_\infty} = \Theta'_{K_\infty} \cap \mathbf{Z}_p[[\mathrm{Gal}(K_\infty/\mathbf{Q})]]$.

We denote by $K_{(\Delta)}$ the fixed subfield of K by Γ_K. Namely, $K_{(\Delta)}$ is the subfield such that $\mathrm{Gal}(K_{(\Delta)}/\mathbf{Q}) = \Delta_K$. Let ω be the Teichmüller character. We consider the ω-component $(\Theta_{K_\infty})^\omega$ which is an ideal of $O_\omega[[\Gamma_{K_\infty}]]$ where $\Gamma_{K_\infty} = \mathrm{Gal}(K_\infty/K_{(\Delta)})$, and $O_\omega = \mathbf{Z}_p$ on which Δ_K acts via ω.

Proposition 2.4 (1) *Let* $c_{K_\infty/K} : O_\omega[[\Gamma_{K_\infty}]] \longrightarrow O_\omega[\Gamma_K]$ *be the natural restriction map. Then we have*

$$c_{K_\infty/K}((\Theta_{K_\infty})^\omega) = (\Theta_K \otimes \mathbf{Z}_p)^\omega.$$

(2) $\mathrm{Fitt}_{O_\omega[\Gamma_K]}(A_K^\omega) = (\Theta_K \otimes \mathbf{Z}_p)^\omega$.

Proof. (1) By the definition of the Stickelberger ideals and Lemma 1.3, it is clear that $c_{K_\infty/K}((\Theta_{K_\infty})^\omega) \subset (\Theta_K \otimes \mathbf{Z}_p)^\omega$.

We prove the other inclusion. Suppose d is in $\mathcal{D}_0$. Since p is ramified in K/K_d, the image of $N_{\mathrm{Gal}(K/K_d)} = \sum_{\sigma \in \mathrm{Gal}(K/K_d)} \sigma$ in $O_\omega[\Gamma_K]$ is zero, so the image of $\nu_{K/K_d}(\theta_{K_d})$ in $O_\omega[\Gamma_K]$ is zero. Hence by Proposition 2.3, $(\Theta_K \otimes \mathbf{Z}_p)^\omega$ is generated by the elements

$$\nu_{K/K_d}(\theta_{K_d} - c_d \nu_{K_d/\mathbf{Q}(\mu_{p^m})}(\theta_{\mathbf{Q}(\mu_{p^m})}))^\omega$$

where $d \in \mathcal{D}_1 \backslash \{p^m\}$, and $\nu_{K/\mathbf{Q}(\mu_{p^m})}(\alpha\theta_{\mathbf{Q}(\mu_{p^m})})^\omega$ with $\alpha \in \mathrm{Ann}_{R_{\mathbf{Q}(\mu_{p^m}),p}}(\mu_{p^m})$.

In the proof of Lemma 3.4 (2) in [8], we proved that

$$\theta_{K_{d,\infty}} - c_d \nu_{K_{d,\infty}/\mathbf{Q}(\mu_{p^\infty})}(\theta_{\mathbf{Q}(\mu_{p^\infty})}) \in \mathbf{Z}_p[[\mathrm{Gal}(K_{d,\infty}/\mathbf{Q})]]. \tag{2.4.1}$$

In fact, the constant $c_F \in \mathbf{Z}_p$ for a CM-field F is defined in the proof of Lemma 3.4 (2) in [8]. Concerning c_{K_d} and $c_{\mathbf{Q}(\mu_{p^m})}$, it follows from the formula on page 53 line 27 in [8] that

$$c_{K_d}/c_{\mathbf{Q}(\mu_{p^m})} = (\prod_{\ell | d'} (1 - \frac{1}{\ell}))[K_d : \mathbf{Q}(\mu_{p^m})]^{-1} = c_d.$$

By the argument on page 53 line 31 in [8], this implies (2.4.1). Therefore, we have

$$\nu_{K_\infty/K_{d,\infty}}(\theta_{K_{d,\infty}} - c_d \nu_{K_{d,\infty}/\mathbf{Q}(\mu_{p^\infty})}(\theta_{\mathbf{Q}(\mu_{p^\infty})})) \in \Theta_{K_\infty},$$

and

$$\nu_{K/K_d}(\theta_{K_d} - c_d \nu_{K_d/\mathbf{Q}(\mu_{p^m})}(\theta_{\mathbf{Q}(\mu_{p^m})})) \in c_{K_\infty/K}(\Theta_{K_\infty}).$$

In the same way, for any $\sigma \in \mathrm{Gal}(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q})$, the fact that $(\sigma - \kappa(\sigma))\theta_{\mathbf{Q}(\mu_{p^\infty})}$ is in $\mathbf{Z}_p[[\mathrm{Gal}(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q})]]$ implies $\nu_{K_\infty/\mathbf{Q}(\mu_{p^\infty})}((\sigma - \kappa(\sigma))\theta_{\mathbf{Q}(\mu_{p^\infty})}) \in \Theta_{K_\infty}$, and

$$\nu_{K/\mathbf{Q}(\mu_{p^m})}((\sigma_{|\mathbf{Q}(\mu_{p^m})} - \kappa(\sigma))\theta_{\mathbf{Q}(\mu_{p^m})}) \in c_{K_\infty/K}(\Theta_{K_\infty}).$$

Since $\{\sigma_{|\mathbf{Q}(\mu_{p^m})} - \kappa(\sigma) \mid \sigma \in \mathrm{Gal}(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q})\}$ generates $\mathrm{Ann}_{R_{\mathbf{Q}(\mu_{p^m}),p}}(\mu_{p^m})$, we obtain $(\Theta_K \otimes \mathbf{Z}_p)^\omega = c_{K_\infty/K}((\Theta_{K_\infty})^\omega)$ by Proposition 2.3.

(2) As usual, we define

$$X_{K_\infty} = \lim_{\leftarrow} A_{K_n}$$

where $A_{K_n} = \mathrm{Cl}_{K_n} \otimes \mathbf{Z}_p$ for the n-th layer K_n of K_∞/K, and the projective limit is taken with respect to the norm maps. By [8] Corollary 0.10, we know

$$\mathrm{Fitt}_{\mathbf{Z}_p[[\mathrm{Gal}(K_\infty/\mathbf{Q})]]}(X^-_{K_\infty}) = (\Theta_{K_\infty})^-,$$

which implies $\mathrm{Fitt}_{O_\omega[[\Gamma_{K_\infty}]]}(X^\omega_{K_\infty}) = (\Theta_{K_\infty})^\omega$. We denote by $I_v(K_\infty/K)$ the inertia subgroup in $\mathrm{Gal}(K_\infty/K)$ of a prime v of K above p. Since p is ramified in K, we have $(\bigoplus_{v|p} I_v(K_\infty/K))^\omega = 0$. Therefore, the natural homomorphism $(X^\omega_{K_\infty})_{\mathrm{Gal}(K_\infty/K)} \longrightarrow A^\omega_K$ is an isomorphism by Lemma 1.2. Hence, by Proposition 2.4 (1) we obtain

$$\begin{aligned}\mathrm{Fitt}_{O_\omega[\Gamma_K]}(A^\omega_K) &= c_{K_\infty/K}(\mathrm{Fitt}_{O_\omega[[\Gamma_{K_\infty}]]}(X^\omega_{K_\infty})) \\ &= c_{K_\infty/K}((\Theta_{K_\infty})^\omega) = (\Theta_K \otimes \mathbf{Z}_p)^\omega.\end{aligned}$$

3 The case p is tamely ramified

In this section, we consider the Fitting ideals in a slightly more general setting. We suppose that k is a totally real base field and K is a CM-field such that K/k is finite and abelian. We suppose that p is an odd prime number and study $A_K = \mathrm{Cl}_K \otimes \mathbf{Z}_p$ as a $\mathbf{Z}_p[\mathrm{Gal}(K/k)]$-module. We use the same notation as in subsection 1.3; in particular, we write $\mathrm{Gal}(K/k) = \Delta_K \times \Gamma_K$ where $\#\Delta_K$ is prime to p and Γ_K is a p-group, and consider A^χ_K which is an $O_\chi[\Gamma_K]$-module for an odd character χ of Δ_K.

Let $K_{(\Delta)}$ be the subfield of K such that $\mathrm{Gal}(K_{(\Delta)}/k) = \Delta_K$. Suppose that $\mathfrak{l}_1,\ldots,\mathfrak{l}_r$ are all the ramifying primes of k in $K/K_{(\Delta)}$, and $P_{\mathfrak{l}_i}$ is the inertia group of a prime above $\mathfrak{l}_i$ in $\mathrm{Gal}(K/K_{(\Delta)})$, namely the Sylow p-subgroup of the inertia group of $\mathfrak{l}_i$ in $\mathrm{Gal}(K/k)$. We assume the following condition

$$(\mathrm{A}_p) \qquad \Gamma_K = \mathrm{Gal}(K/K_{(\Delta)}) = P_{\mathfrak{l}_1} \times \ldots \times P_{\mathfrak{l}_r}.$$

Clearly, if $K/\mathbf{Q}$ satisfies the condition (A) in §1, it satisfies (A_p) for $k = \mathbf{Q}$. In the same way as in the absolutely abelian case, the Stickelberger element $\theta_{K/k}$ is defined by

$$\theta_{K/k} = \sum_{\sigma \in \mathrm{Gal}(K/k)} \zeta(0,\sigma)\sigma^{-1} \in \mathbf{Q}[\mathrm{Gal}(K/k)]$$

where $\zeta(s,\sigma) = \sum_{(\frac{K/k}{\mathfrak{a}})=\sigma} N(\mathfrak{a})^{-s}$ is the partial zeta function. Using subfields F of K and the Stickelberger elements $\theta_{F/k} \in \mathbf{Q}[\mathrm{Gal}(F/k)]$, we define $\Theta'_{K/k}$ to be the $\mathbf{Z}'[\mathrm{Gal}(K/k)]$-submodule of $\mathbf{Q}[\mathrm{Gal}(K/k)]$ generated by all $\nu_{K/F}(\theta_{F/k})$ for intermediate fields F of K/k, and define $\Theta_{K/k} = \Theta'_{K/k} \cap \mathbf{Z}'[\mathrm{Gal}(K/k)]$.

We first introduce some results in [8], which will be used several times later. We define $\Theta'_{K_\infty/k}$ in the total quotient ring of $\mathbf{Z}_p[[\mathrm{Gal}(K_\infty/k)]]$ to be the $\mathbf{Z}_p[[\mathrm{Gal}(K_\infty/k)]]$-module generated by all $\nu_{K_\infty/F_\infty}(\theta_{F_\infty/k})$ for intermediate fields F_∞ of K_∞/k_∞ (cf. the definition when $k=\mathbf{Q}$ in §2 before the proof of Proposition 2.4), and define $\Theta_{K_\infty/k} = \Theta'_{K_\infty/k} \cap \mathbf{Z}_p[[\mathrm{Gal}(K_\infty/k)]]$ (cf. [8] §2, §3).

For any CM-field F which is finite and abelian over k, we know by Deligne and Ribet (cf. [1]) that $\theta^\chi_{F/k} \in O_\chi[\Gamma_F]$ for any odd character χ of Δ_F such that $\chi \neq \omega$ (where ω is the Teichmüller character). We take an odd character χ of Δ_K such that $\chi \neq \omega$. By the above result of Deligne and Ribet, we have $(\Theta'_{K_\infty/k})^\chi = \Theta^\chi_{K_\infty/k}$ and $(\Theta'_{K/k} \otimes \mathbf{Z}_p)^\chi = (\Theta_{K/k} \otimes \mathbf{Z}_p)^\chi$. We consider $X^\chi_{K_\infty}$, which is an $O_\chi[[\Gamma_{K_\infty}]]$-module.

Theorem 3.1 ([8] Theorem 0.9) *We assume that* (A_p)*,* $\chi \neq \omega$*, every prime* $\mathfrak{p}$ *of* k *above* p *is tamely ramified in* K*, and the* μ*-invariant of the cyclotomic* $\mathbf{Z}_p$*-extension* K_∞/K *is zero. Then we have*

$$\mathrm{Fitt}_{O_\chi[[\Gamma_{K_\infty}]]}(X^\chi_{K_\infty}) = \Theta^\chi_{K_\infty/k}.$$

The following lemma follows from Proposition 5.2 in [8].

Lemma 3.2 *Suppose that* $\chi \neq \omega$*. Then the sequence*

$$0 \longrightarrow (\bigoplus_{v|p} I_v(K_\infty/K))^\chi \longrightarrow (X^\chi_{K_\infty})_{\mathrm{Gal}(K_\infty/K)} \longrightarrow A^\chi_K \longrightarrow 0$$

is exact, where v *runs over all primes of* K *above* p*, and* $I_v(K_\infty/K)$ *is the inertia group of* v *in* $\mathrm{Gal}(K_\infty/K)$*.*

Suppose that K satisfies the condition of Theorem 3.1. Moreover, we assume $\chi(\mathfrak{p}) \neq 1$ for all primes $\mathfrak{p}$ of k above p. This assumption implies $\chi(\mathfrak{p}) - 1$ is a unit in O_χ, so we have

$$(\bigoplus_{v|\mathfrak{p}} I_v(K_\infty/K))^\chi \simeq \mathbf{Z}_p[\mathrm{Gal}(K/k)/D_\mathfrak{p}(K/k)]^\chi = 0,$$

where $D_\mathfrak{p}(K/k)$ is the decomposition group of $\mathfrak{p}$ in $\mathrm{Gal}(K/k)$. Therefore, $(\bigoplus_{v|p} I_v(K_\infty/K))^\chi = \bigoplus_{\mathfrak{p}|p}(\bigoplus_{v|\mathfrak{p}} I_v(K_\infty/K))^\chi = 0$, which implies that the natural map

$$(X^\chi_{K_\infty})_{\mathrm{Gal}(K_\infty/K)} \xrightarrow{\simeq} A^\chi_K$$

is an isomorphism by Lemma 3.2. On the other hand, it follows from Lemma 2.1 in [8] (Tate [16] p. 86) that $c_{M_\infty/M}(\theta^\chi_{M_\infty/k}) = (\mathrm{unit})\theta^\chi_{M/k}$ in $O_\chi[\Gamma_M]$ for any subfield M of K under the same condition on χ. Therefore, we have $c_{K_\infty/K}(\Theta^\chi_{K_\infty/k}) = \Theta^\chi_{K/k}$. Hence, as a corollary of Theorem 3.1, we obtain

Corollary 3.3 (cf. [8] Theorem 0.4) *We assume the same conditions of Theorem 3.1, and also assume that $\chi(\mathfrak{p}) \neq 1$ for all $\mathfrak{p}$ of k above p. Then we have*

$$\mathrm{Fitt}_{O_\chi[\Gamma_K]}(A_K^\chi) = (\Theta_{K/k} \otimes \mathbf{Z}_p)^\chi.$$

Thus, the problem lies in the case of characters χ such that $\chi(\mathfrak{p}) = 1$ for some $\mathfrak{p}$ above p. (In this case we say that the trivial zero occurs.)

Proposition 3.4 *For every prime $\mathfrak{p}$ of k above p, we assume that the ramification index of $\mathfrak{p}$ in $k/\mathbf{Q}$ is odd. Suppose also that K/k satisfies the assumption (A_p), the μ-invariant of the cyclotomic $\mathbf{Z}_p$-extension K_∞/K is zero, and all the primes $\mathfrak{p}$ of k above p are unramified in K. Then, for any odd character χ of Δ_K such that $\chi \neq \omega$, we have*

$$\mathrm{Fitt}_{O_\chi[\Gamma_K]}(A_K^\chi) \subset (\Theta_{K/k} \otimes \mathbf{Z}_p)^\chi.$$

Proof. This proposition can be proved by the same method as Theorem 0.6 in [8]. Since the proof is almost the same, we only give a sketch. We use the same technique as Greither [3] and Wiles [19].

We fix a positive integer n. By our assumption on the ramification index of $\mathfrak{p}$, a prime above p is ramified in $K^+(\mu_{p^n})/K^+\mathbf{Q}(\mu_{p^n})^+$, so $K^+(\mu_{p^n})/K^+\mathbf{Q}(\mu_{p^n})^+$ and $K\mathbf{Q}(\mu_{p^n})^+/K^+\mathbf{Q}(\mu_{p^n})^+$ are linearly disjoint. Therefore, using the Chebotarev density theorem, we can choose a prime number r such that r splits in $\mathbf{Q}(\mu_{p^n})$, r is inert in $\mathbf{Q}(\sqrt[p]{p})$, r is unramified in K, and every prime above r is inert in K/K^+ (cf. Proposition 4.1 in Greither [3]; note that we do not need the assumption of "niceness" when we choose such r). Therefore, $r \equiv 1 \pmod{p^n}$, and the Frobenius φ_p of p in $\mathrm{Gal}(k_{r,p^n}/\mathbf{Q})$ generates $\mathrm{Gal}(k_{r,p^n}/\mathbf{Q})$, where k_{r,p^n} denotes the subfield of $\mathbf{Q}(\mu_r)$ with degree p^n. We define E to be the compositum of K and k_{r,p^n}. Note that E also satisfies (A_p). We use the notation A_E, X_{E_∞}, Γ_E, $\Theta_{E/k}$ etc. Since all the primes of k above p are unramified in E, we have

$$\bigoplus_{v|p} I_v(E_\infty/E) \simeq \bigoplus_{\mathfrak{p}|p} \mathbf{Z}_p[\mathrm{Gal}(E/k)]/(\varphi_\mathfrak{p} - 1)$$

where $\varphi_\mathfrak{p}$ is the Frobenius of $\mathfrak{p}$. Using

$$\mathrm{Fitt}_{\mathbf{Z}_p[\mathrm{Gal}(E/k)]}\Big(\bigoplus_{\mathfrak{p}|p} \mathbf{Z}_p[\mathrm{Gal}(E/k)]/(\varphi_\mathfrak{p} - 1)\Big) = \Big(\prod_{\mathfrak{p}|p}(\varphi_\mathfrak{p} - 1)\Big)$$

and the exact sequence

$$0 \longrightarrow \Big(\bigoplus_{v|p} I_v(E_\infty/E)\Big)^\chi \longrightarrow (X_{E_\infty}^\chi)_{\mathrm{Gal}(E_\infty/E)} \longrightarrow A_E^\chi \longrightarrow 0$$

which is obtained from Lemma 3.2, we have

$$(\prod_{\mathfrak{p}|p}(\varphi_{\mathfrak{p}}-1))\,\mathrm{Fitt}_{O_\chi[\Gamma_E]}(A_E^\chi) \subset \mathrm{Fitt}_{O_\chi[\Gamma_E]}((X_{E_\infty}^\chi)_{\mathrm{Gal}(E_\infty/E)}).$$

By our assumption that all the primes of k above p are unramified in E, for any intermediate field F of E/k we have

$$c_{F_\infty/F}(\theta_{F_\infty/k}) = (\prod_{\mathfrak{p}|p}(1-\varphi_{\mathfrak{p}}^{-1}))\theta_{F/k}.$$

Therefore, we have

$$c_{E_\infty/E}(\Theta_{E_\infty/k}^\chi) \subset (\prod_{\mathfrak{p}|p}(\varphi_{\mathfrak{p}}-1))(\Theta_{E/k}\otimes \mathbf{Z}_p)^\chi.$$

Using Theorem 3.1 and the above two inclusions, we obtain

$$(\prod_{\mathfrak{p}|p}(\varphi_{\mathfrak{p}}-1))\,\mathrm{Fitt}_{O_\chi[\Gamma_E]}(A_E^\chi) \subset (\prod_{\mathfrak{p}|p}(\varphi_{\mathfrak{p}}-1))(\Theta_{E/k}\otimes \mathbf{Z}_p)^\chi.$$

Let $f_{\mathfrak{p}}$ be the residue degree of $\mathfrak{p}$ in $k/\mathbf{Q}$. We denote by f the maximum of all $\mathrm{ord}_p(f_{\mathfrak{p}})$ for $\mathfrak{p}$ above p. We take n and M sufficiently large such that $n-M \geq f + \mathrm{ord}_p(\#\,\mathrm{Gal}(K/k))$. Put $\nu = \Sigma_{i=0}^{p^M-1}\sigma^{ip^{n-M}}$ where σ is a generator of $\mathrm{Gal}(E/K)=\mathrm{Gal}(k_{r,p^n}/\mathbf{Q})$. We then have

$$(\prod_{\mathfrak{p}|p}(\varphi_{\mathfrak{p}}-1))\,\mathrm{Fitt}_{O_\chi[\Gamma_E]/(\nu)}(A_E^\chi/(\nu)) \subset (\prod_{\mathfrak{p}|p}(\varphi_{\mathfrak{p}}-1))(\Theta_{E/k}\otimes \mathbf{Z}_p)^\chi \mod (\nu),$$

but the image of the element $(\prod_{\mathfrak{p}|p}(\varphi_{\mathfrak{p}}-1))$ is not a zero divisor in $O_\chi[\Gamma_E]/(\nu)$. Hence we have

$$\mathrm{Fitt}_{O_\chi[\Gamma_E]/(\nu)}(A_E^\chi/(\nu)) \subset (\Theta_{E/k}\otimes \mathbf{Z}_p)^\chi \mod (\nu).$$

Since every prime above r is inert in K/K^+, we have $\chi(\mathfrak{r})\neq 1$ for any prime $\mathfrak{r}$ of k above r. Therefore, the above inclusion implies (cf. [8] page 68)

$$\mathrm{Fitt}_{O_\chi[\Gamma_K]/p^M}(A_K^\chi/p^M) \subset (\Theta_{K/k}\otimes \mathbf{Z}_p)^\chi \mod p^M.$$

This holds for any M, so we obtain the conclusion.

We apply Proposition 3.4 to the following two theorems.

Theorem 3.5 *Suppose that $K/\mathbf{Q}$ is an abelian extension satisfying the condition* (A). *We assume that p is tamely ramified in K. Then*

$$\mathrm{Fitt}_{R_{K,p}^-}(A_K^-) = \Theta_K^-\otimes \mathbf{Z}_p$$

holds.

Proof. We will first prove the inclusion $\mathrm{Fitt}_{R^-_{K,p}}(A^-_K) \subset \Theta^-_K \otimes \mathbf{Z}_p$. To prove this inclusion, it suffices to prove $\mathrm{Fitt}_{R^-_{K,p}}(A^-_K)^\chi \subset (\Theta^-_K \otimes \mathbf{Z}_p)^\chi$ for all odd characters χ of Δ_K.

Let χ be an odd character of Δ_K. We denote by K_χ (resp. $K_{(\Delta),\chi}$) the fixed subfield of K (resp. $K_{(\Delta)}$) by $\mathrm{Ker}\,\chi \subset \Delta_K$. Hence $\mathrm{Gal}(K_\chi/K_{(\Delta),\chi}) = \Gamma_K$ and $\mathrm{Gal}(K_{(\Delta),\chi}/\mathbf{Q}) = \mathrm{Image}\,\chi$. Since $[K : K_\chi]$ is prime to p, the norm map induces an isomorphism $A^\chi_K \xrightarrow{\simeq} A^\chi_{K_\chi}$. Hence $\mathrm{Fitt}_{R^-_{K,p}}(A^-_K)^\chi = \mathrm{Fitt}_{O_\chi[\Gamma_K]}(A^\chi_{K_\chi})$. On the other hand, since $[K : K_\chi]$ is prime to p, we have $c_{K/K_\chi}((\Theta_K \otimes \mathbf{Z}_p)^\chi) = (\Theta_{K_\chi} \otimes \mathbf{Z}_p)^\chi$ by the usual norm argument. This means that $(\Theta_K \otimes \mathbf{Z}_p)^\chi = (\Theta_{K_\chi} \otimes \mathbf{Z}_p)^\chi$ in $O_\chi[\Gamma_K]$. Hence, in order to prove $\mathrm{Fitt}_{R^-_{K,p}}(A^-_K) \subset \Theta^-_K \otimes \mathbf{Z}_p$, it suffices to prove $\mathrm{Fitt}_{O_\chi[\Gamma_K]}(A^\chi_{K_\chi}) \subset (\Theta_{K_\chi} \otimes \mathbf{Z}_p)^\chi$ for all odd χ.

Suppose at first that p is unramified in $K_{(\Delta),\chi}/\mathbf{Q}$. By our assumption of the tameness, p is also unramified in K_χ. So we can apply Proposition 3.4 (note that $\mu = 0$ by [2]) to obtain $\mathrm{Fitt}_{O_\chi[\Gamma_K]}(A^\chi_{K_\chi}) \subset (\Theta_{K_\chi} \otimes \mathbf{Z}_p)^\chi$.

Next, suppose that p is ramified in $K_{(\Delta),\chi}/\mathbf{Q}$. If $\chi \neq \omega$, we have $\chi(p) = 0$, so obtain $\mathrm{Fitt}_{O_\chi[\Gamma_K]}(A^\chi_{K_\chi}) = (\Theta_{K_\chi} \otimes \mathbf{Z}_p)^\chi$ by Corollary 3.3. For $\chi = \omega$, Proposition 2.4 (2) says that the same equality holds.

Therefore, for any odd character χ of Δ_K we have got $\mathrm{Fitt}_{O_\chi[\Gamma_K]}(A^\chi_{K_\chi}) \subset (\Theta_{K_\chi} \otimes \mathbf{Z}_p)^\chi$. This completes the proof of $\mathrm{Fitt}_{R^-_{K,p}}(A^-_K) \subset \Theta^-_K \otimes \mathbf{Z}_p$.

On the other hand, by Lemma 6.3 in [8] we have $(R^-_{K,p} : \mathrm{Fitt}_{R^-_{K,p}}(A^-_K)) \leq \#A^-_K$. Since we know $(R^-_{K,p} : \Theta^-_K \otimes \mathbf{Z}_p) = \#A^-_K$ by Sinnott's theorem ([13] Theorems 2.1 and 5.4), we obtain

$$\mathrm{Fitt}_{R^-_{K,p}}(A^-_K) = \Theta^-_K \otimes \mathbf{Z}_p.$$

Theorem 3.6 *We assume the same conditions as Proposition 3.4, and also that a primitive p-th root of unity is not in K. We put* $R_{K,p} = \mathbf{Z}_p[\mathrm{Gal}(K/k)]$. *Then we have*

$$\mathrm{Fitt}_{R^-_{K,p}}(A^-_K) = \Theta^-_{K/k} \otimes \mathbf{Z}_p.$$

Proof. Proposition 3.4 implies $\mathrm{Fitt}_{R^-_{K,p}}(A^-_K) \subset \Theta^-_{K/k} \otimes \mathbf{Z}_p$. Since $(R^-_{K,p} : \mathrm{Fitt}_{R^-_{K,p}}(A^-_K)) \leq \#A^-_K$ can be proved by the same method as Lemma 6.3 in [8], as in the proof of Theorem 3.5, it is enough to prove

$$(R^-_{K,p} : \Theta^-_{K/k} \otimes \mathbf{Z}_p) = \#A^-_K.$$

This equality can be proved by the same method as Sinnott [12]. It is easy to check that Sinnott's argument works if we assume the condition (A). Now, we are assuming (A_p), and only interested in the p-component A_K, so the same argument as Sinnott's works, as we will explain next.

For any group H, we denote by $N_H = \sum_{\sigma \in H} \sigma$ the norm of H in group algebras. For an intermediate field M of K/k, we define $\mathcal{R}_M$ to be the set of all the ramifying primes of k in M, $H_M = \mathrm{Gal}(K/M)$, and

$$u(M) = N_{H_M} \prod_{\mathfrak{l} \in \mathcal{R}_M} (1 - \varphi_{\mathfrak{l}}^{-1} \frac{N_{I_{\mathfrak{l}}}}{\# I_{\mathfrak{l}}}) \in \mathbf{Q}_p[\mathrm{Gal}(K/k)]$$

where $I_{\mathfrak{l}}$ is the inertia group of $\mathfrak{l}$ in $\mathrm{Gal}(K/k)$, and $\varphi_{\mathfrak{l}}$ is the Frobenius of $\mathfrak{l}$ in $\mathrm{Gal}(K/k)$. We also define U to be the $R_{K,p}$-submodule in $\mathbf{Q}_p[\mathrm{Gal}(K/k)]$ generated by all $u(M)$'s where M runs over all intermediate fields of K/k.

Lemma 3.7 ([12] Proposition 2.2) *U is a free $\mathbf{Z}_p$-submodule in $\mathbf{Q}_p[\mathrm{Gal}(K/k)]$ of rank $[K : k]$.*

Proof. Since U is a finitely generated free $\mathbf{Z}_p$-module, it suffices to show $U \otimes \mathbf{Q}_p = \mathbf{Q}_p[\mathrm{Gal}(K/k)]$. For a character $\psi : \mathrm{Gal}(K/k) \longrightarrow \overline{\mathbf{Q}}_p^{\times}$, we write the induced ring homomorphism $\mathbf{Q}_p[\mathrm{Gal}(K/k)] \longrightarrow \mathbf{Q}_p(\mathrm{Image}\,\psi)$ by the same letter ψ. In order to obtain $U \otimes \mathbf{Q}_p = \mathbf{Q}_p[\mathrm{Gal}(K/k)]$, it suffices to show $\psi(U) \neq 0$ for all characters ψ. Let K_ψ be the fixed field by $\mathrm{Ker}\,\psi$ in K, then we have

$$\psi(u(K_\psi)) = [K : K_\psi] \prod_{\mathfrak{l} \in \mathcal{R}_{K_\psi}} (1 - \psi(\varphi_{\mathfrak{l}}^{-1}) \frac{\psi(N_{I_{\mathfrak{l}}})}{\# I_{\mathfrak{l}}}).$$

In the above product, all $\psi(N_{I_{\mathfrak{l}}})$'s are zero since $I_{\mathfrak{l}} \not\subset H_{K_\psi}$ for any $\mathfrak{l} \in R_{K_\psi}$. Therefore, $\psi(u(K_\psi))$ does not vanish for any character ψ. Q.E.D.

Put

$$w = \sum_{\psi} L(0, \psi^{-1}) e_\psi \in \mathbf{Q}[\mathrm{Gal}(K/k)]$$

where ψ runs over all characters of $\mathrm{Gal}(K/k)$ and e_ψ is the idempotent associated to ψ. (It is well-known that $w \in \mathbf{Q}[\mathrm{Gal}(K/k)]$ (cf. Tate [16] Chap. IV).)

Lemma 3.8 *We have $\Theta_{K/k} \otimes \mathbf{Z}_p = \Theta'_{K/k} \otimes \mathbf{Z}_p = w\,U$.*

Proof. The first equality follows from our assumption that $\mu_p \not\subset K$. For the second equality, it is enough to show that for any intermediate field M in K/k,

$$\nu_{K/M}(\theta_{M/k}) = w N_{H_M} \prod_{\mathfrak{l} \in \mathcal{R}_M} (1 - \varphi_{\mathfrak{l}}^{-1} \frac{N_{I_{\mathfrak{l}}}}{\# I_{\mathfrak{l}}})$$

holds. Let ψ be any character of $\mathrm{Gal}(K/k)$. It is easy to show that if ψ is odd and $\mathrm{Ker}\,\psi \supset H_M$, then the image of the both sides under ψ are

$$[K:M]L(0,\psi^{-1}) \prod_{\mathfrak{l}\in\mathcal{R}_M\setminus\mathcal{R}_{K_\psi}} (1-\psi(\varphi_{\mathfrak{l}}^{-1})),$$

and otherwise are zero. Q.E.D.

Recall that $\Theta_{K/k}\otimes\mathbf{Q}_p = (\Theta_{K/k}\otimes\mathbf{Q}_p)^- = \mathbf{Q}_p[\mathrm{Gal}(K/k)]^-$. Therefore, it follows from Lemmas 3.7 and 3.8 that U^- and wU^- are free $\mathbf{Z}_p$-modules of rank $\frac{1}{2}[K:k]$ in $\mathbf{Q}_p[\mathrm{Gal}(K/k)]^-$. So we can define $(R^-_{K,p}:U^-)$ and $(U^-:wU^-)$ as in Sinnott [12] §1; namely if L and L' are free $\mathbf{Z}_p[\mathrm{Gal}(K/k)]^-$-modules of rank $\frac{1}{2}[K:k]$ contained in $\mathbf{Q}_p[\mathrm{Gal}(K/k)]^-$ and $T:L\longrightarrow L'$ is a surjective linear translation, $(L:L')$ is defined by

$$(L:L') = p^{\mathrm{ord}_p(\det T)}.$$

By Lemma 3.8, we obtain

$$(R^-_{K,p}:(\Theta_{K/k}\otimes\mathbf{Z}_p)^-) = (R^-_{K,p}:U^-)(U^-:wU^-)\,.$$

We will successively determine the indices $(R^-_{K,p}:U^-)$ and $(U^-:wU^-)$.

Lemma 3.9 ([12] (3) in page 118) $(U^-:wU^-) = \#A^-_K$.

Proof. Let T_w be the linear translation on $\mathbf{Q}_p[\mathrm{Gal}(K/k)]^-$ defined by $T_w(x)=wx$. From the facts that T_w is extended to the linear translation on $\overline{\mathbf{Q}}_p[\mathrm{Gal}(K/k)]^-$, $\{e_\psi\}_{\psi:\mathrm{odd}}$ is a basis of $\overline{\mathbf{Q}}_p[\mathrm{Gal}(K/k)]^-$, and $T_w(e_\psi) = L(0,\psi^{-1})e_\psi$, we have

$$\det T_w = \prod_{\psi:\mathrm{odd}} L(0,\,\psi^{-1}) = (2\text{-power})\frac{h^-_K}{w_K}$$

by the class number formula, where $h^-_K = \#\,\mathrm{Coker}(\mathrm{Cl}_{K^+}\longrightarrow\mathrm{Cl}_K)$ is the relative class number of K and w_K is the number of roots of unity in K. Lemma 3.9 follows from our assumption that p is odd and $\mu_p\not\subset K$. Q.E.D.

Lemma 3.10 (cf. [13] Theorem 5.4) $(R^-_{K,p}:U^-)=1$.

Proof. For an odd character χ of Δ_K, we consider the χ-component U^χ. To prove Lemma 3.10, it is enough to prove $(R^\chi_{K,p}:U^\chi)=1$ for any odd character χ of Δ_K. We first describe U^χ in a different way from the definition, using the condition (A_p).

We decompose the inertia group $I_{\mathfrak{l}}$ of a ramifying prime $\mathfrak{l}$ in $\mathrm{Gal}(K/k)$ into $I_{\mathfrak{l}} = \Delta_{\mathfrak{l}} \times P_{\mathfrak{l}}$ where $\Delta_{\mathfrak{l}} \subset \Delta_K$ and $P_{\mathfrak{l}} \subset \Gamma_K$. The character χ induces the ring homomorphism $R_{K,p} = \mathbf{Z}_p[\mathrm{Gal}(K/k)] \longrightarrow O_\chi[\Gamma_K]$ which we write by the same letter χ. We define an $R_{K,p}^\chi$-module $V^\chi(\mathfrak{l}) \subset (R_{K,p} \otimes \mathbf{Q})^\chi$ by

$$V^\chi(\mathfrak{l}) = N_{P_{\mathfrak{l}}} R_{K,p}^\chi + \left(1 - \chi(\varphi_{\mathfrak{l}}^{-1}) \frac{N_{P_{\mathfrak{l}}}}{\# P_{\mathfrak{l}}}\right) R_{K,p}^\chi \ .$$

Recall that $\Gamma_K = P_{\mathfrak{l}_1} \times \cdots \times P_{\mathfrak{l}_r}$ and $P_{\mathfrak{l}_i} \neq 1$ for all $i \in \{1, ..., r\}$. Define $X_\chi = \{\mathfrak{l}_i \mid i \in \{1, ..., r\},\ \Delta_{\mathfrak{l}_i} \subset \mathrm{Ker}\, \chi\}$.

We will prove

$$U^\chi = \prod_{\mathfrak{l} \in X_\chi} V^\chi(\mathfrak{l}) \ . \tag{3.10.1}$$

This can be proved by the same method as Proposition 5.1 in [12]. The generators of the right hand side have the form

$$v(S) = N_{P_S} \prod_{\mathfrak{l} \in X_\chi \setminus S} \left(1 - \chi(\varphi_{\mathfrak{l}}^{-1}) \frac{N_{P_{\mathfrak{l}}}}{\# P_{\mathfrak{l}}}\right)$$

where $P_S = \prod_{\mathfrak{l} \in S} P_{\mathfrak{l}}$ and S runs over all subsets of X_χ. Suppose that M is an intermediate field of K/k. We write $H_M = \mathrm{Gal}(K/M) = \Delta(K/M) \times P(K/M)$ where $\Delta(K/M) \subset \Delta_K$ and $P(K/M) \subset \Gamma_K$. We take $S = \{\mathfrak{l} \in X_\chi \mid \mathfrak{l} \notin \mathcal{R}_M\}$, and

$$T = \{\mathfrak{l} \in \mathcal{R}_M \mid \# I_{\mathfrak{l}} \text{ is prime to } p \text{ and } \Delta_{\mathfrak{l}} \subset \mathrm{Ker}\, \chi\}.$$

Then we have

$$\begin{aligned} \chi\Big(\prod_{\mathfrak{l} \in \mathcal{R}_M} \left(1 - \varphi_{\mathfrak{l}}^{-1} \frac{N_{I_{\mathfrak{l}}}}{\# I_{\mathfrak{l}}}\right)\Big) &= \prod_{\mathfrak{l} \in \mathcal{R}_M,\ \Delta_{\mathfrak{l}} \subset \mathrm{Ker}\, \chi} \left(1 - \chi(\varphi_{\mathfrak{l}}^{-1}) \frac{N_{P_{\mathfrak{l}}}}{\# P_{\mathfrak{l}}}\right) \\ &= \prod_{\mathfrak{l} \in X_\chi \setminus S} \left(1 - \chi(\varphi_{\mathfrak{l}}^{-1}) \frac{N_{P_{\mathfrak{l}}}}{\# P_{\mathfrak{l}}}\right) \prod_{\mathfrak{l} \in T} (1 - \chi(\varphi_{\mathfrak{l}}^{-1})) \ . \end{aligned}$$

The first equality follows from the fact that if $\Delta_{\mathfrak{l}} \not\subset \mathrm{Ker}\, \chi$ then $\chi(N_{I_{\mathfrak{l}}}) = 0$. The second equality follows from $\{\mathfrak{l} \in \mathcal{R}_M \mid \Delta_{\mathfrak{l}} \subset \mathrm{Ker}\, \chi\} = (X_\chi \setminus S) \cup T$. Since $P(K/M) \supset P_S$, $N_{P(K/M)}$ is a multiple of N_{P_S}. Therefore, $\chi(u(M))$ is a multiple of the above $v(S)$. This shows that $U^\chi \subset \prod_{\mathfrak{l} \in X_\chi} V^\chi(\mathfrak{l})$.

Conversely, if S is a subset of X_χ, we take the subfield M of K fixed by the subgroup $\mathrm{Ker}\, \chi \times P_S$ of $\mathrm{Gal}(K/k)$. In this case, the above T is empty and $P(K/M) = P_S$. So we have

$$\chi(u(M)) = \#(\mathrm{Ker}\, \chi) N_{P_S} \prod_{\mathfrak{l} \in X_\chi \setminus S} \left(1 - \chi(\varphi_{\mathfrak{l}}^{-1}) \frac{N_{P_{\mathfrak{l}}}}{\# P_{\mathfrak{l}}}\right) = \#(\mathrm{Ker}\, \chi) v(S) \ .$$

Since $\#(\mathrm{Ker}\,\chi)$ is invertible in $R^{\chi}_{K,p}$, $v(S)$ is a multiple of $\chi(u(M))$. This implies the other inclusion. Thus, we have proved (3.10.1).

Changing the subscripts of $\mathfrak{l}_i$'s, we may write $X_\chi = \{\mathfrak{l}_1, ..., \mathfrak{l}_t\}$. For $j \in \{1, ..., t\}$, we put

$$W^{\chi}_j = \prod_{i=1}^{j} V^{\chi}(\mathfrak{l}_i) \ .$$

We know $W^{\chi}_t = U^{\chi}$ by (3.10.1). Hence we have

$$(R^{\chi}_{K,p} : U^{\chi}) = \prod_{j=1}^{t} (W^{\chi}_{j-1} : W^{\chi}_j)$$

where we are using the convention that $W^{\chi}_0 = R^{\chi}_{K,p}$. Therefore, in order to prove Lemma 3.10, it is enough to prove

$$(W^{\chi}_{j-1} : W^{\chi}_j) = 1 \tag{3.10.2}$$

for all j such that $1 \le j \le t$. We will prove (3.10.2). Let K_χ (resp. $K_{(\Delta),\chi}$) be the fixed subfield by $\mathrm{Ker}\,\chi \subset \Delta_K$ in K (resp. $K_{(\Delta)}$). Now, we are dealing with the extension $K_\chi / K_{(\Delta),\chi}$ whose Galois group is Γ_K and which satisfies (A), so the same method as Sinnott §5 in [12] can be applied.

Put $e_j = N_{P_{\mathfrak{l}_j}} / \# P_{\mathfrak{l}_j}$. We will prove

$$(1 - e_j) W^{\chi}_j = (1 - e_j) W^{\chi}_{j-1} \quad \text{and} \quad (W^{\chi}_j)^{P_{\mathfrak{l}_j}} = (W^{\chi}_{j-1})^{P_{\mathfrak{l}_j}} \ . \tag{3.10.3}$$

From two exact sequences

$$0 \longrightarrow (W^{\chi}_j)^{P_{\mathfrak{l}_j}} \longrightarrow W^{\chi}_j \stackrel{1-e_j}{\longrightarrow} (1 - e_j) W^{\chi}_j \longrightarrow 0$$

$$0 \longrightarrow (W^{\chi}_{j-1})^{P_{\mathfrak{l}_j}} \longrightarrow W^{\chi}_{j-1} \stackrel{1-e_j}{\longrightarrow} (1 - e_j) W^{\chi}_{j-1} \longrightarrow 0,$$

it is clear that (3.10.3) implies (3.10.2). Hence our goal is to prove (3.10.3). Since $(1 - e_j) N_{P_{\mathfrak{l}_j}} = 0$, we have $(1 - e_j) V^{\chi}(\mathfrak{l}_j) = (1 - e_j) R^{\chi}_{K,p}$. Therefore, we have $(1 - e_j) W^{\chi}_j = (1 - e_j) W^{\chi}_{j-1}$. Next, we will prove the second equality in (3.10.3). By definition, W^{χ}_j is generated by $N_{P_{\mathfrak{l}_j}} W^{\chi}_{j-1}$ and $(1 - \chi(\varphi^{-1}_{\mathfrak{l}_j}) \frac{N_{P_{\mathfrak{l}_j}}}{\# P_{\mathfrak{l}_j}}) W^{\chi}_{j-1}$. Therefore, we have

$$(W^{\chi}_j)^{P_{\mathfrak{l}_j}} = N_{P_{\mathfrak{l}_j}} W^{\chi}_{j-1} + (1 - \chi(\varphi^{-1}_{\mathfrak{l}_j}))(W^{\chi}_{j-1})^{P_{\mathfrak{l}_j}} \ .$$

By the completely same method as Sinnott [12] Proposition 5.2, we can show that W^{χ}_{j-1} is a free $O_\chi[P_{\mathfrak{l}_j}]$-module, which implies $N_{P_{\mathfrak{l}_j}} W^{\chi}_{j-1} = (W^{\chi}_{j-1})^{P_{\mathfrak{l}_j}}$.

Thus, we have $(W_j^\chi)^{P_{\mathfrak{l}_j}} = (W_{j-1}^\chi)^{P_{\mathfrak{l}_j}}$, and we have obtained (3.10.3). This completes the proof of Lemma 3.10 and of Theorem 3.6. Q.E.D.

Proof of Theorem 0.3: Since we assumed $\mu = 0$ for K_∞/K and K'/K is a p-extension, we have $\mu = 0$ for K'_∞/K'. So we can apply Theorem 3.6 for K' to obtain $\theta_{K'/k} \in \mathrm{Fitt}_{R_{K',p}}(A_{K'})$ (note that the plus part of $\theta_{K'/k}$ is zero). Since K'/K is unramified, $(A_{K'}^-)_{\mathrm{Gal}(K'/K)} \xrightarrow{\simeq} A_K^-$ is bijective and $c_{K'/K}(\theta_{K'/k}) = \theta_{K/k}$. Therefore, we obtain $\theta_{K/k} \in \mathrm{Fitt}_{R_{K,p}}(A_K)$. Q.E.D.

4 The case p is wildly ramified

We use the same notation as in the previous section. In this section, we assume that p is inert in k. Suppose that L/k is a finite abelian extension satisfying the condition (A_p) in §3. We also assume that the prime $\mathfrak{p} = (p)$ of k is tamely ramified in L/k. We consider the cyclotomic $\mathbf{Z}_p$-extension L_∞/L, and take the m-th layer L_m with some $m \geq 0$. Though this might cause slight confusion, we take $K = L_m$ in this section. Namely, K is the intermediate field of L_∞/L such that $[K : L] = p^m$. The cyclotomic $\mathbf{Z}_p$-extension of K is also denoted by K_∞, so $K_\infty = L_\infty$. We use the notation Δ_K, Γ_K, Δ_L, Γ_L as in §3. Therefore, $\Delta_K = \Delta_L$ and $\Gamma_K = \Gamma_L \times \mathrm{Gal}(K/L)$. We note that K/k also satisfies the condition (A_p). In fact, if $m > 0$, we know

$$\Gamma_K = P_{\mathfrak{p}} \times P_{\mathfrak{l}_1} \times ... \times P_{\mathfrak{l}_r}$$

where $\mathfrak{p}$, $\mathfrak{l}_1$,...,$\mathfrak{l}_r$ are all the ramifying primes of k in $K/K_{(\Delta)}$. We define $\Theta_{K/k}$ by the same method as in §3.

Theorem 4.1 *Assume that p is inert in k, and $\mathfrak{p} = (p)$ is tamely ramified in L. Suppose that L/k satisfies the assumption (A_p), and the μ-invariant of the cyclotomic $\mathbf{Z}_p$-extension L_∞/L is zero. Consider the m-th layer $K = L_m$ for some $m \geq 0$. Then, for any odd character χ of Δ_K which is different from the Teichmüller character ω, we have*

$$\mathrm{Fitt}_{O_\chi[\Gamma_K]}(A_K^\chi) = (\Theta_{K/k} \otimes \mathbf{Z}_p)^\chi.$$

Proof. We define K_χ (resp. L_χ) to be the fixed subfield of $\mathrm{Ker}\,\chi \subset \Delta_K = \Delta_L$ in K (resp. L). Note that L_χ/k also satisfies (A_p), and that $A_K^\chi \xrightarrow{\simeq} A_{K_\chi}^\chi$ and $(\Theta_{K/k} \otimes \mathbf{Z}_p)^\chi = (\Theta_{K_\chi/k} \otimes \mathbf{Z}_p)^\chi$ in $O_\chi[\Gamma_K]$ as we saw in the proof of Theorem 3.5. Hence we may assume $K = K_\chi$ and $L = L_\chi$. Namely, we may assume $K_{(\Delta)}/k$ is cyclic and $\chi : \Delta_K \longrightarrow \overline{\mathbf{Q}}_p^\times$ is faithful.

If $\chi(\mathfrak{p}) \neq 1$, the same argument as the proof of Corollary 3.3 (the standard descent method) implies Theorem 4.1. Thus, we may assume $\chi(\mathfrak{p}) = 1$, so $\mathfrak{p}$ is unramified in L. Suppose that $\varphi_{\mathfrak{p}}$ is the Frobenius of $\mathfrak{p}$ in $\mathrm{Gal}(L/k)$. Since $\mathfrak{p}$ splits completely in $L_{(\Delta)}$, $\varphi_{\mathfrak{p}}$ is in Γ_L. We put $\Gamma' = \Gamma_L/\langle\varphi_{\mathfrak{p}}\rangle$ where $\langle\varphi_{\mathfrak{p}}\rangle$ is the subgroup generated by $\varphi_{\mathfrak{p}}$. We put $\Gamma_m = \mathrm{Gal}(K_\infty/K)$ and $\Gamma_0 = \mathrm{Gal}(K_\infty/L)$. By Lemma 3.2, we have a commutative diagram of exact sequences of $O_\chi[\Gamma_K]$-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\bigoplus_{v|\mathfrak{p}} I_v(K_\infty/K))^\chi & \xrightarrow{f_m} & (X_{K_\infty}^\chi)_{\Gamma_m} & \xrightarrow{g_m} & A_K^\chi & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \rho_1} & & \big\downarrow{\scriptstyle \rho_2} & & \big\downarrow{\scriptstyle \rho_3} & & \\ 0 & \longrightarrow & (\bigoplus_{v|\mathfrak{p}} I_v(K_\infty/L))^\chi & \xrightarrow{f_0} & (X_{K_\infty}^\chi)_{\Gamma_0} & \xrightarrow{g_0} & A_L^\chi & \longrightarrow & 0 \end{array}$$

where v runs over all primes of K_∞ above $\mathfrak{p}$, $I_v(K_\infty/K)$ (resp. $I_v(K_\infty/L)$) is the inertia group of v in $\mathrm{Gal}(K_\infty/K)$ (resp. $\mathrm{Gal}(K_\infty/L)$), ρ_1 and ρ_2 are natural maps, and ρ_3 is the norm map.

As $O_\chi[\Gamma_K]$-modules, both $(\bigoplus_{v|\mathfrak{p}} I_v(K_\infty/K))^\chi$ and $(\bigoplus_{v|\mathfrak{p}} I_v(K_\infty/L))^\chi$ are isomorphic to $O_\chi[\Gamma_L]/(\varphi_{\mathfrak{p}} - 1) \simeq O_\chi[\Gamma']$. We take a generator u of $(\bigoplus_{v|\mathfrak{p}} I_v(K_\infty/L))^\chi$ and define $\overline{e}_1 = f_0(u)$. We also take $\overline{e}_2,\ldots, \overline{e}_n \in (X_{K_\infty}^\chi)_{\Gamma_0}$ such that $g_0(\overline{e}_2),\ldots, g_0(\overline{e}_n)$ generate A_L^χ as an $O_\chi[\Gamma_L]$-module. Next, we take $\mathbf{e}_1,\ldots,\mathbf{e}_n \in X_{K_\infty}^\chi$ such that $\overline{e}_i = \mathbf{e}_i \bmod \Gamma_0$ $(1 \le i \le n)$. Nakayama's lemma implies that $\mathbf{e}_1,\ldots,\mathbf{e}_n$ generate $X_{K_\infty}^\chi$. Put $\Lambda = O_\chi[[\mathrm{Gal}(K_\infty/K_{(\Delta)})]]$. We consider an exact sequence of Λ-modules

$$\Lambda^s \xrightarrow{\Phi} \Lambda^n \longrightarrow X_{K_\infty}^\chi \longrightarrow 0,$$

using the generators $\mathbf{e}_1,\ldots,\mathbf{e}_n$. We denote by $\mathbf{B} = (\mathbf{b}_{ij})_{1\le i\le n, 1\le j\le s}$ the $n \times s$ matrix corresponding to Φ (we regard the elements of Λ^s and Λ^n as column vectors).

We denote by $e_1,\ldots,e_n$ the images of $\mathbf{e}_1,\ldots,\mathbf{e}_n$ in $(X_{K_\infty}^\chi)_{\Gamma_m}$. Put

$$N = N_{\mathrm{Gal}(K/L)} = \Sigma_{\sigma\in\mathrm{Gal}(K/L)}\sigma \in O_\chi[\Gamma_K]$$

(so $N = 1$ if $m = 0$, namely if $K = L$). We will show that Ne_1 generates Image f_m. Since $\mathfrak{p}$ is tamely ramified in L, every prime v above $\mathfrak{p}$ is totally ramified in K_∞/L. In the above commutative diagram, ρ_1 is the natural inclusion map. Hence, when we regard $(\bigoplus_{v|\mathfrak{p}} I_v(K_\infty/K))^\chi$ as a subgroup of $(\bigoplus_{v|\mathfrak{p}} I_v(K_\infty/L))^\chi$, $p^m u$ generates $(\bigoplus_{v|\mathfrak{p}} I_v(K_\infty/K))^\chi$. Let $i_{K/L} : A_L^\chi \longrightarrow A_K^\chi$ be the natural homomorphism. Since

$$g_m(Ne_1) = Ng_m(e_1) = i_{K/L}(g_0(\overline{e}_1)) = i_{K/L}(0) = 0,$$

Ne_1 is in the kernel of g_m, and hence in the image of f_m. Since the image of f_m is isomorphic to $O_\chi[\Gamma']$, we can write $Ne_1 = \alpha f_m(p^m u)$ for some $\alpha \in O_\chi[\Gamma']$. Taking the image of ρ_2 of both sides, we have

$$\rho_2(Ne_1) = p^m\overline{e}_1 = \rho_2(\alpha f_m(p^m u)) = p^m\alpha f_0(u) = p^m\alpha\, \overline{e}_1.$$

Since the image of f_0 is a free $O_\chi[\Gamma']$-module generated by $\overline{e}_1$, this implies that $\alpha = 1$. Hence, Ne_1 generates Image f_m as an $O_\chi[\Gamma_K]$-module.

We denote by $B = (b_{ij}) \in M_{ns}(O_\chi[\Gamma_K])$ the matrix which is the image of $\mathbf{B} \in M_{ns}(\Lambda)$, namely $b_{ij} = \mathbf{b}_{ij} \bmod \Gamma_m$. From the upper exact sequence of the commutative diagram, we know that the $n \times (s+1)$ matrix

$$A = \begin{pmatrix} N & & \\ 0 & & \\ .. & & B \\ 0 & & \end{pmatrix}$$

corresponds to the $O_\chi[\Gamma_K]$-module A_K^χ. Namely, if $\Phi' : O_\chi[\Gamma_K]^{s+1} \longrightarrow O_\chi[\Gamma_K]^n$ is the homomorphism corresponding to A, the cokernel of Φ' is isomorphic to A_K^χ.

We define an $(n-1) \times s$ matrix C by $C = (b_{ij})_{2 \le i \le n, 1 \le j \le s}$, and $\overline{C}$ by $\overline{C} = (\overline{b}_{ij})_{2 \le i \le n, 1 \le j \le s}$ where $\overline{b}_{ij} = b_{ij} \bmod \mathrm{Gal}(K/L)$. For any matrix M with entries in a commutative ring R, we will temporarily denote by $F^{(n)}(M)$ the ideal of R generated by all $n \times n$ minors of M. Then we have

$$\begin{aligned} \mathrm{Fitt}_{O_\chi[\Gamma_K]}(A_K^\chi) &= F^{(n)}(A) = NF^{(n-1)}(C) + F^{(n)}(B) \\ &= \quad \nu_{K/L}(F^{(n-1)}(\overline{C})) + F^{(n)}(B). \end{aligned} \tag{4.1.1}$$

Put $\overline{B} = (\overline{b}_{ij})_{1 \le i \le n, 1 \le j \le s}$ (where $\overline{b}_{ij} = \mathbf{b}_{ij} \bmod \Gamma_0$). Since the matrix

$$\begin{pmatrix} 1 & & \\ 0 & & \\ .. & & \overline{B} \\ 0 & & \end{pmatrix}$$

is a relation matrix of the $O_\chi[\Gamma_L]$-module A_L^χ by the lower exact sequence of the above commutative diagram, $\overline{C}$ is also a relation matrix of A_L^χ. This implies

$$F^{(n-1)}(\overline{C}) = \mathrm{Fitt}_{O_\chi[\Gamma_L]}(A_L^\chi).$$

Note that $\mu_p \not\subset L$ because we are studying the case where $\mathfrak{p} = (p)$ is unramified in L. Hence we can apply Theorem 3.6 to L/k, and obtain

$$F^{(n-1)}(\overline{C}) = (\Theta_{L/k} \otimes \mathbf{Z}_p)^\chi. \tag{4.1.2}$$

By Theorem 3.1, we have $\Theta_{K_\infty/k}^\chi = \mathrm{Fitt}_\Lambda(X_{K_\infty}^\chi) = F^{(n)}(\mathbf{B})$. Therefore, we obtain

$$F^{(n)}(B) = c_{K_\infty/K}(\Theta_{K_\infty/k}^\chi). \tag{4.1.3}$$

Thus, it follows from (4.1.1), (4.1.2), (4.1.3) that

$$\mathrm{Fitt}_{O_\chi[\Gamma_K]}(A_K^\chi) = \nu_{K/L}((\Theta_{L/k} \otimes \mathbf{Z}_p)^\chi) + c_{K_\infty/K}(\Theta_{K_\infty/k}^\chi). \tag{4.1.4}$$

If $m = 0$ (namely if $K = L$), we have obtained the conclusion, so we may assume $m \neq 0$. By the argument of subsection 1.2, we can show that $(\Theta'_{K/k} \otimes \mathbf{Z}_p)^\chi$ is generated by all $\nu_{K/M}(\theta^\chi_{M/k})$'s where M runs over fixed subfields of K by subgroups of Γ_K of the form $P_\mathfrak{p} \times P_{\mathfrak{l}_{i_1}} \times ... \times P_{\mathfrak{l}_{i_s}}$ and $P_{\mathfrak{l}_{i_1}} \times ... \times P_{\mathfrak{l}_{i_s}}$ for some s. Note that $(\Theta_{K/k} \otimes \mathbf{Z}_p)^\chi = (\Theta'_{K/k} \otimes \mathbf{Z}_p)^\chi$ since $\chi \neq \omega$. If M is the fixed subfield of $P_\mathfrak{p} \times P_{\mathfrak{l}_{i_1}} \times ... \times P_{\mathfrak{l}_{i_s}}$, M is a subfield of L. Hence $\nu_{K/M}(\theta^\chi_{M/k})$ is in $\nu_{K/L}((\Theta_{L/k} \otimes \mathbf{Z}_p)^\chi)$. If M is the fixed subfield of $P_{\mathfrak{l}_{i_1}} \times ... \times P_{\mathfrak{l}_{i_s}}$, $\nu_{K/M}(\theta^\chi_{M/k})$ is in $c_{K_\infty/K}(\Theta^\chi_{K_\infty/k})$. Therefore, we have

$$\nu_{K/L}((\Theta_{L/k} \otimes \mathbf{Z}_p)^\chi) + c_{K_\infty/K}(\Theta^\chi_{K_\infty/k}) = (\Theta_{K/k} \otimes \mathbf{Z}_p)^\chi. \tag{4.1.5}$$

Combining (4.1.4) and (4.1.5), we obtain

$$\mathrm{Fitt}_{O_\chi[\Gamma_K]}(A^\chi_K) = (\Theta_{K/k} \otimes \mathbf{Z}_p)^\chi.$$

Corollary 4.2 *We assume the same conditions as in Theorem 4.1, and also that no primitive p-th root of unity is in K. Then we have*

$$\mathrm{Fitt}_{R^-_{K,p}}(A^-_K) = \Theta^-_{K/k} \otimes \mathbf{Z}_p.$$

Proof. We know $\mathrm{Fitt}_{O_\chi[\Gamma_K]}(A^\chi_K) = (\Theta_{K/k} \otimes \mathbf{Z}_p)^\chi$ for *any* odd character χ of Δ_K because μ_p is not in K. This implies the conclusion. Q.E.D.

Proof of Theorem 1.1: We may assume that K satisfies the condition (A). If p is tamely ramified in K, Theorem 3.5 implies the conclusion, so we may assume that p is wildly ramified in K. Let P_p be the Sylow p-subgroup of the inertia subgroup of p in $\mathrm{Gal}(K/\mathbf{Q})$, and let L be the fixed subfield of P_p in K. Then K is an intermediate field of the cyclotomic $\mathbf{Z}_p$-extension L_∞/L, and (K, L) satisfies all the assumptions of Theorem 4.1 (note that $\mu = 0$ by [2]). Therefore, we obtain the conclusion by Theorem 4.1 for $\chi \neq \omega$. If $\chi = \omega$, the conclusion was already proved in Proposition 2.4 (2). Q.E.D.

As we mentioned in §1, Theorem 1.1 implies Theorem 0.1, so we have also proved Theorem 0.1.

We finally prove the following Corollary 4.3. Let p be an odd prime number, and let $\psi : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \longrightarrow \overline{\mathbf{Q}}_p^\times$ be any odd Dirichlet character. (Note that p may divide the order of ψ.) Let K be the imaginary abelian field corresponding to $\mathrm{Ker}\,\psi$. We extend ψ to a ring homomorphism ψ :

$\mathbf{Z}_p[\mathrm{Gal}(K/\mathbf{Q})] \longrightarrow \overline{\mathbf{Q}}_p$, and put $O_\psi = \mathbf{Z}_p[\mathrm{Gal}(K/\mathbf{Q})]/(\mathrm{Ker}\,\psi)$, which we identify with $\mathbf{Z}_p[\mathrm{Image}\,\psi]$. We define the ψ-quotient of A_K by

$$A_K^\psi = A_K \otimes_{R_{K,p}} O_\psi.$$

If $K = \mathbf{Q}(\mu_{p^m})$ for some $m > 0$ and $\psi_{|\Delta_K} = \omega$, we know $A_K^\psi = 0$. We exclude this case, and assume that if $K = \mathbf{Q}(\mu_{p^m})$ for some $m > 0$ then $\psi_{|\Delta_K} \neq \omega$.

Corollary 4.3 (D. Solomon [14]) *Let $B_{1,\psi^{-1}}$ be the generalized Bernoulli number which we regard as an element in $O_\psi = \mathbf{Z}_p[\mathrm{Image}\,\psi]$. Then we have*

$$\# A_K^\psi = \# O_\psi/(B_{1,\psi^{-1}}).$$

Precisely speaking, Solomon proved in [14] the above statement for the ψ-part $A_K(\psi)$ which is defined by

$$A_K(\psi) = \{x \in A_K \mid \alpha x = 0 \text{ for all } \alpha \in \mathrm{Ker}\,\psi \subset \mathbf{Z}_p[\mathrm{Gal}(K/\mathbf{Q})]\}$$

though we prove it for the ψ-quotient A_K^ψ. Since $K/\mathbf{Q}$ is a cyclic extension, it is easy to check that $\# A_K(\psi) = \# A_K^\psi$.

Proof of Corollary 4.3. First of all, we have $B_{1,\psi^{-1}} \in O_\psi = \mathbf{Z}_p[\mathrm{Image}\,\psi]$ by our assumption that $\psi_{|\Delta_K} \neq \omega$ in the case $K = \mathbf{Q}(\mu_{p^m})$. We extend ψ to a ring homomorphism $\psi : \mathbf{Q}_p[\mathrm{Gal}(K/\mathbf{Q})] \longrightarrow \overline{\mathbf{Q}}_p$. Let K' be the abelian field satisfying (A) such that K'/K is unramified everywhere as in §1. For a subfield F of K', if $K \not\subset F$, we have $\psi(c_{K'/K}(\nu_{K'/F}(\theta_F))) = 0$. If $K \subset F$, $\psi(c_{K'/K}(\nu_{K'/F}(\theta_F)))$ is a multiple of $\psi(\theta_K)$. Therefore, we have $\psi(\Theta_K \otimes \mathbf{Z}_p) \subset (\psi(\theta_K))$.

We will show the other inclusion. If $\mu_p \not\subset K'$, we know $\theta_K \in \Theta_K \otimes \mathbf{Z}_p$. If $\mu_p \subset K'$, let N be the conductor of K and $\mathrm{ord}_p(N) = m$. Suppose at first $N \neq p^m$. Since $\mu_p \subset K'$ and $\mathrm{ord}_p(N) = m$, we know $\mu_{p^m} \subset K'$. Since $N \neq p^m$, we also have $K' \supsetneq \mathbf{Q}(\mu_{p^m})$. We use the notation of §2. By Proposition 2.3, we have

$$\theta_{K'} - c_N \nu_{K'/\mathbf{Q}(\mu_{p^m})}(\theta_{\mathbf{Q}(\mu_{p^m})}) \in \Theta_{K'} \otimes \mathbf{Z}_p.$$

Since $c_{K'/K}(\theta_{K'}) = \theta_K$ and $\psi(c_{K'/K}(\nu_{K'/\mathbf{Q}(\mu_{p^m})}(\theta_{\mathbf{Q}(\mu_{p^m})}))) = 0$, $\psi(\theta_K)$ is in $\psi(\Theta_K \otimes \mathbf{Z}_p)$. If $N = p^m$, our assumption $\mu_p \subset K'$ implies $K' = \mathbf{Q}(\mu_{p^m})$. Since $\psi_{|\Delta_K} \neq \omega$ by our assumption, putting $\chi = \psi_{|\Delta_K}$, we know $\theta_K^\chi \in (\Theta_K \otimes \mathbf{Z}_p)^\chi$. This implies $\psi(\theta_K) \in \psi(\Theta_K \otimes \mathbf{Z}_p)$. In any case, we have

$$\psi(\theta_K) \in \psi(\Theta_K \otimes \mathbf{Z}_p).$$

Therefore, we have $\psi(\Theta_K \otimes \mathbf{Z}_p) = (\psi(\theta_K))$. It is well-known (for example, [17] Theorem 4.2) that $\psi(\theta_K) = -B_{1,\psi^{-1}}$. Hence, it follows from Theorem 0.1 that

$$\mathrm{Fitt}_{O_\psi}(A_K^\psi) = (\psi(\theta_K)) = (B_{1,\psi^{-1}}).$$

Since O_ψ is a discrete valuation ring, this is equivalent to $\#A_K^\psi = \#O_\psi/(B_{1,\psi^{-1}})$.

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