Beilinson-Kato and Beilinson-Flach elements, Coleman-Rubin-Stark classes, Heegner points and a Conjecture of Perrin-Riou

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Abstract.

Our first goal in this article is to explain that a weak form of Perrin-Riou's conjecture on the non-triviality of Beilinson–Kato classes follows as an easy consequence of the Iwasawa main conjectures. We also explain that the refined form of this conjecture in the p-supersingular case also follows from the classical Gross–Zagier formula and Kobayashi's p-adic Gross–Zagier formula combined with this simple observation.

Our second goal is to set up a conceptual framework in the context of Λ-adic Kolyvagin systems to treat analogues of Perrin-Riou's conjectures for higher motives of higher rank. We apply this general discussion in order to establish a link between Heegner points on a general class of CM abelian varieties and the (conjectural) Coleman–Rubin–Stark elements we introduce here. This can ben thought of as a higher dimensional version of Rubin's results on rational points on CM elliptic curves.

Contents

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§1. Summary of Contents and Background

We have three goals in this article. Part I is largely expository and we hope it will serve as a motivation for Parts II and III: In this portion, we record a rather straightforward (but somehow overlooked) proof of Perrin-Riou's conjecture (under very mild hypotheses) on the non-vanishing of the p-adic Beilinson–Kato class associated to an elliptic curve $E_{/\mathbb{Q}}$, when E has semistable reduction at p. In a recent preprint [14], this simple observation is used as an input in the proof of Perrin-Riou's refined conjecture in the case of good ordinary reduction. We remark that the most general form of Perrin-Riou's refined conjectures has been announced by Bertolini–Darmon–Venerucci[1] at all primes p of semistable reduction and proved by Venerucci [59] in the case of split-multiplicative reduction. To the best of our knowledge, Bertolini–Darmon–Venerucci utilise a different toolbox.

Also in Part I we explain, in the situation when p is a prime of supersingular reduction, how to deduce an explicit formula for a point of infinite order on $E(\mathbb{Q})$ in terms of the special values of the two (unbounded) p-adic L-functions attached to two p-stabilizations of the associated eigenform. This formula was proved by Perrin-Riou in [47, §3.3.4, Formule (non démontrée)] albeit in a slightly different form, assuming the validity of p-adic Birch and Swinnerton-Dyer formulae[2]. We remark that essentially the same formula in terms of signed p-adic L-functions of Pollack was also proved by Kurihara and Pollack. Even though the argument presented here is well-known to experts[3], we decided to record it in this article for the sake of completeness, particularly because it works equally well in the case of good ordinary reduction (under the assumption that at least one of the two associated p-adic height pairings

[1] Even earlier by Bertolini–Darmon in the case of good ordinary reduction, several years before [14].

[2] As Kobayashi explains in [35], p-adic Birch and Swinnerton-Dyer formulae follow from the truth of Iwasawa main conjectures and his p-adic Gross–Zagier formula.

[3] Also, it is not the most efficient in terms of assumptions: We are grateful to anonymous referee for explaining us an argument to deduce this formula in the case of supersingular reduction, utilising the results of [32, 47, 35], without relying on the truth of Iwasawa main conjectures. In particular, the assumption that the elliptic curve E be semistable can be relaxed.

is non-trivial, which is deduced in [14] as a consequence of the p-adic Gross–Zagier formulae[4] in this set up).

In Part II of this article (Section 3), we first recast this approach relying on the theory of Λ-adic Kolyvagin systems. We explain how this yields a proof of an extension of Perrin-Riou conjecture concerning the non-vanishing of the p-distinguished twists of Beilinson–Flach elements (Corollary 3.1).

In the third and final portion of this paper, we establish a precise link between Heegner points on a general class of CM abelian varieties and the (conjectural) Coleman–Rubin–Stark elements we introduce here associated to these CM abelian varieties (c.f. Theorem 3.9 in the main text). This is a higher dimensional version of Rubin's results on rational points on CM elliptic curves (where he compares elliptic units to Heegner points on CM elliptic curves).

Part I. Perrin-Riou's conjecture for Beilinson–Kato elements. Let E be an elliptic curve defined over $\mathbb{Q}$ and let N denote its conductor. Fix a prime $p > 3$ and let S denote the set consisting of all rational primes dividing Np and the archimedean place. In this set up, Kato [32] has constructed an Euler system $\mathbf{c}^{\mathrm{BK}} = \{c_F^{\mathrm{BK}}\}$ where F runs through abelian extensions of $\mathbb{Q}$, $c_F^{\mathrm{BK}} \in H^1(F, T_p(E))$ is unramified away from the primes dividing Np and $T_p(E)$ is the p-adic Tate module of E. Kato's explicit reciprocity laws show that the class $c_{\mathbb{Q}}^{\mathrm{BK}} \in H^1(\mathbb{Q}, T_p(E))$ is non-crystalline at p (and in particular, non-zero) precisely when $L(E/\mathbb{Q}, 1) \neq 0$, where $L(E/\mathbb{Q}, s)$ is the Hasse-Weil L-function of E. Perrin-Riou in [47, §3.3.2] predicts the following assertion to hold true. Let $\mathrm{res}_p : H^1(G_{\mathbb{Q},S}, T_p(E)) \to H^1(\mathbb{Q}_p, T_p(E))$ denote the restriction map.

Conjecture 1.1. *The class* $\mathrm{res}_p\left(c_{\mathbb{Q}}^{\mathrm{BK}}\right) \in H^1(\mathbb{Q}_p, T_p(E))$ *is non-torsion if and only if* $L(E/\mathbb{Q}, s)$ *has at most a simple zero at* $s = 1$.

This is the conjecture (and its extensions in other settings) we address in the current article.

Theorem 1.1 below concerns the "if" direction in Conjecture 1.1. In Section 2.2, we will deduce it as an easy consequence of the works of Kato [32, Theorem 12.5 and 17.4], Skinner-Urban [57, Theorem 1], Skinner [55, Theorem A] and Wan [63, Theorem 4], [62, Theorem 1.4] on the main conjectures of Iwasawa theory of elliptic curves.

[4]We note that this preprint relies crucially on Kobayashi's forthcoming work on p-adic Gross–Zagier formula for cuspidal eigenforms of arbitrary weight at non-ordinary primes.

Theorem 1.1. *Suppose that E is an elliptic curve such that the residual representation*

$$\overline{\rho}_E : G_{\mathbb{Q},S} \longrightarrow \mathrm{Aut}(E[p])$$

is surjective. Then the "if" part of Perrin-Riou's Conjecture 1.1 holds true in the following cases:

(a) *E has good ordinary reduction at p and one of the following two conditions hold.*
  - *There exists a prime $\ell || N$ such that $p \nmid \mathrm{ord}_\ell(\Delta_\ell)$ for a minimal discriminant Δ_ℓ of E at ℓ.*
  - *There exists a real quadratic field F verifying the conditions of* [63, Theorem 4]*.*

(b) *E has good supersingular reduction at p and N is square-free.*

(c) *E has multiplicative reduction at p and there exists a prime $\ell \mid\mid N$ such that $p \nmid \mathrm{ord}_\ell(\Delta_\ell)$ for a minimal discriminant Δ_ℓ of E at ℓ.*

As per the "only if" direction, one may prove the following result as a rather straightforward corollary (see Section 2.1 for details) of [54, Theorem A′], [58, Theorem 1.1] and [60, Theorem A] in the case of p-ordinary reduction; and [19, Theorem 6.4] together with [62, Theorem 1.4] in the case of p-supersingular reduction. We state it here for the sake of completeness.

Theorem 1.2. *In the situation of Theorem 1.1, the "only if" part of Perrin-Riou's conjecture holds true for all cases* (a)*,* (b) *and* (c) *if we further assume:*

- *in the case of* (a)*, that N is square free and either E has non-split multiplicative reduction at one odd prime or split multiplicative reduction at two odd primes;*
- *in the case of* (b)*, that N is square free, that there exists $q \mid N$ such that $\overline{\rho}_E$ is ramified at q and $\overline{\rho}_E$ is surjective;*
- *in the case of* (c) *that*
  - *p does not divide $\mathrm{ord}_p(\Delta_E)$ and when E split multiplicative reduction at p, the Galois representation $E[p]$ is not finite at p,*
  - *for all primes $\ell \mid\mid N$ such that $\ell \equiv \pm 1 \mod p$, the prime p does not divide $\mathrm{ord}_\ell(\Delta_E)$,*
  - *there exists at least two prime factors $\ell \mid\mid N$ such that p does not divide $\mathrm{ord}_\ell(\Delta_E)$.*

We remark that in the situation of (a), the hypotheses in Theorem 1.2 may be slightly altered if we relied on the work of Zhang [67, Theorem 1.3] on the converse of the Gross–Zagier-Kolyvagin theorem, in place of the work of Skinner. This on one hand would allow us to relax the condition on the conductor N, on the other hand would force us to introduce additional hypothesis (see Theorem 1.1 of loc.cit).

In a variety of cases, one may prove a refined version of Theorem 1.1 and deduce that the square of the logarithm of a suitable Heegner point agrees with the logarithm of the Beilinson–Kato class $\mathbf{BK}_1$ up to an explicit non-zero algebraic factor. We record here the following result in the case when E has good supersingular reduction at p (see Remark 1.1 below for the developments concerning the good ordinary case). For a detailed discussion and proofs (on which we claim no originality), we refer the reader to Section 2.3.

Theorem 1.3. *Suppose that E has good supersingular reduction at p and verifies the hypotheses of Theorem 1.1, as well as that its conductor N is square-free. Then,*

$$\log_E(\mathrm{res}_p(\mathbf{BK}_1)) = -(1-1/\alpha)(1-1/\beta)\cdot C(E)\cdot \log_E(\mathrm{res}_p(P))^2$$

for a suitably chosen Heegner point $P \in E(\mathbb{Q})$, where $\log_E$ stands for the coordinate of the Bloch-Kato logarithm associated to E with respect to a suitably normalized Néron differential on E and $C(E) \in \mathbb{Q}^\times$ is given in (2.4).

A more detailed version of this statement is recorded as Theorem 2.2 below. Even though Theorem 1.3 (in fact, a stronger version stripped off the semistability hypothesis herein) is a direct consequence of the results of Kato [32], Perrin-Riou [47, §3.3.4. Formule (non démontrée)] and Kobayashi [35, Theorem 1.1][5], we decided to include a proof of this corollary in Section 2.3.1 that relies on [6, 8, 1] (which, of course, still requires Kato's explicit reciprocity laws for Beilinson–Kato elements and Kobayashi's p-adic Gross-Zagier formulae as an input), since our later discussion that concerns higher dimensional motives parallels this line of argument.

Remark 1.1. *The treatment of the good ordinary case with the strategy outlined here requires a p-adic Gross–Zagier formula at critical slope. This formula is indeed within reach and it is the subject of* [14].

Theorem 1.3 combined with the Gross–Zagier formula, Kobayashi's p-adic Gross–Zagier formula in this set up and Perrin-Riou's analysis in [47, §2.2.2] yields the following result, which allows us to determine

[5] We are grateful to anonymous referee for explaining this point to us.

a global point in terms of the special values of the associated p-adic L-functions and validates Formula 3.3.4 of [47]. Let $\omega_\alpha, \omega_\beta \in D_{\mathrm{cris}}(V)$ denote the canonical elements given as in Section 2.3.1. We set $\delta_E := [\omega_\beta, \omega_\alpha]/C(E)$, where $[\,,\,] : D_{\mathrm{cris}}(V) \times D_{\mathrm{cris}}(V) \to \mathbb{Q}_p$ is the canonical pairing.

Theorem 1.4. *Let* $\exp_V$ *denote the Bloch-Kato exponential map. Then under the hypotheses of Theorem 1.3,*

$$\exp_V\left(\omega^* \cdot \sqrt{\delta_E\left((1-1/\alpha)^{-2}\cdot L'_{p,\alpha}(E/\mathbb{Q},1) - (1-1/\beta)^{-2}\cdot L'_{p,\beta}(E/\mathbb{Q},1)\right)}\right)$$

is a $\mathbb{Q}$*-rational point on* E *of infinite order.*

A proof of this corollary is recorded at the end of Section 2.3.1.

Remark 1.2. *This result is essentially* [47, §3.3.4, Formule (non démontrée)]*, which is verified assuming the truth of the p-adic Birch and Swinnerton-Dyer conjecture. See* [37, Section 2.7] *for essentially the same formula with the one given in Theorem 1.4 (which is also verified assuming the truth of the p-adic Birch and Swinnerton-Dyer conjecture) in terms of signed p-adic L-functions. Note that Kobayashi's work in* [35, Corollary 1.3] *implies the p-adic Birch and Swinnerton-Dyer conjecture up to non-zero rational constants.*

One may also easily deduce the following version of Theorem 1.3 (relying on Disegni's p-adic Gross-Zagier formula [23, Theorem B] in place of Kobayashi's work and the Rubin-style formula proved in [43, Proposition 11.3.15]) in the case when E has non-split multiplicative reduction at p.

Theorem 1.5 (Theorem 2.3)**.** *Suppose that* E *is an elliptic curve with non-split multiplicative reduction at* p *and verifies the hypotheses of Theorem 1.1, also that there exists a prime* $\ell \mid\mid N$ *such that* $\overline{\rho}_E$ *is ramified at* ℓ*. Assume that* $r_{\mathrm{an}} = 1$ *and further that Nekovář's p-adic height pairing associated to the canonical splitting of the Hodge-filtration on the semi-stable Dieudonné module* $D_{\mathrm{st}}(V)$ *is non-vanishing. Then,*

$$\log_E\left(\mathrm{res}_p(\mathbf{BK}_1)\right)\cdot \log_E\left(\mathrm{res}_p(P)\right)^{-2} \in \overline{\mathbb{Q}}^\times,$$

where P *is any generator of* $E(\mathbb{Q})/E(\mathbb{Q})_{\mathrm{tor}}$ *.*

Remark 1.3. *Bertolini, Darmon and Venerucci have announced that in their forthcoming work* [5]*, they prove a result similar to Corollaries 1.3 and 1.5 in the situation of (a). Also, using techniques different from* [5]*, Venerucci* [59] *gave a proof of the result above when E has split* *multiplicative reduction at p. They may then deduce the conclusions of Theorems 1.1 and 1.2 directly from these results (in the respective setting). In contrast, our exposition here is based on the observation that the weak Perrin-Riou Conjecture 1.1 on the non-vanishing of the Beilinson–Kato class is an immediate consequence of Iwasawa main conjectures. Following Perrin-Riou's original strategy, we then exploit a variety of Gross–Zagier formulae to deduce the full Perrin-Riou conjecture (that relates the Beilinson–Kato class to Heegner points). In particular, the proof of full Perrin-Riou conjecture in the situation of (a) also follows from Theorem 1.1 thanks to a p-adic Gross–Zagier formula for the critical slope p-adic L-function (along with Perrin-Riou's p-adic Gross–Zagier formula for the slope-zero p-adic L-function), whose proof is the subject of* [14].

Concerning Corollaries 1.3 and 1.5, we would like to underline[6] *a common key feature of the three approaches (in* [5, 59] *and the original approach of Perrin-Riou that we take as a base here) towards it, despite their apparent differences: All three works make crucial use of a suitable p-adic Gross–Zagier formula, allowing the comparison of Heegner points with Beilinson–Kato elements. For the approach in* [5]*, this formula is provided by* [4] *(where the relevant p-adic Gross–Zagier formula is proved by exploiting Waldspurger's formula and it resembles Katz's proof of the p-adic Kronecker limit formula) and for Venerucci's approach in* [59]*, it is provided by* [2] *(where the authors use Hida deformations and the Čerednik–Drinfeld uniformization of Shimura curves). In the present article, we rely instead on the p-adic Gross–Zagier formula of Perrin-Riou* [46] *and the critical slope p-adic Gross–Zagier formula in* [14] *in the case of good ordinary reduction; the p-adic Gross–Zagier formulae of Kobayashi* [35] *in the case of good supersingular reduction and the work of Disegni* [23] *when E has non-split-multiplicative reduction at p.*

Remark 1.4. *Our arguments here easily adapt to treat also higher weight p-ordinary eigenforms; however, our conclusion in that situation is not as satisfactory as in the case of elliptic curves. For this reason, here we shall only provide a brief overview of our results towards Perrin-Riou's conjecture in that level of generality. Suppose $f \in S_{2k}(\Gamma_0(N))$ is an eigenform and we let V_f denote the self-dual twist of Deligne's representation attached to f. Let K denote a completion of the Hecke field of f at a prime above p and $\mathfrak{o}_K$ its ring of integers. Fix a Galois-stable $\mathfrak{o}_K$-lattice T_f contained in V_f and let $\Bbbk$ denote the residue field of*

[6]We would like to thank Henri Darmon for an enlightening exchange regarding this point.

$\mathfrak{o}_K$. *We will set* $\rho_f : G_{\mathbb{Q},S} \to \mathrm{GL}(T_f)$ *(where* S *is the set consisting of all rational primes dividing* Np *and the archimedean place) and* $\overline{\rho}_f := \rho_f \otimes \boldsymbol{k}$. *If the conditions that*

- $\overline{\rho}_f$ *is absolutely irreducible,*
- f *is* p*-distinguished (namely, the semi-simplification of* $\overline{\rho}_f\big|_{G_{\mathbb{Q}_p}}$ *is non-scalar),*

simultaneously hold true, then

$$\mathrm{ord}_{s=k}\, L(f,s) = 1 \implies \textit{either } \log_{V_f} \mathbf{BK}_1 \neq 0\,, \\ \textit{or else } \mathrm{res}_p : H^1_{\mathrm{f}}(\mathbb{Q}, V_f) \to H^1_{\mathrm{f}}(\mathbb{Q}_p, V_f) \textit{ is the zero map.}$$

Here $H^1_{\mathrm{f}}(\mathbb{Q}_p, V_f) \subset H^1(\mathbb{Q}_p, V_f)$ *is the image of the Bloch-Kato exponential map* $\exp_{V_f}$ *and* $H^1_{\mathrm{f}}(\mathbb{Q}, V_f)$ *is the Bloch-Kato Selmer group.*

Note in particular that the main result of [1] *towards Perrin-Riou's conjecture for non-p-crystalline semistable modular forms escapes the methods of the current article.*

Part II. $\boldsymbol{\Lambda}$***-adic Kolyvagin systems, Beilinson–Flach elements and Coleman–Rubin–Stark elements*** The general theory of Λ-adic Kolyvagin systems yields a relatively simple criterion to verify the Perrin-Riou conjecture in a great level of generality. This is the content of our Theorem 3.3 below. Combined with the work of Wan [61] and Howard [28, 66], it yields an easy proof of the following statement (which is Corollary 3.1 in the main body of this note):

Theorem 1.6. *Let* $E/\mathbb{Q}$ *be a non-CM elliptic curve that has good ordinary reduction at* p *and assume that the residual representation* $\overline{\rho}_E$ *is absolutely irreducible. Let* $K/\mathbb{Q}$ *be an imaginary quadratic extension that satisfies the weak Heegner hypothesis for* E *and where* p *splits completely. Suppose further that* p *does not divide* $\mathrm{ord}_\ell(j(E))$ *whenever* ℓ *is a prime of split multiplicative reduction. If the twisted L-function* $L(E/K, \alpha, s)$ *associated to the base change of* E/K *and a* p*-distinguished character* α *vanishes at* $s = 1$ *to exact order* 1*, then the corresponding Beilinson–Flach element* $\mathrm{BF}_1 \in H^1_{\mathrm{f}}(K, T_p(E) \otimes \alpha^{-1})$ *is non-trivial.*

We remark that Bertolini and Darmon have announced the proof of an analogue of full Perrin-Riou conjecture in this set up, that enables them to compare the logarithms of Beilinson–Flach elements and Heegner points. Theorem 1.6 is a considerably weaker version of their result. On the other hand, Theorem 1.6 may be extended to cover the case of elliptic curves with good supersingular reduction as well, if one relied on another preprint of Wan on Iwasawa main conjectures at supersingular primes.

The view point offered by the theory of Λ-adic Kolyvagin systems enables us to address similar problems concerning a CM abelian variety A of dimension g. In this situation, a slightly stronger version of Rubin–Stark conjectures equips us with a rank-g Euler system. Making use of Perrin-Riou's *extended logarithm maps* and the methods of [16, 10], we may obtain Kolyvagin systems (Theorem 3.5) out of these classes (that we call the *Coleman–Rubin–Stark Kolyvagin systems*), with which we may apply Theorem 3.3. Our *Explicit Reciprocity Conjecture* 3.2 for these classes is a natural generalization of the Coates–Wiles explicit reciprocity law for elliptic units, and predicts in a rather precise manner how the conjectural Rubin–Stark elements should be related to the Hecke L-values attached to the CM abelian variety A.

All this combined with the results of [16] allows us to prove Theorem 1.7 below, which establishes an explicit link between the Coleman–Rubin–Stark elements and Heegner points. Let A be an abelian variety which has complex multiplication by an order of a CM field K whose index in the maximal order is prime to p and defined over the maximal totally real subfield K_+ of K. We assume that A has good ordinary reduction at every prime above p, that the prime p is unramified in $K_+/\mathbb{Q}$ and that A verifies the non-anomaly hypothesis (3.1) at p. One may than associate A a Hilbert modular CM form ϕ on $\mathrm{Res}_{K_+/\mathbb{Q}}\,\mathrm{GL}_2$ of weight 2. See Section 3.4 for a detailed discussion of these objects.

We will assume that there exists a degree one prime of K_+ above p (we believe that it should be possible to get around of this assumption with more work). Let $\langle\, ,\, \rangle$ denote the p-adic height pairing introduced in Definition 3.16 and *assume that it is non-zero*. Denote by $\mathfrak{C}$ the Coleman–Rubin–Stark element associated to the CM form ϕ (that is given as in Definition 3.12).

Theorem 1.7. *If the Hecke L-series of the associated to CM form ϕ vanishes to exact order* 1 *at $s = 1$, then the Coleman–Rubin–Stark class $\mathfrak{C}$ is non-trivial and we have*

$$\log_\omega(\mathfrak{C}) \equiv \log_\omega(P_\phi)^2 \mod \overline{\mathbb{Q}}^\times,$$

where P_ϕ is a Heegner point on A and $\log_\omega$ is a certain coordinate of the Bloch-Kato logarithm (*with respect to a suitably chosen Néron differential form*) *given as in* (*3.10*) *below.*

See Theorem 3.9 below for a more precise version of this statement. We remark that Burungale and Disegni [3] recently proved the generic non-triviality of p-adic heights. Relying on this result, one may generically by-pass this hypothesis for the twisted variants of Theorem 1.7.

Remark 1.5. *We were informed by D. Disegni that in a future version of* [3]*, the authors will be proving a complementary version of the formula in Theorem 1.7, which will express the right hand side in terms of the special values of a suitable p-adic L-function.*

1.1. Notation

For a number field K, we define K_S to be the maximal extension of K unramified outside a finite set of places S of K that contains all archimedean places as well as all those lying above p. Set $G_{K,S} := \mathrm{Gal}(K_S/K)$.

Let ${}_{p^\infty}$ denote the p-power roots of unity. For a complete local noetherian $\mathbb{Z}_p$-algebra R and an $R[[G_{\mathbb{Q},S}]]$-module X which is free of finite rank over R, we define $X^* := \mathrm{Hom}(X, {}_{p^\infty})$ and refer to it as the Cartier dual of X. For any ideal I of R, we denote by $X[I]$ the R-submodule of X killed by all elements of I.

Let A/K be an abelian variety of dimension g which has good reduction outside S. We let $T_p(A)$ denote the p-adic Tate-module of A; this is a free $\mathbb{Z}_p$-module of rank $2g$ which is endowed with a continuous $G_{K,S}$-action. We define the *cyclotomic deformation* $\mathbb{T}_p(A)$ of $T_p(A)$ by setting $\mathbb{T}_p(A) := T_p(A) \otimes \Lambda$ (where we let $G_{K,S}$ act diagonally), where $\Lambda := \mathbb{Z}_p[[\Gamma]]$ (with $\Gamma = \mathrm{Gal}(K_{\mathrm{cyc}}/K)$ is the Galois group of the cyclotomic $\mathbb{Z}_p$-extension K_{cyc}/K) is the cyclotomic Iwasawa algebra. We let K_n/K denote the unique subextension of K_{cyc}/K of degree p^n (and Galois group $\Gamma_n := \Gamma/\Gamma^{p^n} \cong \mathbb{Z}/p^n\mathbb{Z}$). When $K = \mathbb{Q}$, we write $\mathbb{Q}_\infty$ in place of $\mathbb{Q}_{\mathrm{cyc}}$.

In the first part of this article the number field K will be $\mathbb{Q}$ whereas in the second part, it will be either a more general totally real field or a CM field. In Part II, we will also work with a general continuous G_K-representation T which unramified outside S and which is free of finite rank over a finite flat extension $\mathfrak{o}$ of $\mathbb{Z}_p$. We will denote by L the field of fractions of $\mathfrak{o}$ and by $\mathbb{T}$ the $G_{K,S}$-representation $T \otimes \Lambda$. We will also set $\Lambda_{\mathfrak{o}} := \Lambda \otimes_{\mathbb{Z}_p} \mathfrak{o} = \mathfrak{o}[[\Gamma]]$.

Fix a topological generator γ of Γ. We will denote by pr_0 the augmentation map $\Lambda \to \mathbb{Z}_p$ (which induced from $\gamma \mapsto 1$) and by slight abuse, also any map induced by it.

1.1.1. *Selmer structures* Given a general Galois representation $\mathbb{T}$, we let $\mathcal{F}_\Lambda$ denote the *canonical Selmer structure* on $\mathbb{T}$ defined by setting $H^1_{\mathcal{F}_\Lambda}(K_\lambda, \mathbb{T}) = H^1(K_\lambda, \mathbb{T})$ for every prime λ of K.

In the notation of [42, Definition 2.1.1] we have $\Sigma(\mathcal{F}) = S$ for each of the Selmer structures above.

Definition 1.1. *For every prime λ of K, there is a perfect local Tate pairing*

$$\langle\, , \rangle_{\lambda,\mathrm{Tate}} : H^1(K_\lambda, X) \times H^1(K_\lambda, X^*) \to H^2(K_\lambda, {}_{p^\infty}) \xrightarrow{\sim} \mathbb{Q}_p/\mathbb{Z}_p\,.$$

For a Selmer structure $\mathcal{F}$ on X, define the dual Selmer structure $\mathcal{F}^*$ *on X^* by setting $H^1_{\mathcal{F}^*}(\mathbb{Q}_\ell, X^*) := H^1_{\mathcal{F}}(\mathbb{Q}_\ell, X)^\perp$, the orthogonal complement of $H^1_{\mathcal{F}}(\mathbb{Q}_\ell, X)$ with respect to the local Tate pairing.*

Given a Selmer structure $\mathcal{F}$ on X, we may propagate it to the subquotients of X (via [42, Example 1.1.2]). We denote the propagation of $\mathcal{F}$ to a subquotient still by $\mathcal{F}$.

§2. Part I. Perrin-Riou's conjecture for elliptic curves over $\mathbb{Q}$

Let $E/\mathbb{Q}$ be an elliptic curve as above and recall the Beilinson–Kato Euler system $\mathbf{c}^{\mathbf{BK}} = \{c_F^{\mathbf{BK}}\}$. We write

$$\mathbb{BK}_1 := \{c_{\mathbb{Q}_n}^{\mathrm{BK}}\} \in \varprojlim H^1(G_{\mathbb{Q}_n,S}, T_p(E)) = H^1(\mathbb{Q}, \mathbb{T}_p(E)),$$

where the last equality follows from [42, Lemma 5.3.1(iii)]. We also set $\mathbf{BK}_1 := c_{\mathbb{Q}}^{\mathrm{BK}}$ so that $\mathrm{pr}_0(\mathbb{BK}_1) = \mathbf{BK}_1$. It follows from the non-vanishing results of Rohrlich [51] and Kato's explicit reciprocity laws for the Beilinson–Kato elements that $\mathbb{BK}_1$ never vanishes.

Let us denote the order of vanishing of the Hasse-Weil L-function $L(E/\mathbb{Q}, s)$ by r_{an} and call it the *analytic rank of E*.

2.1. Preliminaries

We first explain how the "only if" part of Perrin-Riou's conjecture (Theorem 1.2) may be deduced as an easy corollary of the work of Skinner, Skinner-Zhang and Venerucci. The argument we present here involves some of the reduction steps which we rely on for the proof of the "if" part of the conjecture and we pin these down in this portion of our article.

Proof of Theorem 1.2. Suppose first that $\mathrm{res}_p^s(\mathbf{BK}_1) \neq 0$, where res_p^s is the singular projection given as the compositum of the arrows

$$\begin{aligned} H^1(\mathbb{Q}, T_p(E)) &\to H^1(\mathbb{Q}_p, T_p(E)) \\ &\twoheadrightarrow H^1(\mathbb{Q}_p, T_p(E))/H^1_{\mathrm{f}}(\mathbb{Q}_p, T_p(E)) =: H^1_s(\mathbb{Q}_p, T_p(E)). \end{aligned}$$

Kato's explicit reciprocity law shows that $r_{\mathrm{an}} = 0$. We may therefore assume without loss of generality that $\mathbf{BK}_1$ is crystalline at p, namely that $\mathbf{BK}_1 \in H^1_{\mathrm{f}}(\mathbb{Q}, T_p(E))$.

Since $\mathbf{BK}_1 \neq 0$, it follows from [42, Corollary 5.2.13(i)] that $H^1_{\mathcal{F}^*_{\mathrm{can}}}(\mathbb{Q}, T_p(E)^*)$ is finite. Recall that $\mathcal{F}_{\mathrm{str}}$ denotes the Selmer structure on $T_p(E)$ given by

- $H_{\mathcal{F}_{\mathrm{str}}}(\mathbb{Q}_\ell, T_p(E)) = H^1_{\mathrm{f}}(\mathbb{Q}_\ell, T_p(E))$
$:= \ker\left(H^1(\mathbb{Q}_\ell, T_p(E)) \to H^1(\mathbb{Q}_\ell^{\mathrm{ur}}, T_p(E)\otimes\mathbb{Q}_p)\right)$, if $\ell \neq p$,
- $H_{\mathcal{F}_{\mathrm{str}}}(\mathbb{Q}_p, T_p(E)) = 0.$

We contend to verify that $H^1_{\mathcal{F}_{\mathrm{str}}}(\mathbb{Q}, T_p(E)) = 0$. Assume on the contrary that $H^1_{\mathcal{F}_{\mathrm{str}}}(\mathbb{Q}, T_p(E))$ is non-trivial. Since module $H^1(G_{\mathbb{Q},S}, T_p(E))$ is torsion free under our running hypothesis on the image of $\overline{\rho}_E$, this amounts to saying that $H^1_{\mathcal{F}_{\mathrm{str}}}(\mathbb{Q}, T_p(E))$ has positive rank.

Recall further that the propagation of the Selmer structure $\mathcal{F}_{\mathrm{str}}$ to the quotients $T_p(E)/p^nT_p(E)$ (in the sense of [42]) is still denoted by $\mathcal{F}_{\mathrm{str}}$. Recall that $T_p(E)^* \cong E[p^\infty]$ and note for any positive integer n that we may identify the quotient $T_p(E)/p^nT_p(E)$ with $E[p^n]$. By [42, Lemma 3.7.1], we have an injection

$$\begin{aligned} &H_{\mathcal{F}_{\mathrm{str}}}(\mathbb{Q}, T_p(E))/p^nH_{\mathcal{F}_{\mathrm{str}}}(\mathbb{Q}, T_p(E)) \\ &\qquad\qquad \hookrightarrow H_{\mathcal{F}_{\mathrm{str}}}(\mathbb{Q}, T_p(E)/p^nT_p(E)) = H_{\mathcal{F}_{\mathrm{str}}}(\mathbb{Q}, E[p^n]) \end{aligned}$$

induced from the projection $T \to T/p^nT$. This shows that

$$\mathrm{length}_{\mathbb{Z}_p}\left(H_{\mathcal{F}_{\mathrm{str}}}(\mathbb{Q}, E[p^n])\right) \geq n. \tag{2.1}$$

As above, we let $\mathcal{F}_{\mathrm{can}}$ denote the canonical Selmer structure on $T_p(E)$, given by

- $H_{\mathcal{F}_{\mathrm{can}}}(\mathbb{Q}_\ell, T_p(E)) = H^1_{\mathrm{f}}(\mathbb{Q}_\ell, T_p(E))$, if $\ell \neq p$,
- $H_{\mathcal{F}_{\mathrm{can}}}(\mathbb{Q}_p, T_p(E)) = H^1(\mathbb{Q}_p, T_p(E)).$

We then have an inclusion

$$H_{\mathcal{F}_{\mathrm{str}}}(\mathbb{Q}_\ell, E[p^n]) \subset H_{\mathcal{F}^*_{\mathrm{can}}}(\mathbb{Q}_\ell, E[p^n]),\ \ \forall \ell. \tag{2.2}$$

Here, $E[p^n]$ is identified with $T_p(E)/p^nT_p(E)$ on the left and with $T_p(E)^*[p^n]$ on the right. In particular, $H_{\mathcal{F}_{\mathrm{str}}}(\mathbb{Q}_\ell, E[p^n])$ is the image of $H^1_{\mathrm{f}}(\mathbb{Q}_\ell, T_p(E))$ under the map induced from $T_p(E) \to T_p(E)/p^nT_p(E)$ whereas $H_{\mathcal{F}^*_{\mathrm{can}}}(\mathbb{Q}_\ell, E[p^n])$ is the inverse image of $H_{\mathcal{F}^*_{\mathrm{can}}}(\mathbb{Q}_\ell, E[p^\infty])$ under the map induced from $E[p^n] \to E[p^\infty]$.

We explain the containment (2.2). When $\ell \neq p$, it follows from [53, Lemma I.3.8(i)] that

$$H_{\mathcal{F}_{\mathrm{str}}}(\mathbb{Q}_\ell, E[p^n]) = H_{\mathcal{F}^*_{\mathrm{can}}}(\mathbb{Q}_\ell, E[p^n])\,.$$

When $\ell = p$, we have $H_{\mathcal{F}_{\mathrm{str}}}(\mathbb{Q}_\ell, E[p^n]) = 0$ so the claimed containment trivially holds in this case as well.

It follows from (2.1) and the inclusion (2.2) that

$$\text{length}_{\mathbb{Z}_p}\left(H_{\mathcal{F}^*_{\text{can}}}(\mathbb{Q}, E[p^n])\right) \geq n. \tag{2.3}$$

However, $H_{\mathcal{F}^*_{\text{can}}}(\mathbb{Q}, E[p^\infty])$ is finite and therefore the length of

$$H_{\mathcal{F}^*_{\text{can}}}(\mathbb{Q}, E[p^n]) \cong H_{\mathcal{F}^*_{\text{can}}}(\mathbb{Q}, E[p^\infty])[p^n]$$

(where the isomorphism is thanks to [42, Lemma 3.5.3], which holds true here owing to our assumption on the image of $\overline{\rho}_E$) is bounded independently of n. This contradicts (2.3) and shows that $H^1_{\mathcal{F}_{\text{str}}}(\mathbb{Q}, T_p(E)) = 0$. Thence, the map

$$\text{res}_p : H^1_{\text{f}}(\mathbb{Q}, T_p(E)) \longrightarrow H^1_{\text{f}}(\mathbb{Q}_p, T_p(E))$$

is injective. The module $H^1_{\text{f}}(\mathbb{Q}_p, T_p(E))$ is free of rank one and we conclude that the module $H^1_{\text{f}}(\mathbb{Q}, T_p(E))$ has also rank one. When we are in the situation of (a) or (c), the proof now follows from the converse of the Kolyvagin-Gross–Zagier theorem proved in [54, 58]. In the situation of (b), let us choose an imaginary quadratic field F where the prime p splits, where even number of primes dividing N, including q in the statement of the theorem, remain inert in F and such that $L(E^{(F)}, 1) \neq 0$ (here, $E^{(F)}$ denotes the qudratic twist of E). An imaginary quadratic field F with these properties exist by Dirichlet's theorem on arithmetic progressions and generic non-vanishing results of [9]. We deduce by Kato's theorem [32, Theorem 14.2] that $H^1_{\text{f}}(\mathbb{Q}, T_p(E^{(F)})) = 0$ and hence,

$$H^1_{\text{f}}(F, T_p(E)) = H^1_{\text{f}}(\mathbb{Q}, T_p(E^{(F)})) \oplus H^1_{\text{f}}(\mathbb{Q}, T)$$

has rank one. The result follows using [19, Theorem 6.4].

Q.E.D.

Lemma 2.1. *Let $\mathcal{F}_{\text{str}}$ and $\mathcal{F}_{\text{can}}$ be the Selmer structures given as in the proof of Theorem 1.2. Then the index of $H_{\mathcal{F}_{\text{str}}}(\mathbb{Q}_\ell, E[p^n])$ within $H_{\mathcal{F}^*_{\text{can}}}(\mathbb{Q}_\ell, E[p^n])$ is bounded independently of n and the choice of the prime ℓ.*

Proof. The argument above that explains the containment

$$H_{\mathcal{F}_{\text{str}}}(\mathbb{Q}_\ell, E[p^n]) \subset H_{\mathcal{F}^*_{\text{can}}}(\mathbb{Q}_\ell, E[p^n])$$

shows that the said index is bounded by the order of $H_{\mathcal{F}^*_{\text{can}}}(\mathbb{Q}_p, E[p^n])$. By definition, $H_{\mathcal{F}^*_{\text{can}}}(\mathbb{Q}_p, E[p^\infty]) = 0$ and hence,

$$H_{\mathcal{F}^*_{\text{can}}}(\mathbb{Q}_p, E[p^n]) = \ker(H^1(\mathbb{Q}_p, E[p^n]) \longrightarrow H^1(\mathbb{Q}_p, E[p^\infty])),$$

which in turn equals the image of $E(\mathbb{Q}_p)[p^\infty]$ under the connecting homomorphism of cohomology. Our proof is complete since $E(\mathbb{Q}_p)[p^\infty]$ has finite cardinality. Q.E.D.

Proposition 2.1. *If $r_{\rm an} \leq 1$, then $H^1_{\mathcal{F}^*_{\rm can}}(\mathbb{Q}, T_p(E)^*)$ is finite.*

Proof. If $r_{\rm an} = 0$, it follows that $\mathrm{res}_p^s(\mathbf{BK})$ and in particular that $\mathbf{BK}_1 \neq 0$. The conclusion of the proposition follows from [42, Corollary 5.2.13(i)]. Suppose now that $r_{\rm an} = 1$. In this case,

$$\begin{aligned} H^1_{\mathcal{F}_{\rm str}}(\mathbb{Q}, T_p(E)) &= \ker(H^1_{\rm f}(\mathbb{Q}, T_p(E)) \longrightarrow H^1_{\rm f}(\mathbb{Q}_p, T_p(E))) \\ &= \ker\left(E(\mathbb{Q})\,\widehat{\otimes}\,\mathbb{Z}_p \longrightarrow E(\mathbb{Q}_p)\,\widehat{\otimes}\,\mathbb{Z}_p\right) = 0 \end{aligned}$$

where the second equality follows from the finiteness of the Tate–Shafarevich group [36] and the final equality by the Gross–Zagier theorem. We conclude using Lemma 2.1 that the order of $H_{\mathcal{F}^*_{\rm can}}(\mathbb{Q}, T^*)[p^n] \cong H_{\mathcal{F}^*_{\rm can}}(\mathbb{Q}, E[p^n])$ is bounded independently of n. The proof follows. Q.E.D.

2.2. Main conjectures and Perrin-Riou's conjecture

We recall in this section Kato's formulation of the Iwasawa main conjecture for the elliptic curve E and record results towards this conjecture. It follows from Kato's reciprocity laws and Rohrlich's [51] non-vanishing theorems that the class $\mathbb{BK}_1$ is non-vanishing and the Λ-module $H^1(\mathbb{Q}, \mathbb{T}_p(E))$ is of rank one (as it was predicted by the weak Leopoldt conjecture for E).

For two ideals $I, J \subset \Lambda$, we write $I \doteqdot J$ to mean that $I = p^e J$ for some integer e.

Conjecture 2.1.

$$\mathrm{char}\left(H^1_{\mathcal{F}^*_\Lambda}(\mathbb{Q}, \mathbb{T}_p(E)^*)^\vee\right) \doteqdot \mathrm{char}\left(H^1(\mathbb{Q}, \mathbb{T}_p(E))/\Lambda \cdot \mathbb{BK}_1\right)$$

.

This assertion is (up to powers of p) what Kato calls in [32, §12] *Iwasawa main conjectures without p-adic L-functions*. It is equivalent (via Poitou–Tate global duality and Kato's reciprocity laws) to the classical formulation of Iwasawa main conjecture for E up to powers of p.

Theorem 2.1 (Kato, Kobayashi, Skinner-Urban, Wan)**.** *In the setting of Theorem 1.1, Conjecture 2.1 holds true in the following cases:*

(a) *E has good ordinary reduction at p and one of the following two conditions hold.*

– *There exists a prime $\ell || N$ such that $p \nmid \mathrm{ord}_\ell(\Delta_\ell)$ for a minimal discriminant Δ_ℓ of E at ℓ.*
– *There exists a real quadratic field F verifying the conditions of* [63, Theorem 4].

(b) *E has good supersingular reduction at p and N is square-free.*
(c) *E has multiplicative reduction at p and there exists a prime $\ell \mid\mid N$ such that $p \nmid \mathrm{ord}_\ell(\Delta_\ell)$ for a minimal discriminant Δ_ℓ of E at ℓ.*

Proof. In the setting of (a), see the works of Kato and Skinner-Urban [32, 57] (as well as the enhancement of the latter due to Wan [63], that allows to alter a certain local hypothesis of Skinner and Urban) and in the situation of (b), the works of Kobayashi and Wan [34, 62]. In the setting of (c), the works of Skinner and Kato [55, 32] yields the desired conclusion. Note that Kato has stated his divisibility result towards Conjecture 2.1 only when E has good ordinary reduction at p. We refer the reader to [52] for the slightly more general version of his theorem required to treat the non-crystalline semistable case as well. Q.E.D.

We are now ready to present a proof of Theorem 1.1, which is a direct consequence of Theorem 2.1.

Proof of Theorem 1.1. The natural map

$$H^1_{\mathcal{F}^*_\Lambda}(\mathbb{Q}, \mathbb{T}_p(E)^*)^\vee/(\gamma-1)\cdot H^1_{\mathcal{F}^*_\Lambda}(\mathbb{Q}, \mathbb{T}_p(E)^*)^\vee \longrightarrow H^1_{\mathcal{F}^*_{\mathrm{can}}}(\mathbb{Q}, T_p(E)^*)^\vee$$

has finite kernel and cokernel by the proof of Proposition 5.3.14 of [42] (applied with the height one prime $\mathfrak{P} = (\gamma-1)$). We note that the requirement on the Λ-module $H^2(\mathbb{Q}_S/\mathbb{Q}, \mathbb{T}_p(E))[\gamma-1]$ is not necessary for the portion of this proposition concerning us. We further remark that since $H^0(\mathbb{Q}_p, T^*)$ is finite, it follows that $H^2(\mathbb{Q}_p, \mathbb{T}_p(E))[\gamma-1]$ is pseudo-null and the proposition indeed applies. It follows from Proposition 2.1 that $(\gamma-1)$ is prime to the characteristic ideal of $H^1_{\mathcal{F}^*_\Lambda}(\mathbb{Q}, \mathbb{T}_p(E)^*)^\vee$, and by Theorem 2.1 that it is also prime to $\mathrm{char}\left(H^1(\mathbb{Q}, \mathbb{T}_p(E))/\Lambda\cdot\mathbb{BK}_1\right)$. This tells us that

$$\left\{c^{\mathbf{BK}}_{\mathbb{Q}_n}\right\} = \mathbb{BK}_1 \notin (\gamma-1)H^1(\mathbb{Q}, \mathbb{T}_p(E)) = \ker\left(H^1(\mathbb{Q}, \mathbb{T}_p(E)) \longrightarrow H^1(\mathbb{Q}, T_p(E))\right),$$

which amounts to saying that $c^{\mathbf{BK}}_{\mathbb{Q}} \neq 0$. Furthermore, our running hypothesis that $r_{\mathrm{an}} = 1$ together with Kato's explicit reciprocity laws implies that $c^{\mathbf{BK}}_{\mathbb{Q}} \in H^1_{\mathrm{f}}(\mathbb{Q}, T_p(E))$. The proof of Proposition 2.1 now shows that $\mathrm{res}_p(c^{\mathbf{BK}}_{\mathbb{Q}}) \neq 0$ (as otherwise, the module $H^1_{\mathcal{F}_{\mathrm{str}}}(\mathbb{Q}, T_p(E))$ would have been non-zero). Q.E.D.

2.3. Logarithms of Heegner points and Beilinson–Kato classes

As before, we suppose that E is an elliptic curve such that the residual representation $\overline{\rho}_E$ is surjective. Assume further that p does not divide $\mathrm{ord}_\ell(j(E))$ whenever $\ell \mid N$ is a prime of split multiplicative reduction. Assume also that one of the following conditions hold true.

(a) E has good ordinary reduction at p.
(b) E has good supersingular reduction at p and N is square-free.
(c) E has multiplicative reduction at p and there exists a prime $\ell \mid\mid N$ such that there $\overline{\rho}_E$ is ramified at ℓ.

With this set up, we will be able refine the conclusion of Theorem 1.1 in a variety of cases and deduce that the square of the logarithm of a suitable Heegner point agrees with the logarithm of the Beilinson–Kato class $\mathbf{BK}_1$ up to an explicit non-zero algebraic factor. In these cases, we will therefore justify some of the hypothetical conclusions in [47, §3.3.3] (where we claim almost no originality; see the paragraph following the statement of Theorem 1.3).

We fix a Weierstrass minimal model $\mathcal{E}_{/\mathbb{Z}}$ of E. Let $\omega_{\mathcal{E}}$ be a Néron differential that is normalized as in [49, §3.4] and is such that we have $\Omega_E^+ := \int_{E(\mathbb{C})^+} \omega_{\mathcal{E}} > 0$ for the real period Ω_E^+. Suppose till the end of Section 2.3 that $L(E,s)$ has a simple zero at $s=1$. In this situation, $E(\mathbb{Q})$ has rank one and the Néron-Tate height $\langle P,P\rangle_\infty$ of any generator P of the free part of $E(\mathbb{Q})$ is related via the Gross–Zagier theorem to the first derivative of $L(E,s)$ at $s=1$:

$$\frac{L'(E,1)}{\Omega_E^+} = C(E)\cdot\langle P,P\rangle_\infty \tag{2.4}$$

with $C(E)\in\mathbb{Q}^\times$.

Let $D_{\mathrm{cris}}(V)$ be the crystalline Dieudonné module of $V := T_p(E)\otimes\mathbb{Q}_p$ and we define the element $\omega_{\mathrm{cris}}\in D_{\mathrm{cris}}(V)$ as that corresponds[7] to $\omega_{\mathcal{E}}$ under the comparison isomorphism. We let $\mathcal{H}\subset\mathbb{Q}_p[[\Gamma]]$ denote Perrin-Riou's ring of tempered distributions and let

$$\mathfrak{Log}_V : H^1(\mathbb{Q}_p,\mathbb{T}_p(E))\otimes_\Lambda\mathcal{H}\longrightarrow\mathcal{H}\otimes D_{\mathrm{cris}}(V)$$

[7] When E has multiplicative reduction at p, the vector $\omega_E\in\mathrm{Fil}^0 D_{\mathrm{dR}}(V) = \mathrm{Fil}^0 D_{\mathrm{st}}(V)$ that corresponds to $\omega_{\mathcal{E}}$ does not belong to the crystalline Dieudonné module. In this case, we define ω_{cris} is defined as the projection of ω_E to $D_{\mathrm{st}}(V)^{\varphi=\pm1}$ (where the sign is determined according to whether the reduction type of E is split or non-split), parallel to the orthogonal complement $D_{\mathrm{cris}}(V)^\perp$ of $D_{\mathrm{cris}}(V)$ under the canonical pairing $D_{\mathrm{dR}}(V)\otimes D_{\mathrm{dR}}(V)\xrightarrow{\langle\,,\rangle}\mathbb{Q}_p$.

be Perrin-Riou's extended dual exponential map and write $\mathcal{L}_{\mathbf{BK}}$ as a shorthand for the element $\mathfrak{Log}_V\left(\mathrm{res}_p(\mathbb{BK}_1)\right) \in \mathcal{H} \otimes D_{\mathrm{cris}}(V)$. We also let

$$\log_V : H^1_{\mathrm{f}}(\mathbb{Q}_p, V) \xrightarrow{\sim} D_{\mathrm{dR}}(V)/\mathrm{Fil}^0 D_{\mathrm{dR}}(V)$$

denote Bloch-Kato logarithm.

2.3.1. *E has good reduction at p* In this case, $D_{\mathrm{cris}}(V)$ is a two dimensional vector space. Let $\alpha^{-1}, \beta^{-1} \in \overline{\mathbb{Q}}_p$ be the eigenvalues of the crystalline Frobenius φ acting on $D_{\mathrm{cris}}(V)$. Extending the base field if necessary, let D_α and D_β denote corresponding eigenspaces. Set $\omega_{\mathrm{cris}} = \omega_\alpha + \omega_\beta$ with $\omega_\alpha \in D_\alpha$ and $\omega_\beta \in D_\beta$. On projecting $\mathcal{L}_{\mathbf{BK}}$ onto either of these vector spaces we obtain $\mathcal{L}_{\mathbf{BK},?} \in \mathcal{H}$ (so that $\mathcal{L}_{\mathbf{BK},?} \cdot \omega_?$ is the projection of $\mathcal{L}_{\mathbf{BK}}$ onto $\mathcal{H} \otimes D_?$) for $? = \alpha, \beta$.

Theorem 2.2. *Suppose that E has good supersingular reduction at p and N is square-free. For $? = \alpha, \beta$,*

(i) *the Amice transform of the distribution $\mathcal{L}_{\mathbf{BK},?}$ is the Manin-Vishik, Amice-Velu p-adic L-function $L_{p,?}(E/\mathbb{Q}, s)$ associated to the pair $(E, D_?)$;*
(ii) *when $r_{\mathrm{an}} = 1$, one of the two p-adic L-functions vanishes at $s = 1$ to degree 1;*
(iii) *still when $r_{\mathrm{an}} = 1$, at least one of the associated p-adic height pairings $\langle\, ,\, \rangle_{p,?}$ is non-degenerate,*
(iv) $\log_E\left(\mathrm{res}_p(\mathbf{BK}_1)\right) = -(1-1/\alpha)(1-1/\beta) \cdot C(E) \cdot \log_E\left(\mathrm{res}_p(P)\right)^2$.

Note that the quantity $(1 - 1/\alpha)(1 - 1/\beta) = (1 + 1/p)$ belongs to $\mathbb{Q}^\times$.

Remark 2.1. *There are a number of constructions of p-adic heights. In the proof of Theorem 2.2, two of these make a direct appearance, so it is important to settle the comparison between all these constructions. The first is Benois' p-adic height pairing, which he introduced in* [6]. *Benois denotes this height pairing by $h^{\mathrm{sel}}_{V,\mathbb{D}_?,1}$ (where $? \in \{\alpha, \beta\}$ as above) and shows in* [6, Theorem III] *that it agrees with Nekovář's height pairing (which we will denote by $h^{\mathrm{spl}}_{V,\mathbb{D}_?}$, following Benois) introduced in* [Nek93] *in terms of the associated Hodge splitting. The second is the height pairing*

$$(E(\mathbb{Q}) \otimes \mathbb{Q}_p) \otimes (E(\mathbb{Q}) \otimes \mathbb{Q}_p) \longrightarrow \mathbb{Q}_p(\sqrt{-p})$$

that appears in Kobayashi's Gross–Zagier formulae. Kobayashi uses three different definitions of this pairing when proving his formulae, and shows that all three different formulations are equivalent. One of them *is* $h^{\rm spl}_{V,\mathbb{D}_?}=h^{\rm sel}_{V,\mathbb{D}_?,1}$*. Note that we have* $E(\mathbb{Q})\otimes\mathbb{Q}_p=H^1_{\rm f}(\mathbb{Q},V)$ *when* $r_{\rm an}=1$.

Thanks to this discussion, we will not have to distinguish among these height pairings (whenever the choice of the submodule $D_?$ *is fixed) and* $\langle\,,\,\rangle_{p,?}$ *will stand for any one of these.*

Remark 2.2. *Kobayashi has in fact proved in* [35, Corollaries 1.3(ii) and 4.9] *that when* $r_{\rm an}=1$*, both* p*-adic* L*-functions vanish at* $s=1$ *to degree* 1 *and* both *of the associated* p*-adic height pairings* $\langle\,,\,\rangle_{p,?}$ *are non-degenerate. The reason why we recall his results in this weaker form is that we apply the strategy here to prove Theorem 2.2(iv) also in the case when* E *has good ordinary reduction at* p *in* [14]*. In that case, if we had a* p*-adic Gross–Zagier formula for the critical slope* p*-adic* L*-function, we could have had proceeded precisely in this manner, even though we do not know that both* p*-adic height pairings in this set up are non-trivial. See Remark 2.3 below for a further discussion concerning this point.*

Proof of Theorem 2.2. The first assertion is due to Kato[8], see [32]. It follows from [47, Proposition 2.2.2] and Theorem 1.1 that $\mathcal{L}'_{\mathbf{BK}}(1)\neq 0$. Thence, for at least one of α or β (say it is α) we have

$$\mathrm{ord}_{s=1}L_{p,\alpha}(E/\mathbb{Q},s)=1\,.$$

This completes the proof of the second assertion. The third follows from the p-adic Gross–Zagier formula of Kobayashi in [35]. We will deduce[9] the last portion from the discussion in [47, §3.3.3], recast in view of [8, 6] and combined with Kobayashi's p-adic Gross–Zagier formula.

Let $\omega_{\rm cris}$ and $\omega_\alpha,\omega_\beta$ be as above. Let us write $\mathfrak{Log}_{V,\alpha}\in \mathrm{Hom}_\Lambda(H^1(\mathbb{Q},\mathbb{T}),\mathcal{H})$ for the ω_α-coordinate of the projection of the image of $\mathfrak{Log}_V$ parallel to D_β. We likewise define $\mathfrak{Log}_{V,\beta}$; note that

$$\mathfrak{Log}_V=\mathfrak{Log}_{V,\alpha}\cdot\omega_\alpha+\mathfrak{Log}_{V,\beta}\cdot\omega_\beta\,.$$

Let $\omega^*\in D_{\rm cris}(V)/\mathrm{Fil}^0D_{\rm cris}(V)$ denote the unique element such that $[\omega,\omega^*]=1$, where

$$[\,,\,]:D_{\rm cris}(V)\otimes D_{\rm cris}(V)\longrightarrow\mathbb{Q}_p$$

[8] This is accomplished after suitably normalizing $\mathbb{BK}_1$ and throughout this work, we implicitly assume that we have done so.

[9] Although it is possible to prove (iv) combining the fundamental works [47, 32, 35] (without relying on [8, 6]), we decided that we will stick to the current exposition as our later discussion that concerns higher dimensional motives is in line with the approach we present here.

the canonical pairing. Define $\log_{\omega^*}(\mathbf{BK}_1)$ so that

$$\log_V(\mathrm{res}_p(\mathbf{BK}_1)) = \log_{\omega^*}(\mathrm{res}_p(\mathbf{BK}_1)) \cdot \omega^* .$$

Note that with our choice of ω, we have $\log_{\omega^*} = \log_E$. The dual basis of $\{\omega_\alpha, \omega_\beta\}$ with respect to the pairing $[\,,\,]$ is $\{\omega_\beta^*, \omega_\alpha^*\}$, where ω_β^* (respectively, ω_α^*) is the image of ω^* under the inverse of the isomorphism $s_{D_\beta} : D_\beta \xrightarrow{\sim} D_{\mathrm{cris}}(V)/\mathrm{Fil}^0 D_{\mathrm{cris}}(V)$ (respectively, under the inverse of s_{D_α}). Suppose that α is such that $\langle\,,\,\rangle_{p,\alpha}$ is non-degenerate. Then,

$$\begin{aligned}
&(1-1/\alpha)^2 \cdot C(E) \cdot \langle P, P\rangle_{p,\alpha} = \frac{d\, L_{p,\alpha}(E/\mathbb{Q}, s)}{ds}\Big|_{s=1} && (2.5)\\
&= \mathfrak{Log}_{V,\alpha}(\partial_\alpha \mathbb{BK}_1)(1) && (2.6)\\
&= \left[\exp^*(\partial_\alpha \mathbf{BK}_1), (1-p^{-1}\varphi^{-1})(1-\varphi)^{-1} \cdot \omega_\beta^*\right]\\
&= \frac{1-p^{-1}\beta}{1-1/\beta} \times [\exp^*(\partial_\alpha \mathbf{BK}_1), \omega^*]\\
&= \frac{1-1/\alpha}{1-1/\beta} \times \frac{[\exp^*(\partial_\alpha \mathbf{BK}_1), \log_V(\mathrm{res}_p(\mathbf{BK}_1))]}{\log_{\omega^*}(\mathrm{res}_p(\mathbf{BK}_1))}\\
&= -\frac{1-1/\alpha}{1-1/\beta} \times \frac{\langle \mathbf{BK}_1, \mathbf{BK}_1\rangle_{p,\alpha}}{\log_{\omega^*}(\mathrm{res}_p(\mathbf{BK}_1))} && (2.7)
\end{aligned}$$

where (2.5) is Kobayashi's formula and the second will follow from the definition of $\mathfrak{Log}_{V,\alpha}$ and the fact that it maps to Beilinson–Kato class to the Amice-Velu, Manin-Vishik distribution as soon as we define the *derivative* $\partial\mathbb{BK}_1$ of the Beilinson–Kato class; we take care of this at the very end. The third equality will also from the explicit reciprocity laws of Perrin-Riou (as proved by Colmez) (c.f., the discussion in [11, Section 2.1]) once we define projection $\partial\mathbf{BK}_1 \in H^1_s(\mathbb{Q}_p, V)$ of the derived Beilinson–Kato class $\partial\mathbb{BK}_1$. Fourth and fifth equalities follow from definitions (and using the fact that $\alpha\beta = p$). We now explain (2.7)[10] together with the definitions of the objects $\partial\mathbb{BK}_1$ and $\partial\mathbf{BK}_1$.

Let $D^\dagger_{\mathrm{rig}}(V)$ denote the (φ,Γ)-module attached to V and $\mathbb{D}_\alpha \subset D^\dagger_{\mathrm{rig}}(V)$ the saturated (φ,Γ)-submodule of $D^\dagger_{\mathrm{rig}}(V)$ attached to D_α by Berger. Set $\widetilde{\mathbb{D}}_\alpha := D^\dagger_{\mathrm{rig}}(V)/\mathbb{D}_\alpha$, which is also a (φ,Γ)-module of rank one.

Given a (φ,Γ)-module $\mathbb{D}$, one may define the cohomology $H^1(\mathbb{D})$ (respectively, Iwasawa cohomology $H^1_{\mathrm{Iw}}(\mathbb{D})$) group of $\mathbb{D}$ making use of the Fontaine-Herr complex associated to $\mathbb{D}$ (c.f., [8, Section 1.2]). A

[10]This identity can also be deduced from [47, Proposition 2.3.4 (bis)].

well-known result of Herr yields canonical isomorphisms

$$H^1_{\mathrm{Iw}}(D^\dagger_{\mathrm{rig}}(V)) \cong H^1(\mathbb{Q}_p, \mathbb{T}_p(E)) \otimes \mathcal{H}\ ,\ \ H^1(D^\dagger_{\mathrm{rig}}(V)) \cong H^1(\mathbb{Q}_p, V)\,.$$

Furthermore, we have an exact sequence

$$0 \to H^1_{\mathrm{Iw}}(\mathbb{D}_\alpha) \longrightarrow H^1_{\mathrm{Iw}}(D^\dagger_{\mathrm{rig}}(V)) \xrightarrow{\pi_{/f}} H^1_{\mathrm{Iw}}(\widetilde{\mathbb{D}}_\alpha)$$

and we set $\mathrm{Res}_p^{/f}(\mathbb{BK}_1) := \pi_{/f} \circ \mathrm{res}_p(\mathbb{BK}_1) \in H^1_{\mathrm{Iw}}(\widetilde{\mathbb{D}}_\alpha)$. It turns also that we have the following commutative diagram:

$$\begin{array}{ccc} H^1_{\mathrm{Iw}}(D^\dagger_{\mathrm{rig}}(V)) & \xrightarrow{\pi_{/f}} & H^1_{\mathrm{Iw}}(\widetilde{\mathbb{D}}_\alpha) \\ {\scriptstyle \mathrm{pr}_0}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{pr}_0} \\ H^1(\mathbb{Q}_p, V) & \longrightarrow & H^1_{/f}(\mathbb{Q}_p, V) \end{array}$$

so that $\mathrm{pr}_0 \circ \mathrm{Res}_p^{/f}(\mathbb{BK}_1) = \mathrm{res}_p^{/f}(\mathbf{BK}_1) = 0$. Here, $H^1_{/f}(\mathbb{Q}_p, V) := H^1(\mathbb{Q}_p, V)/H^1_f(\mathbb{Q}_p, V)$ is the singular quotient. The vanishing of $\mathrm{pr}_0 \circ \mathrm{Res}_p^{/f}(\mathbb{BK}_1)$ shows that

$$\mathrm{Res}_p^{/f}(\mathbb{BK}_1) \in \ker(\mathrm{pr}_0 : H^1_{\mathrm{Iw}}(\widetilde{\mathbb{D}}_\alpha) \to H^1_{/f}(\mathbb{Q}_p, V)) = (\gamma - 1)\cdot H^1_{\mathrm{Iw}}(\widetilde{\mathbb{D}}_\alpha),$$

so that there is an element $\partial\mathbb{BK}_1 \in H^1_{\mathrm{Iw}}(\widetilde{\mathbb{D}}_\alpha)$ with the property that

$$\log_p \chi_{\mathrm{cyc}}(\gamma)\cdot \mathrm{Res}_p^{/f}\,(\mathbb{BK}_1) = (\gamma - 1)\cdot \partial\mathbb{BK}_1\,,$$

and as a matter of fact, this element is uniquely determined in our set up. Berger's reinterpretation of the Perrin-Riou map $\mathfrak{Log}_{V,\alpha}$ shows that it factors through $\pi_{/f}$ and therefore also that

$$\mathfrak{Log}_{V,\alpha}(\partial_\alpha\mathbb{BK}_1)(1) = \frac{d}{ds}\, L_{p,\alpha}(E/\mathbb{Q}, s)\Big|_{s=1}$$

as we have claimed in (2.6). The element $\partial\mathbf{BK}_1 \in H^1(\mathbb{Q}_p, V)$ is simply $\mathrm{pr}_0(\partial\mathbb{BK}_1)$.

Also attached to $\mathbb{D}_\alpha$, one may construct an extended Selmer group $\mathbf{R}^1\Gamma(V, \mathbb{D}_\alpha)$, which is the cohomology of a Selmer complex $\mathbf{R}\Gamma(V, \mathbb{D}_\alpha)$ (c.f., [8, Section 2.3]). It follows from [8, Proposition 11] that this Selmer group agrees in our set up with the classical Bloch-Kato Selmer group. It comes equipped with a p-adic height pairing $\langle\, ,\, \rangle_{p,\alpha}$ (see Section 4.2 of loc. cit.; note that Benois denotes this pairing by $h^{\mathrm{sel}}_{V,\mathbb{D}_\alpha,1}$) and the identity (2.7) now follows from the Rubin-style formula proved in [1, Theorem 4.13] for this height pairing. We recall that the comparison of height pairings that appear in Kobayashi's formula (2.5) and the Rubin-style formula (2.7) is explained in Remark 2.1.

Using now the fact that $\langle\cdot,\cdot\rangle_{p,\alpha}$ and $\log_{\omega^*}(\mathrm{res}_p\,(\,\cdot\,))^2$ are both non-trivial quadratic forms on the one dimensional $\mathbb{Q}_p$-vector space $E(\mathbb{Q})\otimes\mathbb{Q}_p$ and combining with (2.7), we conclude that

$$\frac{\langle P,P\rangle_{p,\alpha}}{\log_{\omega^*}(P)^2}=\frac{\langle\mathbf{BK}_1,\mathbf{BK}_1\rangle_{p,\alpha}}{\log_{\omega^*}(\mathbf{BK}_1)^2}=-(1-1/\alpha)(1-1/\beta)\cdot C(E)\cdot\frac{\langle P,P\rangle_{p,\alpha}}{\log_{\omega^*}(\mathbf{BK}_1)}.$$

Q.E.D.

Proof of Theorem 1.4. It follows from [47, §3.3.2] that

$$[\log_V(\mathrm{res}_p(\mathbf{BK}_1)),\omega]=\left[(1-\varphi)^{-1}(1-p^{-1}\varphi^{-1})\,\mathcal{L}'_{\mathbf{BK}}(1),\omega\right]\,. \tag{2.8}$$

On unwinding definitions and using Theorem 2.2(i), we infer that

$$\begin{aligned}\left[(1-\varphi)^{-1}(1-p^{-1}\varphi^{-1})\,\mathcal{L}'_{\mathbf{BK}}(1),\omega\right]&=\left(1-\frac{1}{\alpha}\right)^{-1}\left(1-\frac{1}{\beta}\right)L'_{p,\alpha}(E,1)\,[\omega_\alpha,\omega]\\&\quad+\left(1-\frac{1}{\beta}\right)^{-1}\left(1-\frac{1}{\alpha}\right)L'_{p,\beta}(E,1)\,[\omega_\beta,\omega]\\&=\left(1-\frac{1}{\alpha}\right)^{-1}\left(1-\frac{1}{\beta}\right)L'_{p,\alpha}(E,1)\,[\omega_\alpha,\omega_\beta]\\&\quad-\left(1-\frac{1}{\beta}\right)^{-1}\left(1-\frac{1}{\alpha}\right)L'_{p,\beta}(E,1)\,[\omega_\alpha,\omega_\beta]\end{aligned}\tag{2.9}$$

Noting $\log_{\omega^*}(\mathrm{res}_p(\mathbf{BK}_1))=[\log_V(\mathrm{res}_p(\mathbf{BK}_1)),\omega]$ and using Theorem 1.3, it follows on combining (2.8) and (2.9) that

$$\log_{\omega^*}(\mathrm{res}_p(P))^2=\delta_E\Big((1-1/\alpha)^{-2}\,L'_{p,\alpha}(E,1)-(1-1/\beta)^{-2}\,L'_{p,\beta}(E,1)\Big),$$

where we recall that $\delta_E=[\omega_\beta,\omega_\alpha]/C(E)$ and $P\in E(\mathbb{Q})$ is a point of infinite order. We may rewrite the final equality as

$$\log_V(\mathrm{res}_p(P))=\omega^*\sqrt{\delta_E\left((1-1/\alpha)^{-2}\;L'_{p,\alpha}(E,1)-(1-1/\beta)^{-2}\;L'_{p,\beta}(E,1)\right)}\,. \tag{2.10}$$

The result follows on exponentiating (2.10) and using the fact that the compositum of the arrows

$$E(\mathbb{Q})/E(\mathbb{Q})_{\mathrm{tor}}\xrightarrow{\mathrm{res}_p}H^1_{\mathrm{f}}(\mathbb{Q}_p,T)\xrightarrow{\log_V}D_{\mathrm{cris}}(V)/\mathrm{Fil}^0D_{\mathrm{cris}}(V)$$

is injective. Q.E.D.

Remark 2.3. *Much of what we have recorded above for a supersingular prime p applies verbatim for a good ordinary prime as well. Suppose that α is the root of the Hecke polynomial which is a p-adic unit, so that $v_p(\beta) = 1$. In this case, we again have two p-adic L-functions: The projection of $\mathcal{L}_{\mathbf{BK}}$ to D_α yields the Mazur-Tate-Teitelbaum p-adic L-function $L_{p,\alpha}(E/\mathbb{Q}, s)$, whereas its projection to D_β should agree*[11] *with the* critical slope *p-adic L-function $L_{p,\beta}(E/\mathbb{Q}, s)$ of Bellaïche and Pollack-Stevens. The analogous statements to those in Theorem 2.2 therefore reduces to check that one of the following holds true:*

a) *There exists a p-adic Gross–Zagier formula for the critical slope p-adic L-function $L_{p,\beta}(E/\mathbb{Q}, s)$, or that*
b) $\mathrm{ord}_{s=1} L_{p,\beta}(f, s) \geq \mathrm{ord}_{s=1} L_{p,\alpha}(f, s)$.

We suspect that the latter statement may be studied through a critical slope *main conjecture and its relation with the ordinary main conjecture. We will pursue this direction in a future joint work with R. Pollack.*

2.3.2. *E has non-split multiplicative reduction at p* Let us write $D_{\mathrm{st}}(V)$ for the semi-stable Dieudonné module of V and $q_E \in \mathbb{Q}_p$ for the Tate parameter of $E_{/\mathbb{Q}_p}$. The 2-dimensional $\mathbb{Q}_p$-vector space $D_{\mathrm{st}}(V)$ is a filtered (φ, N)-module with a $\mathbb{Q}_p$-basis $\{v_0, v_1\}$ with the following properties:

- $\varphi v_1 = -v_1,\ \ \varphi v_0 = -p^{-1} v_0$.
- $N v_1 = \mathrm{ord}_p(q_E) v_0,\ \ N v_0 = 0$.
- $\mathrm{Fil}^{-1} D_{\mathrm{st}}(V) = D_{\mathrm{st}}(V),\ \ \mathrm{Fil}^0 D_{\mathrm{st}}(V) = \mathrm{span}\{v_1 - \log_p(q_E) v_0\}$, $\mathrm{Fil}^1 D_{\mathrm{st}}(V) = 0$.

Notice also that the natural containment $D_{\mathrm{st}}(V) \subset D_{\mathrm{dR}}(V)$ is in fact an equality and

$$D_{\mathrm{cris}}(V) = D_{\mathrm{st}}(V)^{N=0} = \mathrm{span}\{v_0\} = D_{\mathrm{st}}(V)^{\varphi = -p^{-1}} .$$

Moreover, the natural map

$$D_{\mathrm{cris}}(V) \longrightarrow D_{\mathrm{st}}(V)/\mathrm{Fil}^0 D_{\mathrm{st}}(V)$$

is an isomorphism. Let $\eta \in D_{\mathrm{cris}}(V)$ denote the unique element such that $\langle \eta, \omega_E \rangle = 1$. We finally let $\omega_{\mathrm{cris}} \in D_{\mathrm{st}}(V)^{\varphi=-1}$ for the unique

[11] We remark that this conclusion does not formally follow directly from Kato's explicit reciprocity laws. See [39]; also [26] where a proof of this was announced.

vector for which we have

$$\omega_E - \omega_{\mathrm{cris}} \in D_{\mathrm{cris}}(V)^{\perp},$$

(where $D_{\mathrm{cris}}(V)^{\perp}$ is the orthogonal complement of $D_{\mathrm{cris}}(V)$ under the canonical pairing $D_{\mathrm{dR}}(V) \otimes D_{\mathrm{dR}}(V) \xrightarrow{\langle\,,\rangle} \mathbb{Q}_p$) so that we have $\langle \eta, \omega_{\mathrm{cris}} \rangle = 1$.

In this set up, we have $\mathcal{L}_{\mathbf{BK}} = \mathcal{L}_{\mathrm{MTT}} \cdot \omega_{\mathrm{cris}}$ (see [32, Theorem 16.5] and [7, §4.3.1]), where $\mathcal{L}_{\mathrm{MTT}} \in \Lambda$ is the Mazur-Tate-Teitelbaum measure.

Theorem 2.3. *Suppose that Nekovář's p-adic height pairing associated to the canonical splitting of the Hodge-filtration on the semi-stable Dieudonné module $D_{\mathrm{st}}(V)$ is non-vanishing. Then,*

$$\log_V \left(\mathrm{res}_p(\mathbf{BK}_1)\right) \cdot \log_V \left(\mathrm{res}_p(P)\right)^{-2} \in \overline{\mathbb{Q}}^{\times}.$$

Proof. Relying on the discussion in Section 2.3.2, one proceeds as in the proof of Theorem 2.2 to deduce this assertion as a consequence of the p-adic Gross–Zagier formula (c.f., [23, Theorem B]), the Rubin-style formula proved in [43, Proposition 11.3.15] and our Theorem 1.1. Q.E.D.

Remark 2.4. *Our argument so far easily adapt to prove that analogous conclusions hold true for a* essentially self-dual *elliptic modular form f which verifies the hypotheses of Remark 1.4 and for which the natural map*

$$\mathrm{res}_p : H^1_{\mathrm{f}}(\mathbb{Q}, V_f) \to H^1_{\mathrm{f}}(\mathbb{Q}_p, V_f)$$

is non-zero. Here, V_f is the self-dual twist of Deligne's representation, as in Remark 1.4.

§3. Part II. Λ-adic Kolyvagin systems, Beilinson–Flach elements, Coleman–Rubin–Stark elements and Heegner points

In this section, we recast the proof of Theorem 1.1 in terms of the theory of Λ-adic Kolyvagin systems (and in great generality), with the hope that it will provide us with further insights to analyse the analogs of Perrin-Riou's predictions in other situations.

3.1. The set up

Let $\mathscr{M}$ denote a self-dual motive over either a totally real or CM number field K with coefficients in a number field F. We fix and isomorphism $\mathrm{j} : \mathbb{C} \xrightarrow{\sim} \mathbb{C}_p$ as well as embeddings $\iota_\infty : F \hookrightarrow \mathbb{C}$ and $\iota_p : F \hookrightarrow \mathbb{C}_p$

such that $\mathrm{j}\circ\iota_\infty = \iota_p$. Let $L(\mathscr{M},s) := L(\iota_\infty(\mathscr{M}),s)$ denote its L-function and write $r_{\mathrm{an}}(\mathscr{M})$ for its order of vanishing at the central critical point (which is a quantity conditional on the expected functional equation and analytic continuation). Let V denote its p-adic realization (which is a finite dimensional vector space over E, the completion of F at the prime above p induced by ι_p, endowed with a continuous G_K-action) and $T \subset V$ a $\mathfrak{o}_E$-lattice. We write $\overline{T}$ for $T/\mathfrak{m}_E T$, where $\mathfrak{m}_E$ is the maximal ideal of $\mathfrak{o}_E$ and $T^* := \mathrm{Hom}(T, {}_{p^\infty})$ for the Cartier dual of T. We will set $2r := [K:\mathbb{Q}]\dim_E V$ (note by the self-duality of $\mathscr{M}$ that the quantity on the right is indeed even) and let S denote the finite set of places of K which consists of all primes above p, all archimedean places and all places at which V is ramified. We will assume throughout Section 3 that V is *critical Panchishkin ordinary* in the sense that for each prime $\mathfrak{p}$ of K above p, the following three conditions hold true:

(CP0) There is a direct summand $F_\mathfrak{p}^+T \subset T$ (as an $\mathfrak{o}_E$-submodule) of rank $r_\mathfrak{p}$, which is stable under the $G_{K_\mathfrak{p}}$-action and such that $\sum_{\mathfrak{p}|p}[K_\mathfrak{p}:\mathbb{Q}_p]r_\mathfrak{p} = r$.

(CP1) The Hodge-Tate weights of the subspace $F_\mathfrak{p}^+V := F_\mathfrak{p}^+T\otimes E$ are strictly negative.

(CP2) The Hodge-Tate weights of the quotient $V/F_\mathfrak{p}^+V$ are non-negative.

Here, our convention is that the Hodge-Tate weight of the cyclotomic character is -1.

Remark 3.1. *When $r = 1$, we may in fact drop the critical Panchishkin condition on T without any further work.*

Example 3.1. *Suppose that A/K is an abelian variety of dimension g, which has good ordinary reduction at all primes above p. Then $r = g[K:\mathbb{Q}]$. The p-adic realization of the motive $h^1(A)(1)$ (which is a self-dual motive over K, with coefficients in $\mathbb{Q}$) is $V_p(A) := T_p(A)\otimes\mathbb{Q}_p$ is the p-adic Tate-module and $V_p(A)$ is critical Panchishkin ordinary.*

Recall from Section 1.1 that K stands for a totally real or CM number field and $\mathfrak{o}$ is the ring of integers of a finite extension L of $\mathbb{Q}_p$. Recall also that T is a free $\mathfrak{o}$-module of finite rank, which is endowed with a continuous action of G_K and that is unramified outside a finite set of primes. Furthermore, $\Lambda_\mathfrak{o} := \mathfrak{o}[[\Gamma]]$ is the Iwasawa algebra, $\mathbb{T} := T\otimes_{\mathbb{Z}_p}\Lambda$ and pr_0 the augmentation map $\Lambda_\mathfrak{o} \to \mathfrak{o}$. For positive integers α and k, we set $R_{k,\alpha} := \Lambda/(p^k,(\gamma-1)^\alpha)$ and $T_{k,\alpha} := \mathbb{T}\otimes_{\mathbb{Z}_p} R_{k,\alpha}$.

3.2. Module of Λ-adic Kolyvagin systems

Throughout this section, we shall assume that the hypotheses **(H.0)-(H.3)** of [42, Section 3.5] are in effect as well as the following two hypotheses:
(H.nA) $H^0(K_\mathfrak{p}, \overline{T}) = 0$ for every prime $\mathfrak{p}$ of K above p.
(H.Tam) We have $H^0\left(G_{K_\lambda}/I_\lambda, H^0(I_\lambda, V/T)\big/H^0(I_\lambda, V/T)_{\mathrm{div}}\right) = 0$ for every prime $\lambda \in S$ (where $I_\lambda \subset G_{K_\lambda}$ stands for the inertia subgroup).

Definition 3.1. *We define the Greenberg Selmer structure $\mathcal{F}_{\mathrm{Gr}}$ by the local conditions*

$$H^1_{\mathcal{F}_{\mathrm{Gr}}}(K_\mathfrak{p}, \mathbb{T}) := \mathrm{im}\left(H^1(K_\mathfrak{p}, F^+_\mathfrak{p}T \otimes \Lambda) \to H^1(K_\mathfrak{p}, \mathbb{T})\right)$$

for every prime $\mathfrak{p}$ above p, and by setting $H^1_{\mathcal{F}_{\mathrm{Gr}}}(K_\lambda, \mathbb{T}) := H^1(K_\lambda, \mathbb{T})$ for $\lambda \nmid p$.

Under the hypothesis **(H.nA)** (and the self-duality assumption on T), it follows that $H^1(K_p, \mathbb{T})$ is a free $\Lambda_\mathfrak{o}$-module of rank $2r$ and

$$H^1_{\mathcal{F}_{\mathrm{Gr}}}(K_p, \mathbb{T}) := \oplus_{\mathfrak{p}|p} H^1_{\mathcal{F}_{\mathrm{Gr}}}(K_\mathfrak{p}, \mathbb{T}) \subset H^1(K_p, \mathbb{T})$$

is a direct summand of rank r. We fix a direct summand $H^1_+(K_p, \mathbb{T}) \supset H^1_{\mathcal{F}_{\mathrm{Gr}}}(K_p, \mathbb{T})$ of rank $r+1$.

We will write $H^1_{\mathrm{f}}(K_p, T)$ for the image of $H^1_{\mathcal{F}_{\mathrm{Gr}}}(K_p, \mathbb{T})$ under pr_0 and also denote the Selmer group determined by the propagation of the Selmer structure $\mathcal{F}_{\mathrm{Gr}}$ to T by $H^1_{\mathrm{f}}(K, T)$. Let $H^1_+(K_p, T)$ denote the image of $H^1_+(K_p, \mathbb{T})$ under pr_0. The $\mathfrak{o}_E$-module $H^1_+(K_p, T)$ is a direct summand of $H^1(K_p, T)$ of rank $r+1$ (thanks to our hypothesis **H.nA**). Using the fact that $\mathcal{F}_{\mathrm{Gr}}$ is self-dual, this determines (via local Tate duality) a direct summand $H^1_-(K_p, T) \subset H^1_{\mathcal{F}_{\mathrm{Gr}}}(K_p, T)$ of rank $r-1$ and allows us to define a Selmer structure $\mathcal{F}_-$ on T (which is given by the local conditions determined by $H^1_-(K_p, T) \subset H^1(K_p, T)$ at p, and by propagating of $\mathcal{F}_{\mathrm{Gr}}$ at any other place).

Definition 3.2. *We define the Selmer structures $\mathcal{F}_+$ and $\mathcal{F}_\Lambda$ by the local conditions*

$$H^1_{\mathcal{F}_\Lambda}(K_p, \mathbb{T}) = H^1(K_p, \mathbb{T}) \quad , \quad H^1_{\mathcal{F}_+}(K_p, \mathbb{T}) = H^1_+(K_p, \mathbb{T})$$

and by setting $H^1_{\mathcal{F}_\Lambda}(K_\lambda, \mathbb{T}) = H^1_{\mathcal{F}_+}(K_\lambda, \mathbb{T}) := H^1(K_\lambda, \mathbb{T})$ for $\lambda \nmid p$.

Given a Selmer structure $\mathcal{F}$ on $\mathbb{T}$, we let $\chi(\mathcal{F}, \mathbb{T}) := \chi(\mathcal{F}, \overline{T})$ denote the core Selmer rank (in the sense of [42, Definition 4.1.11]) of the propagation of the Selmer structure $\mathcal{F}$ to $\overline{T}$. It follows from the discussion in [42, §4.1 and §5.2] that $\chi(\mathcal{F}_\Lambda, \mathbb{T}) = r$.

Proposition 3.1. *We have* $\chi(\mathcal{F}_{\mathrm{Gr}}, \mathbb{T}) = 0$*; whereas* $\chi(\mathcal{F}_+, \mathbb{T}) = 1$*.*

Proof. These follow using global Euler characteristic formulae, together with the fact that $\chi(\mathcal{F}_\Lambda, \mathbb{T}) = r$. Q.E.D.

Definition 3.3. *For an integer* $j \geq 2$*, we let* $\mathcal{P}_j$ *denote the set of* Kolyvagin primes*, given as in* [15, Section 2.4]*. Also as in op. cit., we set* $\mathcal{P} := \mathcal{P}_2$*.*

Given an integer $j \geq k+\alpha$, one may define the module $\mathbf{KS}(\mathcal{F}_+, T_{k,\alpha}, \mathcal{P}_j)$ of Kolyvagin systems for the Selmer structure $\mathcal{F}_+$ on the artinian module $T_{k,\alpha}$ as in the paragraph following Definition 3.3 in [15] (after replacing the Selmer structure $\mathcal{F}_{\mathrm{can}}$ by our more general Selmer structure $\mathcal{F}_+$).

Definition 3.4. *The* $\Lambda_{\mathfrak{o}}$*-module*

$$\mathbf{KS}(\mathcal{F}_+, \mathbb{T}, \mathcal{P}) := \varprojlim_{k,\alpha} \left(\varinjlim_{j \geq k+\alpha} \mathbf{KS}(\mathcal{F}_+, T_{k,\alpha}, \mathcal{P}_j) \right)$$

is called the module of Λ*-adic Kolyvagin systems.*

Theorem 3.1. *Under the hypotheses* **(H.0)** - **(H.3)** *of* [42, Section 3.5] *and assuming* **(H.Tam)** *and* **(H.nA)***, the natural maps*

$$\mathbf{KS}(\mathcal{F}_+, \mathbb{T}, \mathcal{P}) \longrightarrow \mathbf{KS}(\mathcal{F}_+, T, \mathcal{P}) \longrightarrow \mathbf{KS}(\mathcal{F}_+, \overline{T}, \mathcal{P})$$

are both surjective, the $\Lambda_{\mathfrak{o}}$*-module* $\mathbf{KS}(\mathcal{F}_+, \mathbb{T}, \mathcal{P})$ *is free of rank one and its generated by any* Λ*-adic Kolyvagin system whose projection to* $\mathbf{KS}(\mathcal{F}_+, \overline{T}, \mathcal{P})$ *is non-zero.*

Proof. This is the main theorem of [15]; one only needs to replace $\mathcal{F}_{\mathrm{can}}$ in loc. cit. with $\mathcal{F}_+$ (but that entails no complications). Q.E.D.

Remark 3.2. *When* $T = T_p(E)$ *is the* p*-adic Tate module of an elliptic curve* $E/\mathbb{Q}$*, the hypothesis* **(H.Tam)** *is equivalent to the requirement that* p *does not divide* $\mathrm{ord}_\ell(j(E))$ *whenever* ℓ *is a prime of split multiplicative reduction.*

Still when $T = T_p(E)$*, the hypothesis that* $H^2(\mathbb{Q}_p, \overline{T}) = 0$ *is equivalent to the requirement that* $E(\mathbb{Q}_p)[p] = 0$*. This is precisely the condition that the prime* p *be* non-anomalous *for* E *in the sense of* [41]*. It is easy to see that all primes* $p > 5$ *at which*

- E *has supersingular reduction, or*
- E *has good ordinary reduction with* $a_p(E) \neq 1$*, or*
- E *has non-split-multiplicative reduction, or*
- E *has split-multiplicative reduction with* $p \nmid \mathrm{ord}_p(j_E) = -\mathrm{ord}_p(q_E)$ *(where* q_E *is the Tate-parameter), or*

- *E has good ordinary reduction with $p > 7$ and $E(\mathbb{Q})_{\mathrm{tor}}$ is non-trivial*

are non-anomalous. Indeed, the asserted property in the first two cases follow from the Hasse bound, in the third and fourth using the Kodaira-Néron theorem and in the final case by [24, Proposition 2.1].

In our supplementary note [17, Appendix A], *we are able to lift the non-anomaly hypothesis on $T_p(E)$ and prove the following in this setting:*

Theorem 3.2. *Suppose that E is an elliptic curve such that the residual representation*

$$\overline{\rho}_E : G_{\mathbb{Q},S} \longrightarrow \mathrm{Aut}(E[p])$$

is surjective. Assume further that p does not divide $\mathrm{ord}_\ell(j(E))$ whenever $\ell \mid N$ is a prime of split multiplicative reduction. Then,

i) *the Λ-module $\mathbf{KS}(\mathbb{T}_p(E))$ of Λ-adic Kolyvagin systems contains a free Λ-module of rank one with finite index;*
ii) *there exists a Λ-adic Kolyvagin system $\boldsymbol{\kappa} \in \mathbf{KS}(\mathbb{T}_p(E))$ with the property that $\mathrm{pr}_0(\boldsymbol{\kappa}) \in \mathbf{KS}(T_p(E))$ is non-zero.*

Let $\mathrm{res}_{+/f}$ denote the compositum of the arrows

$$H^1_{\mathcal{F}_+}(K,\mathbb{T}) \longrightarrow H^1_+(K_p,\mathbb{T}) \longrightarrow H^1_+(K_p,\mathbb{T})/H^1_{\mathcal{F}_{\mathrm{Gr}}}(K_p,\mathbb{T})\,.$$

Definition 3.5. *Given a Λ-adic Kolyvagin system $\in \mathbf{KS}(\mathcal{F}_+,\mathbb{T},\mathcal{P})$ for which ${}_1 \neq 0$, we define its* defect *by setting*

$$\delta() := \mathrm{char}\left(H^1_{\mathcal{F}_+}(K,\mathbb{T})/\Lambda\cdot{}_1\right)/\mathrm{char}\left(H^1_{\mathcal{F}_+^*}(K,\mathbb{T}^*)^\vee\right)\,.$$

Whenever $\mathrm{res}_{+/f}\,({}_1) \neq 0$, we also define the Kolyvagin constructed p-adic L-function

$$\mathfrak{L}_p() := \mathrm{char}\left(H^1_{+/f}(K_p,\mathbb{T})/\mathrm{res}_{+/f}({}_1)\right)\,.$$

Observe that $\delta() \subset \Lambda_{\mathfrak{o}}$ thanks to the general Kolyvagin system machinery (c.f. [42, Theorem 5.3.10], which is enhanced in [10, Appendix A] and [13, Theorem 3.6.3] to apply with the Selmer structure $\mathcal{F}_+$). Note, using Poitou-Tate global duality, that the defect of is given as the quotient $\delta() = \mathfrak{L}_p()/\mathrm{char}(H^1_{\mathcal{F}^*_{\mathrm{Gr}}}(K,\mathbb{T}^*)^\vee)$ when $\mathrm{res}_{+/f}\,({}_1) \neq 0$.

Remark 3.3. *It follows from Theorem 3.1 that any generator of the module $\mathbf{KS}(\mathcal{F}_+,\mathbb{T},\mathcal{P})$ of Λ-adic Kolyvagin systems is Λ-primitive (in the sense of* [42, Definition 5.3.9]*). It then follows from* [42, 5.3.10(iii)] *that for any generator of $\mathbf{KS}(\mathcal{F}_+,\mathbb{T},\mathcal{P})$ we have $\delta() = \Lambda$.*

Example 3.2. *Suppose $E/\mathbb{Q}$ is an elliptic curve and $T = T_p(E)$. Let* $^{\mathbf{BK}} \in \mathbf{KS}(\mathcal{F}_\Lambda, \mathbb{T})$ *denote the Λ-adic Kolyvagin system associated to E, that descend from the Beilinson–Kato elements via* [42, Theorem 5.3.3]. *It follows from Theorem 2.1 (in all cases that it applies), $\delta(^{\mathbf{BK}})$ is generated by a power of p and in particular, it is prime to $(\gamma - 1)$.*

Recall the direct summand $H^1_-(K_p, T) \subset H^1_{\mathcal{F}_{\mathrm{Gr}}}(K_p, T)$ of rank $r-1$ and define the map $\mathrm{res}_{f/-}$ as the compositum of the arrows

$$H^1_{\mathrm{f}}(K, T) \longrightarrow H^1_{\mathrm{f}}(K_p, T) \longrightarrow H^1_{\mathrm{f}}(K_p, T)\big/H^1_-(K_p, T)\,.$$

Theorem 3.3. *Assume that* **(H.Tam)**, **(H.nA)** *and the hypotheses* **(H.0)** - **(H.3)** *of Mazur and Rubin in* [42] *hold true.*
i) *Suppose that the map $\mathrm{res}_{f/-} : H^1_{\mathrm{f}}(K, T) \to H^1_{f/-}(K_p, T)$ is injective.*

(a) *For any non-trivial $\in \mathbf{KS}(\mathcal{F}_+, \mathbb{T}, \mathcal{P})$, we have $_1 \neq 0$.*
(b) *For any whose defect $\delta()$ is prime to $(\gamma - 1)$, we have $\kappa_1 \neq 0$.*

ii) *Conversely, if $\kappa_1 \neq 0$ for some $\kappa \in \mathbf{KS}(\mathcal{F}_+, T, \mathcal{P})$, then the map $\mathrm{res}_{f/-}$ is injective.*

See the discussion in Remark 3.4 and Conjecture 3.1 pertaining to the injectivity of the map $\mathrm{res}_{f/-}$.

Proof. The requirement that the map $\mathrm{res}_{f/-}$ be injective is equivalent to asking that $H^1_{\mathcal{F}_-}(K, T) = 0$. The proof of Theorem 1.2 adapts without difficulty (on replacing $\mathcal{F}_{\mathrm{str}}$ with $\mathcal{F}_-$ and $\mathcal{F}^*_{\mathrm{can}}$ with $\mathcal{F}^*_+$) to show that $H^1_{\mathcal{F}^*_+}(K, T^*)$ is finite. It follows from [42, Corollary 5.2.13(i)] that for every non-trivial Kolyvagin system $\kappa \in \mathbf{KS}(\mathcal{F}_+, T, \mathcal{P})$, we have $\kappa_1 \neq 0$ for its initial term. Theorem 3.1 shows that κ lifts to a Λ-adic Kolyvagin system $\in \mathbf{KS}(\mathcal{F}_+, \mathbb{T}, \mathcal{P})$, and we have $_1 \neq 0$ (as we have $\mathrm{pr}_0(_1) = \kappa_1 \neq 0$). Since the $\Lambda_{\mathfrak{o}}$-module $\mathbf{KS}(\mathcal{F}_+, \mathbb{T}, \mathcal{P})$ is cyclic and $H^1(K, \mathbb{T})$ is torsin-free under our running assumptions, (a) follows.

Let now be a Λ-primitive Kolyvagin system (such exists and generates the module $\mathbf{KS}(\mathcal{F}_+, \mathbb{T}, \mathcal{P})$ by Theorem 3.1). Let $g \in \Lambda_{\mathfrak{o}}$ be such that $= g \cdot$. It follows from [42, Theorem 5.3.10(iii)] and the choice of that g is prime to $(\gamma - 1)$. Furthermore, since $\mathrm{pr}_0(_1) \neq 0$ by the discussion above, it follows that $\kappa_1 = \mathrm{pr}_0(g) \cdot \mathrm{pr}_0(_1) \neq 0$. This completes the proof of (b).

To prove (ii), note that $H^1_{\mathcal{F}^*_+}(K, T^*)$ is finite by our assumption and [42, Theorem 5.2.2]. The proof of Theorem 1.2 shows (after suitable alterations, as we have pointed out in the first paragraph of this proof) that this implies the vanishing of $H^1_{\mathcal{F}_-}(K, T)$. This is precisely what we desired to prove. Q.E.D.

Remark 3.4. *For $T = T_p(E)$ for an elliptic curve as in Example 3.2 above, the $\mathrm{res}_{f/-}$ in the statement of Theorem 3.3 is simply the localization map at p. It is easy to see using the work of Gross–Zagier, Kolyvagin and Skinner (under additional mild hypothesis) that this map is injective if and only if $r_{\mathrm{an}}(E) = 1$. In particular, Perrin-Riou's conjecture in this set up follows from Theorem 3.3 and the discussion in Example 3.2.*

The discussion in Remark 3.4 leads us to the following prediction:

Conjecture 3.1. *There exists a choice of the direct summand $H^1_+(K_p, \mathbb{T})$ such that the map $\mathrm{res}_{f/-}$ is injective iff $r_{\mathrm{an}}(\mathscr{M}) \leq 1$.*

3.3. Example: Perrin-Riou's conjecture for Beilinson–Flach elements

Let $E/\mathbb{Q}$ be a non-CM elliptic curve with square-free conductor N and let $f_E \in S_2(\Gamma_0(N))$ denote the associated newform. Assume that the residual representation $\overline{\rho}_E : G_K \to \mathrm{Aut}(E[p])$ is absolutely irreducible and suppose that E has good ordinary reduction at the prime p. Fix an embedding $\iota_p : \overline{\mathbb{Q}} \to \mathbb{C}_p$. Suppose K is an imaginary quadratic extension of $\mathbb{Q}$ that satisfies the weak Heegner hypothesis for E, so that the order of vanishing $r_{\mathrm{an}}(E/K) := \mathrm{ord}_{s=1} L(E/K, s)$ is odd. Suppose further that p does not divide $\mathrm{ord}_\ell(j(E))$ whenever $\ell | N$ is a prime of split multiplicative reduction.

We will assume throughout this subsection that the prime p splits in K and write $p = \wp\wp^c$ as a product of primes of K, where the prime $\wp$ is induced from ι_p. Fix forever an auxiliary modulus $\mathfrak{f}$ of K which is prime to p and the ray class group of K modulo $\mathfrak{f}$ is prime to p. Fix also a ring class character α modulo $\mathfrak{f}p^\infty$ of finite order, for which we have $\alpha(\wp) \neq \alpha(\wp^c)$. We let $\mathfrak{o}$ denote the finite flat extension of $\mathbb{Z}_p$ in which α takes its values and write $T = T_p(E) \otimes \alpha^{-1}$ for the free $\mathfrak{o}$-module of rank 2 on which G_K acts diagonally. Set $\mathscr{D}(T) := \mathrm{Hom}(T, \mathfrak{o})(1) \cong T_p(E) \otimes \alpha$. Let $F^+T \subset T$ denote the Greenberg subspaces of T; this is a free $\mathfrak{o}$-module of rank one such that the $G_{\mathbb{Q}_p}$-action on the quotient T/F^+T is unramified.

In this set up, one may define the Greenberg Selmer structure $\mathcal{F}_{\mathrm{Gr}}$ on $\mathbb{T}$ as above, as well as modify this Selmer structure appropriately to apply Theorem 3.1.

Definition 3.6. *We define the Selmer structure $\mathcal{F}_+$ on $\mathbb{T}$ by relaxing the local conditions at the prime $\wp$.*

It follows from the discussion in [12, Section 3.2] (with the choice $f_1 = f_E$ in loc.cit.), Beilinson–Flach element Euler system of Lei-Loeffler-Zerbes [38] gives rise to a Kolyvagin system $^{\mathrm{BF}} = \{\kappa_\eta^{\mathrm{BF}}\}_\eta \in \mathbf{KS}(\mathcal{F}_+, \mathbb{T})$

(where the indices run through square free ideals η of $\mathfrak{o}_K$ that are products of appropriately chosen Kolyvagin primes) which we call the *Beilinson-Flach Kolyvagin system.* Then $\mathbb{BF}_1 := \kappa_1^{\mathrm{BF}} \in H^1_{\mathcal{F}_+}(K, \mathbb{T})$ and we write $\mathrm{BF}_1 \in H^1_{\mathcal{F}_+}(K,T)$ for its image. It follows from Theorems 6.1.3 and 6.4.1 of [38] that $\mathrm{res}^s_\wp(\mathrm{BF}_1) \neq 0$ iff $r_{\mathrm{an}}(E,\alpha) := \mathrm{ord}_{s=1} L(E/K,\alpha)$ equals 0. Here,

$$\mathrm{res}^s_\wp : H^1(K,T) \longrightarrow H^1(K_\wp,T)/H^1_{\mathcal{F}_{\mathrm{Gr}}}(K_\wp,T)$$

is the singular projection. We also remark that BF_1 here corresponds to $z_1^{f_E,\alpha}$ in the notation of [38]. In this particular situation, the map $\mathrm{res}_{f/-}$ is simply the map

$$\mathrm{res}_{f/-} = \mathrm{res}_\wp :\ H^1_{\mathrm{f}}(K,T) \longrightarrow H^1_{\mathcal{F}_{\mathrm{Gr}}}(K_\wp,T).$$

When $r_{\mathrm{an}}(E,\alpha) = 0$, then $\mathrm{res}^s_\wp(\mathrm{BF}_1) \neq 0$ and the Kolyvagin system machinery shows that $H^1_{\mathrm{f}}(K,T) = 0$ and the $\mathrm{res}_\wp$ is injective for this trivial reason.

Proposition 3.2. *If* $r_{\mathrm{an}}(E,\alpha) = 1$*, then the map* $\mathrm{res}_\wp \colon H^1_{\mathrm{f}}(K,T) \to H^1_{\mathcal{F}_{\mathrm{Gr}}}(K_\wp,T)$ *is injective.*

Proof. Let $V_p(E) := T_p(E) \otimes \mathbb{Q}_p$ and $V := T \otimes \mathbb{Q}_p$. We let M denote the finite abelian extension of K that is cut by α. Note that we have

$$H^1_{\mathrm{f}}(K,V) \xrightarrow{\sim} H^1_{\mathrm{f}}(M,V)^{G_K} \xrightarrow{\sim} \left(H^1_{\mathrm{f}}(M,V_p(E)) \otimes \alpha^{-1}\right)^{G_K} \xrightarrow{\sim} H^1_{\mathrm{f}}(M,V_p(E))^\alpha$$

where the first arrow follows from the inflation-restriction sequence and the rest are self-evident. It follows from [66] together with a mild extension[12] of the main results of [28] (where one utilizes the twisted Heegner point Euler system for V, in place of the usual Heegner points in op. cit.) that $H^1_{\mathrm{f}}(M,V_p(E))^\alpha$ is 1-dimensional as a $\mathrm{Frac}(\mathfrak{o}) = L$-vector space. Furthermore,

$$E(M)^\alpha \xrightarrow{\sim} H^1_{\mathrm{f}}(M,V_p(E))^\alpha$$

[12] One may also appeal to [18, Theorem C] together with the Gross–Zagier formula proved in [66] for the type of Rankin-Selberg products we consider here. Even though the result in [18] covers far more ground (in a certain sense of the word), it is stated under the following additional hypotheses: (1) K verifies the strong Heegner hypothesis (i.e., every prime dividing N splits completely in $K/\mathbb{Q}$), (2) $p \nmid \phi(N)$, where (only here) ϕ stands for Euler's totient function. In the particular set up here, these additional assumptions are not necessary.

where for an abelian group X, we write $X^{\alpha} := \left(X \hat{\otimes} L(\alpha^{-1})\right)^{G_K}$ with $L(\alpha^{-1})$ being the 1-dimensional L-vector space on which G_K acts by α^{-1}. We therefore have the following commutative diagram

$$\begin{tikzcd}
E(M)^{\alpha} \arrow[r, "\sim"] \arrow[d, "\mathrm{res}_{\wp}"] & H^1_{\mathrm{f}}(M, V_p(E))^{\alpha} \arrow[r, "\sim"] \arrow[d, "\mathrm{res}_{\wp}"] & H^1_{\mathrm{f}}(K, V) \arrow[d, "\mathrm{res}_{\wp}"] \\
\left(\bigoplus_{\mathfrak{p}|\wp} E(M_{\mathfrak{p}})\right)^{\alpha} \arrow[r, "\sim"] & H^1_{\mathcal{F}_{\mathrm{Gr}}}(M_{\wp}, V_p(E))^{\alpha} \arrow[r, "\sim"] & H^1_{\mathcal{F}_{\mathrm{Gr}}}(K_{\wp}, V)
\end{tikzcd}$$

The left-most arrow is evidently injective (as its source is spanned by an M-rational point) and hence, the right-most arrow is injective as well. Q.E.D.

The following statement is the Perrin-Riou conjecture for Beilinson–Flach elements.

Corollary 3.1. *Assume that the residual representation $\overline{\rho}_E$ (afforded by $E[p]$) is absolutely irreducible. Suppose also that there exists a rational prime $q \in S$ which does not split in $K/\mathbb{Q}$ and $\overline{\rho}_E$ is ramified at q. If $r_{\mathrm{an}}(E, \alpha) = 1$, then $\mathrm{res}_{\wp}(\mathrm{BF}_1)$ is non-zero.*

Proof. The main theorem in [61] (which applies in our set up) shows that the two-variable Iwasawa main conjecture holds true for $V = T \otimes \mathbb{Q}_p$ (over the $\mathbb{Z}_p^2$-extension of K). We refer the readers the paragraph preceding [12, Theorem 3.24] and Theorem 3.19 in op. cit., where we explain how to translate Wan's work to the standard form of Iwasawa main conjectures. This, together with a descent argument (as in the proof of [57, Corollary 3.28]) shows that the cyclotomic Iwasawa main conjecture (for $V = T \otimes \mathbb{Q}_p$) holds true. We in turn infer that the defect $\delta(^{\mathrm{BF}})$ of the Beilinson–Flach Kolyvagin system is prime to $(\gamma - 1)$ and the proof follows from Theorem 3.3(i) and Proposition 3.2. Q.E.D.

3.4. CM Abelian Varieties and Perrin-Riou-Stark elements

Let K be a CM field and K_+ its maximal totally real subfield that has degree g over $\mathbb{Q}$. Fix a complex conjugation $c \in \mathrm{Gal}(\overline{\mathbb{Q}}/K_+)$ lifting the generator of $\mathrm{Gal}(K/K_+)$. Fix forever an odd prime p unramified in $K/\mathbb{Q}$ and an embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$.

3.4.1. *CM types and p-ordinary abelian varities* Fix a *p-ordinary CM-type* Σ; this means that the embeddings $\Sigma_p := \{\iota_p \circ \sigma\}_{\sigma \in \Sigma}$ induce exactly half of the places of K over p. Identify Σ_p with the associated subset of primes $\{\mathfrak{p}_1, \cdots, \mathfrak{p}_s\}$ of K above p and $\Sigma_p^c = \{\mathfrak{p}_1^c, \cdots \mathfrak{p}_s^c\}$. Note that the disjoint union $\Sigma_p \sqcup \Sigma_p^c$ is the set of all primes of K above p. It follows that there then exists an abelian variety that has CM by K

and has good ordinary reduction at p, and its CM-type is Σ. Fix such an abelian variety A and assume that the index of the order $\mathrm{End}_K(A)$ inside the maximal order $\mathcal{O}_K$ is prime to p. *We will also assume*[13] *that A is principally polarized and that it arises as the base-change of an abelian variety defined over K_+ (which we still denote by A, by slight abuse) that has real multiplication by K_+.* Let $\mathcal{A} = \mathcal{A}_{/\mathfrak{o}_K}$ denote the Néron model of A.

3.4.2. *Grössencharacters of CM abelian varieties* The p-adic Tate-module $T_p(A)$ of A is a free $\mathbb{Z}_p$-module of rank $2g$ on which G_K acts continuously. As explained in the Remark on page 502 of [56], $T_p(A)$ is free of rank one over $\mathcal{O}_K \otimes \mathbb{Z}_p = \prod_{\mathfrak{q}} \mathfrak{o}_{\mathfrak{q}}$, where the product is over the primes of K that lie above p and $\mathfrak{o}_{\mathfrak{q}}$ stands for the valuation ring of $K_{\mathfrak{q}}$. We thus have a decomposition $T_p(A) = \bigoplus_{\mathfrak{q}} T_{\mathfrak{q}}(A)$, where each $T_{\mathfrak{q}}(A) = \varprojlim A[\mathfrak{q}^n]$ is a free $\mathfrak{o}_{\mathfrak{q}}$-module of rank one. The G_K-action on $T_p(A)$ gives rise to characters $\psi_{\mathfrak{q}} : G_K \to \mathcal{O}_{\mathfrak{q}}^{\times}$. It follows from [50, §2] that each $\psi_{\mathfrak{q}}$ is surjective for p large enough; we fix until the end a prime p satisfying this condition. We thence obtain a decomposition

$$T_p(A) \otimes_{\mathbb{Z}_p} \overline{\mathbb{Q}}_p = \bigoplus_{\mathfrak{q}|p} \bigoplus_{\sigma : K_{\mathfrak{q}} \hookrightarrow \overline{\mathbb{Q}}_p} V_{\mathfrak{q}}^{\sigma}$$

where $V_{\mathfrak{q}}^{\sigma}$ is the one-dimensional $\overline{\mathbb{Q}}_p$-vector space on which G_K acts via the character $\psi_{\mathfrak{q}}^{\sigma}$, which is the compositum

$$G_K \xrightarrow{\psi_{\mathfrak{q}}} \mathfrak{o}_{\mathfrak{q}}^{\times} \xrightarrow{\sigma} \overline{\mathbb{Q}}_p^{\times}.$$

Fix embeddings $j_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $j_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ extending ι_p. We write $\mathscr{S} = \Sigma \cup \Sigma^c$ for the set of all embeddings of K into $\overline{\mathbb{Q}}$. Theory of CM associates a Grössencharacter character

$$: \mathbb{A}_K/K^{\times} \longrightarrow K^{\times},$$

to A, which in turn induces Hecke characters

$$\psi_\tau \ : \mathbb{A}_K/K^{\times} \xrightarrow{\Psi} K^{\times} \xrightarrow{j_\infty \circ \tau} \mathbb{C}^{\times}$$

as well as gives rise to its p-adic avatars

$$\psi_\tau^{(p)} \ : \mathbb{A}_K/K^{\times} \xrightarrow{\Psi} K^{\times} \xrightarrow{j_p \circ \tau} \mathbb{C}_p^{\times}.$$

[13] These assumptions are rather stringent: Note, for example, that K is forced to contain the reflex field of the CM pair (K, Σ).

Furthermore, the two sets $\{\mathrm{rec} \circ \psi_\tau^{(p)}\}_{\tau \in \mathscr{S}}$ and $\{\psi_\mathfrak{q}^\sigma\}_{\mathfrak{q},\sigma}$ of p-adic Hecke characters may be identified, where $\mathrm{rec} : \mathbb{A}_K/K^\times \to G_K$ is the reciprocity map. Since we assumed that the field K contains the reflex field of (K, Σ), the Hasse-Weil L-function $L(A/K, s)$ of A then factors into a product of Hecke L-series[14]

$$L(A/K, s + 1/2) = \prod_{\tau \in \mathscr{S}} L(s, \psi_\tau^u) \in K \otimes \mathbb{C} \cong \mathbb{C}^{\mathscr{S}}$$

where ψ^u is the unitarization of ψ. Fix $\varepsilon \in \Sigma$ and identify K with K^ε. This choice (together with the chosen embeddings j_∞ and j_p) in turn fixes a prime $\mathfrak{p} \in \Sigma_p$ and $\sigma : K_\mathfrak{p} \hookrightarrow \overline{\mathbb{Q}}_p$ in a way that $\mathrm{rec} \circ \psi_\varepsilon^{(p)} = \psi_\mathfrak{p}^\sigma$. We set $L = \sigma(K_\mathfrak{p})$ and $\mathfrak{o} := \sigma(\mathcal{O}_\mathfrak{p})$ and define the p-adic Hecke character $\psi := \psi_\mathfrak{p}^\sigma : G_K \twoheadrightarrow \mathfrak{o}^\times$. We set $\psi^* = \chi_{\mathrm{cyc}} \psi^{-1}$ and $T := \mathfrak{o}(\psi^*)$; note that we have $T^* \cong A[\varpi^\infty]$ (where $\varpi \in \mathfrak{m}_L$ is a uniformizer). In this situation, our non-anomaly hypothesis **(H.nA)** translates into the requirement that

$$A(K_\mathfrak{q})[\varpi] = 0 \text{ for every prime } \mathfrak{q} \text{ of } K \text{ above } p \tag{3.1}$$

which we assume throughout this subsection.

This condition implies that is self-dual, in the sense that for each $\tau \in \mathscr{S}$ there is a functional equation with sign $\epsilon(/K) = \pm 1$ (that does not depend on the choice of τ) relating the value $L(s, \psi_\tau)$ to the value $L(1 - s, \psi_\tau)$.

We let $\mathfrak{P} = \mathfrak{p}\mathfrak{p}^c$ denote the prime of K_+ below the prime $\wp$ we have fixed above and let $\varpi_+ := \varpi\varpi^c \in K_{+,\mathfrak{P}}$ be a uniformizer. We write $T_\mathfrak{P}(A) := \varprojlim A[\varpi_+^n]$ for the $\mathfrak{P}$-adic Tate module of A and set $\mathbb{T}_\mathfrak{P}(A) := T_\mathfrak{P}(A) \otimes \Lambda$. By the theory of complex multiplication and our assumption on A, we have $T_\mathfrak{P}(A) = \mathrm{Ind}_{K/K_+} T$. For every prime $\mathfrak{Q}$ of K_+ above p, we have a p-ordinary filtration $F_\mathfrak{Q}^+ T_\mathfrak{P}(A) \subset T_\mathfrak{P}(A)$ (that also gives rise to the filtration $F_\mathfrak{Q}^+ \mathbb{T}_\mathfrak{P}(A) \subset \mathbb{T}_\mathfrak{P}(A)$) when $T_\mathfrak{P}(A)$ is restricted to $G_{K_\mathfrak{Q}}$.

We finally let ϕ denote the Hilbert modular CM form of weight two associated to ψ_ϵ, $L(\phi, s)$ the associated Hecke L-function and $r_{\mathrm{an}}(\phi)$ the order of vanishing at the central critical point $s = 1$. In what follows, we write $\Lambda_\mathfrak{o}$ in place of $\Lambda \otimes \mathfrak{o}$.

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[14]Until the end of this article we shall write $L(\star, s)$ for the motivic L-functions and $L(s, \star)$ for the automorphic L-functions (so that the latter is centered at $s = 1/2$).

3.4.3. *Perrin-Riou-Coleman maps and Selmer structures* We introduce the Selmer structures we shall apply our theory in Section 3.2 with.

Definition 3.7. *We define the Greenberg-submodule*

$$H^1_{\mathrm{Gr}}(K_p,\mathbb{T}) := \bigoplus_{\mathfrak{q}\in\Sigma_p^c} H^1(K_{\mathfrak{q}},\mathbb{T}) \subset H^1(K_p,\mathbb{T})\,.$$

The semi-local Shapiro's lemma induces an isomorphism

$$\mathfrak{sh}:\ H^1(K_{+,p},\mathbb{T}_{\mathfrak{P}}(A)) \xrightarrow{\sim} H^1(K_p,\mathbb{T}) \tag{3.2}$$

under which

$$H^1_{\mathrm{Gr}}(K_{+,p},\mathbb{T}_{\mathfrak{P}}(A)) := \bigoplus_{\wp|p} \mathrm{im}\left(H^1(K_{+,\wp},F^+_{\wp}\mathbb{T}_{\mathfrak{P}}(A)) \to H^1(K_{+,\wp},\mathbb{T}_{\mathfrak{P}}(A)\right)$$

maps isomorphically onto $H^1_{\mathrm{Gr}}(K_p,\mathbb{T})$*.*

For each prime $\wp$ of K_+ above p, we let $D_\wp(T_{\mathfrak{P}}(A))$ denote the Dieudonné module of $T_{\mathfrak{P}}(A)$ considered as a $G_{K_{+,\wp}}$-representation. As explained in [10], we may (and we will) think of this as an $\mathfrak{o}$-module of rank $2f_\wp$ (where $f_\wp = [K_{+,\wp}:\mathbb{Q}_p]$). We also let

$$\exp^*:\ H^1_{/f}(K_{+,\wp},T_{\mathfrak{P}}(A)) \longrightarrow \mathrm{Fil}^0\, D_\wp(T_{\mathfrak{P}}(A))$$

denote the Bloch-Kato dual exponential map,

$$\log_{A,\wp}: H^1_f(K_{+,\wp},T_{\mathfrak{P}}(A)) \longrightarrow D_\wp(T_{\mathfrak{P}}(A))/\mathrm{Fil}^0\, D_\wp(T_{\mathfrak{P}}(A))$$

the inverse of the Bloch-Kato exponential map and

$$[\,,\,]: D_\wp(T_{\mathfrak{P}}(A)) \times D_\wp(T_{\mathfrak{P}}(A)) \longrightarrow \mathfrak{o} \tag{3.3}$$

the canonical alternating perfect pairing induced from the Weil-pairing (thanks to which we have an identification $D_\wp(T_{\mathfrak{P}}(A))^* := D_\wp(T_{\mathfrak{P}}(A)^{\mathcal{D}}(1)) \cong D_\wp(T_{\mathfrak{P}}(A))$), where for a $\mathbb{Z}_p$-module M, we write $M^{\mathcal{D}}$ for its $\mathbb{Z}_p$-linear dual. We let $D_\wp(T_{\mathfrak{P}}(A))_{[-1]} = D_\wp(F^+T_{\mathfrak{P}}(A))$ denote the subspace of $D_\wp(T_{\mathfrak{P}}(A))$ on which the crystalline Frobenius φ acts with slope -1 and let

$$\omega_\wp = \{\omega_{i,\wp}\}_{i=1}^{f_\wp} \subset \mathrm{Fil}^0 D_\wp(T_{\mathfrak{P}}(A)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

denote a fixed basis corresponding (under the comparison isomorphism) to the Néron differential[15] on A. Since we have

$$\mathrm{Fil}^0 D_\wp(T_{\mathfrak{P}}(A)) \cap D_\wp(T_{\mathfrak{P}}(A))_{[-1]} = 0,$$

we may choose a basis $\omega_\wp^* = \{\omega_{j,\wp}^*\}_{j=1}^{f_\wp}$ of $D_\wp(T_{\mathfrak{P}}(A))_{[-1]} = D_\wp(F^+T_{\mathfrak{P}}(A))$ such that $[\omega_{i,\wp}, \omega_{j,\wp}^*] = \delta_{i,j}$, where $\delta_{i,j}$ is the Kronecker-delta. Let ω denote the collection $\{\omega_\wp^*\}_{\wp \mid p}$ of these distinguished bases.

Definition 3.8. *Let us write π_{Gr} for the projection*

$$\pi_{\mathrm{Gr}} : \mathbb{T}_{\mathfrak{P}}(A) \longrightarrow \mathbb{T}_{\mathfrak{P}}(A)/F^+\mathbb{T}_{\mathfrak{P}}(A)$$

as well as all maps induced from it in the level of Galois cohomology.

Notice that

$$\pi_{\mathrm{Gr}} : H^1(K_{+,\wp}, \mathbb{T}_{\mathfrak{P}}(A)) \longrightarrow H^1(K_{+,\wp}, \mathbb{T}_{\mathfrak{P}}(A)/F^+\mathbb{T}_{\mathfrak{P}}(A))$$

is surjective since we assume (3.1).

Definition 3.9. *Let $K_{+,\wp,n}$ denote the nth layer in the cyclotomic tower of $K_{+,\wp}$ and set $\Gamma_n := \mathrm{Gal}(K_{+,\wp,n}/K_{+,\wp})$.*

Theorem 3.4 (Perrin-Riou)**.** *There exists a unique $\Lambda_{\mathfrak{o}}$-linear morphism*

$$\mathfrak{L}_{(\wp)} : H^1(K_{+,\wp}, \mathbb{T}_{\mathfrak{P}}(A)/F^+\mathbb{T}_{\mathfrak{P}}(A)) \longrightarrow \Lambda \otimes D_\wp(T_{\mathfrak{P}}(A)/F^+T_{\mathfrak{P}}(A))$$

with the following property: For any

$$\begin{aligned} z_\infty = \{z_n\}_n \in \varprojlim_n H^1(K_{+,\wp,n}, T_{\mathfrak{P}}(A)/F^+T_{\mathfrak{P}}(A)) \\ = H^1(K_{+,\wp}, \mathbb{T}_{\mathfrak{P}}(A)/F^+\mathbb{T}_{\mathfrak{P}}(A)), \end{aligned}$$

all $j = 1, \cdots, f_\wp$ and faithful characters θ of Γ_n, we have

$$\theta\left([\mathfrak{L}_{(\wp)}(z), \omega_{j,\wp}^*]\right) = \begin{cases} [\exp^*(z_0), (1-p^{-1}\varphi^{-1})(1-\varphi)^{-1}\omega_{j,\wp}^*] & , \textit{ if } n = 0, \\ \\ \frac{1}{\tau(\theta^{-1})}[\sum_{\sigma\in\Gamma_n} \theta^{-1}(\sigma)\exp^*(z_n^\sigma), \varphi^{-n}\omega_{j,\wp}^*] & , \textit{ if } n \geq 1, \end{cases}$$

where $\tau(\theta^{-1})$ is the Gauss sum.

---

[15]This is a top degree invariant form on a Néron model of A and as such, does only determine the top-degree exterior product $\wedge\omega^* := \omega_{1,\wp}^* \wedge \cdots \wedge \omega_{f_\wp,\wp}^*$ uniquely.

Furthermore, the kernel of the map

$$\mathfrak{L}_\omega^{(\wp)} : H^1(K_{+,\wp}, \mathbb{T}_{\mathfrak{P}}(A)) \xrightarrow{[\mathfrak{L}_{(\wp)}\circ \pi_{\mathrm{Gr}}(-),\, \omega_\wp^*]} \Lambda_{\mathfrak{o}}^{f_\wp}$$

is precisely the Greenberg submodule $H^1_{\mathrm{Gr}}(K_{+,\wp}, \mathbb{T}_{\mathfrak{P}}(A))$ *and it is pseudo-surjective.*

Proof. Set $\underline{\mathbf{T}} := T_{\mathfrak{P}}(A)/F^+T_{\mathfrak{P}}(A)$ and $\underline{\mathbf{V}} := \underline{\mathbf{T}} \otimes \mathbb{Q}_p$. Note that $\underline{\mathbf{V}}$ is crystalline and that all slopes in its Newton and Hodge polygons are zero. The map $\mathfrak{L}_{(\wp)}$ is obtained from Perrin-Riou's construction in [48] (which, in our set up, she directly deduces from Coleman's work [20]; see Section 4.3 in op. cit.), on setting

$$\mathfrak{L}_{(\wp)}(z_\infty) := \langle z_\infty, \Omega_{\underline{\mathbf{V}}^*}(1 \otimes \omega_\wp^*)\rangle$$

for each $z_\infty \in H^1(K_{+,\wp}, \mathbb{T}_{\mathfrak{P}}(A)/F^+\mathbb{T}_{\mathfrak{P}}(A))$. Here, $\underline{\mathbf{V}}^* = \underline{\mathbf{T}}^* \otimes \mathbb{Q}_p$ and $\underline{\mathbf{V}}^* = \mathrm{Hom}(\underline{\mathbf{T}}, \mathbb{Z}_p(1))$ is the Tate-dual of $\underline{\mathbf{T}}$ (and it is canonically isomorphic via the Weil pairing to $F^+T_{\mathfrak{P}}(A)$) and

$$\Omega_{\underline{\mathbf{V}}^*} = \Omega^{\varepsilon}_{\underline{\mathbf{V}}^*,1} : \Lambda \otimes D_\wp(\underline{\mathbf{T}}^*) \longrightarrow H^1(K_{+,\wp}, \underline{\mathbf{T}}^* \otimes \Lambda)$$

is Perrin-Riou's big exponential map constructed in [48] (see in particular paragraphs 4.1.2 and 4.1.3 in op. cit. for a discussion relevant to our setting) for the $G_{K_{+,\wp}}$-representation $\underline{\mathbf{V}}^*$ with $h = 1$ and a fixed choice of a generator ε of $\mathbb{Z}_p(1)$.

The assertion concerning the kernel of $\mathfrak{L}_\omega^{(\wp)}$ is clear. The fact that $\mathfrak{L}_\omega^{(\wp)}$ is pseudo-surjective follows from the validity of Perrin-Riou's $\delta_{\mathbb{Z}_p}(\underline{\mathbf{V}}^*)$-conjecture. In our set up, this conjecture is proved in [48, Proposition 4.2.5]. Q.E.D.

Definition 3.10. *i) Let us write* $\mathfrak{L}_\omega^{(\wp,i)}$ *for the map*

$$\mathfrak{L}_\omega^{(\wp,i)} : H^1(K_{+,\wp}, \mathbb{T}_{\mathfrak{P}}(A)) \longrightarrow \Lambda_{\mathfrak{o}}^{\oplus(f_\wp-1)}$$

obtained from $\mathfrak{L}_\omega^{(\wp)}$ *by omitting the summand in the target which corresponds to* $\omega_{i,\wp}^*$.
ii) We set

$$\mathfrak{L}_\omega := \oplus_{\wp|p}\, \mathfrak{L}_\omega^{(\wp)} := H^1(K_{+,p}, \mathbb{T}_{\mathfrak{P}}(A)) \longrightarrow \Lambda_{\mathfrak{o}}^{g},$$

where $g = [K_+ : \mathbb{Q}]$ *(it is also the dimension of the abelian variety A).*

We note that $\ker(\mathfrak{L}_\omega) = H^1_{\mathrm{Gr}}(K_{+,p}, \mathbb{T}_{\mathfrak{P}}(A))$.

Definition 3.11. *We fix a prime $\mathfrak{Q}$ of K_+ above p, as well as an integer $1 \leq i \leq f_{\mathfrak{Q}}$. We define the map*

$$\mathfrak{L}_{(\mathfrak{Q},i)} = \bigoplus_{\mathfrak{Q}\neq\wp \mid p} \mathfrak{L}_\omega^{(\wp)} \oplus \mathfrak{L}_\omega^{(\mathfrak{Q},i)} \,:\, H^1(K_{+,p}, \mathbb{T}_{\mathfrak{P}}(A)) \longrightarrow \Lambda_{\mathfrak{o}}^{g-1}\,.$$

We set $H^1_+(K_{+,p}, \mathbb{T}_{\mathfrak{P}}(A)) := \ker(\mathfrak{L}_{(\mathfrak{Q},i)})$ and define $H^1_+(K_p, \mathbb{T})$ as the isomorphic image of $H^1_+(K_{+,p}, \mathbb{T}_{\mathfrak{P}}(A))$ under Shapiro's morphism $\mathfrak{sh}$ (given as in (3.2)*). We define the Selmer structure $\mathcal{F}_+$ on $\mathbb{T}$ by the requiring that*

$$H^1_{\mathcal{F}_+}(K_p, \mathbb{T}) = H^1_+(K_p, \mathbb{T}) \text{ and } H^1_{\mathcal{F}_+}(K_\lambda, \mathbb{T}) = H^1(K_\lambda, \mathbb{T}) \text{ for } \lambda \nmid p\,.$$

In this situation, Theorem 3.1 applies since we assumed (3.1). Furthermore, *if we assume the truth of Perrin-Riou-Stark Conjecture* proposed in [10] (which is a slightly strong form of Rubin–Stark conjectures) *as well as Leopoldt's conjecture for all subextensions of $K(A[\varpi])/K$*, we may obtain a natural generator of this module using the main results of [16, 10]; this is what we explain in the next paragraph.

3.4.4. *The Coleman–Rubin–Stark element* Let $K_{\mathrm{cyc}} = K\mathbb{Q}_\infty$ denote the cyclotomic $\mathbb{Z}_p$-extension of K. Since we assumed that p is unramified in $K/\mathbb{Q}$, we may canonically identify Γ with $\mathrm{Gal}(K_{\mathrm{cyc}}/K)$. Let K_∞ denote the maximal $\mathbb{Z}_p$-power extension of $K$and $\Gamma_K = \mathrm{Gal}(K_\infty/K)$ its Galois group over K.

Let ω_ψ denote the character of G_K giving its action on $A[\varpi]$. It is the unique character of G_K which has the properties that the character $\langle\psi\rangle := \psi\omega_\psi^{-1}$ factors through Γ_K and that it is trivial on Γ_K. Let $T_{\omega_\psi} = \mathfrak{o}(1) \otimes \omega_\psi^{-1}$ and define $H^1_\infty(K, T_{\omega_\psi}) := \varprojlim H^1(F, T_{\omega_\psi})$, where the projective limit is over all finite sub-extensions of K_∞/K. Assume the truth of the *Perrin-Riou-Stark Conjecture 4.14* in [10] (with the Dirichlet character ω_ψ) and let[16] $\mathfrak{S}_\infty^{\omega_\psi} \in \wedge^g H^1_\infty(K, T_{\omega_\psi})$ denote the element whose existence is predicted by the said conjecture. As in Definition 4.16 of loc. cit., we may twist this element to obtain the twisted Perrin-Riou-Stark elements $\mathfrak{S}_\infty \in \wedge^g H^1_\infty(K, T)$ as well as their projections

$$\mathfrak{S}_{\mathrm{cyc}} = \mathfrak{S}_{\mathrm{cyc}}^{(1)} \wedge \cdots \wedge \mathfrak{S}_{\mathrm{cyc}}^{(g)} \in \wedge^g H^1(K, \mathbb{T}) \cong \wedge^g H^1(K_+, \mathbb{T}_{\mathfrak{P}}(A))$$

(where we denote the image of $\mathfrak{S}_{\mathrm{cyc}}$ under the isomorphism above still by the same symbol). We finally set

$$\mathbb{RS}^\psi_{/f} := \mathrm{res}_{/f}^{\otimes g}(\mathfrak{S}_{\mathrm{cyc}}) \in \wedge^g H^1_{/f}(K_p, \mathbb{T}) \cong \wedge^g H^1_{/f}(K_p, \mathbb{T}_{\mathfrak{P}}(A))\,.$$

[16]We invite the interested reader to consult [10, Remark 4.13] and the discussion that precedes this remark for the desired integrality properties of the Rubin–Stark elements.

Theorem 3.5. *There exists a generator* ${}^{\mathrm{CRS}}$ *(the Coleman-adapted Rubin–Stark Kolyvagin system) of the free Λ-module $\mathbf{KS}(\mathcal{F}_+,\mathbb{T},\mathcal{P})$ whose initial term $\kappa_1^{\mathrm{CRS}} \in H^1_{\mathcal{F}_+}(K_p,\mathbb{T})$ has the following property:*

$$\mathfrak{L}_{(\mathfrak{Q},i)}\left(\mathrm{res}_{/f}\left(\kappa_1^{\mathrm{CRS}}\right)\right)=\mathfrak{L}_{\omega}^{\otimes g}\left(\mathbb{RS}^{\psi}_{/f}\right)$$

Proof. This is precisely the content of Theorem A.11 and Proposition A.8 of [10] (except that for our purposes, it is sufficient to restrict our attention to the case when Λ in loc. cit. has also Krull dimension 2). In order to apply these results, we simply choose $\mathfrak{L}_\omega$ in place of Ψ and the summand in the target of $\mathfrak{L}_\omega^{(\mathfrak{Q})}$ that corresponds to $\omega_{i,\mathfrak{Q}}$ in place of $L(1)$ in loc. cit. Note in this case that the Selmer structure $\mathcal{F}_+$ above corresponds to the Selmer structure denoted by $\mathcal{F}_L$ in loc. cit. Q.E.D.

Definition 3.12. *We let $\mathfrak{C}_\infty \in H^1_{\mathcal{F}_+}(K_+,\mathbb{T}_{\mathfrak{P}}(A))$ denote the element that corresponds to κ_1^{CRS} under the isomorphism $\mathfrak{sh}$ and let $\mathfrak{C} \in H^1_{\mathcal{F}_+}(K_+,T_{\mathfrak{P}}(A))$ (that we call the* Coleman–Rubin–Stark class*) its obvious projection.*

The following follows as an direct consequence of Theorem 3.3(ii) and Theorem 3.5:

Corollary 3.2. *The map $\mathrm{res}_{f/-}: H^1_{\mathrm{f}}(K_+,T_{\mathfrak{P}}(A)) \to H^1_{f/-}(K_p,T_{\mathfrak{P}}(A))$ is injective if and only if the Coleman–Rubin–Stark class $\mathfrak{C}$ is non-trivial.*

Proof. We only need to verify that the hypotheses of Theorem 3.3(ii) hold true under our running assumptions and that is what we shall carry out in what follows.

(H.Tam) We need to check that $H^0\left(G_{K_\lambda}/I_\lambda, H^0(I_\lambda,V/T)/H^0(I_\lambda,V/T)_{\mathrm{div}}\right) = 0$ for every prime $\lambda \in S$ which is coprime to p (where $I_\lambda \subset G_{K_\lambda}$ stands for the inertia subgroup), where $T := \mathfrak{o}(\psi^*)$ and $\psi^* = \chi_{\mathrm{cyc}}\psi^{-1}$ as before. This is clear.

(H.nA) The truth of this hypotheses is already assumed.

(H.0) T is a free $\mathfrak{o}$-module of finite rank, as required.

(H.1) We need to check that the G_K-representation $T/\mathfrak{m}_L T$ is absolutely irreducible. This is evident.

(H.2) This condition holds true with $\tau = \mathrm{id}$.

(H.3) We need to verify that

$$H^1(K(T,{}_{p^\infty})/K,T/\mathfrak{m}_L T)=0=H^1(K(T,{}_{p^\infty})/K,T^*[\mathfrak{m}_L]), \tag{3.4}$$

where $K(T,{}_{p^\infty}) \subset \overline{K}$ is the fixed field of $\ker\left(G_K \longrightarrow \mathrm{GL}(T)\oplus\mathrm{GL}({}_{p^\infty})\right)$. Set $G := \mathrm{Gal}(K(T,{}_{p^\infty})/K)$ and observe that G is an abelian group. Let $G^{(p)} < G$ denote its maximal pro-p subgroup; note that $G^{(p)}$ is a finitely generated $\mathbb{Z}_p$-module and we can write $G^{(p)} = \Gamma_1 \times \cdots \times \Gamma_s$ as a product of pro-cyclic pro-p groups. By the inflation-restriction sequence, it suffices to prove that

$$H^1(G^{(p)}, T/\mathfrak{m}_L T) = 0 = H^1(G^{(p)}, T^*[\mathfrak{m}_L])\,.$$

Let us put W to denote any one of $T/\mathfrak{m}_L T$ and $T^*[\mathfrak{m}_L]$. Since W is a non-trivial Galois module which is one-dimensional as a $\mathfrak{o}/\mathfrak{m}_L$-vector space, there exists $i \in \{1, \cdots, s\}$ such that $W^{\Gamma_i} = 0$. Let us fix such i. Again using the inflation-restriction sequence, we may reduce to checking that $H^1(\Gamma_i, W) = 0$. Let $\gamma_i \in \Gamma_i$ denote any topological generator, so that we have $H^1(\Gamma_i, W) = W/(\gamma_i - 1)W$. The exactness of the sequence

$$0 \longrightarrow W^{\Gamma_i} \longrightarrow W \xrightarrow{[\gamma_i - 1]} W \longrightarrow W/(\gamma_i - 1)W \longrightarrow 0$$

shows that $\#W/(\gamma_i - 1)W = \#W^{\Gamma_i} = 0$ and (3.4) follows.

Q.E.D.

3.4.5. *Katz' p-adic L-function an explicit reciprocity conjecture for Rubin–Stark elements* We recall here the definition the p-adic L-function of Katz and Hida-Tilouine and propose an extension of the Coates–Wiles reciprocity law for elliptic units to a reciprocity law concerning the Perrin-Riou-Stark elements.

Definition 3.13. *A pair (m_0, d) (where $m_0 \in \mathbb{Z}$ and $d = \sum_{\sigma\in\Sigma} d_\sigma \sigma \in \mathbb{Z}^\Sigma$) is called* Σ-critical *if either $m_0 > 0$ and $d_\sigma \geq 0$ or else $m_0 \leq 1$ and $d_\sigma \geq 1 - m_0$ for every $\sigma \in \Sigma$.*

Likewise, a Grössencharacter is called Σ-critical *if its infinity type equals the expression $\sum_{\sigma\in\Sigma}(m_0 + d_\sigma)\sigma - d_\sigma\sigma^c \in \mathbb{Z}^{\mathscr{S}}$ for some Σ-critical pair (m_0, d).*

Let $\mathscr{O}$ denote the p-adic completion of the maximal unramified extension of $\mathbb{Q}_p$. Let Γ_K denote the Galois group of the maximal $\mathbb{Z}_p$-power extension of K. By class field theory, note that we have $\Gamma_K \cong \mathbb{Z}_p^{1+g+\delta}$, where δ is Leopoldt's defect (which equals zero whenever we assume Leopoldt's conjecture).

The following statement was proved by Hida and Tilouine in [30], extending a previous construction due to Katz. In our discussion below, we will mostly stick to the exposition in [29] and we shall rely on Hsieh's notation (except perhaps minor alterations). Let $\mathfrak{f} \subset \mathfrak{o}_K$ denote the conductor of . Write $\mathfrak{f} = \mathfrak{f}^+\mathfrak{f}^-$ where $\mathfrak{f}^+$ (resp., $\mathfrak{f}^-$) is a product of primes that split (resp., that remain inert or ramify) over K_+. The following is Proposition 4.9 in [29].

Theorem 3.6 (Katz, Hida-Tilouine). *There exists an element* $\mathscr{L}_p^{\Sigma} \in \mathscr{O}[[\Gamma_K]]$ *(Katz' p-adic L-function) that is uniquely determined by the following interpolation property on the p-adic avatars* $^{(p)} = (\lambda_\tau^{(p)})$ *of the* Σ*-critical characters* $= (\lambda_\tau)$ *of infinity type* $m_0\Sigma + (1-c)d \in \mathbb{Z}^{\mathscr{S}}$ *and of conductor dividing* $\mathfrak{f}p^\infty$*:*

$$\frac{\mathscr{L}_p^{\Sigma}(^{(p)})}{\Omega_p^{m_0\Sigma+2d}} = t \cdot \frac{\pi^d\,\Gamma_\Sigma(m_0\Sigma + d)}{\sqrt{|D_{K_+}|_{\mathbb{R}}}\,\mathrm{im}(\delta)^d} \cdot \mathscr{E}_p()\mathscr{E}_{\mathfrak{f}^+}() \cdot \prod_{\mathfrak{q}|\mathfrak{f}p}(1-(\mathfrak{q})) \cdot \frac{L(m_0/2,{}^u)}{\Omega_\infty^{m_0\Sigma+2d}}$$

where the equality takes place in $\iota_p(\overline{\mathbb{Q}})^{\mathscr{S}}$ *and where*

- $\Omega_p = (\Omega_p(\sigma))_\sigma \in \mathscr{O}^{\Sigma}$ *and* $\Omega_\infty = (\Omega_\infty(\sigma))_\sigma \in \mathbb{C}^{\Sigma}$ *are the periods which are attached to a Néron differential* ω *on the abelian scheme* $\mathcal{A}$*, as in* [31, Chapter II]*;*
- t *is a certain fixed power* 2 *(which can be made explicit);*
- $\delta \in K_+$ *is the element chosen as in* [30, 0.9a-b] *and* [29, Section 3.1]*;*
- $\mathscr{E}_p()$ *and* $\mathscr{E}_{\mathfrak{f}^+}()$ *are products of modified Euler factors defined in* [29, (4.16)] *and denoted by* Eul_p*,* $Eul_{\mathfrak{f}^+}$ *in loc. cit.,*
- $^u := /||_{\mathbb{A}}$ *is the unitarization of* *(where this terminology is borrowed from Hida and Tilouine* [30, pp. 231-232]*).*

In particular, when the Grössencharacter has infinity type Σ (so that $m_0 = 1$ and $d = 0$) and conductor $\mathfrak{f}$, the interpolation formula simplifies to

$$\frac{\mathscr{L}_p^{\Sigma}(^{(p)})}{\Omega_p^{\Sigma}} = t \cdot \mathscr{E}_p()\mathscr{E}_{\mathfrak{f}^+}() \cdot \prod_{\wp|p}(1-(\wp)) \cdot \frac{L(1/2,{}^u)}{\sqrt{|D_{K_+}|_{\mathbb{R}}}\,\Omega_\infty^{\Sigma}}$$

We call the pullback $\mathscr{L}_{p,\Psi}^{\Sigma}$ of $\mathscr{L}_p^{\Sigma}$ along the character (where is the Grössencharacter associated to A) the *-branch of* $\mathscr{L}_p^{\Sigma}$. In more concrete terms we have

$$\mathscr{L}_{p,\Psi}^{\Sigma}(\chi) := \mathscr{L}_p^{\Sigma}(\chi^{(p)})$$

for every character χ of finite order.

Recall the p-adic Hecke character ψ we have fixed above, which is associated to our choice of $\epsilon \in \Sigma$ (and fixed embeddings j_∞ and j_p). We will fix the modulus $\mathfrak{f}$ to be chosen as the conductor of ψ.

Corollary 3.3. *For every character* χ *of* Γ_K *of finite order we have,*

$$\frac{\mathscr{L}_p^{\Sigma}(\chi\psi)}{\Omega_p(\epsilon)} = \mathscr{E}(\chi\psi)\prod_{\wp|p}(1-\chi_\epsilon\psi_\epsilon(\wp)) \cdot \frac{L(1/2,\chi_\epsilon\psi_\epsilon^u)}{\Omega_\infty(\epsilon)}$$

where $\mathscr{E}(\chi\psi) := t \cdot \mathscr{E}_p(\chi\psi) \cdot \mathscr{E}_{\mathfrak{f}^+}(\chi\psi)$ is a product of Euler factors up to an explicit power of 2.

Remark 3.5. *Note that the Euler like factors $1 - \chi_\epsilon\psi_\epsilon(\wp)$ are equal to 1 so long as χ is not the trivial character.*

Definition 3.14. *We let $\mathscr{L}_{\mathrm{cyc}}^{\Sigma} \in \mathscr{O}[[\Gamma]]$ denote the measure obtained by restricting the -branch $\mathscr{L}_{p,\psi}^{\Sigma}$ of the Katz p-adic L-function to the cyclotomic characters of finite order. On replacing the CM-type Σ with Σ^c, we also obtain a p-adic L-function $\mathscr{L}_{\mathrm{cyc}}^{\Sigma^c}$, which is the restriction of the the ^{c}-branch of Katz p-adic L-function $\mathscr{L}_{p,\psi^c}^{\Sigma^c}$ to the cyclotomic characters. The p-adic L-function $\mathscr{L}_{p,\psi^c}^{\Sigma^c}$ is characterized by an interpolation formula analogous to one given in Corollary 3.3; note however that one needs to replace the embedding ϵ with ϵ^c.*

Definition 3.15. *Let n be a positive integer and $z = \{z_n\} \in H^1(K_+, \mathbb{T}_{\mathfrak{P}}(A))$ be an arbitrary element and let χ be a primitive character of Γ_n. Fix a prime $\wp$ of K_+ lying above p. Set $K_n^+ := K_+\mathbb{Q}_n$ and denote by $\wp_n$ the unique prime of K_n^+ above $\wp$. For every $1 \le j \le f_\wp$, we define the* Perrin-Riou symbol $[z, \wp, j, \chi]$ *by setting*

$$[z, \wp, j, \chi] := \frac{1}{\tau(\chi)} \left[\sum_{\gamma \in \Gamma_n} \chi(\gamma) \exp_n^*(\mathrm{res}_{\wp_n}(z_n)^\gamma) \, , \, \varphi^{-n}(\omega_{j,\wp}) \right]$$

where $\tau(\chi)$ is the Gauss sum and

$$\exp_n^* : H^1(K_{n,\wp_n}, T_{\mathfrak{P}}(A)) \longrightarrow \mathbb{Q}_{p,n} \otimes_{\mathbb{Z}_p} \mathrm{Fil}^0 D_\wp(T_{\mathfrak{P}}(A))$$

is the dual exponential map. On fixing an ordering of primes of K_+ above p, we let $\mathbb{R}_\chi(z, \omega, \chi)$ denote the $1 \times g$ matrix given by

$$\mathbb{R}_\chi(z, \omega, \chi) = ([z, \wp, j, \chi])_{\substack{\wp \mid p \\ 1 \le j \le f_\wp}} .$$

For an element $\mathbf{z} = z_1 \wedge \cdots \wedge z_g \in \wedge^g H^1(K_+, \mathbb{T}_{\mathfrak{P}}(A))$, we further define the $g \times g$ matrix $\mathbb{M}(\mathbf{z}, \omega, \chi)$ by setting

$$\mathbb{M}(\mathbf{z}, \omega, \chi) = (\mathbb{R}_\chi(z_i, \omega, \chi))_{i=1}^g$$

We remark that this definition actually depends only on the element $\wedge\omega^*$ (that corresponds to a Néron differential on A) and not the choice of a basis that represents this differential.

We propose the following explicit reciprocity conjecture for the twisted Perrin-Riou-Stark element

$$\mathfrak{S}_{\mathrm{cyc}} = \mathfrak{S}_{\mathrm{cyc}}^{(1)} \wedge \cdots \wedge \mathfrak{S}_{\mathrm{cyc}}^{(g)} \in \wedge^g H^1(K_+, \mathbb{T}_{\mathfrak{P}}(A))$$

as a natural extension of Coates–Wiles explicit reciprocity law for elliptic units.

Conjecture 3.2. *There exists a choice of a Néron differential on A so that we have*

$$\det \mathbb{M}(\mathfrak{S}_{\mathrm{cyc}}, \omega, \chi) = \mathscr{E}(\chi\psi) \cdot \frac{L(1/2, \chi_\epsilon \psi_\epsilon)}{\Omega_\infty(\epsilon)}$$

for every primitive character χ of Γ_n.

Whenever we assume the truth of Conjecture 3.2, we shall implicitly assume also that we are working with a basis ω as comes attached to the appropriate choice of a Néron differential.

Corollary 3.4. *If the Explicit Reciprocity Conjecture 3.2 holds true, we have*

$$\mathfrak{L}_\omega^{\otimes g}\left(\mathbb{RS}^\psi_{/f}\right) = \frac{\mathscr{L}^\Sigma_{\mathrm{cyc}}}{\Omega_p(\epsilon)}\,.$$

Proof. This follows at once making use of the displayed equation (5) in [11]. Q.E.D.

3.4.6. *Perrin-Riou's conjecture for CM abelian varieties* We may finally turn our attention to Conjecture 3.1 in this set up. *We will assume until the end of Section 3.5 that there exists a degree one prime of K_+ above p* (mostly for brevity; we expect that one should be able to get around of this assumption with more work) and we choose the prime $\mathfrak{Q}$ above that we work with as this prime of degree one. Note in this case that $i = f_{\mathfrak{Q}} = 1$ and also that

$$\ker\left(\mathfrak{L}^{(\mathfrak{Q},1)}\right) = H^1(K_{+,\mathfrak{Q}}, \mathbb{T}_{\mathfrak{P}}(A)) \oplus \bigoplus_{\mathfrak{Q}\neq\mathfrak{p}|p} H^1_{\mathcal{F}_{\mathrm{Gr}}}(K_{+,\mathfrak{p}}, \mathbb{T}_{\mathfrak{P}}(A)).$$

Also in this situation, note that the set $\omega_{\mathfrak{Q}}$ is a singleton, and by slight abuse, we denote its only element also by $\omega_{\mathfrak{Q}}$.

Definition 3.16. *We let*

$$\langle\, ,\, \rangle : H^1_{\mathrm{f}}(K_+, T_{\mathfrak{P}}(A)) \otimes H^1_{\mathrm{f}}(K_+, T_{\mathfrak{P}}(A)) \longrightarrow L$$

denote the p-adic height pairing of Perrin-Riou [45, Section 2.3] *associated to the canonical unit root Hodge splitting, the cyclotomic character and Iwasawa's branch of the p-adic logarithm.*

The following is the version of Perrin-Riou conjecture in this set up:

Theorem 3.7. *If the Explicit Reciprocity Conjecture 3.2 holds true and if either $r_{\mathrm{an}}(\phi) = 0$ or else $r_{\mathrm{an}}(\phi) = 1$ and the p-adic height pairing $\langle\, ,\, \rangle$ is non-zero, then the Coleman–Rubin–Stark class $\mathfrak{C}$ is non-trivial.*

Remark 3.6. *If one could extend the main theorem of Kolyvagin and Logachev* [33] *to cover CM abelian varieties, one may prove Theorem 3.7 without assuming either the explicit reciprocity conjecture or the non-triviality of the p-adic height pairing.*

Proof of Theorem 3.7. When $r_{\mathrm{an}}(\phi) = 0$, it follows from the interpolation formula for the p-adic L-function that $\mathbb{1}(\mathscr{L}^{\Sigma}_{\mathrm{cyc}}) \neq 0$. The assertion in this case follows on combining Theorem 3.5 and Corollary 3.4.

Suppose now that $r_{\mathrm{an}}(\phi) = 1$ and the p-adic height pairing $\langle\, ,\, \rangle$ is non-zero. Let $J \subset \mathscr{O}[[\Gamma]]$ denote the augmentation ideal. On choosing an auxiliary totally imaginary extension F/K_+ in suitable manner (in a way that the non-vanishing results of Friedberg and Hoffstein [25] apply) and relying on the Gross–Zagier formula of Yuan-Zhang-Zhang in [65] and its p-adic variant due to Disegni [23], we conclude that $\mathscr{L}^{\Sigma}_{\mathrm{cyc}} \not\equiv 0 \mod J^2$. Combined with the standard application of the Coleman–Rubin–Stark Λ-adic Kolyvagin system $^{\mathrm{CRS}}$ (and a control argument for Greenberg Selmer groups), it follows that the $\mathfrak{o}$-module $H^1_{\mathrm{f}}(K_+, T_{\mathfrak{P}}(A))$ is of rank one and that $\text{Ш}(A/K_+)[\mathfrak{P}^\infty]$ is finite. Thence the map $\mathrm{res}_{f/-}$ may be given explicitly via the commutative diagram

$$\begin{tikzcd}
H^1_{\mathcal{F}_{\mathrm{Gr}}}(K_+, T_{\mathfrak{P}}(A)) \arrow[r, "\mathrm{res}_{f/-}"] & H^1_{\mathcal{F}_{\mathrm{Gr}}}(K_{+,\mathfrak{Q}}, T_{\mathfrak{P}}(A)) \\
A(K_+)\widehat{\otimes}\, \mathfrak{o} \arrow[r, "\mathrm{res}_{f/-}"] \arrow[u, "\cong"'] & A(K_{+,\mathfrak{Q}})\widehat{\otimes}\, \mathfrak{o} \arrow[u, "\cong"']
\end{tikzcd}$$

and it is evidently injective. The proof follows by Corollary 3.2. Q.E.D.

In what follows, we let $D_{\mathfrak{P}}(A)$ be a shorthand for the Dieudonné module of the $G_{K_{+,\mathfrak{Q}}}$-representation $V_{\mathfrak{P}}(A)$. Observe that $K_{+,\mathfrak{Q}} = \mathbb{Q}_p$ thanks to our choice of $\mathfrak{Q}$.

Definition 3.17. *Fix a quadratic and purely imaginary extension E of K_+ such that the relative discriminant Δ_{E/K_+} is totally odd and relatively prime to $D_{K_+}Np$ (where D_{K_+} is the discriminant of $K_+/\mathbb{Q}$). We let $\eta = \eta_{E/K_+}$ the quadratic character associated to E/K_+. Let N denote the level of the normalised new Hilbert eigenform ϕ of parallel weight 2 and suppose that $\eta(N) = (-1)^{g-1}$. Let ϕ_η denote the twisted weight 2 form, and suppose that $r_{\mathrm{an}}(\phi_\eta) = 0$. We note that there are infinitely many choices for the field E simultaneously verifying all these conditions (thanks to* [25]*); we pick and fix one forever.*

Definition 3.18. *Let $L_p(\phi,\cdot)\in\Lambda_{\mathfrak{o}}$ be the p-adic L-function that Manin* [40, Sections 5 and 6] *and Dabrowski* [21] *associate to the p-ordinary stabilisation of ϕ. It is characterized by the following interpolation property (c.f., Theorem 4.4.1 of* [22]*): For every non-trivial character of Γ of finite order*[17] *and conductor $\mathfrak{f}_\chi$*

$$L_p(\phi,\chi)=\chi(D_{K_+})\,\tau(\bar\chi)\,\mathbf{N}(\mathfrak{f}_\chi)^{1/2}\,\alpha_{\mathfrak{f}_\chi}^{-1}\cdot\frac{L(\phi,\bar\chi,1)}{\Omega_\phi^+}\tag{3.5}$$

where

- *$\tau(\bar\chi)$ is a Gauss sum that is normalized by Disegni in loc. cit.;*
- *$\alpha_{\mathfrak{f}_\chi}:=\prod_{\mathfrak{q}|p}a_{\mathfrak{q}}(\phi)^{v_{\mathfrak{q}}(\mathfrak{f}_\chi)}$ and $a_{\mathfrak{q}}(\phi)$ the p-unit root of the Hecke polynomial at $\mathfrak{q}$;*
- *Ω_ϕ^+ is the real period defined by Shimura and Yoshida* [64].

Let E be as in Definition 3.17 and let $L_p(\phi_E,\cdot)\in\Lambda_{\mathfrak{o}}$ denote the p-adic L-function attached (by Panchishkin [44]*, Hida* [27] *and Disegni* [22, Theorem 4.3.4]*) to the base change ϕ_E of ϕ.*

There is likewise a p-adic L-function $L_p(\phi_\eta,\cdot)$ associated to a p-ordinary stabilisation of the twisted form ϕ_η (and a corresponding real period $\Omega^+_{\phi_\eta}$). The Artin formalism yields a factorization

$$L_p(\phi_E,\chi\circ N_{E/K_+})=\chi(\Delta_{E/K_+})^2\,\frac{\Omega_\phi^+\Omega_{\phi_\eta}^+}{D_E^{-1/2}\Omega_\phi}L_p(\phi,\chi)L_p(\phi_\eta,\chi)\tag{3.6}$$

for every character χ as above, where $\Omega_\phi=(8\pi)^2\langle\phi,\phi\rangle_N$ is the Shimura period. We remark that the ratio $\Omega_\phi^+\Omega_{\phi_\eta}^+/\Omega_\phi$ is always an algebraic number (in fact, it belongs to the Hecke field). Moreover, comparing the respective interpolation formulae and the periods that intervene (as in [3, Lemma 2.4]), it follows that

$$L_p(\phi_E)\doteq\frac{\mathscr{L}^{\Sigma}_{\rm cyc}}{\Omega_p(\epsilon)}\cdot\frac{\mathscr{L}^{\Sigma^c}_{\rm cyc}}{\Omega_p(\epsilon^c)}\,.\tag{3.7}$$

Here, $\doteq$ signifies equality up to a factor from $\overline{\mathbb{Q}}^\times$ and the p-adic L-functions $\mathscr{L}^{\Sigma}_{\rm cyc}$ and $\mathscr{L}^{\Sigma^c}_{\rm cyc}$ are given as in Definition 3.14.

Theorem 3.8. *The Explicit Reciprocity Conjecture 3.2 implies the "only if" portion of Conjecture 3.1 whenever the p-adic height pairing of Definition 3.16 is non-zero, p is prime to $w_2(K_+):=H^0(K_+,\mathbb{Q}/\mathbb{Z}(2))$ and $D_{\mathfrak{P}}(A)^{\varphi=1}=0$.*

[17] Since we assumed that $K_+/\mathbb{Q}$ is unramified, it follows that any non-trivial character is ramified at all all primes of K_+ above p.

Proof. By Corollary 3.2, we may assume that the Coleman–Rubin–Stark element $\mathfrak{S}$ is non-trivial and given that, we contend to prove that $r_{\mathrm{an}}(\phi) \leq 1$ in our set up.

As above, let $\omega_{\mathcal{A}}$ denote a Néron differential on $\mathcal{A}$ and let

$$\omega_{\mathfrak{Q}}^{*} \in \mathrm{Fil}^0 D_{\mathfrak{P}}(A)^{*} \cong \mathrm{Fil}^0 D_{\mathfrak{P}}(A)$$

denote the element that corresponds to ω_A under the comparison isomorphism. Let $D_{[-1]} \subset D_{\mathfrak{P}}(A)$ denote the subspace of $D_{\mathfrak{P}}(A)$ on which φ acts with slope -1. Then the space $D_{[-1]}$ is one-dimensional and $D_{[-1]} \cap \mathrm{Fil}^0 D_{\mathfrak{P}}(A) = \{0\}$. Since $\mathrm{Fil}^0 D_{\mathfrak{P}}(A)$ is the exact orthogonal compliment of $\mathrm{Fil}^0 D_{\mathfrak{P}}(A)^{*}$ under the pairing $[\,,\,]$ above, there exists a unique element $\omega_{\mathfrak{Q}} \in D_{[-1]}$ with $[\omega_{\mathfrak{Q}}, \omega_{\mathfrak{Q}}^{*}] = 1$ (which in fact spans $D_{[-1]}$ as an L-vector space). We will denote the image of $\omega_{\mathfrak{Q}}$ under the isomorphism $D_{[-1]} \xrightarrow{\sim} D_{\mathfrak{P}}(A)/\mathrm{Fil}^0 D_{\mathfrak{P}}(A)$ by $\overline{\omega}_{\mathfrak{Q}}$. As explained in [11, Section 2.1], we have

$$\begin{aligned} 1\left(\mathfrak{L}^{\mathfrak{Q},1}(z_\infty)\right) &= \left[\exp^{*}(z_0), (1-p^{-1}\varphi^{-1})(1-\varphi)^{-1}\omega_{\mathfrak{Q}}\right] \\ &= (1-1/\alpha)(1-\alpha/p)^{-1}\left[\exp^{*}(z_0), \omega_{\mathfrak{Q}}\right] \end{aligned} \tag{3.8}$$

for every $z_\infty = \{z_n\}_{n\geq 0} \in H^1_{/f}(K_{+,\mathfrak{Q}}, \mathbb{T}_{\mathfrak{P}}(A))$, where α is the p-unit eigenvalue for φ acting on $D_{\mathfrak{P}}(A)$. Note that $\alpha \neq 1$ by assumption.

Case 1. $\mathrm{res}_{/f}(\mathfrak{C}) \neq 0$. Under our running hypotheses, it follows from Corollary 3.4 and (3.8) that $1(\mathscr{L}^{\Sigma}_{\mathrm{cyc}}) \neq 0$. The interpolation property for this p-adic L-function now shows that $r_{\mathrm{an}}(\phi) = 0$.

Case 2. $\mathrm{res}_{/f}(\mathfrak{C}) = 0$. This means that $\mathfrak{S} \in H^1_{\mathrm{f}}(K_+, T_{\mathfrak{P}}(A))$ and in turn, also that

$$\begin{aligned} \mathfrak{S}_\infty &\in \ker\left(H^1_{/f}(K_{+,\mathfrak{Q}}, \mathbb{T}_{\mathfrak{P}}(A)) \to H^1_{/f}(K_{+,\mathfrak{Q}}, T_{\mathfrak{P}}(A))\right) \\ &= (\gamma - 1)H^1_{/f}(K_{+,\mathfrak{Q}}, \mathbb{T}_{\mathfrak{P}}(A)) \end{aligned}$$

It follows by Theorem 3.5 that $1(\mathscr{L}^{\Sigma}_{\mathrm{cyc}}) = 0$ and the interpolation formula shows (as non of the Euler-like factors in its statement vanish) that $L(1/2, \psi_\varepsilon) = 0$. In this case, the discussion in [43, §11.3.14] applies and allows us to construct the derivative $\mathfrak{d}\mathfrak{S}_\infty \in H^1_{/f}(K_{+,\mathfrak{Q}}, T_{\mathfrak{P}}(A))$ of the class $\mathfrak{S}_\infty$ (our element $\mathfrak{d}\mathfrak{S}_\infty$ corresponds to the image of the element $(Dx_{\mathrm{Iw}})_{\mathfrak{Q}}$ in loc. cit. under the cyclotomic character).

It then follows from Nekovář's Rubin-style formula [43, Proposition 11.3.15] (also, his p-adic height compares via §11.3 to those introduced by Perrin-Riou) and shows that

$$\langle \mathfrak{S}, \mathfrak{S} \rangle = -\left[\exp^{*}\left(\mathfrak{d}\mathfrak{S}_\infty\right), \log_{A,\mathfrak{Q}}(\mathfrak{S})\right]_{D_{\mathfrak{P}}(A)} \tag{3.9}$$

For $c \in H^1_{\mathrm{f}}(K_{+,\mathfrak{Q}}, T_{\mathfrak{P}}(A))$, let us define $\log_\omega(c) \in L$ so that

$$\log_{A,\mathfrak{Q}}(c) = \log_\omega(c) \cdot \varpi_{\mathfrak{Q}}\,. \tag{3.10}$$

Combining (3.8), (3.9), Corollary 3.4 and the defining property of the Coleman–Rubin–Stark elements in Theorem 3.5, we conclude that

$$-\log_\omega(\mathfrak{S}) \frac{\left(\mathscr{L}^{\Sigma}_{\mathrm{cyc}}\right)'(1)}{\Omega_p(\epsilon)} = (1-1/\alpha)(1-\alpha/p)^{-1}\langle \mathfrak{S}, \mathfrak{S}\rangle\,. \tag{3.11}$$

Here, $\left(\mathscr{L}^{\Sigma}_{\mathrm{cyc}}\right)'(1) := \lim\limits_{s\to 1} \chi^{s-1}_{\mathrm{cyc}}(\mathscr{L}^{\Sigma}_{\mathrm{cyc}})/(s-1)$ is the derivative of the cyclotomic restriction of the Katz' p-adic L-function, along the cyclotomic character. By our assumption that the p-adic height pairing is non-vanishing, it follows that $\mathscr{L}^{\Sigma}_{\mathrm{cyc}} \notin J^2$, where we recall that $J \subset \mathscr{O}[[\Gamma]]$ is the augmentation ideal.

The Kolyvagin system method applied with ${}^{\mathrm{CRS}} \in \mathbf{KS}(\mathcal{F}_+, \mathbb{T}, \mathcal{P})$ implies (using [42], Theorem 5.2.2) in our situation that $H^1_{\mathcal{F}_+}(K_+, T_{\mathfrak{P}}(A))$ has rank one, whereas the dual Selmer group $H^1_{\mathcal{F}^*_+}(K_+, T_{\mathfrak{P}}(A)^*)$ has finite cardinality. These facts combined with Poitou-Tate global duality (and our assumption that $\mathrm{res}_{/f}(\mathfrak{C}) = 0$) show that

$$H^1_{\mathrm{f}}(K_+, T_{\mathfrak{P}}(A)) = H^1_{\mathcal{F}_{\mathrm{Gr}}}(K_+, T_{\mathfrak{P}}(A))$$

has rank one. By Nekovář's result in [43, Theorem 12.2.8 (3)] on the parity conjecture, it follows that the sign of the functional equation for $L(s,\phi)$ equals -1. Let us choose a totally imaginary extension E/K_+ as in Definition 3.17. Using (3.7), the fact that $\mathscr{L}^{\Sigma}_{\mathrm{cyc}} \notin J^2$, the choice of the CM field E and Disegni's p-adic Gross–Zagier formula in [23], we conclude that there exists a non-trivial Heegner point $P \in A(E)$. The main results of [65] implies that $r_{\mathrm{an}}(\phi_E) = 1$, which in turn shows that $r_{\mathrm{an}}(\phi) = 1$ as well. Q.E.D.

3.5. Logarithms of Heegner points and Perrin-Riou-Stark elements

We continue with our discussion on CM abelian varieties and our aim in this subsection is to compare the Bloch-Kato logarithm of the Coleman–Rubin–Stark element $\mathfrak{C}$ to the square of the logarithm of a global point on our abelian variety up to a non-zero algebraic factor. This is a generalized form of Perrin-Riou's predictions for elliptic curves (and Beilinson–Kato elements) that we revisited in Section 2.3 below in the context of elliptic curves defined over $\mathbb{Q}$.

Throughout this section we assume that $r_{\mathrm{an}}(\phi) = 1$, where ϕ is the Hilbert modular form of parallel weight 2 associated to ψ_ϵ. We also keep working with our assumptions and conventions we have set at the start of Sections 3.4.1, 3.4.4, 3.4.6 and throughout Section 3.4.2. We recall the periods $\Omega^+_\phi, \Omega^+_{\phi_\eta}, \Omega_\phi$ attached to ϕ, as well as the CM period $\Omega_\infty(\epsilon)$ we have introduced in Sections 3.4.5 and 3.4.6 above.

Let E/K_+ be as in Definition 3.17. The work of Yuan-Zhang-Zhang [65] applies in this situation and equips us with a *Heegner point* $P_\phi \in A(K_+) \subset A(E)$.

$$\begin{aligned} C(\phi, E, \epsilon) &:= \mathscr{E}_{\mathfrak{f}^+}(\psi_\epsilon) \prod_{\mathfrak{q}|p} (1 - 1/a_{\mathfrak{q}}(\phi))^2 \; D_F^{-1} D_E^{1/2} \frac{\Omega_\phi}{\Omega_\infty(\epsilon)} \cdot L(\phi_\eta, 1)^{-1} \\ &= \mathscr{E}_{\mathfrak{f}^+}(\psi_\epsilon) \prod_{\mathfrak{q}|p} (1 - 1/a_{\mathfrak{q}}(\phi))^2 \; D_F^{-1} D_E^{1/2} \frac{\Omega_\phi}{\Omega^+_\phi \Omega^+_{\phi_\eta}} \, \frac{\Omega^+_\phi}{\Omega_\infty(\epsilon)} \, \Omega^+_{\phi_\eta} L(\phi_\eta, 1)^{-1} \in \overline{\mathbb{Q}}^\times. \end{aligned}$$

where $\mathscr{E}_{\mathfrak{f}^+}(\psi_\epsilon)$ is as above and is a product of certain modified root numbers at the primes above p (and they are given as in [3, Section 2.3]).

Theorem 3.9. $\log_\omega(\mathfrak{C}) = (1-1/\alpha)^{-1}(1-\alpha/p)\cdot C(\phi, E, \epsilon)\cdot \log_\omega(P_\phi)^2$.

Proof. We start observing that

$$\begin{aligned} \frac{\langle P_\phi, P_\phi \rangle}{\log_\omega(P_\phi)^2} = \frac{\langle \mathfrak{C}, \mathfrak{C} \rangle}{\log_\omega(\mathfrak{C})^2} &= -\frac{\left(\mathscr{L}^\Sigma_{\mathrm{cyc}}\right)'(1)}{\Omega_p(\epsilon) \cdot \log_\omega(\mathfrak{C})} \cdot (1 - 1/\alpha)^{-1}(1 - \alpha/p) \\ &= \mathscr{E}_{\mathfrak{f}^+}(\psi_\epsilon)\,(1 - 1/\alpha)^{-1}(1 - \alpha/p) \cdot \frac{\Omega^+_\phi}{\Omega_\infty(\epsilon)} \cdot \frac{L'_p(\phi, 1)}{\log_\omega(\mathfrak{C})} \end{aligned}$$

where the first equality on the first line is because $H^1_{\mathrm{f}}(K_+, T_{\mathfrak{P}}(A))$ has rank one by the proof of Theorem 3.7 and $\langle \cdot, \cdot \rangle / \log_\omega(\cdot)^2$ is a non-trivial quadratic form on this space (thanks to our assumption that the p-adic height pairing is non-zero); the second equality on the first line is (3.11) and finally, the equality on the second line follows from the Claim below. Indeed, the factor $\mathscr{E}_{\mathfrak{f}^+}(\chi_\epsilon \psi_\epsilon)\,\chi(D_{K_+})$ that appear in the statement of Claim varies analytically in χ and tend to $\mathscr{E}_{\mathfrak{f}^+}(\psi_\epsilon)$ as χ tends to the trivial character 1.

We therefore conclude that,

$$\frac{L'_p(\phi, 1)}{\langle P_\phi, P_\phi \rangle} = \mathscr{E}_{\mathfrak{f}^+}(\psi_\epsilon)^{-1}\,(1 - 1/\alpha)(1 - \alpha/p)^{-1}\,\frac{\Omega_\infty(\epsilon)}{\Omega^+_\phi}\,\frac{\log_\omega(\mathfrak{C})}{\log_\omega(P_\phi)^2}\,. \tag{3.12}$$

On the other hand,

$$
\begin{aligned}
\frac{L_p'(\phi,1)}{\langle P_\phi, P_\phi\rangle} &= \frac{L_p'(\phi_E,1)}{\langle P_\phi, P_\phi\rangle}\cdot \frac{D_E^{1/2}\Omega_\phi}{\Omega_\phi^+}\cdot \left(\Omega_{\phi_\eta}^+\cdot L_p(\phi_\eta,1)\right)^{-1}\\
&= \frac{L_p'(\phi_E,1)}{\langle P_\phi, P_\phi\rangle}\cdot \frac{D_E^{1/2}\Omega_\phi}{\Omega_\phi^+}\cdot \left(\prod_{\mathfrak{q}|p}\left(1-\frac{1}{\eta(\mathfrak{q})a_{\mathfrak{q}}(\phi)}\right)^2 L(\phi_\eta,1)\right)^{-1}\\
&= \prod_{\mathfrak{q}|p}\left(1-1/a_{\mathfrak{q}}(\phi)\right)^2 D_F^{-1}D_E^{1/2}\,\frac{\Omega_\phi}{\Omega_\phi^+}\,L(\phi_\eta,1)^{-1} \qquad (3.13)
\end{aligned}
$$

where the first equality follows from the factorization (3.6) of the base-change p-adic L-function, the second from the interpolation formula for the twisted p-adic L-function and the last from the p-adic Gross–Zagier formula of Disegni [23, Theorem B]. Using (3.12) together with (3.13), we conclude that

$$
\begin{aligned}
\frac{\log_\omega(\mathfrak{C})}{\log_\omega(P_\phi)^2} = (1-1/\alpha)^{-1}(1-\alpha/p)\,\mathscr{E}_{\mathfrak{f}^+}(\psi_\epsilon)\times&\\
\prod_{\mathfrak{q}|p}\left(1-1/a_{\mathfrak{q}}(\phi)\right)^2\; D_F^{-1}D_E^{1/2}\,\frac{\Omega_\phi}{\Omega_\infty(\epsilon)}\cdot L(\phi_\eta,1)^{-1}&
\end{aligned}
$$

and the proof follows. Q.E.D.

Claim. *For all sufficiently ramified characters χ of Γ the following identity holds:*

$$
\frac{\mathscr{L}^{\Sigma}_{\mathrm{cyc}}(\chi)}{\Omega_p(\epsilon)} = \mathscr{E}_{\mathfrak{f}^+}(\chi_\epsilon\psi_\epsilon)\,\chi(D_{K_+})\,\Omega_\phi^+\,L_p(\phi,\chi^{-1})\,.
$$

Proof. This will follow once we match the interpolation factors in Corollary 3.3 for $\mathscr{L}^{\Sigma}_{\mathrm{cyc}}(\chi)$ and (3.5) for $L_p(\phi,\chi^{-1})$. More precisely, we would like to verify that $\mathscr{E}_p(\chi_\epsilon\psi_\epsilon)$ equals $\tau(\chi)\,\mathbf{N}(\mathfrak{f}_\chi)^{1/2}\,\alpha_{\mathfrak{f}_\chi}^{-1}$ for all sufficiently ramified characters χ. The proof of this claim is essentially contained in [3, Section 2.3] (although we also rely on Hsieh's exposition in [29, Section 4.7]), we provide an outline here for the sake of completeness. For all such χ, the local L-factors are trivial and by Tate's local functional equation, it follows that

$$
\mathscr{E}_{\mathfrak{f}^+}(\chi_\epsilon\psi_\epsilon) = \prod_{\wp\in\Sigma_p^c}\tau(\psi_{\epsilon,\wp}\chi_{\epsilon,\wp}\,,\Psi_\wp)\,,
$$

where $\tau(\psi_{\epsilon,\wp}\chi_{\epsilon,\wp},\Psi_\wp)$ are the Gauss sums which are (un)normalized as in [23] and $\Psi_\wp$ are a suitably determined local additive characters. Let $\alpha_{\mathfrak{P}}$ be the local character of $K^\times_{+,\mathfrak{P}}$ such that the local constituent of ϕ at $\mathfrak{P}=\wp\wp^c$ is the irreducible principal series $\pi(\alpha_{\mathfrak{P}},\beta_{\mathfrak{P}})$ for some other local character $\beta_{\mathfrak{P}}$. Then $\tau(\psi_{\epsilon,\wp}\chi_{\epsilon,\wp},\Psi_\wp)=\tau(\alpha_{\mathfrak{P}}\chi_{\epsilon,\mathfrak{P}},\Psi_{\mathfrak{P}})$, where we write $\chi_{\epsilon,\mathfrak{P}}$ for the local character $\chi_{\epsilon,\wp}$ of $K^\times_{+,\mathfrak{P}}=K^\times_\wp$ and we similarly define the additive character $\Psi_{\mathfrak{P}}$. We therefore infer that

$$\mathscr{E}_{\mathfrak{f}^+}(\chi_\epsilon\psi_\epsilon)=\prod_{\mathfrak{P}|p}\tau(\alpha_{\mathfrak{P}}\chi_{\epsilon,\mathfrak{P}},\Psi_{\mathfrak{P}})\,.$$

The $\tau(\chi)$ of [22] is precisely the normalization of $\prod_{\mathfrak{P}|p}\tau(\chi_{\mathfrak{P}},\Psi_{\mathfrak{P}})$, so that

$$\tau(\chi)\,\mathbf{N}(\mathfrak{f}_\chi)^{1/2}=\prod_{\mathfrak{P}|p}\tau(\chi_{\epsilon,\mathfrak{P}},\Psi_{\mathfrak{P}})\,.$$

Finally, as explained in the proof of Lemma A.1.1 of [23] we have

$$\tau(\chi_{\epsilon,\mathfrak{P}},\Psi_{\mathfrak{P}})=a_{\mathfrak{q}}(\phi)^{v_{\mathfrak{q}}(\mathfrak{f}_\chi)}\,\tau(\alpha_{\mathfrak{P}}\chi_{\epsilon,\mathfrak{P}},\Psi_{\mathfrak{P}})$$

and hence,

$$\begin{aligned}\alpha_{\mathfrak{f}_\chi}^{-1}\,\tau(\chi)\,\mathbf{N}(\mathfrak{f}_\chi)^{1/2}&=\alpha_{\mathfrak{f}_\chi}^{-1}\prod_{\mathfrak{P}|p}\tau(\chi_{\epsilon,\mathfrak{P}},\Psi_{\mathfrak{P}})\\&=\prod_{\mathfrak{P}|p}\tau(\alpha_{\mathfrak{P}}\chi_{\epsilon,\mathfrak{P}},\Psi_{\mathfrak{P}})=\mathscr{E}_{\mathfrak{f}^+}(\chi_\epsilon\psi_\epsilon)\,.\end{aligned}$$

Q.E.D.

Remark 3.7. *Assuming that the sign of the functional equation for the Hecke character $\psi^*_\epsilon:=\psi_\epsilon\circ c$ equals $+1$, Burungale and Disegni proved in* [3] *that the p-adic height pairing*

$$\langle\,,\,\rangle_\chi : H^1_{\mathrm{f}}(K,T_\wp(A)\otimes\chi)\otimes H^1_{\mathrm{f}}(K,T_{\wp^c}(A)\otimes\chi^{-1})\longrightarrow M$$

for almost all anticyclotomic Hecke characters χ of K of finite order (where M is a finite extension of $\mathbb{Q}_p$ in which χ takes its values).

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