The second syzygy of the trivial G-module, and an equivariant main conjecture

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In honor of K. Iwasawa

Abstract.

For a finite abelian p-extension K/k of a totally real field and the cyclotomic $\mathbb{Z}_p$-extension K_∞/K, we prove a strong version of an equivariant Iwasawa main conjecture by determining completely the Fitting ideal over $\mathbb{Z}_p[[\mathrm{Gal}(K_\infty/k)]]$ of the classical Iwasawa module $X_{K_\infty,S}$, which is the Galois group of the maximal pro-p abelian S-ramified extension of K_∞, where S contains all ramified primes in K_∞/k. To do this, we prove a conjecture which was proposed in a previous paper by the first and second author, concerning the minors of a relation matrix for the second syzygy module of the trivial module $\mathbb{Z}$ over a suitable group ring.

§1. Introduction

In a recent paper [2] the first two authors described the Fitting ideals of certain Iwasawa modules in terms of ideals generated by minors of a relation matrix for a second syzygy module of $\mathbb{Z}$ over abelian group rings. Moreover, in §1 (Remark to Proposition 1.5) of that paper an explicit description of those ideals was stated, but one of the two inclusions remained conjectural. Our aim in this paper is to prove the conjectural inclusion as well. This means that these ideals of minors are now completely determined.

In §1.1 of this introduction, we explain the conjecture on the ideals of minors. Then in §1.2 we give a consequence of the conjecture (now a theorem) in Iwasawa theory. Indeed, we are now able to state and prove an equivariant Iwasawa main conjecture.

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1.1. A second syzygy of $\mathbb{Z}$

Let G be a finite abelian group such that

$$G \simeq \mathbb{Z}/n_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/n_s\mathbb{Z},$$

where n_i is a divisor of n_{i+1} for each $i = 1,...,s-1$. Suppose that $\sigma_i \in G$ is a generator of the i-th component of the above decomposition for each i. We put $R = \mathbb{Z}[G]$, which is the group ring of G over $\mathbb{Z}$, and consider the R-module $\mathbb{Z}$ with trivial action of G. Using the method in [2] §1.2, we obtain a free resolution

$$R^{s(s+1)(s+2)/6} \xrightarrow{\Phi_3} R^{s(s+1)/2} \xrightarrow{\Phi_2} R^s \xrightarrow{\Phi_1} R \longrightarrow \mathbb{Z} \longrightarrow 0$$

of $\mathbb{Z}$. The full description of this exact sequence will be given in §2, and we just mention here that Φ_1 is defined by $\Phi_1(x_i) = \sigma_i - 1$, where $(x_i)_{1\leq i\leq s}$ is the standard basis of R^s. We put $\Omega^2 = \operatorname{Ker}\Phi_1$, which is a second syzygy of $\mathbb{Z}$. Thus we have an exact sequence

$$R^{s(s+1)(s+2)/6} \xrightarrow{\Phi_3} R^{s(s+1)/2} \longrightarrow \Omega^2 \longrightarrow 0\ .$$

We denote by $\tilde{M}_s$ the matrix which corresponds to the R-homomorphism Φ_3, and which can be written down explicitly (see §2.1).

Put $c = s(s+1)/2$.

For an integer v such that $0 \leq v \leq c$, we define $\mathfrak{m}_v = \mathrm{Min}_v(\tilde{M}_s)$ to be the ideal of R generated by all v-minors of the matrix $\tilde{M}_s$. We note that $\mathfrak{m}_v$ is equal to the higher Fitting ideal $\mathrm{Fitt}_{c-v,R}(\Omega^2)$ (concerning the definition of Fitting ideals, see [5]). The matrix $\tilde{M}_s$ is rather sparse, but the calculation of the relevant minors is still difficult. The entries are elements of the form $\pm\tau_i$ and $\pm\nu_i$ where $\tau_i = \sigma_i - 1 \in R$ and $\nu_i = 1 + \sigma_i + \ldots + \sigma_i^{n_i-1} \in R$ (recall that n_i is the order of σ_i in G). We note here the important relation $\tau_i\nu_i = 0$.

We can easily see $\mathfrak{m}_v = 0$ for any v such that $s(s-1)/2+1 < v \leq c$, using $\mathfrak{m}_v = \mathrm{Fitt}_{c-v,R}(\Omega^2)$ (see Proposition 1.5 (a) in [2]). Now we state our main theorem in which we determine all $\mathfrak{m}_v$.

We define H to be the ideal of R generated by $\tau_1, \ldots, \tau_s$; differently put, H is the augmentation ideal of R. We will define the notion "admissible monomial" in §2.2, and denote by $\mathfrak{n}_d$ the ideal generated by all admissible monomials of degree d, with $0 < d \leq s(s-1)/2$. We also define $\mathfrak{n}_0 = R$ and $\mathfrak{n}_{\frac{s(s-1)}{2}+1} = 0$. In this paper we prove the following.

Theorem 1.1. *For any integer v such that $0 \leq v \leq \frac{s(s-1)}{2} + 1$, we have*

$$\mathfrak{m}_v = \mathrm{Fitt}_{c-v,R}(\Omega^2) = \sum_{d=0}^{v} H^{v-d}\mathfrak{n}_d\ .$$

In particular, we have

$$\mathfrak{m}_{\frac{s(s-1)}{2}+1} = H \sum_{d=0}^{s(s-1)/2} H^{\frac{s(s-1)}{2}-d}\mathfrak{n}_d = H\mathfrak{m}_{\frac{s(s-1)}{2}} \ .$$

In [2] Proposition 1.5 (b) we proved that the left hand side is contained in the right hand side of the above equality, and stated this equality as a conjecture. For $s \leq 4$ we had verified the conjecture by hand, which was already a sizable calculation for $s = 4$. The third author checked it for $s = 5$ in [7].

In this paper we prove the converse inclusion. To do this, we exhibit many square submatrices of $\tilde{M}_s$ which are lower triangular; so their determinants can be evaluated at once (product of the diagonal entries), and in this way we are able to produce all monomials in the elements τ_i and ν_i that are required to establish the desired converse inclusion. The arguments are entirely elementary and combinatorial, but some of the details are a bit tricky. Even though the entire motivation for this work comes from [2], it is not necessary to be familiar with that paper for reading the proof of Theorem 1.1.

1.2. An equivariant main conjecture for $X_{K_\infty,S}$

Now we state one of the consequences in Iwasawa theory, which is a main motivation of the above theorem. Let K/k be a finite abelian extension of totally real number fields, p an odd prime, and K_∞/K the cyclotomic $\mathbb{Z}_p$-extension. We consider the maximal pro-p abelian extension $M_{\infty,S}/K_\infty$ which is unramified outside S, where S is a finite set of primes of k containing all ramified primes in K_∞/k, and hence in particular all p-adic primes. We put $X_{K_\infty,S} = \mathrm{Gal}(M_{\infty,S}/K_\infty)$. It is well-known that this is a finitely generated torsion $\Lambda_{K_\infty} = \mathbb{Z}_p[[\mathrm{Gal}(K_\infty/k)]]$-module.

The celebrated main conjecture in Iwasawa theory, which is a theorem of Wiles [8], states a relationship between the characteristic ideal of a character component of $X_{K_\infty,S}$ and the p-adic L-function of Deligne-Ribet. Let $\Theta_{K_\infty/k,S}$ be the pseudo-measure of $\mathrm{Gal}(K_\infty/k)$ in the sense of Serre [6], corresponding to the p-adic L-function of Deligne-Ribet, which interpolates the values $L_S(1-n,\chi)$ of S-truncated L-functions for characters χ of $\mathrm{Gal}(K/k)$. Suppose that I_{K_∞} is the augmentation ideal $\mathrm{Ker}(\Lambda_{K_\infty} = \mathbb{Z}_p[[\mathrm{Gal}(K_\infty/k)]] \longrightarrow \mathbb{Z}_p)$; then we know $I_{K_\infty}\Theta_{K_\infty/k,S} \subset \Lambda_{K_\infty}$ (see [6]). Especially, if γ is a generator of $\mathrm{Gal}(K_\infty/K) \simeq \mathbb{Z}_p$ and $T = \gamma - 1 \in \Lambda_{K_\infty}$, then we have $T\Theta_{K_\infty/k,S} \in \Lambda_{K_\infty}$. For simplicity, assume $K \cap k_\infty = k$ where k_∞ is the cyclotomic $\mathbb{Z}_p$-extension, put $G = \mathrm{Gal}(K/k)$, $\Gamma = \mathrm{Gal}(K_\infty/K)$, and $\mathcal{G} = \mathrm{Gal}(K_\infty/k)$, so $\mathcal{G} = G \times \Gamma$.

For any character χ of G and any $\mathbb{Z}_p[G]$-module M, we define the χ-component M_χ by $M_\chi = M \otimes_{\mathbb{Z}_p[G]} \mathcal{O}_\chi$, where $\mathcal{O}_\chi = \mathbb{Z}_p[\mathrm{Image}\ \chi]$ on which G acts via χ. If M is a $\mathbb{Z}_p[[\mathcal{G}]]$-module, M_χ is an $\mathcal{O}_\chi[[\Gamma]]$-module. We note that M_χ is a quotient of M. We denote the image of an element $x \in M$ in M_χ by x_χ. The main conjecture asserts that for each character χ of G, the characteristic ideal of an $\mathcal{O}_\chi[[\Gamma]]$-module $(X_{K_\infty,S})_\chi$ is generated by the χ-component $(\Theta_{K_\infty/k,S})_\chi$ of $\Theta_{K_\infty/k,S}$ if χ is not the trivial character, and by $(T\Theta_{K_\infty/k,S})_\chi$ if χ is the trivial character.

Our goal in this paper is to establish a more refined relationship than the main conjecture between the two sides, that is $X_{K_\infty,S}$ and $\Theta_{K_\infty/k,S}$, *without taking the character components*.

There exist several equivariant main conjectures, but they all use modified Iwasawa modules (e.g. cohomology groups; for example, see [1] and [4]), and there has been no equivariant theory for $X_{K_\infty,S}$ itself before our work. The difficulty in studying $X_{K_\infty,S}$ comes from the fact that $X_{K_\infty,S}$ does not have projective dimension ≤ 1 over Λ_{K_∞} when p divides the order of G.

In this paper we study the classical object $X_{K_\infty,S}$ itself, and prove a kind of equivariant main conjecture for it. For any ring R and any R-module M, we denote the initial Fitting ideal $\mathrm{Fitt}_{0,R}(M)$ simply by $\mathrm{Fitt}_R(M)$. We will determine the Fitting ideal $\mathrm{Fitt}_{\Lambda_{K_\infty}}(X_{K_\infty,S})$ completely, by which we get more information than the characteristic ideals of the character components. For example, we obtain information on the size of the $\mathcal{O}_\chi$-torsion submodule of $(X_{K_\infty,S})_\chi$ from our knowledge of $\mathrm{Fitt}_{\Lambda_{K_\infty}}(X_{K_\infty,S})$ (see also Remark 1.4 (2) and the argument just after Remark 1.4), but first and foremost, we get *the exact relationship between the Iwasawa module $X_{K_\infty,S}$ and the p-adic L-function $\Theta_{K_\infty/k,S}$*.

In previous work [3] by the first two authors, we proved in Theorem 4.1 in [3] that

$$\mathrm{Fitt}_{\Lambda_{K_\infty}}(X_{K_\infty,S}) = \mathfrak{A}\Theta_{K_\infty/k,S}$$

with a certain ideal $\mathfrak{A}$ of Λ_{K_∞}. What we do in this paper is performing further computations on the ideal $\mathfrak{A}$ in order to describe it completely. We also note that the description given here is more general and explicit than the appendix in [2].

The essential case is that K/k is a nontrivial p-extension, so we assume it (for the general case, see [2]). We still assume that $K \cap k_\infty = k$ and put $G = \mathrm{Gal}(K/k)$. We change the notation slightly from the previous subsection, and write

$$G \simeq \mathbb{Z}/p^{n_1}\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/p^{n_s}\mathbb{Z}$$

with $0 < n_1 \leq \ldots \leq n_s$ for some $s \in \mathbb{Z}_{>0}$. We need the auxiliary quadratic function

$$\varphi(\alpha) = \alpha(2s - \alpha - 1)/2 \qquad \text{(for any } \alpha \in \mathbb{Z} \text{ such that } 0 \leq \alpha \leq s-1).$$

This is an increasing function in the above range, and $\varphi(0) = 0$, $\varphi(1) = s-1$, $\varphi(2) = 2s-3$,..,and $\varphi(s-1) = s(s-1)/2$. We define a sequence $(m_v)_{0 \leq v \leq s(s-1)/2}$ of integers as follows. We define $m_0 = 0$, and if v satisfies $\varphi(\alpha) < v \leq \varphi(\alpha+1)$ for some integer $\alpha \in \mathbb{Z}$ with $0 \leq \alpha \leq s-2$, then m_v is defined by

$$m_v = (s-1)n_1 + \ldots + (s-\alpha)n_\alpha + (v - \varphi(\alpha))n_{\alpha+1}.$$

In particular, $m_v = vn_1$ for $0 \leq v \leq s-1$, and

$$m_{\frac{s(s-1)}{2}} = (s-1)n_1 + (s-2)n_2 + \ldots + n_{s-1} \ .$$

Recall that I_{K_∞} is the augmentation ideal of $\Lambda_{K_\infty} = \mathbb{Z}_p[[\mathcal{G}]]$, with $\mathcal{G} = \mathrm{Gal}(K_\infty/k)$. We define an ideal $\mathfrak{a}_G$ of Λ_{K_∞} by

$$\mathfrak{a}_G = \sum_{v=0}^{\frac{s(s-1)}{2}} p^{m_v} I_{K_\infty}^{\frac{s(s-1)}{2}-v},$$

which is determined only by the structure of G as an abelian group. Our equivariant main conjecture which we prove in this paper is

Theorem 1.2. *Assume that the μ-invariant of k_∞/k vanishes, and that S contains all ramified primes in K_∞/k (as we already mentioned). Then we have*

$$\mathrm{Fitt}_{\Lambda_{K_\infty}}(X_{K_\infty,S}) = \mathfrak{a}_G I_{K_\infty} \Theta_{K_\infty/k,S} \ .$$

In order to understand the ideal $\mathfrak{a}_G$, let us consider simple cases. Suppose that G is homogeneous, that is, $n_1 = \ldots = n_s = n$. Put

$$J_{K_\infty} = p^n \Lambda_{K_\infty} + I_{K_\infty}.$$

If $n = 1$, J_{K_∞} coincides with the maximal ideal $\mathfrak{m}_{K_\infty}$ of Λ_{K_∞}. It is easy to check that $m_v = nv$ and

$$\begin{aligned} \mathfrak{a}_G &= \sum_{v=0}^{s(s-1)/2} p^{nv} I_{K_\infty}^{\frac{s(s-1)}{2}-v} = (p^n, I_{K_\infty})^{s(s-1)/2} \\ &= J_{K_\infty}^{s(s-1)/2}. \end{aligned}$$

Therefore, we get

Corollary 1.3. *Assume further that* $n_1 = \ldots = n_s = n$. *Then we have*

$$\mathrm{Fitt}_{\Lambda_{K_\infty}}(X_{K_\infty,S}) = J_{K_\infty}^{s(s-1)/2} I_{K_\infty} \Theta_{K_\infty/k,S} \ .$$

Also, if $n = 1$, *we have*

$$\mathrm{Fitt}_{\Lambda_{K_\infty}}(X_{K_\infty,S}) = \mathfrak{m}_{K_\infty}^{s(s-1)/2} I_{K_\infty} \Theta_{K_\infty/k,S},$$

where $\mathfrak{m}_{K_\infty}$ *is the maximal ideal of* Λ_{K_∞}.

Remark 1.4. (1) At some stage, there was a guess that

$$\mathrm{Fitt}_{\Lambda_{K_\infty}}(X_{K_\infty,S}) = I_{K_\infty} \Theta_{K_\infty/k,S}.$$

Our theorem shows that this is only true for $s = 1$, and the bigger s gets, the more badly it fails, because $\mathfrak{a}_G$ becomes smaller and smaller for $s \to \infty$.

(2) Since the ideal $\mathfrak{a}_G$ is of finite index in Λ_{K_∞}, its image in $\Lambda_{K_\infty,\chi}$ is also of finite index in $\Lambda_{K_\infty,\chi}$. Thus Theorem 1.2 implies the usual main conjecture, and in fact refines it.

Theorem 1.2 also gives information on the size of the $\mathcal{O}_\chi$-torsion submodule of $(X_{K_\infty,S})_\chi$. Suppose for simplicity that we are in the situation of Corollary 1.3, and also assume that χ is nontrivial. Then we know that the characteristic ideal of $(X_{K_\infty,S})_\chi$ is generated by $(\Theta_{K_\infty/k,S})_\chi$. This shows that

$$\chi(I_{K_\infty})\chi(J_{K_\infty})^{s(s-1)/2} = \mathrm{Fitt}_{\mathcal{O}_\chi[[\Gamma]]}(((X_{K_\infty,S})_\chi)_{\mathrm{tors}}).$$

where $((X_{K_\infty,S})_\chi)_{\mathrm{tors}}$ is the $\mathcal{O}_\chi$-torsion part. Since the left hand side becomes fairly small if s becomes large (see also Remark 1.4 (4) below), the above formula also shows that $((X_{K_\infty,S})_\chi)_{\mathrm{tors}}$ is fairly big if s is large. In this way our Theorem 1.2 also sheds light on the torsion part of Iwasawa modules.

(3) For an abelian CM-extension L/k which is unramified outside p and the cyclotomic $\mathbb{Z}_p$-extension L_∞/L, we can determine the Fitting ideal of the Pontrjagin dual of the minus part of the p-part of the class group of L_∞, using Theorem 1.2. For the details, see [2].

(4) It appears that the (finite) quotient module $Q(G) = I_{K_\infty}/\mathfrak{a}_G I_{K_\infty}$ can quickly become very large. Arguments from commutative algebra and calculations show that for example in the homogeneous case with $n = 1$, $p = 3$ and $s = 5$ ($s = 6$), the length of $Q(G)$ is at least 1230 (13710 respectively). It also appears that in the homogeneous case with $n = 1$ and any p, the length of $Q(G)$ grows at least as fast as a positive constant times p^s for $s \to \infty$. We will treat this question in more detail in future work.

(5) The method to get Theorem 1.2 could probably be applied to certain cohomology groups of more general p-adic representations (or of motives) satisfying suitable properties. We will come back to this problem in the future.

(6) We can prove the case $p = 2$ of Theorem 1.2 by the same method if we can establish the usual main conjecture for each character of G, replacing $\Theta_{K_\infty/k,S}$ by $2^{-[k:\mathbb{Q}]}\Theta_{K_\infty/k,S}$ under the assumption that $\mu = 0$. And indeed, it seems that Wiles' argument in [8] implies the usual main conjecture even for the case $p = 2$ if $\mu = 0$. This means that Theorem 1.2 should hold even for the case $p = 2$, which we hope to be able to study in future work.

Let us explain in which respect our equivariant main conjecture produces information that goes beyond the usual main conjecture, which treats the characteristic ideal of the χ-component of $X_{K_\infty,S}$ for any character χ of $\mathrm{Gal}(K/k)$. If we denote by K_χ the *cyclic* extension of k corresponding to χ, the characteristic ideal of the χ-component is determined by $X_{K_\chi,\infty,S}$. In other words, the usual main conjecture contains only information on $X_{K_\infty,S}$ for cyclic K/k, without even determining it completely as a module. Our equivariant main conjecture gives more precise information.

We will explain that our equivariant main conjecture even contains information on objects attached to the finite extension K/k. Let M_S/K be the maximal abelian pro-p extension which is unramified outside S. We define $\mathcal{G}_S$ by

$$\mathcal{G}_S = \mathrm{Ker}(\mathrm{Gal}(M_S/K) \longrightarrow \mathrm{Gal}(K_\infty/K)),$$

which is a $\mathbb{Z}_p[\mathrm{Gal}(K/k)]$-module. Let $\mathrm{Res}_{K_\infty/K} : \Lambda_{K_\infty} \to \mathbb{Z}_p[\mathrm{Gal}(K/k)]$ be the natural map induced by the restriction. Then, we get from Theorem 1.1

Corollary 1.5. $\mathrm{Fitt}_{\mathbb{Z}_p[\mathrm{Gal}(K/k)]}(\mathcal{G}_S) = \mathrm{Res}_{K_\infty/K}(\mathfrak{a}_G I_{K_\infty}\Theta_{K_\infty/k,S})$.

In fact, taking the Galois coinvariants of $X_{K_\infty,S}$, we can apply the argument of Corollary 4.1 in [2] to get the above corollary. Note also that both sides of the above equation are in principle numerically computable.
Example. Take $k = \mathbb{Q}$, $p = 3$, and $\mathbb{Q}(\ell)$ the unique cubic extension of conductor ℓ for any prime $\ell \equiv 1 \pmod 3$.

In general, suppose that $G = \mathrm{Gal}(K/k) = (\mathbb{Z}/p\mathbb{Z})^{\oplus 2}$ (so $s = 2$) and write

$$\Theta_{K_\infty/k,S} = \alpha_{-1}\frac{N_G}{T} + \alpha_0 + \alpha_1 T + \alpha_2 T^2 + \dots$$

with $\alpha_{-1} \in \mathbb{Z}_p$, and $\alpha_i \in \mathbb{Z}_p[G]$ for all $i \geq 0$, where $N_G = \Sigma_{\sigma \in G}\sigma$. In this case, we have $\mathfrak{a}_G = \mathfrak{m}_{K_\infty}$, which implies

$$\mathrm{Res}_{K_\infty/K}(\mathfrak{m}_{K_\infty} I_{K_\infty} \Theta_{K_\infty/k,S}) = \alpha_{-1} p N_G \mathbb{Z}_p[G] + \mathfrak{m}_G I_G \alpha_0,$$

where $\mathfrak{m}_G$ and I_G are the maximal ideal and the augmentation ideal of $\mathbb{Z}_p[G]$, respectively.

We now take K to be the composite field of $\mathbb{Q}(7)$ and $\mathbb{Q}(13)$, so K is the $(\mathbb{Z}/3\mathbb{Z})^{\oplus 2}$-extension of conductor 91. We denote by σ and τ a generator of $\mathrm{Gal}(\mathbb{Q}(7)/\mathbb{Q})$ and $\mathrm{Gal}(\mathbb{Q}(13)/\mathbb{Q})$ respectively, and regard σ, τ as generators of G. (Warning: Only in the present example, we use σ, τ instead of σ_1, σ_2 for generators of G. Notation will change later, and the generators of an abelian p-group will be written $\sigma_1, \ldots, \sigma_s$, so σ becomes σ_1 and τ becomes σ_2, and we will set $\tau_i = \sigma_i - 1$.) In this concrete example, α_{-1} is a unit, and α_0 is computed as

$$\alpha_0 \equiv 48 + 42\sigma + 38\sigma^2 + 62\tau + 42\sigma\tau + 80\sigma^2\tau + 45\tau^2 + 44\sigma\tau^2 + 70\sigma^2\tau^2 \pmod{3^4}$$

up to a unit factor. Put $x = \sigma - 1$ and $y = \tau - 1$ (again, x, y will be τ_1, τ_2, later). As a result, we can compute the right hand side of Corollary 1.5 as

$$\mathrm{Res}_{K_\infty/K}(\mathfrak{m}_{K_\infty} I_{K_\infty} \Theta_{K_\infty/k,S}) = (27, 9x, 3y, xy^2, 6x^2 + x^2 y). \tag{1}$$

Note that this is an ideal of finite index, and that ideals of this kind do not usually appear in Iwasawa theory.

Now we compute the left hand side of Corollary 1.5 directly. Using PARI-GP we find that

$$\mathcal{G}_S \simeq \mathbb{Z}/9\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z},$$

and the actions of σ and τ are described by the matrices

$$\begin{pmatrix} 1 & 4 & -3 & 0 \\ 0 & -2 & 3 & 0 \\ 0 & -4 & 4 & 0 \\ 3 & 3 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} -2 & -3 & 3 & 0 \\ 0 & 4 & 0 & 0 \\ -3 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

respectively. This means that when we take generators e_1, e_2, e_3, e_4 of $\mathcal{G}_S$ corresponding to the above decomposition, the actions of σ and τ are

$$\sigma(e_1) = e_1 + 4e_2 - 3e_3, \qquad \tau(e_1) = -2e_1 - 3e_2 + 3e_3, \ldots \qquad \text{etc.}$$

We would like to thank J. Nomura for computing the actions for us. Then some further computations show that $\mathcal{G}_S$ has the relation matrix

$$\begin{pmatrix} 7x^2-6x & 3x-3 & 0 \\ x & x & 0 \\ 3+3x & 0 & -x \\ 3+y+3x & -3 & 0 \\ 7xy-3x & 3y & 0 \\ 6+3x & -y & 0 \\ 0 & 0 & y \\ 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

as a $\mathbb{Z}_p[G]$-module. Thus, $\mathrm{Fitt}_{\mathbb{Z}_p[G]}(\mathcal{G}_S)$ is the ideal generated by all 3×3-minors of the above matrix. We get

$$\mathrm{Fitt}_{\mathbb{Z}_p[G]}(\mathcal{G}_S) = (27, 9x, 3y, xy^2, 6x^2+x^2y). \tag{2}$$

For example, the determinant of the submatrix obtained by picking the 4-th, 6-th and 7-th row gives $3y$ times a unit. In fact, $-3y^2-y^3-3xy^2+18y+9xy = 3y(7+3x-xy)$ since we know $y^3=-3y-3y^2$, using $(1+y)^3-1=0$. In this way, we can check Corollary 1.5 numerically, using (1) and (2). It does come out correctly, and this demonstrates that our theorem contains rather delicate information on $\mathcal{G}_S$.

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§2. The matrix $\tilde{M}_s$ and the minors

2.1. A free resolution of $\mathbb{Z}$

We consider the group G and the group ring R as in §1.1. The free resolution of $\mathbb{Z}$ as an R-module

$$R^{s(s+1)(s+2)/6} \xrightarrow{\Phi_3} R^{s(s+1)/2} \xrightarrow{\Phi_2} R^s \xrightarrow{\Phi_1} R \longrightarrow \mathbb{Z} \longrightarrow 0$$

constructed in [2] §1.2 from the tensor products of group rings of cyclic groups can be described in the following way. We write $(x_i)_{1\le i\le s}$ for

the standard basis of R^s, $(x_ix_j)_{1\le i\le j\le s}$ for a basis of $R^{s(s+1)/2}$, and $(x_ix_jx_k)_{1\le i\le j\le k\le s}$ for a basis of $R^{s(s+1)(s+2)/6}$. Then Φ_1 is the homomorphism such that $\Phi_1(x_i)=\tau_i$, and Φ_2 is defined by $\Phi_2(x_i^2)=\nu_i x_i$ and $\Phi_2(x_ix_j)=\tau_i x_j-\tau_j x_i$ if $i<j$. The homomorphism Φ_3 is defined by

$$\Phi_3(x_i^3)=\tau_i x_i^2,$$

$$\Phi_3(x_i^2x_j)=\tau_j x_i^2+\nu_i x_i x_j \quad \text{if} \quad i<j,$$

$$\Phi_3(x_ix_j^2)=\tau_i x_j^2-\nu_j x_i x_j \quad \text{if} \quad i<j,$$

and

$$\Phi_3(x_ix_jx_k)=\tau_i x_jx_k-\tau_j x_ix_k+\tau_k x_ix_j \quad \text{if} \quad i<j<k.$$

We define an $s(s+1)(s+2)/6$ by $s(s+1)/2$ matrix $\tilde{M}_s$ as the matrix corresponding to Φ_3. The rows of $\tilde{M}_s$ are labeled by $x_ix_jx_k$, and the columns are labeled by $x_\ell x_n$. In any one row there are at most 3 nonzero entries, so $\tilde{M}_s$ is sparse. For an example, we refer the reader to subsection 1.2 of [2], where the matrix $\tilde{M}_3$ is written out completely. We will study submatrices and minors of $\tilde{M}_s$. As in §1.1 we use the notation $\mathfrak{m}_v$ to denote the ideal of R generated by all v-minors of the matrix $\tilde{M}_s$.

2.2. Admissible polynomials

We consider monomials $\nu=\nu_1^{f_1}\cdot\ldots\cdot\nu_s^{f_s}$ in $\nu_1,\ldots,\nu_s\in R$. We say ν is *ordered* if $f_1\ge\ldots\ge f_s$. Also, we say it is *ordered and admissible* if it is ordered and the inequalities

$$\sum_{j=1}^{i} f_j \le \sum_{j=1}^{i}(s-j)$$

are satisfied for all i such that $1\le i\le s$. The right hand side equals the quantity $\varphi(i)$ introduced in subsection 1.2.

We say $\nu=\nu_1^{f_1}\cdot\ldots\cdot\nu_s^{f_s}$ is *admissible* if it is obtained from an ordered admissible monomial by permuting the ν_i. If we take $s=4$, then for example $\nu_1^3\nu_2^2\nu_3$ is ordered and admissible, $\nu_1\nu_2^3\nu_3^2$ is admissible but not ordered, and $\nu_2^3\nu_3^3$ is not admissible.

Let $\mathcal{D}$ be the set of all doubletons in $\{1,2,\ldots,s\}$, and let $\phi:\mathcal{D}\longrightarrow\{1,2,\ldots,s\}$ any map satisfying $\phi(D)\in D$ for all $D\in\mathcal{D}$. We called every such map a *selector* in [2]. A partial selector ψ is, by definition, a map from a subset $\mathcal{D}_\psi$ of $\mathcal{D}$ to $\{1,2,\ldots,s\}$, again satisfying the condition $\psi(D)\in D$ whenever $\psi(D)$ is defined. We define a monomial $\nu(\psi)$ by $\nu(\psi)=\prod_{D\in\mathcal{D}_\psi}\nu_{\psi(D)}$.

Lemma 2.1. *Suppose that $\nu = \nu_1^{f_1} \cdot \ldots \cdot \nu_s^{f_s}$ is admissible. Then there is a partial selector $\psi : \mathcal{D}_\psi \longrightarrow \{1, 2, \ldots, s\}$ such that $\nu = \nu(\psi)$.*

Proof. Since ν is admissible, we can take an admissible monomial ν' which is of degree $s(s-1)/2$ and which is a multiple of ν. By permuting the ν_i of ν', we have an ordered and admissible ν''. By [2] Proposition 1.2, we know that there is a selector ϕ such that $\nu'' = \nu(\phi)$. This implies that there is a selector ϕ' satisfying $\nu' = \nu(\phi')$, and that there is a partial selector ψ satisfying $\nu = \nu(\psi)$. Q.E.D.

We note that our definition of "admissibility" in this paper is slightly different from that we gave in [2], but they are equivalent by Lemma 2.1 (see page 950 in [2]).

By Proposition 1.4 in [2] and the above Lemma 2.1, we know that if $\nu = \nu_1^{f_1} \cdot \ldots \cdot \nu_s^{f_s}$ is admissible, then $\pm\nu$ appears as a minor of $\tilde{M}_s$.

For an integer $0 < d \leq s(s-1)/2$, let $\mathfrak{n}_d$ denote the ideal of R generated by all admissible monomials in $\nu_1, \ldots, \nu_s$ of degree d. We also define $\mathfrak{n}_0 = R$ and $\mathfrak{n}_{s(s-1)/2+1} = 0$. Now we have explained all the notation in §1.1.

§3. Proof of Theorem 1.2

In this section we prove Theorem 1.2, assuming Theorem 1.1.

Let K/k, S, Λ_{K_∞}, $X_{K_\infty,S}$, $\Theta_{K_\infty/k,S}$ be as in §1.2. We write

$$\Lambda_{K_\infty} = \mathbb{Z}_p[G][[\mathrm{Gal}(K_\infty/K)]] = \mathbb{Z}_p[G][[T]],$$

where $1+T$ is a generator of $\mathrm{Gal}(K_\infty/K)$. Let $\sigma_1, \ldots, \sigma_s$ be generators of G corresponding to the decomposition $G \simeq \bigoplus_{i=1}^s \mathbb{Z}/p^{n_i}\mathbb{Z}$. We define $\tau_i = \sigma_i - 1$, $\nu_i = N_{<\sigma_i>}$ as above, and define $\mathfrak{m}_v$ to be the ideal of $\mathbb{Z}_p[G]$ generated by all v-minors of the matrix $\tilde{M}_s$. We note that $\mathfrak{m}_v$ is an ideal of $\mathbb{Z}_p[G]$ (not of $\mathbb{Z}[G]$) in this section. We define an ideal $\mathfrak{A}_G$ of Λ_{K_∞} by

$$\mathfrak{A}_G = (\mathfrak{m}_{\frac{s(s-1)}{2}+1} T^{s-1} + \mathfrak{m}_{\frac{s(s-1)}{2}} T^s + \ldots + \mathfrak{m}_1 T^{\frac{s(s+1)}{2}-1} + T^{\frac{s(s+1)}{2}} \mathbb{Z}_p[G])\Lambda_{K_\infty}.$$

Suppose that $\Omega^2 = \mathrm{Ker}(\Phi_1 : \mathbb{Z}_p[G]^s \longrightarrow \mathbb{Z}_p[G])$ where Φ_1 is defined as in the previous section, and regard Ω^2 as a Λ_{K_∞}-module with the trivial action of T ($Tx = 0$ for all $x \in \Omega^2$). Then the meaning of this ideal $\mathfrak{A}_G$ is explained by $\mathfrak{A}_G = \mathrm{Fitt}_{\Lambda_{K_\infty}}(\Omega^2)$.

We proved in [3] Theorem 4.1 that

$$\mathrm{Fitt}_{\Lambda_{K_\infty}}(X_{K_\infty,S}) = T^{1-s}\mathfrak{A}_G\Theta_{K_\infty/k,S}.$$

Thus what we have to prove is the following

Proposition 3.1. *Theorem 1.1 implies* $T^{1-s}\mathfrak{A}_G = \mathfrak{a}_G I_{K_\infty}$.

Let $H = (\tau_1, \ldots, \tau_s)$ be the ideal of $\mathbb{Z}_p[G]$ generated by all $\tau_i = \sigma_i - 1$, and m_v as defined in §1.2. We suppose that $\mathfrak{n}_d$ is an ideal of $\mathbb{Z}_p[G]$ rather than of $\mathbb{Z}[G]$, namely it is an ideal of $\mathbb{Z}_p[G]$ generated by all admissible ν-monomials of degree d. We first prove two lemmas.

Lemma 3.2. *Let* $\mathrm{aug} : \mathbb{Z}_p[G] \longrightarrow \mathbb{Z}_p$ *be the augmentation homomorphism. Then for any* v *such that* $0 \leq v \leq s(s-1)/2$ *we have*

$$\mathrm{aug}(\mathfrak{n}_v) = p^{m_v}\mathbb{Z}_p \ .$$

Proof. If $v = 0$, this follows from $\mathfrak{n}_0 = \mathbb{Z}_p[G]$ and $m_0 = 0$. We suppose $v > 0$. We note that for every $1 \leq \alpha \leq s$,

$$\nu_\alpha = 1 + (1+\tau_\alpha) + \ldots + (1+\tau_\alpha)^{p^{n_\alpha}-1} = p^{n_\alpha} + \tau_\alpha\beta_\alpha$$

for some $\beta_\alpha \in \mathbb{Z}_p[G]$. Thus we have $\mathrm{aug}(\nu_\alpha) = p^{n_\alpha}$. For $\nu = \nu_{a_1} \cdot \ldots \cdot \nu_{a_v}$, we have $\mathrm{aug}(\nu) = p^{c_\nu}$ where $c_\nu = \sum_{w=1}^{v} n_{a_w}$. Thus if $\varphi(\alpha) < v \leq \varphi(\alpha+1)$ for some α with $0 \leq \alpha \leq s-2$ and

$$\nu = \nu_1^{s-1} \cdot \ldots \cdot \nu_\alpha^{s-\alpha}\nu_{\alpha+1}^{v-\varphi(\alpha)}, \tag{3}$$

then $\mathrm{aug}(\nu) = p^{m_v}$. Note that the above ν is ordered and admissible.

On the other hand, if ν is admissible and $\mathrm{aug}(\nu) = p^{c_\nu}$, it follows from the definitions of m_v and of admissibility that $c_\nu \geq m_v$. Thus we get Lemma 3.2. Q.E.D.

Lemma 3.3. *We assume Theorem 1.1. Then for any integer* v *such that* $0 \leq v \leq s(s-1)/2$, *we have*

$$\mathfrak{m}_v = \sum_{d=0}^{v} p^{m_d} H^{v-d}.$$

Proof. By Theorem 1.1 we have

$$\mathfrak{m}_v = \sum_{d=0}^{v} \mathfrak{n}_d H^{v-d}.$$

Therefore, it is enough to prove

$$\sum_{d=0}^{v} \mathfrak{n}_d H^{v-d} = \sum_{d=0}^{v} p^{m_d} H^{v-d}. \tag{4}$$

We prove this equality by induction on v. First of all, if $v = 0$, then both sides are trivial and we get equality. Suppose that $\nu = \nu_{a_1} \cdot \ldots \cdot \nu_{a_v}$

is admissible. Since $\mathrm{aug}(\nu_{a_w}) = p^{n_{a_w}}$ for any w with $1 \leq w \leq v$ as in the last lemma, we have

$$\nu = \nu_{a_1} \cdot \ldots \cdot \nu_{a_v} \in \prod_{w=1}^{v} (p^{n_{a_w}} \mathbb{Z}_p[G] + H) \subset \sum_{d=0}^{v} p^{m_d} H^{v-d} \tag{5}$$

by the argument of the proof of Lemma 3.2. Note that the second inclusion comes from $\mathrm{aug}(\mathfrak{n}_d) = p^{m_d}\mathbb{Z}_p$ $(0 \leq d \leq v)$, which is nothing but Lemma 3.2. This shows that ν is in the right hand side of (4). Therefore, $\mathfrak{n}_d$ is in the right hand side of (4). We have

(6)
$$\sum_{d=0}^{v-1} \mathfrak{n}_d H^{v-d} = H \sum_{d=0}^{v-1} \mathfrak{n}_d H^{v-1-d} = H \sum_{d=0}^{v-1} p^{m_d} H^{v-1-d} = \sum_{d=0}^{v-1} p^{m_d} H^{v-d}$$

where we used the inductive hypothesis to get the second equality. Thus we know that the left hand side of (4) is contained in the right hand side of (4).

On the other hand, if we take ν as in (3), then $\mathrm{aug}(\nu) = p^{m_v}$ and (5) together with the equation (6) show that p^{m_v} is in the left hand side of (4). This together with (6) implies that the converse inclusion holds. This completes the proof of Lemma 3.3. Q.E.D.

Now we prove Proposition 3.1. First of all, it follows from Theorem 1.1 that $\mathfrak{m}_{\frac{s(s-1)}{2}+1} = H\mathfrak{m}_{\frac{s(s-1)}{2}}$. Thus by the definition of $\mathfrak{A}_G$ and Lemma 3.3, we have

$$\begin{aligned} T^{1-s}\mathfrak{A}_G &= (\sum_{v=0}^{\frac{s(s-1)}{2}} \mathfrak{m}_v T^{\frac{s(s-1)}{2}+1-v} + H\mathfrak{m}_{\frac{s(s-1)}{2}})\Lambda_{K_\infty} \\ &= (\sum_{v=0}^{\frac{s(s-1)}{2}} \sum_{d=0}^{v} p^{m_d} H^{v-d} T^{\frac{s(s-1)}{2}+1-v} + H\sum_{d=0}^{\frac{s(s-1)}{2}} p^{m_d} H^{\frac{s(s-1)}{2}-d})\Lambda_{K_\infty}. \end{aligned}$$

Since I_{K_∞} is generated by H and T, we have

$$\begin{aligned} T^{1-s}\mathfrak{A}_G &= T \sum_{d=0}^{\frac{s(s-1)}{2}} p^{m_d} I_{K_\infty}^{\frac{s(s-1)}{2}-d} + H \sum_{d=0}^{\frac{s(s-1)}{2}} p^{m_d} H^{\frac{s(s-1)}{2}-d}\Lambda_{K_\infty} \\ &= I_{K_\infty} \sum_{d=0}^{\frac{s(s-1)}{2}} p^{m_d} I_{K_\infty}^{\frac{s(s-1)}{2}-d} \\ &= I_{K_\infty}\mathfrak{a}_G \ . \end{aligned}$$

Thus we have proved Proposition 3.1 and Theorem 1.2, assuming Theorem 1.1.

§4. Outline of the proof of Theorem 1.1

In the rest of this paper we prove Theorem 1.1. We compute the higher Fitting ideals of Ω^2, using the relation matrix of Φ_3. As we mentioned in §1.1, we only have to prove $\mathfrak{m}_v \supset \sum_{d=0}^{v} H^{v-d}\mathfrak{n}_d$. To show this inclusion, for an element x of the right hand side, we construct an explicit lower triangular submatrix of the relation matrix of Φ_3, whose determinant is $\pm x$. Thus we may neglect the signs of entries of the relation matrix, and think of the matrix $\tilde{M}_s$ in the following way. The row and the column labels of $\tilde{M}_s$ are $x_i x_j x_k$ and $x_\ell x_n$, respectively, which we denote, for simplicity, by ijk and ℓn, respectively.

In the $i^3 = i \cdot i \cdot i$-row, there is only one nonzero entry τ_i in the $i^2 = i \cdot i$-column. In the $i^2 \cdot j = i \cdot i \cdot j$-row with $i \neq j$, there are only two nonzero entries: τ_j in the $i^2 = i \cdot i$-column, and ν_i in the $i \cdot j$-column. In the $i \cdot j \cdot k$-row with $i < j < k$, there are only three nonzero entries: τ_k in the $i \cdot j$-column, τ_j in the $i \cdot k$-column, and τ_i in the $j \cdot k$-column. Therefore, $\tilde{M}_s$ is roughly of the following form:

$$
\begin{array}{c}
\\
1{\cdot}1{\cdot}1 \\
\cdots \\
s{\cdot}s{\cdot}s \\
1{\cdot}1{\cdot}2 \\
\cdots \\
(s{-}1){\cdot}s{\cdot}s \\
1{\cdot}2{\cdot}3 \\
\\
\cdots \\
\\
\\
(s{-}2){\cdot}(s{-}1){\cdot}s
\end{array}
\begin{array}{c}
\begin{array}{ccccccccccc}
1{\cdot}1 & \cdots & s{\cdot}s & 1{\cdot}2 & 1{\cdot}3 & \cdots & 2{\cdot}3 & \cdots & (s{-}1){\cdot}s
\end{array} \\
\left(\begin{array}{ccccccccc}
\tau_1 & & & & & & & & \\
& \cdots & & & & & & & \\
& & \tau_s & & & & & & \\
\tau_2 & & & \nu_1 & & & & & \\
& \cdots & & & & \cdots & & \cdots & \\
& & \tau_{s-1} & & & & & & \nu_s \\
& & & \tau_3 & \tau_2 & & \tau_1 & & \\
& & & & & & & & \\
& & & & & \cdots & & \cdots & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & \cdots & \tau_{s-2}
\end{array}\right)
\end{array}.
$$

In the rest of this paper we assume $s \geq 3$. Put

$$t = \frac{s(s-1)}{2} + 1.$$

We fix $0 \leq v \leq t$ and consider elements $x \in H^{v-d}\mathfrak{n}_d$ (with $0 \leq d \leq v$) of the form $x = \tau(x)\nu(x)$, where $\tau(x)$ is any monomial in $\tau_1, \ldots, \tau_s$ of degree $v - d$ and $\nu(x)$ is any admissible monomial in $\nu_1, \ldots, \nu_s$ of

degree d. Sometimes $\tau(x)$ (and $\nu(x)$) will be called the τ-part (ν-part respectively) of x. For the proof of Theorem 1.1, we may assume that there is no index i such that $\tau(x)$ contains the factor τ_i and $\nu(x)$ contains the factor ν_i, since then x would be zero.

For any such monomial x, it is our plan to exhibit a square submatrix $\mathcal{M}(x)$ of $\tilde{M}_s$ which is lower triangular and whose determinant is x. Note that then the determinant is the product of all diagonal entries; these entries are of the form τ_i or ν_j, and hence the size of the matrix must equal the degree v of the given monomial x. It turns out that the essential difficulty is to handle the case where $v = t$ (the maximal possible value); it will only take one paragraph at the very end of §9 to deduce the case $v < t$ from this.

However, handling the case $v = t$ is an arduous task and requires a series of intermediate steps. Let us try to outline these steps. We call x a τ-monomial if $x = \tau(x)$ in the above notation, that is, $\nu(x) = 1$. Similarly, x is a ν-monomial if $x = \nu(x)$. In general, x will be a (τ, ν)-monomial, meaning that it contains factors τ_i as well as ν_j.

We first explain our proof for τ-monomials. We consider the submatrix C_s of $\tilde{M}_s$ whose row labels are ijk such that i, j, k are all distinct, and whose column labels are ℓn with $\ell \neq n$. This submatrix is in a fairly large southeastern part of $\tilde{M}_s$. We systematically construct lower triangular submatrices of C_s, which "realize" τ-monomials x of relatively small degree.

As a first step (§5), we will construct lower triangular submatrices $N_{j,S_j} = N_{j,S_j}(a_1, ..., a_m)$, which realize τ-monomial of degree $m \leq s - 2$, where j is an integer $3 \leq j \leq s$, a_1,...,a_m are certain integers in $\{1, \ldots, j-1\}$, and S_j is a subset of $\{1, \ldots, s\}$. Two important properties of this matrix is that (i) $\det N_{j,S_j}(a_1, ..., a_m) = \tau_{a_1} \cdot \ldots \tau_{a_m}$, and (ii) the column labels are bj for some b's. There is one more important property of this matrix, which we do not state here, see Proposition 5.2.

Using these submatrices in §5, we will realize in §7 an arbitrary τ-monomial τ of degree $(s-1)(s-2)/2$ by constructing a submatrix $T(\tau)$ of C_s (see Theorem 7.3). By the property (ii) we mentioned in the previous paragraph, the column labels of N_{3,S_3},...,N_{s,S_s} are all distinct. Noting this, we construct $T(\tau)$ by combining N_{3,S_3},...,N_{s,S_s}. In this section there are two subcases. Case I occurs when τ does not involve every factor $\tau_1, \ldots, \tau_s$; then a fairly straightforward combination of the submatrices N_{j,S_j} with S_j empty, produces the desired result. Case II is the case that all $\tau_1, \ldots, \tau_s$ divide τ. This case is much harder and we need to make a careful choice of S_s with the help of graph theory. We give no details here but refer the reader to §7 and in particular to the example given there.

Next, in §8 we are able to raise the degree to the required maximum, namely to t, constructing a submatrix $M(\tau)$ of $\tilde{M}_s$. Now we use submatrices outside C_s to construct $M(\tau)$, but the result in §7 we mentioned above plays the most important role.

We explain our proof for a general monomial x. If x is a ν-monomial, the conclusion was already proved in the end of §2. So we assume that x is neither a τ-monomial nor a ν-monomial. In §6, we first slightly modify and extend the construction of §5: here we construct a submatrix $N'_{j,S_j} = N'_{j,S_j}(a_1, \dots, a_{m'})$ of $\tilde{M}_s$, which realizes a monomial whose τ-part is as in §5, and whose ν-part is of the form ν_j^f for arbitrary j (as in §5) and suitable exponent f. More precisely, we get $\det N'_{j,S_j}(a_1, \dots, a_{m'}) = \nu_j^f \tau_{a_1} \cdot \dots \tau_{a_{m'}}$.

Finally, in §9 we combine the outcome of §§6 and 8 to treat general (τ, ν)-monomials x of degree t. To construct a submatrix $\mathcal{M}(x)$ which realizes a general monomial x, we have to combine several N'_{j,S_j} with careful choices of S_j's, $M(\tau)$ in §8 for some τ dividing the τ-part of x, and several other submatrices of $\tilde{M}_s$. This will then complete the proof of Theorem 1.1.

§5. Small τ-monomials

In this section we deal with certain τ-monomials of relatively small degree. For this, we consider the submatrix C_s of $\tilde{M}_s$ whose row labels are ijk with $1 \le i < j < k \le s$ and whose column labels are ℓn with $1 \le \ell < n \le s$. This matrix C_s has $s(s-1)(s-2)/6$ rows and $s(s-1)/2$ columns. It occupies a large region in the south-east of the entire matrix $\tilde{M}_s$, which (let us recall) has format $s(s+1)(s+2)/6$ by $s(s+1)/2$.

Suppose that j is an integer such that $3 \le j \le s$ and fix it in this section. Let S be a subset of $\{1, 2, \dots, j-1\}$. We put $m = j - 2 - \#S$, and assume $m > 0$. The subset S may be empty. We consider a sequence of integers $a_1, \dots, a_m$. We say in this paper that an m-tuple $(a_\mu)_{1 \le \mu \le m}$ is *cautiously increasing* if it is non-decreasing and never jumps by more than one; namely $a_{\mu+1} - a_\mu = 0$ or 1 for all $1 \le \mu \le m-1$. Suppose that $(a_\mu)_{1 \le \mu \le m}$ is cautiously increasing, $1 \le a_1 \le a_m \le j-1$, and $\{a_1, \dots, a_m\} \cap S$ is empty. We will construct in this section a certain submatrix $N_{j,S}(a_1, \dots, a_m)$ of C_s (and hence of $\tilde{M}_s$), which has determinant $\tau_{a_1} \cdot \dots \cdot \tau_{a_m}$.

We arrange the numbers 1, 2, …, $j-1$ in the following order:

$$a_1, a_1 + 1, \dots, j-1, 1, 2, \dots, a_1 - 1,$$

and remove the elements in S. Let us call the resulting sequence

$$b_0, b_1, \ldots, b_m.$$

Especially, $b_0 = a_1$ since a_1 is not in S. Note that $(b_\mu)_{1\leq\mu\leq m}$ depends only on a_1 and S. We use $(b_\mu)_{1\leq\mu\leq m}$ for the construction of $N_{j,S}(a_1, \ldots, a_m)$, but not b_0. We note that by definition if $\mu \neq \rho$, we have $b_\mu \neq b_\rho$.

Lemma 5.1. *For any μ and ρ such that $1 \leq \mu \leq m$ and $\mu \leq \rho \leq m$, we have $a_\mu \neq b_\rho$.*

Proof. Suppose at first that $a_1 < b_\rho \leq a_m$. Since $a_1, \ldots, a_m$ are not in S and cautiously increasing, we have $b_1 = a_1 + 1$, $b_2 = a_1 + 2$, $\ldots$, $b_\rho = a_1 + \rho$. Since (a_μ) is cautiously increasing, we have $a_\mu \leq a_1 + \mu - 1$. Thus we obtain

$$a_\mu \leq a_1 + \mu - 1 \leq a_1 + \rho - 1 < a_1 + \rho = b_\rho,$$

which implies $a_\mu \neq b_\rho$.

Next, if $b_\rho > a_m$, it is clear that $a_\mu \neq b_\rho$ because $a_\mu \leq a_m < b_\rho$.

Finally, suppose that $b_\rho \leq a_1$. Since $\rho \neq 0$, we know $b_\rho \neq a_1$ and get $b_\rho < a_1 \leq a_\mu$. Thus we obtain the conclusion. Q.E.D.

By Lemma 5.1 we have $a_\mu \neq b_\mu$ for all $1 \leq \mu \leq m$. Since $a_\mu, b_\mu < j$, we know $\#\{a_\mu, b_\mu, j\} = 3$ and $\#\{b_\mu, j\} = 2$. Therefore, we can pick a submatrix $N = N_{j,S}(a_1, \ldots, a_m)$ of C_s by specifying the row labels a_1b_1j, a_2b_2j, $\ldots$, a_mb_mj and the column labels b_1j, b_2j, $\ldots$, b_mj, in this exact order. Then the diagonal term at position $(a_\mu b_\mu j, b_\mu j)$ is τ_{a_μ}, since the μ-th row label is $a_\mu b_\mu j$ and the μ-th column label is $b_\mu j$. Thus $N_{j,S}(a_1, ..., a_m)$ is the matrix of the form

$$\begin{array}{c} \\ a_1b_1j \\ a_2b_2j \\ \\ ... \\ \\ a_mb_mj \end{array} \begin{array}{c} \begin{array}{cccc} b_1j & b_2j & ... & b_mj \end{array} \\ \begin{pmatrix} \tau_{a_1} & & & \\ & \tau_{a_2} & & 0 \\ & & & \\ & & ... & \\ & & & \\ * & & & \tau_{a_m} \end{pmatrix} \end{array}.$$

If $\mu < \rho$, the $(a_\mu b_\mu j, b_\rho j)$-entry of N is zero, since $b_\rho \neq b_\mu$ and $b_\rho \neq a_\mu$ by Lemma 5.1. This shows that the matrix N is lower triangular, and the product over the diagonal is $\tau_{a_1} \cdots \tau_{a_m}$. Thus we have obtained

Proposition 5.2. *Take $S \subset \{1,2,...,j-1\}$, and a cautiously increasing sequence a_1,...,a_m of elements in $\{1,2,...,j-1\} \setminus S$ such that $m = j-2-\#S > 0$. Then the matrix $N_{j,S}(a_1,...,a_m)$ is lower triangular and*

$$\det N_{j,S}(a_1,...,a_m) = \prod_{\mu=1}^{m} \tau_{a_\mu} \ .$$

This matrix has no column labels $a_1 j$, nor kj with $k \in S$. In other words, the column labels of $N_{j,S}(a_1,...,a_m)$ are bj with $b \in \{1,...,j-1\} \setminus (S \cup \{a_1\})$.

Example. Take $j = 10$, $S = \{1,7\}$ and $N_{10,S}(3,3,3,4,4,5)$. Then, since

$$a_1 = a_2 = a_3 = 3 < a_4 = a_5 = 4 < a_6 = 5,$$

we find

$$b_1 = 4,\ b_2 = 5,\ b_3 = 6,\ b_4 = 8,\ b_5 = 9,\ b_6 = 2.$$

Therefore, the matrix $N_{10,S}(3,3,3,4,4,5)$ is

$$\begin{array}{c} \\ 3\cdot 4\cdot 10 \\ 3\cdot 5\cdot 10 \\ 3\cdot 6\cdot 10 \\ 4\cdot 8\cdot 10 \\ 4\cdot 9\cdot 10 \\ 5\cdot 2\cdot 10 \end{array} \begin{array}{c} \begin{array}{cccccc} 4\cdot 10 & 5\cdot 10 & 6\cdot 10 & 8\cdot 10 & 9\cdot 10 & 2\cdot 10 \end{array} \\ \begin{pmatrix} \tau_3 & & & & & \\ & \tau_3 & & & & 0 \\ & & \tau_3 & & & \\ \tau_8 & & & \tau_4 & & \\ \tau_9 & & & & \tau_4 & \\ & \tau_2 & & & & \tau_5 \end{pmatrix} \end{array}.$$

§6. Introducing a power of ν_j

We construct a submatrix $N'_{j,S}(a_1,...,a_{m'})$ of $\tilde{M}_s$, which is a modification of $N_{j,S}(a_1,...,a_m)$ in the previous section. Note that this is no longer a submatrix of C_s since we will now use row labels of type bjj, that is, with repeated numbers. The column labels we use, however, will still be "square-free". The point will be that the determinant of our submatrix is a τ-monomial as in the previous section multiplied by a power of a single element ν_j. This will be used later to assemble much bigger matrices whose determinant involves powers of several ν_j.

Let j, $S \subset \{1,2,...,j-1\}$ be as in the previous section. Suppose that a_1,...,$a_{m'}$ is a cautiously increasing sequence of elements in $\{1,2,...,j-1\} \setminus S$ with $m' \leq m = j-2-\#S$.

Using a_1 and S, we define a sequence $b_0,\ldots,b_m$ as in the previous section. Recall that $\mu \mapsto b_\mu$ is a bijection from $\{0,1,\ldots,m\}$ to

$\{1, 2, \ldots, j-1\} \setminus S$. Put $f = m - m' + 1$. We consider the square matrix $N' = N'_{j,S}(a_1, \ldots, a_{m'})$ of size $m+1$ whose column labels are $b_0 j$, $b_1 j$, $\ldots$, $b_m j$ and whose row labels are $b_0 j^2$, $b_1 j^2$, $\ldots$, $b_{f-1} j^2$, $a_1 b_f j$, $a_2 b_{f+1} j$, $\ldots$, $a_{m'} b_m j$, in this exact order. Note that $f + m' - 1 = m$. We define $(a'_\mu)_{1 \le \mu \le m}$ by $a'_\mu = a_1$ for $\mu = 1, \ldots, f$, and $a'_\mu = a_{\mu - f + 1}$ for any μ such that $f < \mu \le m$. Then N' has row labels $b_0 j^2$, $b_1 j^2$, $\ldots$, $b_{f-1} j^2$, $a'_f b_f j$, $a'_{f+1} b_{f+1} j$, $\ldots$, $a'_m b_m j$. By definition, $(a'_\mu)_{1 \le \mu \le m}$ is cautiously increasing. We have

$$a'_\mu \neq b_\rho$$

for any μ and ρ such that $f \le \mu \le \rho \le m$ by Lemma 5.1. Therefore, $\#\{a'_\mu, b_\mu, j\} = 3$ and N' is certainly a submatrix of $\tilde{M}_s$. We see that N' is of the form

$$\begin{array}{c} \\ b_0 j^2 \\ b_1 j^2 \\ \cdots \\ b_{f-1} j^2 \\ a'_f b_f j \\ a'_{f+1} b_{f+1} j \\ \cdots \\ a'_m b_m j \end{array} \begin{array}{c} \begin{array}{cccccccc} b_0 j & b_1 j & \ldots & b_{f-1} j & b_f j & b_{f+1} j & \ldots & b_m j \end{array} \\ \left(\begin{array}{cccccccc} \nu_j & & & & & & & \\ & \nu_j & & & & & & 0 \\ & & \cdots & & & & & \\ & & & \nu_j & & & & \\ & & & & \tau_{a_1} & & & \\ & & & & & \tau_{a_2} & & \\ & & & & & & \cdots & \\ * & & & & & & & \tau_{a_{m'}} \end{array} \right). \end{array}$$

In the rows having labels $b_0 j^2$, $b_1 j^2$, $\ldots$, $b_{f-1} j^2$, there is only one nonzero entry ν_j at the diagonal position because $b_\mu \neq j$. Therefore, the above inequality $a'_\mu \neq b_\rho$ implies that N' is lower triangular. Thus we have obtained

Proposition 6.1. *Take $S \subset \{1, 2, \ldots, j-1\}$, and a cautiously increasing sequence $a_1, \ldots, a_{m'}$ of elements in $\{1, 2, \ldots, j-1\} \setminus S$ with $m' \le m = j - 2 - \#S$. Then the matrix $N'_{j,S}(a_1, \ldots, a_{m'})$ is lower triangular and*

$$\det N'_{j,S}(a_1, \ldots, a_{m'}) = \nu_j^f \prod_{\mu=1}^{m'} \tau_{a_\mu},$$

where $f = m - m' + 1$. This matrix $N'_{j,S}(a_1, \ldots, a_{m'})$ has no column labels kj with $k \in S$.

§7. Bigger τ-monomials

In this section we exhibit lower triangular submatrices whose determinants are τ-monomials of much higher degree than in section 5. We

do not quite reach the necessary maximal degree $t = s(s-1)/2+1$ yet, but we save some room for later, only working within the submatrix C_s introduced earlier. The matrix $T(\tau)$ constructed in this section plays an important role in later sections. We put

$$t_0 = \frac{(s-1)(s-2)}{2} .$$

Let τ be a monomial of degree t_0 in $\tau_1, \ldots, \tau_s$;

$$\tau = \tau_1^{e_1}\tau_2^{e_2}\cdots\tau_s^{e_s}.$$

We will construct a submatrix of C_s which is lower triangular and whose determinant is τ. We say τ is ordered if $e_1 \geq \ldots \geq e_s$ is satisfied.

We may and will assume τ is ordered. Since the degree of τ is t_0, we have

$$\sum_{i=1}^{s} e_i = t_0 = \frac{(s-1)(s-2)}{2} . \tag{7}$$

For a monomial $x = \tau_{a_1}\tau_{a_2}\cdots\tau_{a_q}$ such that $a_1 \leq \ldots \leq a_q$ is satisfied, if $y = \tau_{a_1}\cdots\tau_{a_\mu}$ and $z = \tau_{a_{\mu+1}}\cdots\tau_{a_q}$ for some μ with $1 \leq \mu \leq q$, we say that $x = yz$ is an *ordered decomposition.* We make the ordered decomposition

$$\tau = \tau^{(3)}\tau^{(4)}\cdots\tau^{(s)}$$

such that the degree of $\tau^{(j)}$ is $j-2$ for any j with $3 \leq j \leq s$. This means that if we write

$$\tau = \tau_{a_1}\tau_{a_2}\cdots\tau_{a_{t_0}}$$

with $a_1 \leq a_2 \leq \ldots \leq a_{t_0}$, then $\tau^{(3)}, \ldots, \tau^{(s)}$ are defined by

$$\tau^{(3)} = \tau_{a_1},\ \tau^{(4)} = \tau_{a_2}\tau_{a_3},\ \ldots,\ \tau^{(j)} = \tau_{a_{d_j+1}}\tau_{a_{d_j+2}}\cdots\tau_{a_{d_j+j-2}},$$
$$\ldots,\ \tau^{(s)} = \tau_{a_{d_s+1}}\tau_{a_{d_s+2}}\cdots\tau_{a_{t_0}}$$

where $d_j = (j-2)(j-3)/2$. Since τ is ordered, $(a_{d_j+n})_{1\leq n\leq j-2}$ is cautiously increasing for all j.

Lemma 7.1. *For any j such that $3 \leq j \leq s-1$, $\tau^{(j)}$ consists of τ_i's with $i \leq j-1$.*

Proof. It is enough to prove

$$\sum_{i=1}^{j-1} e_i \geq d_{j+1} = \sum_{i=1}^{j-2} i = \frac{(j-1)(j-2)}{2} .$$

Since τ is ordered, it follows from (7) that

$$\begin{aligned}\sum_{i=1}^{j-1} e_i &\geq \frac{(s-1)(s-2)}{2}\cdot\frac{j-1}{s} = \frac{(s-3)(j-1)}{2} + \frac{j-1}{s} \\ &> \frac{(s-3)(j-1)}{2} \geq \frac{(j-1)(j-2)}{2}.\end{aligned}$$

This completes the proof. Q.E.D.

Case I. We assume $e_s = 0$.

For any j such that $3 \leq j \leq s$, suppose that

$$\tau^{(j)} = \tau_{a_{d_j+1}}\tau_{a_{d_j+2}}\cdots\tau_{a_{d_j+j-2}}$$

as above. Then by Lemma 7.1 we have $a_{d_j+1} \leq a_{d_j+2} \leq \ldots \leq a_{d_j+j-2} \leq j-1$ if $j \leq s-1$. For $j = s$, since $e_s = 0$ by our assumption , $\tau^{(j)}$ consists of τ_m's with $m \leq s-1$.

Therefore, noting that $(a_{d_j+n})_{1\leq n\leq j-2}$ is cautiously increasing, we can construct $N_{j,S}(a_{d_j+1}, a_{d_j+2}, \ldots, a_{d_j+j-2})$ with $S = \emptyset$ (see §5) for any j such that $3 \leq j \leq s$. We denote this matrix by $N_j(\tau^{(j)})$.

We define

$$T(\tau) = \begin{pmatrix} N_3(\tau^{(3)}) & & & 0 \\ & N_4(\tau^{(4)}) & & \\ & & \cdots & \\ * & & & N_s(\tau^{(s)}) \end{pmatrix}$$

which is a submatrix of C_s.

If abj is a row label of $T(\tau)$ for some a, $b < j$, then it is a row label of $N_j(\tau^{(j)})$. For any j' such that $j' > j$ and any k, the entry of (abj, kj') is zero because $\{k, j'\} \not\subset \{a, b, j\}$. This together with Proposition 5.2 shows that $T(\tau)$ is lower triangular. It is clear from Proposition 5.2 that

$$\det T(\tau) = \tau^{(3)}\tau^{(4)}\cdots\tau^{(s)} = \tau.$$

Case II. We assume $e_s > 0$. We note that this condition together with the condition that τ is ordered implies $s \geq 5$, since $t_0 < s$ if $s \leq 4$. Note that no case $s > 4$ was covered by the verifications done in [2]. By Lemma 7.1 we can define $N_j(\tau^{(j)})$ for any j such that $3 \leq j \leq s-1$ as in Case I.

In our arguments so far, we used row labels abj with $a, b < j \leq s$ to produce factors τ_a in the determinant. So this excludes $a = s$, which

means that in order to produce factors τ_s, we have to resort to tricks. The row labels used for this will be *abs*, with corresponding column labels *ab*, and the pairs *ab* used for this have to be chosen with the utmost care, to control the interference with other rows and columns (some row labels *abs* have probably been used up already). We will have to use some graph theory at this point.

We put $z = (s-2)(s-3)/2 = d_s$ and write

$$\tau^{(s)} = \tau_{a_{z+1}} \tau_{a_{z+2}} \cdots \tau_{a_{z+s-2}},$$

where $a_{z+1}, a_{z+2}, \ldots, a_{z+s-2}$ is cautiously increasing. By our assumption, we know $a_{z+s-2} = a_{t_0} = s$. Put $s - \ell = a_{z+1}$ for some $\ell \in \mathbb{Z}$. Since $a_{z+1}, a_{z+2}, \ldots a_{z+s-2}$ is cautiously increasing, we get

$$a_{z+1} \geq s - (s-2) + 1 = 3,$$

which implies $\ell \leq s-3$. Also, by Lemma 7.1 we know that $a_{z+1} \leq s-1$. Actually, we can show more; the inequality in the proof of Lemma 7.1 shows that $\Sigma_{i=1}^{s-2} e_i > z$, which implies $a_{z+1} \leq s-2$. Therefore, we have $2 \leq \ell \leq s-3$.

Let m be the positive integer such that $a_{z+m} = s-1$ and $a_{z+m+1} = s$. Since $a_{z+m+1} = \ldots = a_{z+s-2} = s$, we have

$$e_s = s - 2 - m \ . \tag{8}$$

Since $a_{z+1} = s - \ell$ and $a_{z+m+1} = s$, we know $\ell = a_{c+m+1} - a_{c+1} \leq m$, which implies

$$e_s + 1 = s - m - 1 \leq s - \ell - 1 < s - \ell \ .$$

Let j be an integer satisfying $3 \leq j \leq s-1$. We note that $\tau^{(j)}$ begins with $\tau_{a_{d_j+1}}$, namely there is an ordered decomposition $\tau^{(j)} = \tau_{a_{d_j+1}} \tau'$ for some τ'. Then by Proposition 5.2, $a_{d_j+1} j$ is not a column label of $N_j(\tau^{(j)})$ for any j.

For an integer k such that $3 \leq k \leq s-1$, we consider an undirected graph $G(k)$ whose vertices $V(k)$ and edges $E(k)$ are defined as follows:

$$V(k) = \{1, \ldots, k\}, \quad E(k) = \{\{a_{d_j+1} j\} \mid 3 \leq j \leq k\} \cup \{\{12\}\}.$$

By induction on k we can show that $G(k)$ has no closed path. Therefore, it is a *tree* and connected (see for example, [9] Theorem 3.1).

We consider the graph $G(k)$ with $k = s - \ell$, namely $G(s-\ell)$. Using the inequality $s - \ell > e_s + 1$ which we showed above and the fact that $G(s-\ell)$ is connected, we can take a subgraph G' of $G(s-\ell)$ such that

- G' is a tree (hence connected),
- $s - \ell$ is a vertex of G',
- the set $V(G')$ of vertices of G' consists of $e_s + 1$ elements, and
- the set $E(G')$ of edges of G' consists of e_s elements.

We write

$$E(G') = \{\{h_1 g_1\}, \ldots, \{h_{e_s} g_{e_s}\}\}.$$

Example. Take $s = 7$ and $\tau = \tau_1^3\tau_2^2\tau_3^2\tau_4^2\tau_5^2\tau_6^2\tau_7^2$. The degree of τ is 15. By the ordered decomposition of τ, we have $\tau^{(3)} = \tau_1$, $\tau^{(4)} = \tau_1^2$, $\tau^{(5)} = \tau_2^2\tau_3$, $\tau^{(6)} = \tau_3\tau_4^2\tau_5$, and $\tau^{(7)} = \tau_5\tau_6^2\tau_7^2$. Therefore, $\ell = \#\{5,6,7\} - 1 = 3 - 1 = 2$. The graph $G(6)$ has edges

$$E(6) = \{\{12\}, \{13\}, \{14\}, \{25\}, \{36\}\}.$$

Certainly, $G(6)$ is a tree. We consider $G(5)$. In this case there is a unique connected subgraph G' of $G(5)$, which has 3 vertices, and of which 5 is a vertex. Namely, G' has vertices $V(G') = \{1, 2, 5\}$ and edges

$$E(G') = \{\{12\}, \{25\}\}.$$

So we can take $h_1 = 1$, $g_1 = 2$, $h_2 = 2$, $g_2 = 5$.

In what follows, some of the row and column labels become a little complex. For better visibility, we sometimes put them between curly brackets { }, that is, we sometimes use $\{abc\}$, $\{ab\}$ instead of abc, ab for row and column labels.

We consider a matrix $\mathcal{N}_{\tau,s}(G')$ whose row labels are $\{h_\mu g_\mu s\}_{1\le\mu\le e_s}$ and whose column labels are $\{h_\mu g_\mu\}_{1\le\mu\le e_s}$. Since $\{h_\mu g_\mu\}$ are all different, this is a submatrix of C_s. It is a diagonal matrix $\tau_s I$ where I is the identity matrix;

$$\mathcal{N}_{\tau,s}(G') = \begin{array}{c} \\ h_1g_1s \\ h_2g_2s \\ \cdots \\ h_{e_s}g_{e_s}s \end{array} \begin{array}{c} \begin{array}{cccc} h_1g_1 & h_2g_2 & \cdots & h_{e_s}g_{e_s} \end{array} \\ \begin{pmatrix} \tau_s & & & \\ & \tau_s & & 0 \\ & & & \\ & & \cdots & \\ 0 & & & \tau_s \end{pmatrix} \end{array}.$$

Next, we set $S_{G'} = V(G')\backslash\{s-\ell\}$. Then we have $\#S_{G'} = e_s+1-1 = e_s$. It follows from (8) that

$$m = s - 2 - e_s = s - 2 - \#S_{G'}.$$

Therefore, we can define $N_{s,S_{G'}}(a_{z+1}, a_{z+2}, \ldots, a_{z+m})$, which was constructed in §5 for $j = s$ and $S = S_{G'}$. We put

$$N_{s,G'}(\tau^{(s)}) = N_{s,S_{G'}}(a_{z+1}, a_{z+2}, \ldots, a_{z+m}).$$

Lemma 7.2. (i) *For any μ such that $1 \leq \mu \leq e_s$, $\{h_\mu g_\mu\}$ does not appear in the column labels of $N_3(\tau^{(3)})$, $N_4(\tau^{(4)})$, …, $N_{s-1}(\tau^{(s-1)})$, $N_{s,G'}(\tau^{(s)})$.*

(ii) *For any μ such that $1 \leq \mu \leq e_s$, $\{h_\mu g_\mu s\}$ does not appear in the row labels of $N_3(\tau^{(3)})$, $N_4(\tau^{(4)})$, …, $N_{s-1}(\tau^{(s-1)})$, $N_{s,G'}(\tau^{(s)})$.*

(iii) *For any μ such that $1 \leq \mu \leq e_s$, neither $\{h_\mu s\}$ nor $\{g_\mu s\}$ appears in the column labels of $N_3(\tau^{(3)})$, $N_4(\tau^{(4)})$, …, $N_{s-1}(\tau^{(s-1)})$, $N_{s,G'}(\tau^{(s)})$.*

Proof. (i) By Proposition 5.2 and the definition of $G(s-\ell)$, no edge of $G(s-\ell)$ appears in the column labels of $N_3(\tau^{(3)})$, $N_4(\tau^{(4)})$, …, $N_{s-1}(\tau^{(s-1)})$. Since h_μ, $g_\mu \leq s - \ell < s$ and the column labels of $N_{s,G'}(\tau^{(s)})$ are of the form $\{bs\}$, $\{h_\mu g_\mu\}$ is not a column label of $N_{s,G'}(\tau^{(s)})$. Thus we get the conclusion.

(ii) Since the row labels of $N_j(\tau^{(j)})$ are of the form $\{abj\}$, $\{h_\mu g_\mu s\}$ cannot appear in the row labels of $N_j(\tau^{(j)})$ for any $j < s$. Suppose that $\{a_\rho b_\rho s\}$ is a row label of $N_{s,G'}(\tau^{(s)})$. Then by the definition of b_ρ, it is not in $S_{G'} \cup \{s-\ell\}$. Therefore, we have

$$b_\rho \notin \{h_\mu, g_\mu \mid \mu = 1, 2, \ldots, e_s\}. \tag{9}$$

Thus we get $\{h_\mu, g_\mu, s\} \neq \{a_\rho, b_\rho, s\}$, which implies (ii).

(iii) Since the column labels of $N_j(\tau^{(j)})$ are of the form $\{bj\}$, neither $\{h_\mu s\}$ nor $\{g_\mu s\}$ appears in the column labels of $N_j(\tau^{(j)})$ for any $j < s$. For a column label $\{b_\rho s\}$ of $N_{s,G'}(\tau^{(s)})$, by (9) we have $h_\mu \neq b_\rho$ and $g_\mu \neq b_\rho$ for any μ such that $1 \leq \mu \leq e_s$. Thus neither $\{h_\mu s\}$ nor $\{g_\mu s\}$ appears in the column labels of $N_{s,G'}(\tau^{(s)})$. Q.E.D.

By Lemma 7.2 (i) and (ii), we can define $T(\tau)$ by

$$T(\tau) = \begin{pmatrix} \mathcal{N}_{\tau,s}(G') & & & & & & 0 \\ & N_3(\tau^{(3)}) & & & & & \\ & & N_4(\tau^{(4)}) & & & & \\ & & & \cdots & & & \\ & & & & N_{s-1}(\tau^{(s-1)}) & & \\ * & & & & & & N_{s,G'}(\tau^{(s)}) \end{pmatrix},$$

which is a submatrix of C_s. It follows from Lemma 7.2 (iii) that there is only one nonzero entry in a $h_\mu g_\mu s$-row of the matrix $T(\tau)$. For $j' > j$,

the (abj, kj')-entry is zero. Therefore, using Proposition 5.2, we know that $T(\tau)$ is lower triangular.

Combining Cases I and II, we get

Theorem 7.3. *For any τ which is a monomial in $\tau_1, \ldots, \tau_s$ of degree $t_0 = (s-1)(s-2)/2$, we can construct a submatrix $T(\tau)$ of C_s, which is lower triangular and whose determinant is τ. For the case τ is ordered, an explicit construction of $T(\tau)$ is given above.*

Example. Consider $\tau = \tau_1^3\tau_2^2\tau_3^2\tau_4^2\tau_5^2\tau_6^2\tau_7^2$ for $s = 7$ as above. Then $T(\tau)$ is described as

	12	25	23	24	34	35	45	15	46	56	16	26	67	37	47
127	τ_7														
257		τ_7													
123	τ_3		τ_1												
124	τ_4			τ_1											
134					τ_1										
235		τ_3	τ_5			τ_2									
245		τ_4		τ_5			τ_2								
315						τ_1		τ_3							
346					τ_6				τ_3						
456							τ_6		τ_5	τ_4					
416									τ_1		τ_4				
526		τ_6								τ_2		τ_5			
567										τ_7			τ_5		
637													τ_3	τ_6	
647									τ_7				τ_4		τ_6

§8. The final step for τ-monomials

So far we have obtained τ-monomials of degree $t_0 = (s-1)(s-2)/2$. In a final round we now consider a monomial in $\tau_1, \ldots, \tau_s$ of degree t. Recall that

$$t = \frac{s(s-1)}{2} + 1 = t_0 + s \ ,$$

and that t is the final degree required in Theorem 1.1. We will construct a submatrix $M(\tau)$ of $\tilde{M}_s$ of size t, which is lower triangular and whose determinant is τ.

As in the previous section, we may assume

$$\tau = \tau_1^{e_1}\tau_2^{e_2}\cdots\tau_s^{e_s}$$

is ordered, namely $e_1 \geq \ldots \geq e_s$. Note that we are assuming $\Sigma_{i=1}^{s} e_i = t$. We define r to be the positive integer such that $e_r > 0$ and $e_{r+1} = 0$ if $e_s = 0$. If $e_s > 0$, we define $r = s$. We put

$$\tau' = \tau_1^{e_1-1}\tau_2^{e_2-1}\cdots\tau_r^{e_r-1},$$

which is also ordered.

At first we assume $e_s = 0$. Then $r < s$ holds. We take the ordered decomposition $\tau' = \tau(1)\tau(2)$ such that $\tau(1)$, $\tau(2)$ are monomials of degree t_0, $s-r$, respectively.

We define a diagonal matrix A_r by

$$A_r = \begin{array}{c} \\ 1\cdot1\cdot1 \\ 2\cdot2\cdot2 \\ \\ r\cdot r\cdot r \end{array} \begin{array}{c} \begin{array}{cccc} 1\cdot1 & 2\cdot2 & \ldots. & r\cdot r \end{array} \\ \begin{pmatrix} \tau_1 & & & \\ & \tau_2 & & 0 \\ & & & \\ 0 & & & \tau_r \end{pmatrix} \end{array}$$

which is a submatrix of $\tilde{M}_s$.

Suppose that $\tau(2) = \tau_{a_1^{(2)}}\tau_{a_2^{(2)}}\cdots\tau_{a_{s-r}^{(2)}}$. Let us now define a diagonal matrix $B_{r+1,s}(\tau(2)) = B_{r+1,s}(a_1^{(2)},\ldots,a_{s-r}^{(2)})$ by

$$B_{r+1,s}(\tau(2)) = \begin{array}{c} \\ a_1^{(2)}\cdot(r+1)^2 \\ a_2^{(2)}\cdot(r+2)^2 \\ \\ a_{s-r}^{(2)}\cdot s^2 \end{array} \begin{array}{c} \begin{array}{cccc} (r+1)^2 & (r+2)^2 & \ldots & s^2 \end{array} \\ \begin{pmatrix} \tau_{a_1^{(2)}} & & & \\ & \tau_{a_2^{(2)}} & & 0 \\ & & & \\ 0 & & & \tau_{a_{s-r}^{(2)}} \end{pmatrix} \end{array}$$

where we wrote j^2 for $j\cdot j$. We note that $a_n^{(2)}$ with $1 \leq n \leq s-r$ satisfies $a_n^{(2)} < r+1$, which implies that $B_{r+1,s}(\tau(2))$ is a submatrix of $\tilde{M}_s$. Clearly, the determinant of $B_{r+1,s}(\tau(2))$ is $\tau(2)$.

We now have

$$\tau = \tau_1\cdots\tau_r\tau(1)\tau(2)$$

with $\deg\tau(1) = t_0$ and $\deg\tau(2) = s-r$. We also note that $\tau(1)$ is ordered. Therefore, as we explained in §7, we can construct a lower triangular matrix $T(\tau(1))$ with $\det T(\tau(1)) = \tau(1)$. We define $M(\tau)$ by

$$M(\tau) = \begin{pmatrix} A_r & C_1 & C_2 \\ C_3 & T(\tau(1)) & C_4 \\ C_5 & * & B_{r+1,s}(\tau(2)) \end{pmatrix}.$$

Since the rows of A_r, $T(\tau(1))$, $B_{r+1,s}(\tau(2))$ are all distinct and the columns of these three matrices are also all distinct, $M(\tau)$ is a submatrix of $\tilde{M}_s$. Since $\{i^2\}$ does not appear in the column labels of $T(\tau(1))$ for any i with $1 \leq i \leq s$, we know $C_1 = 0$. If $1 \leq i \leq r$, $\{i^2\}$ does not appear in the column labels of $B_{r+1,s}(\tau(2))$, which implies $C_2 = 0$. Also, no row label of the form $\{i^2 j\}$ appears among the row labels of $T(\tau(1))$ for any i, j with $1 \leq i, j \leq s$, which implies $C_3 = 0$ and $C_4 = 0$. If $1 \leq i \leq r$, no row label of the form $\{i^2 j\}$ appears among the row labels of $B_{r+1,s}(\tau(2))$. This shows that $C_5 = 0$. It follows that $M(\tau)$ is lower triangular, and

$$\begin{aligned} \det M(\tau) &= \det A_r \det T(\tau(1)) \det B_{r+1,s}(\tau(2)) \\ &= \tau_1 \cdots \tau_r \tau(1)\tau(2) = \tau \ . \end{aligned}$$

Next, we assume $e_s > 0$. In this case, we have $r = s$, and the degree of $\tau' = \tau_1^{e_1 - 1} \cdots \tau_s^{e_s - 1}$ is t_0, Therefore, by the method of §7 we can construct a lower triangular matrix $T(\tau')$ with $\det T(\tau') = \tau'$.

We let A_s be the matrix A_r defined above with $r = s$, and define $M(\tau)$ by

$$M(\tau) = \begin{pmatrix} A_s & C_1 \\ C_2 & T(\tau') \end{pmatrix},$$

which is a submatrix of $\tilde{M}_s$. By the same method as in the case $e_s = 0$, we can show that $C_1 = 0$, $C_2 = 0$, and $M(\tau)$ is lower triangular. Also, we have

$$\det M(\tau) = \tau_1 \cdots \tau_s \tau' = \tau.$$

For any monomial τ in $\tau_1, \ldots, \tau_s$ of degree t without assuming it is ordered, it is clear by symmetry that we can construct a submatrix of $\tilde{M}_s$, which is lower triangular, and whose determinant is τ. Thus we obtain

Proposition 8.1. *For any τ which is a monomial in $\tau_1, \ldots, \tau_s$ of degree $t = \frac{s(s-1)}{2} + 1$, we can construct a submatrix $M(\tau)$ of $\tilde{M}_s$, which is lower triangular, and whose determinant is τ. Above, we gave an explicit method of constructing $M(\tau)$, assuming τ is ordered.*

§9. Synthesis: General (τ, ν)-monomials

We now assemble our previous constructions, as the final step of the proof of Theorem 1.1. We now work on monomials x in $\tau_1, \ldots, \tau_s$ and $\nu_1, \ldots, \nu_s$ of degree $t = \frac{s(s-1)}{2} + 1$. Every such monomial x factors

uniquely as $x = \tau(x)\nu(x)$ where $\tau(x)$ is a product of factors τ_i and $\nu(x)$ is a product of factors ν_j. We call $\tau(x)$ and $\nu(x)$ the τ-part and the ν-part of x, respectively. *Recall that* $\tau_i\nu_i = 0$ for all i with $1 \leq i \leq s$. As said earlier, the case of monomials of degree less than t will be dealt with at the very end.

For a ν-monomial $\nu = \nu_1^{f_1}\nu_2^{f_2}\cdots\nu_s^{f_s}$, we call ν *anti-ordered admissible* if $f_1 \leq \ldots \leq f_s$ and

$$\sum_{j=i}^{s} f_j \leq \sum_{j=1}^{s+1-i} (s-j)$$

for all $1 \leq i \leq s$. Clearly, a monomial which is anti-ordered admissible is admissible.

It is our goal to construct, for every x of degree t such that $\nu(x)$ is admissible, a lower triangular submatrix of $\tilde{M}_s$ whose determinant is x. To achieve our goal, we may assume that both $\tau(x)$ and $\nu(x)$ are nontrivial, because the cases $\nu(x) = 1$, $\tau(x) = 1$ were already proved in Proposition 8.1 and in Proposition 1.4 in [2] (see also the end of §2), respectively. Also, by symmetry and $\tau_i\nu_i = 0$, we may assume

$$x = \tau_1^{e_1}\tau_2^{e_2}\cdots\tau_r^{e_r}\nu_g^{f_g}\nu_{g+1}^{f_{g+1}}\cdots\nu_s^{f_s}$$

with $1 \leq r < g \leq s$. By symmetry we may also assume

$$\begin{aligned} &e_1 \geq e_2 \geq \ldots \geq e_r > 0\ ,\ 0 < f_g \leq f_{g+1} \leq \ldots \leq f_s\ , \\ &\text{and } \nu(x) = \nu_g^{f_g}\nu_{g+1}^{f_{g+1}}\cdots\nu_s^{f_s} \text{ is anti-ordered admissible.} \end{aligned} \tag{10}$$

Assuming this, we will construct a submatrix $M(x)$ of $\tilde{M}_s$, which is lower triangular, and whose determinant is x.

For $\nu = \nu(x) = \nu_g^{f_g}\nu_{g+1}^{f_{g+1}}\cdots\nu_s^{f_s}$, the existence of a partial selector for ν implies the following lemma.

Lemma 9.1. *The ν-part of x can be written as*

$$\nu = \nu(x) = \nu^{(g)}\nu^{(g+1)}\cdots\nu^{(s)}$$

such that for all j satisfying $g \leq j \leq s$,

$$\deg \nu^{(j)} \leq j - 1$$

and

$$\nu^{(j)} = \nu_{a_{j1}}\nu_{a_{j2}}\cdots\nu_{a_{jr_j}}\nu_j^{f_j'}$$

for some non-negative integer $r_j \in \mathbb{Z}_{\geq 0}$ and positive integer $f_j' \in \mathbb{Z}_{>0}$, and some $(a_{jk})_{1\leq k\leq r_j}$ such that $g \leq a_{j1} < a_{j2} < \ldots < a_{jr_j} < j$.

Proof. By Lemma 2.1, there is a partial selector $\psi : \mathcal{D}_\psi \longrightarrow \{g, g+1, \ldots, s\}$ corresponding to ν. If it is needed, since $g \geq 2$, we can change ψ to ψ' such that $\nu(\psi') = \nu(\psi) = \nu$ and that for any j such that $g \leq j \leq s$, there is i with $1 \leq i < j$, $\{i,j\} \in \mathcal{D}_{\psi'}$, and $\psi'(\{i,j\}) = j$. In fact, if for some j there is no i satisfying the above, then since $f_j > 0$ and $\{i,j\} \in \mathcal{D}_\psi$ for some i with $\psi(\{i,j\}) = j$, we can take ψ' obtained from ψ by removing $\{i,j\}$ from $\mathcal{D}_\psi$ and adding $\{1,j\} \in \mathcal{D}_{\psi'}$ with $\psi'(\{1,j\}) = j$.

For any j with $g \leq j \leq s$, we define

$$\nu^{(j)} = \prod_{\substack{i=1 \\ \{i,j\} \in \mathcal{D}_{\psi'}}}^{j-1} \nu_{\psi'(\{i,j\})}.$$

Then $\nu^{(j)}$ satisfies the conditions in Lemma 9.1, in particular we have $\deg \nu^{(j)} \leq j-1$ and $f'_j > 0$. Q.E.D.

Example. Consider $\nu = \nu_2\nu_3^4\nu_4^4\nu_5^4\nu_6^4\nu_7^4$ for $s = 7$. This ν is admissible. For this ν, we can decompose

$$\nu^{(2)} = \nu_2,\ \nu^{(3)} = \nu_3^2,\ \nu^{(4)} = \nu_4^3,\ \nu^{(5)} = \nu_5^4,\ \nu^{(6)} = \nu_3\nu_6^4,\ \nu^{(7)} = \nu_3\nu_4\nu_7^4.$$

We decompose ν as in Lemma 9.1, and put

$$m_j = j - 1 - \deg \nu^{(j)}$$

for all j such that $g \leq j \leq s$. Noting $f'_j > 0$ in Lemma 9.1, we have $\deg \nu^{(j)} > 0$, which implies

$$0 \leq m_j \leq j-2.$$

We put

$$\tau = \tau(x) = \tau_1^{e_1}\tau_2^{e_2}\cdots\tau_r^{e_r}.$$

Then we get

$$\begin{aligned} \sum_{j=g}^{s} m_j &= \sum_{j=g}^{s}(j-1) - \deg\nu = \sum_{j=g}^{s}(j-1) - (\frac{s(s-1)}{2} + 1 - \deg\tau) \\ &= \deg\tau - (\frac{(g-1)(g-2)}{2} + 1)\ . \end{aligned}$$

Therefore, we can write down the ordered decomposition

$$\tau = \tau^{(g-1)}\tau^{(g)}\tau^{(g+1)}\cdots\tau^{(s)}$$

such that $\deg \tau^{(g-1)} = \frac{(g-1)(g-2)}{2} + 1$ and

$$\deg \tau^{(j)} = m_j$$

for any j such that $g \leq j \leq s$.

For each j with $g \leq j \leq s$, we take $\tau^{(j)}$, $\nu^{(j)}$ as above, in particular $\nu^{(j)}$ as in Lemma 9.1;

$$\nu^{(j)} = \nu_{a_{j1}} \nu_{a_{j2}} \cdots \nu_{a_{jr_j}} \nu_j^{f'_j}.$$

We write

$$\tau^{(j)} = \tau_{n_{j1}} \tau_{n_{j2}} \cdots \tau_{n_{jm_j}}$$

with $n_{j1} \leq \ldots \leq n_{jm_j}$.

We define S_j by

$$S_j = \{a_{j1}, \ldots, a_{jr_j}\},$$

which is a set of r_j elements. By definition, we have

$$\begin{aligned} m_j + f'_j - 1 &= j - 1 - \deg \nu^{(j)} + f'_j - 1 \\ &= j - 1 - (f'_j + r_j) + f'_j - 1 \\ &= j - 2 - \#S_j \ . \end{aligned}$$

Therefore, $m_j \leq j - 2 - \#S_j$, and we can consider $N'_{j,S_j}(n_{j1}, \ldots, n_{jm_j})$, which is defined in §6. It is lower triangular and

$$\det N'_{j,S_j}(n_{j1}, \ldots, n_{jm_j}) = \nu_j^{f'_j} \tau_{n_{j1}} \tau_{n_{j2}} \cdots \tau_{n_{jm_j}}$$

by Proposition 6.1 and $f'_j = (j - 2 - \#S_j) - m_j + 1$.

To the matrix $N'_{j,S_j}(n_{j1}, \ldots, n_{jm_j})$, we now add rows labeled $\{a_{j\mu}^2 j\}$ and also columns labeled $\{a_{j\mu} j\}$ for $1 \leq \mu \leq r_j$ as follows:

$$\begin{array}{c} \\ a_{j1}^2 j \\ \ldots \\ a_{jr_j}^2 j \end{array} \begin{array}{c} \begin{array}{ccc} a_{j1} j & \ldots & a_{jr_j} j \end{array} \\ \begin{pmatrix} \nu_{a_{j1}} & & \\ & \ldots & \\ & & \nu_{a_{jr_j}} \\ & & \\ * & & \end{pmatrix} \end{array} \begin{array}{c} \\ \\ \\ \\ \begin{pmatrix} 0 \\ \\ \\ N'_{j,S_j}(n_{j1}, \ldots, n_{jm_j}) \end{pmatrix} \end{array}.$$

First of all, the labels $\{a_{j1} j\}, \ldots, \{a_{jr_j} j\}$ do not appear among the column labels of $N'_{j,S_j}(n_{j1}, ..., n_{jm_j})$, and the labels $\{a_{j1}^2 j\}, \ldots, \{a_{jr_j}^2 j\}$ do

not appear among the row labels of $N'_{j,S_j}(n_{j1},...,n_{jm_j})$. Therefore, the above matrix is a submatrix of $\tilde{M}_s$. We call this matrix $\mathcal{M}_j(\nu^{(j)},\tau^{(j)})$.

In the rows with index $\{a_{ji}^2 j\}$ the only nonzero entry sits at the diagonal position. This together with the fact that $N'_{j,S_j}(n_{j1},\ldots,n_{jm_j})$ is lower triangular implies that $\mathcal{M}_j(\nu^{(j)},\tau^{(j)})$ is also lower triangular. We have

$$\begin{aligned}\det \mathcal{M}_j(\nu^{(j)},\tau^{(j)}) &= \nu_{a_{j1}}\nu_{a_{j2}}\cdots\nu_{a_{jr_j}}\nu_j^{f'_j}\tau_{n_{j1}}\tau_{n_{j2}}\cdots\tau_{n_{jm_j}}\\ &= \nu^{(j)}\tau^{(j)}\ .\end{aligned}$$

Since $\tau^{(g-1)}$ is of degree $\frac{(g-1)(g-2)}{2}+1$ and ordered, by Proposition 8.1 we can build a lower triangular matrix $M(\tau^{(g-1)})$ of degree $\frac{(g-1)(g-2)}{2}+1$, whose row labels are of the form $\{abc\}$ with a, b, $c\le g-1$, and whose column labels are of the form $\{bc\}$ with b, $c\le g-1$.

For x as in (10) we define $\mathcal{M}(x)$ by

$$\mathcal{M}(x)=\begin{pmatrix} M(\tau^{(g-1)}) & & & & \\ & \mathcal{M}_g(\nu^{(g)},\tau^{(g)}) & & & 0 \\ & & \cdots & & \\ & & & \cdots & \\ * & & & & \mathcal{M}_s(\nu^{(s)},\tau^{(s)}) \end{pmatrix}.$$

Suppose that $\{abc\}$ is a row label of $M(\tau^{(g-1)})$. Then as we explained, since a, b, $c\le g-1$, we have $j\notin\{a,b,c\}$ for any j such that $g\le j\le s$. This shows that the column labels of $\mathcal{M}(x)$ are all distinct, and so are the row labels. Thus $\mathcal{M}(x)$ is a submatrix of $\tilde{M}_s$. Also, for any row label $\{abc\}$ of $M(\tau^{(g-1)})$ we have $\{n,j\}\not\subset\{a,b,c\}$ for any j such that $g\le j\le s$ and any n, and so $\mathcal{M}(x)$ is lower triangular. By the construction of $\mathcal{M}(x)$, we get

$$\begin{aligned}\det \mathcal{M}(x) &= \det M(\tau^{(g-1)})\prod_{j=g}^{s}\det \mathcal{M}_j(\nu^{(j)},\tau^{(j)})\\ &= \tau^{(g-1)}\prod_{j=g}^{s}\nu^{(j)}\tau^{(j)}=\tau\nu=x\ .\end{aligned}$$

Thus we finally obtain the following theorem.

Theorem 9.2. *Let x be a monomial in $\tau_1,\ldots,\tau_s$ and $\nu_1,\ldots,\nu_s$ of degree $t=\frac{s(s-1)}{2}+1$ satisfying the condition (10). Then we can construct a submatrix $\mathcal{M}(x)$ of $\tilde{M}_s$, which is lower triangular, and whose determinant is x.*

By symmetry this theorem implies that any monomial x in $\tau_1, \dots, \tau_s$ and $\nu_1, \dots, \nu_s$ of degree t such that the ν-part of x is admissible, we can construct a submatrix $\mathcal{M}(x)$ of $\tilde{M}_s$, which is lower triangular, and whose determinant is x. Thus we have proved that

$$\mathfrak{m}_t \supset \sum_{d=0}^{t} H^{t-d}\mathfrak{n}_d \ .$$

To finish the proof, we now discuss the case where the monomial x has degree $v < t$. Let $\tau(x)$ and $\nu(x)$ be the τ-part and the ν-part of x respectively as usual. Let d be the degree of $\nu(x)$. Then $d \le v \le t-1 = s(s-1)/2$. Let $e = v - d$ be the degree of $\tau(x)$, so $e + d = v$. First let us note that the case $e = 0$ need not be considered for the following reason. If $e = 0$, then x is a ν-monomial, and admissible. It was already shown in Proposition 1.4 of [2] and Lemma 2.1 that all admissible ν-monomials can be obtained as minors of $\tilde{M}_s$ (see the end of §2). Hence we may assume that $e > 0$, so the τ-part of x contains a factor τ_i, say. We then define $\tilde{x} = \tau_i^{t-v} \cdot x$. This is then of degree t. (Note that $\tilde{x}$ will not be zero since the ν-part of x has no factor ν_i, otherwise we would already have $x = 0$.) By Theorem 9.2 and the remark following it, there is a lower triangular $t \times t$ submatrix of $\tilde{M}_s$ whose determinant (= product over the diagonal) is $\tilde{x}$. Since x is a monomial of degree v dividing $\tilde{x}$, we can extract a $v \times v$ matrix $\mathcal{M}(x)$ from $\mathcal{M}(\tilde{x})$ such that $\det(\mathcal{M}(x)) = x$, simply by deleting rows and columns corresponding to factors τ_i that sit on the diagonal of $\mathcal{M}(\tilde{x})$ and that appear in $\tilde{x}$ but not in x. This means that we have established the inclusion $\mathfrak{m}_v \supset \sum_{d=0}^{v} H^{v-d}\mathfrak{n}_d$ for any $v \le t$, and this completes at last the proof of Theorem 1.1.

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