Height functions for motives, II

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Abstract.

This is the Part II of our paper "Height functions for motives". We consider more general period domains and the height functions on more general sets of motives. We also consider the corresponding Hodge theoretic variant of Nevanlinna theory.

Contents.

§0. Introduction

This is a sequel of our paper [19] concerning height functions for mixed motives over number fields, which we call Part I. The new subjects in this Part II are as follows.

1. We consider more general period domains $X(\mathbb{C})$ and more general sets $X(F)$ of motives over number fields F than Part I. See Section 1.2 for the definition of $X(\mathbb{C})$ and Section 1.3 for the definition of $X(F)$ of this paper.

2. We start a Hodge theoretic variant of Nevanlinna theory, which we would call *Hodge-Nevanlinna theory.*

Vojta compares number theory and Nevanlinna theory (see [36], for example) . He compares height functions

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(1) $V(F) \to \mathbb{R}_{>0}$ in number theory,
(2) $\mathcal{M}(B, V(\mathbb{C})) \to \mathbb{R}$ in Nevanlinna theory.

Here F is a number field, V is an algebraic variety over F, $V(F)$ and $V(\mathbb{C})$ denote the sets of F-points and $\mathbb{C}$-points of V, respectively, B is a connected one-dimensional complex analytic manifold (that is, a connected Riemann surface) endowed with a finite flat morphism $B \to \mathbb{C}$, and we denote by $\mathcal{M}(B, V(\mathbb{C}))$ the set of meromorphic functions from B to $V(\mathbb{C})$. If V is a dense open subvariety of a projective variety $\bar{V}$, a meromorphic function from B to $V(\mathbb{C})$ is nothing but a holomorphic function $f : B \to \bar{V}(\mathbb{C})$ such that $f(B) \not\subset \bar{V}(\mathbb{C}) \smallsetminus V(\mathbb{C})$. For $f \in \mathcal{M}(B, V(\mathbb{C}))$, Nevanlinna theory asks how often f has singularities on B, that is, how often we have $f(z) \in \bar{V}(\mathbb{C}) \smallsetminus V(\mathbb{C})$ $(z \in B)$, relating this question to height functions of f. (In the classical Nevanlinna theory, $B = \mathbb{C}$ endowed with the identity morphism $B \to \mathbb{C}$, $\bar{V} = \mathbf{P}^1(\mathbb{C})$, and Nevanlinna theory asks how often a meromorphic function on $\mathbb{C}$ has values in a given finite subset of $\mathbf{P}^1(\mathbb{C})$.)

In this paper, we compare the height functions in the above (1) and (2) and our height functions

(I) $X(F) \to \mathbb{R}_{>0}$ in number theory,
(II) $\mathcal{M}_{\mathrm{hor}}(B, X(\mathbb{C})) \to \mathbb{R}$ in Hodge-Nevanlinna theory,
(III) $\mathcal{M}_{\mathrm{hor}}(C, X(\mathbb{C})) \to \mathbb{R}$ in Hodge theory.

Here F and B are as above, C in (III) is a connected compact Riemann surface (that is, a smooth projective curve over $\mathbb{C}$), $X(F)$ is a set of motives over F, $X(\mathbb{C})$ is a period domain which classifies Hodge structures, and for $Y = B$ or C, $\mathcal{M}_{\mathrm{hor}}(Y, X(\mathbb{C}))$ denotes the set of horizontal meromorphic functions f from Y to $X(\mathbb{C})$. See 1.2.26 for the precise definition of $\mathcal{M}_{\mathrm{hor}}(Y, X(\mathbb{C}))$. If $\bar{X}(\mathbb{C}) \supset X(\mathbb{C})$ denotes the toroidal partial compactification of $X(\mathbb{C})$ consisting of the classes of nilpotent orbits of rank ≤ 1 ([23], [21] Part III, [24], [22]), $\mathcal{M}_{\mathrm{hor}}(Y, X(\mathbb{C}))$ is identified with the set of horizontal morphisms $f : Y \to \bar{X}(\mathbb{C})$ such that $f(Y) \not\subset \bar{X}(\mathbb{C}) \smallsetminus X(\mathbb{C})$. In our Hodge-Nevanlinna theory (II), for $Y = B$, we ask how often f has singularities as a meromorphic function from B to $X(\mathbb{C})$, that is, how often we have $f(z) \in \bar{X}(\mathbb{C}) \smallsetminus X(\mathbb{C})$ $(z \in B)$, relating this question to Hodge theoretic height functions of f.

(I) and (III) were already considered in Part I, but as is said above, we consider more general situation in this Part II. (In Part I, we used the notation $\mathrm{Mor}_{\mathrm{hor}}(C, \bar{X}(\mathbb{C}))$ for $\mathcal{M}_{\mathrm{hor}}(C, X(\mathbb{C}))$ in (III), where Mor stands for morphisms whereas $\mathcal{M}$ stands for meromorphic maps, regarding elements of $\mathcal{M}_{\mathrm{hor}}(C, X(\mathbb{C}))$ as horizontal morphisms $C \to \bar{X}(\mathbb{C})$.) In this Part II, $X(F)$ is the set of G-mixed motives over F and $X(\mathbb{C})$ is the set of G-mixed Hodge structures, where G is a linear algebraic group over $\mathbb{Q}$. Here a G-mixed motive (resp. G-mixed Hodge structure) means an exact $\otimes$-functor from the category of linear representations of G to the category of mixed motives (resp. mixed Hodge structures). Mumford-Tate domains ([12]) appear as standard examples of $X(\mathbb{C})$, and higher Albanese manifolds [15] also appear as examples of $X(\mathbb{C})$. For various Shimura varieties, the set of their $\mathbb{C}$-points give special cases of $X(\mathbb{C})$ and their F-points give points of $X(F)$. In (II) (resp. (III)), for $Y = B$ (resp. C), $\mathcal{M}_{\mathrm{hor}}(Y, X(\mathbb{C}))$ is understood as set of isomorphism classes of variations of G-mixed Hodge structure with logarithmic degeneration on Y.

We expect that the comparisons of (1), (2), (I), (II), (III) shed new lights to the arithmetic of the world of motives. Since $X(F)$ is usually not the set of F-points of an algebraic variety, the study of (I) is not reduced to the study of (1). Since $\bar{X}(\mathbb{C})$ is usually not a complex analytic space (it is a "logarithmic manifold" in the sense of [23], 3.5.7), the study of (II) is not reduced to the study of (2). As in Section 3.7, the height functions in (I), (II), (III) are related not only philosophically, but also actually via asymptotic formulas.

The organization of this paper is as follows. In Section 1, we define $X(\mathbb{C})$ and $X(F)$. Section 2 is a preparation for Hodge-Nevanlinna theory. There we study some Hodge theoretic curvature forms (theorems 2.3.3, 2.4.3). In Section 3, we define height functions in (I), (II), (III). In Section 4, we present questions.

We expect that our theory is related to Iwasawa theory (4.3.10).

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§1. Period domains and motives

In Section 1.2 (resp. Section 1.3), we define the set $X(\mathbb{C})$ (resp. $X(F)$) which appears in Section 0. Section 1.1 is a preparation for Section 1.2. In Section 1.4, we give some examples. The contents of Sections 1.1 and 1.2 are contained in the theory of Mumford-Tate domains in [12] if the linear algebraic group G is reductive and if $\mathcal{G} = G$ (1.1.10).

1.1. The period domain D

In this Section 1.1, we consider period domains $D(G, \Upsilon)$ (see 1.1.8 for the definition) and their generalizations $D(G, \mathcal{G}, H_b)$ (see 1.1.11 for the definition). In the case G is a reductive algebraic group, $D(G, \Upsilon)$ is the Mumford-Tate domain studied in [12]. For a linear algebraic group G in general, $D(G, \Upsilon)$ is the period domain considered in [22].

1.1.1. In 1.1.1–1.1.3, we prepare notation.

For a linear algebraic group G over $\mathbb{Q}$, let G_u be the unipotent radical of G and let G_{red} be the reductive group G/G_u.

Let $\text{Rep}(G)$ be the category of finite-dimensional linear representations of G over $\mathbb{Q}$.

1.1.2. Let $\mathbb{Q}$MHS (resp. $\mathbb{R}$MHS) be the category of mixed $\mathbb{Q}$ (resp. $\mathbb{R}$)-Hodge structures.

Let $\mathbb{Q}$HS (resp. $\mathbb{R}$HS) be the full subcategory of $\mathbb{Q}$MHS (resp. $\mathbb{R}$MHS) consisting of objects which are direct sums of pure objects.

For $H \in \mathbb{Q}$MHS (resp. $\mathbb{R}$MHS), let $H_{\mathbb{Q}}$ (resp. $H_{\mathbb{R}}$) be the underlying $\mathbb{Q}$ (resp. $\mathbb{R}$)-vector space of H.

On the other hand, for $H \in \mathbb{Q}$MHS, let $\mathbb{R} \otimes_{\mathbb{Q}} H$ be the associated mixed $\mathbb{R}$-Hodge structure. For such H, $H_{\mathbb{R}}$ denotes the $\mathbb{R}$-vector space $\mathbb{R} \otimes_{\mathbb{Q}} H_{\mathbb{Q}}$ underlying the mixed $\mathbb{R}$-Hodge structure $\mathbb{R} \otimes_{\mathbb{Q}} H$ though this may be a bit confusing.

1.1.3. For a commutative ring R, let $\text{Mod}_{ff}(R)$ be the category of finitely generated free R-modules. In particular, for a field k, $\text{Mod}_{ff}(k)$ is the category of finite-dimensional k-vector spaces.

1.1.4. In the rest of this paper, let G be a linear algebraic group over $\mathbb{Q}$. We assume that we are given a homomorphism $w : \mathbf{G}_m \to G_{\text{red}} = G/G_u$ whose image is in the center of G_{red} such that for some (equivalently, for any) lifting $\mathbf{G}_m \to G$ of w, the adjoint action of $\mathbf{G}_m$ on $\text{Lie}(G_u)$ is of weight ≤ -1.

Note that any $V \in \text{Rep}(G)$ is endowed with a canonical G-stable increasing filtration, which we denote by $W_\bullet V$ and call the weight filtration, such that for any lifting $\mathbf{G}_m \to G$ of w and for $i \in \mathbb{Z}$, $W_i V$ is the part of weight $\leq i$ of V for the action of $\mathbf{G}_m$.

Lemma 1.1.5. *Let $H : \text{Rep}(G) \to \mathbb{Q}MHS$ be an exact $\otimes$-functor. Then for any $V \in \text{Rep}(G_{\text{red}}) \subset \text{Rep}(G)$, we have $H(V) \in \mathbb{Q}HS \subset \mathbb{Q}MHS$.*

Proof. Since any representation of the reductive group G_{red} is semisimple, the weight filtration of $V \in \text{Rep}(G)$ splits. Q.E.D.

1.1.6. Let $S_{\mathbb{C}/\mathbb{R}}$ be the Weil restriction of the multiplicative group $\mathbb{G}_m$ from $\mathbb{C}$ to $\mathbb{R}$, which represents the functor $R \mapsto (\mathbb{C} \otimes_{\mathbb{R}} R)^\times$ for commutative rings R over $\mathbb{R}$. Let $w : \mathbb{G}_{m,\mathbb{R}} \to S_{\mathbb{C}/\mathbb{R}}$ be the homomorphism which represents the natural maps $R^\times \to (\mathbb{C} \otimes_{\mathbb{R}} R)^\times$ for commutative rings R over $\mathbb{R}$.

As in [8], the category $\mathbb{R}$HS (1.1.2) is equivalent to the category of finite-dimensional linear representations of $S_{\mathbb{C}/\mathbb{R}}$ over $\mathbb{R}$. For a finite-dimensional representation V of $S_{\mathbb{C}/\mathbb{R}}$ over $\mathbb{R}$, the corresponding object of $\mathbb{R}$HS has V as the underlying $\mathbb{R}$-structure and has the Hodge decomposition

$$V_{\mathbb{C}} := \mathbb{C} \otimes_{\mathbb{R}} V = \bigoplus_{p,q \in \mathbb{Z}} V_{\mathbb{C}}^{p,q},$$

where

$$V_{\mathbb{C}}^{p,q} = \{v \in V_{\mathbb{C}} \mid [z]v = z^p \bar{z}^q v \text{ for } z \in \mathbb{C}^\times\}.$$

Here $[z]$ denotes z regarded as an element of $S_{\mathbb{C}/\mathbb{R}}(\mathbb{R}) = \mathbb{C}^\times$. For a finite-dimensional representation V of $S_{\mathbb{C}/\mathbb{R}}$ over $\mathbb{R}$, the part of V of weight w of the corresponding Hodge structure coincides with the part of V of weight w for the action of $\mathbf{G}_{m,\mathbb{R}}$ via $\mathbf{G}_{m,\mathbb{R}} \overset{w}{\to} S_{\mathbb{C}/\mathbb{R}}$.

1.1.7. Consider a $G_{\text{red}}(\mathbb{R})$-conjugacy class Υ of homomorphisms $S_{\mathbb{C}/\mathbb{R}} \overset{h}{\to} G_{\text{red},\mathbb{R}}$ of algebraic groups over $\mathbb{R}$! such that the composition $\mathbf{G}_{m,\mathbb{R}} \overset{w}{\to} S_{\mathbb{C}/\mathbb{R}} \overset{h}{\to} G_{\text{red},\mathbb{R}}$ coincides with the homomorphism induced by $w : \mathbf{G}_m \to G_{\text{red}}$ (1.1.4).

1.1.8. For Υ as in 1.1.7, we define the period domain $D(G, \Upsilon)$ to be the set of isomorphism classes of exact $\otimes$-functors $H : \text{Rep}(G) \to \mathbb{Q}$MHS preserving the underlying $\mathbb{Q}$-vector spaces with weight filtrations satisfying the following condition (i).

(i) The homomorphism $S_{\mathbb{C}/\mathbb{R}} \to G_{\text{red},\mathbb{R}}$ corresponding to $\text{Rep}(G_{\text{red}})$ $overset H \to \mathbb{R}$HS via the theory of Tannakian categories belongs to Υ.

In the case G is reductive, this set $D(G, \Upsilon)$ is identified with Υ itself, and it is a Mumford-Tate domain studied in [12]. In general, this set $D(G, \Upsilon)$ coincides with the set which is denoted by D in [22] associated to Υ.

1.1.9. The set $D(G, \Upsilon)$ is not empty. In fact, there is a homomorphism $S_{\mathbb{C}/\mathbb{R}} \to G_{\mathbb{R}}$ such that the composition $S_{\mathbb{C}/\mathbb{R}} \to G_{\mathbb{R}} \to G_{\text{red},\mathbb{R}}$ belongs to Υ. This homomorphism defines an exact $\otimes$-functor $\text{Rep}(G) \to \mathbb{Q}$MHS which is an element of $D(G, \Upsilon)$.

1.1.10. Consider an algebraic normal subgroup $\mathcal{G}$ of G (defined over $\mathbb{Q}$). Let $\mathcal{Q} = G/\mathcal{G}$.

Consider an exact ⊗-functor

$$H_b : \mathrm{Rep}(G) \to \mathbb{Q}\mathrm{MHS}$$

which keeps the underlying $\mathbb{Q}$-vector spaces with weight filtrations. (The subscript b in H_b presents the role of H_b as a base point.)

1.1.11. For $(\mathcal{G}, H_b)$ as in 1.1.10, we define the period domain $D(G, \mathcal{G}, H_b)$ as follows.

Let Υ be the $G_{\mathrm{red}}(\mathbb{R})$-conjugacy class of the homomorphism $S_{\mathbb{C}/\mathbb{R}} \to G_{\mathrm{red},\mathbb{R}}$ corresponding to the exact ⊗-functor $\mathrm{Rep}(G_{\mathrm{red}}) \to \mathbb{R}\mathrm{HS}$ induced by H_b (1.1.5). Let $H_b|_{\mathcal{Q}}$ be the composite exact ⊗-functor $\mathrm{Rep}(\mathcal{Q}) \to \mathrm{Rep}(G) \overset{H_b}{\to} \mathbb{Q}\mathrm{MHS}$ and let $\Upsilon_{\mathcal{Q}}$ be the $\mathcal{Q}_{\mathrm{red}}(\mathbb{R})$-conjugacy class of homomorphisms $S_{\mathbb{C}/\mathbb{R}} \to \mathcal{Q}_{\mathrm{red},\mathbb{R}}$ induced by $H_b|_{\mathcal{Q}}$. Define $D = D(G, \mathcal{G}, H_b) \subset D(G, \Upsilon)$ to be the inverse image of $\mathrm{class}(H_b|_{\mathcal{Q}}) \in D(\mathcal{Q}, \Upsilon_{\mathcal{Q}})$ under the canonical map $D(G, \Upsilon) \to D(\mathcal{Q}, \Upsilon_{\mathcal{Q}})$; $\mathrm{class}(H) \mapsto \mathrm{class}(H|_{\mathcal{Q}})$.

Note that we have $\mathrm{class}(H_b) \in D$.

1.1.12. In fact, $D(G, \Upsilon)$ is regarded as the case $\mathcal{G} = G$ of $D(G, \mathcal{G}, H_b)$. For Υ as in 1.1.7, by fixing any $H_b \in D(G, \Upsilon)$, we have $D(G, \Upsilon) = D(G, G, H_b)$. On the other hand, for H_b as in 1.1.10, we have $D(G, G, H_b) = D(G, \Upsilon)$ where Υ is determined by H_b as in 1.1.11.

We consider the complex analytic structures of these period domains. We first review the complex analytic structure of $D(G, \Upsilon)$ which is given in [12], [22].

Lemma 1.1.13. *$G(\mathbb{R})G_u(\mathbb{C})$ acts on $D(G, \Upsilon)$ transitively.*

Proof. [22] Prop. 3.2.5. Q.E.D.

1.1.14. The set $D(G, \Upsilon)$ is regarded as a complex analytic manifold as follows.

Note that $H \in D(G, \Upsilon)$ is regarded as an exact ⊗-functor from $\mathrm{Rep}(G)$ to the category $\mathcal{C} \supset \mathbb{Q}\mathrm{MHS}$ of finite-dimensional $\mathbb{Q}$-vector spaces V such that V is endowed with an increasing filtration (called the weight filtration) and $V_{\mathbb{C}}$ is endowed with a decreasing filtration (called the Hodge filtration). By 1.1.13, in the set of isomorphism classes of exact ⊗-functors from $\mathrm{Rep}(G)$ to $\mathcal{C}$, we have a unique $G(\mathbb{C})$-orbit $\check{D}(G, \Upsilon)$ containing $D(G, \Upsilon)$. Then $G(\mathbb{C})$ acts on $\check{D}(G, \Upsilon)$ transitively and the isotropy group in $G(\mathbb{C})$ of each point of $\check{D}(G, \Upsilon)$ is an algebraic subgroup of $G(\mathbb{C})$. Hence $\check{D}(G, \Upsilon)$ is a complex analytic manifold.

By the following lemma, $D(G, \Upsilon)$ is also a complex analytic manifold.

Lemma 1.1.15. *$D(G, \Upsilon)$ is open in $\check{D}(G, \Upsilon)$*

Proof. Let $x \in D = D(G, \Upsilon)$. Since the Hodge filtration $F(x)^\bullet \mathrm{Lie}(G_{\mathrm{red}})_{\mathbb{C}}$ is pure of weight 0, the map

$$\mathrm{Lie}(G_{\mathrm{red}})_{\mathbb{R}} \to \mathrm{Lie}(G_{\mathrm{red}})_{\mathbb{C}}/F(x)^0\mathrm{Lie}(G_{\mathrm{red}})_{\mathbb{C}}$$

is surjective. Hence the map

$$\mathrm{Lie}(G)_{\mathbb{R}} + \mathrm{Lie}(G_u)_{\mathbb{C}} \to \mathrm{Lie}(G)_{\mathbb{C}}/F(x)^0\mathrm{Lie}(G)_{\mathbb{C}}$$

is surjective. Since $\mathrm{Lie}(G)_{\mathbb{C}}/F(x)^0\mathrm{Lie}(G)_{\mathbb{C}}$ is the tangent space of $\check{D} = \check{D}(G, \Upsilon)$ at x, the last surjectivity shows that $G(\mathbb{R})G_u(\mathbb{C})x$ is a neighborhood of x in $\check{D}$. Q.E.D.

(This Lemma is Proposition 3.2.7 of [22]. The proof of it given there is wrong.)

1.1.16. Let $\mathcal{A}$ be the category of complex analytic spaces in the sense of Grothendieck (the structure sheaf can have non-zero nilpotent sections). For a commutative ring R and for $Y \in \mathcal{A}$, let $\mathrm{Mod}_{ff}(R, Y)$ be the category of locally constant sheaves of finitely generated free R-modules on Y. Let $\mathbb{Q}\mathrm{MHS}(Y)$ be the category of pairs $(\mathcal{H}_{\mathbb{Q}}, \mathrm{fil})$, where $\mathcal{H}_{\mathbb{Q}} \in \mathrm{Mod}_{ff}(\mathbb{Q}, Y)$ endowed with an increasing filtration by locally constant $\mathbb{Q}$-subsheaves (called the weight filtration) and fil is a decreasing filtration on $\mathcal{H}_{\mathcal{O}} := \mathcal{O}_Y \otimes_{\mathbb{Q}} \mathcal{H}_{\mathbb{Q}}$ by subbundles (called the Hodge filtration) such that for each $y \in Y$, the fiber $(\mathcal{H}_{\mathcal{Q},y}, \mathrm{fil}(y))$ at y is a $\mathbb{Q}$-mixed Hodge structure.

Then the complex analytic manifold $D(G, \Upsilon)$ represents the functor $\mathcal{A} \to$ (Sets) which sends $Y \in \mathcal{A}$ to the set of isomorphism classes of exact $\otimes$-functors $\mathcal{H} : \mathrm{Rep}(G) \to \mathbb{Q}\mathrm{MHS}(Y)$ such that $\mathcal{H}(V)_{\mathbb{Q}}$ with the weight filtration for $V \in \mathrm{Rep}(G)$ coincides with the constant sheaf V with the weight filtration and such that the following (i) is satisfied.

(i) For any $y \in Y$, the homomorphism $S_{\mathbb{C}/\mathbb{R}} \to G_{\mathrm{red},\mathbb{R}}$ induced by the $\otimes$-functor $\mathrm{Rep}(G_{\mathrm{red}}) \to \mathbb{R}\mathrm{HS}$; $V \mapsto \mathbb{R} \otimes_{\mathbb{Q}} \mathcal{H}(V)(y)$ belongs to Υ.

For $D = D(G, \Upsilon)$, consider the universal object $\mathcal{H}_D \in \mathbb{Q}\mathrm{MHS}(D)$. Note that this object need not satisfy the Griffiths transversality, that is, it need not be a variation of mixed Hodge structure. For $V \in \mathrm{Rep}(G)$, we have the universal Hodge filtration on $\mathcal{O}_D \otimes_{\mathbb{Q}} \mathcal{H}_D(V)_{\mathbb{Q}} = \mathcal{O}_D \otimes_{\mathbb{Q}} V$.

Proposition 1.1.17. *Let $D = D(G, \Upsilon)$. Concerning the tangent bundle T_D of D, we have a canonical isomorphism*

$$T_D \cong (\mathcal{O}_D \otimes_{\mathbb{Q}} \mathrm{Lie}(G))/\mathrm{fil}^0.$$

Here fil *denotes the universal Hodge filtration on $\mathcal{O}_D \otimes_{\mathbb{Q}} \mathcal{H}_D(\mathrm{Lie}(G))_{\mathbb{Q}} = \mathcal{O}_D \otimes_{\mathbb{Q}} \mathrm{Lie}(G)$ where* $\mathrm{Lie}(G)$ *is endowed with the adjoint action of G.*

Proof. Let U be an open set of D and let $\tilde{U} = U[\epsilon]/(\epsilon^2)$ be the complex analytic space whose underlying topological space is the same as that of U and whose structure sheaf is $\mathcal{O}_U[\epsilon]/(\epsilon^2)$ (with ϵ an indeterminate). By the usual infinitesimal understanding of the tangent bundle, $T_D(U)$ is identified with the set of all morphisms $f : \tilde{U} \to D$ such that the composition $U \to \tilde{U} \overset{f}{\to} D$ is the inclusion morphism. Since D represents the functor described in 1.1.16, this set is identified with the set $\Gamma(U, (\mathcal{O}_U \otimes_{\mathbb{Q}} \mathrm{Lie}(G))/\mathrm{fil}^0)$ as follows. Let h be an element of the last set. Then h corresponds to f which sends $V \in \mathrm{Rep}(G)$ to $(V, \mathrm{fil}) \in \mathbb{Q}\mathrm{MHS}(\tilde{U})$ where fil is as follows. Let fil$'$ be the universal filtration on $\mathcal{O}_U \otimes_{\mathbb{Q}} V$. Locally, lift h to $\mathcal{O}_{\tilde{U}} \otimes_{\mathbb{Q}} \mathrm{Lie}(G)$. Then $\mathrm{fil}^r = \{x + \epsilon h(x) \mid x \in (\mathrm{fil}')^r\} + \epsilon(\mathrm{fil}')^r$, where $h(x)$ is defined by the action of $\mathrm{Lie}(G)$ on V. Q.E.D.

Proposition 1.1.18. *The morphism* $D(G, \Upsilon) \to D(\mathcal{Q}, \Upsilon_{\mathcal{Q}})$ *is smooth.*

Proof. This follows from the surjectivity of the induced maps of tangent spaces (1.1.17), which follows from the surjectivity of $\mathrm{Lie}(G) \to \mathrm{Lie}(\mathcal{Q})$. Q.E.D.

1.1.19. By definition, $D(G, \mathcal{G}, H_b)$ is a closed complex analytic subspace of $D(G, \Upsilon)$.

Proposition 1.1.20. *Let* $D = D(G, \mathcal{G}, H_b)$.

(1) D *is a complex analytic manifold.*

(2) The tangent bundle of D *is canonically isomorphic to* $(\mathcal{O}_D \otimes_{\mathbb{Q}} \mathrm{Lie}(\mathcal{G}))_{\mathcal{O}}/\mathrm{fil}^0$. *Here* fil *is the Hodge filtration on* $\mathcal{O}_D \otimes_{\mathbb{Q}} \mathcal{H}_D(\mathrm{Lie}(\mathcal{G}))_{\mathbb{Q}} = \mathcal{O}_D \otimes_{\mathbb{Q}} \mathrm{Lie}(\mathcal{G})$ *where* $\mathrm{Lie}(\mathcal{G})$ *is endowed with the adjoint action of* G.

Proof. This follows from 1.1.18. Q.E.D.

Proposition 1.1.21. $D(G, \mathcal{G}, H_b)$ *is a finite disjoint union of* $\mathcal{G}(\mathbb{R})\mathcal{G}_u(\mathbb{C})$*-orbits.*

Proof. First we consider the case G is reductive. For $x \in D(G, \mathcal{G}, H_b)$, let $\varphi_x : S_{\mathbb{C}/\mathbb{R}} \to G(\mathbb{R})$ be the homomorphism associated to x. Let $x \in D(G, \mathcal{G}, H_b)$ be the class of H_b, and let J (resp. $J_{\mathcal{Q}}$) be the algebraic subgroup of G (resp. $\mathcal{Q}$) consisting of elements a such that $a\varphi_x(z)a^{-1} = \varphi_x(z)$ (resp. $a\varphi_{x,\mathcal{Q}}(z)a^{-1} = \varphi_{x,\mathcal{Q}}(z)$ where $\varphi_{x,\mathcal{Q}}$ denotes the composition $S_{\mathbb{C}/\mathbb{R}} \overset{\varphi_x}{\to} G_{\mathbb{R}} \to \mathcal{Q}_{\mathbb{R}}$), and let I be the inverse image of $J_{\mathcal{Q}}$ in G. Then $J \subset I$ and $\mathcal{G} \subset I$.

Claim 1. We have $I(\mathbb{R})/J(\mathbb{R}) \overset{\cong}{\to} D(G, \mathcal{G}, H_b)$; $g \mapsto gx$.

In fact, if $y \in D(G, \mathcal{G}, H_b)$, $y = gx$ for some $g \in G(\mathbb{R})$ in $D(G, \Upsilon)$ where Υ is associated to x. Since φ_x and φ_y induce the same homomorphism $S_{\mathbb{C}/\mathbb{R}} \to \mathcal{Q}_{\mathbb{R}}$, we see that the image of g in $\mathcal{Q}(\mathbb{R})$ belongs to $J_{\mathcal{Q}}(\mathbb{R})$. This proves Claim 1.

Claim 2. The map $I(\mathbb{R}) \to D(G, \mathcal{G}, H_b)$ induces a surjection $I(\mathbb{R})/I(\mathbb{R})^\circ \to \mathcal{G}(\mathbb{R})\backslash D(G, \mathcal{G}, H_b)$ where $I(\mathbb{R})^\circ$ denotes the connected component of $I(\mathbb{R})$ containing 1.

Proposition 1.1.21 in the case G is reductive follows from Claim 2 and the finiteness of $I(\mathbb{R})/I(\mathbb{R})^\circ$.

We prove Claim 2. Let $g_1, g_2 \in I(\mathbb{R})$ and assume $g_2 = g_1 g$ for some $g \in I(\mathbb{R})^\circ$. We prove $g_2 \in \mathcal{G}(\mathbb{R}) g_1 J(\mathbb{R})$. The image $g_{\mathcal{Q}}$ of g in $J_{\mathcal{Q}}(\mathbb{R})$ belongs to the connected component of $J_{\mathcal{Q}}(\mathbb{R})$ containing 1 and hence $g_{\mathcal{Q}} = \exp(A)$ for some $A \in \mathrm{Lie}(J_{\mathcal{Q}})_{\mathbb{R}}$. Since G is reductive, the surjection $\mathrm{Lie}(G) \to \mathrm{Lie}(\mathcal{Q})$ is regarded as the projection to a direct factor of the Lie algebra $\mathrm{Lie}(G)$, and hence the map $\mathrm{Lie}(J) \to \mathrm{Lie}(J_{\mathcal{Q}})$ is surjective. Take an element $\tilde{A} \in \mathrm{Lie}(J)_{\mathbb{R}}$ which is sent to A and let $a = \exp(\tilde{A}) \in J(\mathbb{R})$. Then $h := ga^{-1}$ belongs to $\mathcal{G}(\mathbb{R})$. We have $g_2 = g_1 h a = g_1 h g_1^{-1} g_1 a$. Since $\mathcal{G}$ is normal in G, $g_1 h g_1^{-1} \in \mathcal{G}(\mathbb{R})$.

Now we do not assume G is reductive. By the case G is reductive treated above, it is sufficient to prove that if $y_1, y_2 \in D(G, \mathcal{G}, H_b)$ and if the images of y_1 and y_2 in $D(G_{\mathrm{red}}, \Upsilon)$ coincide, then $y_2 = gy_1$ for some $g \in \mathcal{G}_u(\mathbb{C})$. Take a homomorphism $h : S_{\mathbb{C}/\mathbb{R}} \to G_{\mathbb{R}}$ which lifts the common homomorphism $S_{\mathbb{C}/\mathbb{R}} \to G_{\mathrm{red},\mathbb{R}}$ induced by y_1 and by y_2 and let H be the element of $D(G, \Upsilon)$ corresponding to h. Then H induces $\mathrm{Rep}(G) \to \mathbb{R}\mathrm{HS}$. By [6] 2.20, $y_j = e^{i\delta(y_j)} u(y_j) H$ for unique $u(y_j) \in G_u(\mathbb{R})$ and $\delta(y_j) \in \mathrm{Lie}(G_u)_{\mathbb{R}}$ such that for any $V \in \mathrm{Rep}(G)$ and for any $p, q \in \mathbb{Z}$, $\delta(y_i)$ sends the (p, q)-Hodge component of $u(y_j)H(V)$ into the sum of (p', q')-Hodge components of $u(y_j)H(V)$ with $p' < p$, $q' < q$. Since $y_{1,\mathcal{Q}} = y_{2,\mathcal{Q}}$ in $D(\mathcal{Q}, \Upsilon_{\mathcal{Q}})$, we have $u(y_1)_{\mathcal{Q}} = u(y_2)_{\mathcal{Q}}$ in $\mathcal{Q}_u(\mathbb{R})$ and $\delta(y_1)_{\mathcal{Q}} = \delta(y_2)_{\mathcal{Q}}$ in $\mathrm{Lie}(\mathcal{Q}_u)_{\mathbb{R}}$. Thus $y_2 = gy_1$ with $g = e^{i\delta(y_2)} u(y_2) u(y_1)^{-1} e^{-i\delta(y_1)}$ and $g_{\mathcal{Q}} = 1$. Hence $g \in \mathcal{G}_u(\mathbb{C})$. Q.E.D.

1.1.22. The action of $\mathcal{G}(\mathbb{R})\mathcal{G}_u(\mathbb{C})$ on $D(G, \mathcal{G}, H_b)$ need not be transitive.

Example. Let $G = J \cup J\sigma \subset GL_2$, where

$$J = \{\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \in GL_2\}, \quad \sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let $w : \mathbf{G}_m \to G$ be the canonical embedding as scaler matrices. Consider the $G(\mathbb{R})$-conjugacy class $\Upsilon = \{h_+, h_-\}$ where $h_\pm : S_{\mathbb{C}/\mathbb{R}} \to J \subset G$

is defined by

$$h_{\pm}(a+bi) = \begin{pmatrix} a & \mp b \\ \pm b & a \end{pmatrix}.$$

We have $D(G,\Upsilon) = \{h_+, h_-\}$. Let $\mathcal{G} = G \cap SL_2$ and let $H_b = h_+ \in D(G,\Upsilon)$. Then $\det : \mathcal{Q} = G/\mathcal{G} \overset{\cong}{\to} \mathbf{G}_m$ and hence $D(\mathcal{Q}, \Upsilon_{\mathcal{Q}})$ consists of one element. Hence $D(G,\mathcal{G},H_b)$ consists of two elements h_+ and h_-. But $\mathcal{G}(\mathbb{R}) \cong SO(2,\mathbb{R})$ is connected and $\mathcal{G}_u = \{1\}$, and hence the action of $\mathcal{G}(\mathbb{R})\mathcal{G}_u(\mathbb{C})$ on $D(G,\mathcal{G},H_b)$ is trivial and is not transitive.

1.2. The period domain $X(\mathbb{C})$

In this Section 1.2, we consider the period domain $X(\mathbb{C}) = X_{G,\mathcal{G},H_b,K}(\mathbb{C})$ (see 1.2.3 for the definition). We also describe a special case $X_{G,\Upsilon,K}(\mathbb{C})$ (see 1.2.2) of $X_{G,\mathcal{G},H_b,K}(\mathbb{C})$ hoping that it works as a guide to the more complicated object $X_{G,\mathcal{G},H_b,K}(\mathbb{C})$.

1.2.1. Let $\mathbf{A}^f_{\mathbb{Q}}$ be the non-archimedean component of the adele ring of $\mathbb{Q}$.

1.2.2. Assume we are given Υ as in 1.1.7. Let K be an open compact subgroup of $G(\mathbf{A}^f_{\mathbb{Q}})$. Define

$$X_{G,\Upsilon,K}(\mathbb{C}) := G(\mathbb{Q})\backslash(D(G,\Upsilon) \times (G(\mathbf{A}^f_{\mathbb{Q}})/K)).$$

Here $G(\mathbb{Q})$ acts on $D(G,\Upsilon) \times (G(\mathbf{A}^f_{\mathbb{Q}})/K)$ diagonally from the left.

1.2.3. Assume that we are given $(\mathcal{G}, H_b)$ as in 1.1.10. Let K be an open compact subgroup of $\mathcal{G}(\mathbf{A}^f_{\mathbb{Q}})$.

Define

$$X(\mathbb{C}) = X_{G,\mathcal{G},H_b,K}(\mathbb{C}) := \mathcal{G}(\mathbb{Q})\backslash(D(G,\mathcal{G},H_b) \times (\mathcal{G}(\mathbf{A}^f_{\mathbb{Q}})/K)),$$

where $\mathcal{G}(\mathbb{Q})$ acts on $D(G,\mathcal{G},H_b) \times (\mathcal{G}(\mathbf{A}^f_{\mathbb{Q}})/K)$ diagonally from the left.

1.2.4. $X_{G,\Upsilon,K}(\mathbb{C})$ is the case $\mathcal{G} = G$ of $X_{G,\mathcal{G},H_b,K}(\mathbb{C})$ (1.1.12).

1.2.5. As is explained in 1.2.6 below, $X_{G,\Upsilon,K}(\mathbb{C})$ is identified with the set of isomorphism classes of pairs (H,λ), where

H is an exact $\otimes$-functor from $\mathrm{Rep}(G)$ to $\mathbb{Q}$MHS

such that for some isomorphism of $\otimes$-functors from $\mathrm{Rep}(G)$ to $\mathrm{Mod}_{ff}(\mathbb{Q})$

$$\theta : (V \mapsto H(V)_{\mathbb{Q}}) \overset{\cong}{\to} (V \mapsto V) \quad \text{preserving the weight filtrations,}$$

the homomorphism $S_{\mathbb{C}/\mathbb{R}} \to G_{\mathrm{red}}(\mathbb{R})$ induced by the restrictions of H and θ to $\mathrm{Rep}(G_{\mathrm{red}})$ belongs to Υ, and

λ (called the K-level structure) is a mod K class of an isomorphism of $\otimes$-functors from $\mathrm{Rep}(G)$ to $\mathrm{Mod}_{ff}(\mathbf{A}^f_{\mathbb{Q}})$

$$\tilde{\lambda} : (V \mapsto \mathbf{A}^f_{\mathbb{Q}} \otimes_{\mathbb{Q}} V) \overset{\cong}{\to} (V \mapsto \mathbf{A}^f_{\mathbb{Q}} \otimes_{\mathbb{Q}} H(V)_{\mathbb{Q}}) \quad \text{preserving the weight filtrations.}$$

1.2.6. In 1.2.5, we identify the class of a pair (H, λ) as above with $\mathrm{class}(H', g) \in X_{G,\Upsilon,K}(\mathbb{C})$, where $H' = H$ endowed with the identification of $H(V)_{\mathbb{Q}}$ with V via θ, and $g = (\mathbf{A}^f_{\mathbb{Q}} \otimes \theta) \circ \tilde{\lambda} \in G(\mathbf{A}^f_{\mathbb{Q}})$. Conversely, we identify $\mathrm{class}(H', g) \in X_{G,\Upsilon,K}(\mathbb{C})$ ($\mathrm{class}(H') \in D$, $g \in G(\mathbf{A}^f_{\mathbb{Q}})$) with the class of the pair (H, λ), where $H = H'$, $\tilde{\lambda} = g$.

1.2.7. As is explained in 1.2.8 below, the set $X_{G,\mathcal{G},H_b,K}(\mathbb{C})$ is identified with the set of isomorphism classes of triples (H, ξ, λ), where

H is an exact $\otimes$-functor from $\mathrm{Rep}(G)$ to $\mathbb{Q}$MHS,

ξ is an isomorphism of $\otimes$-functors from $\mathrm{Rep}(\mathcal{Q})$ to $\mathbb{Q}$MHS

$$H|_{\mathcal{Q}} \overset{\cong}{\to} H_b|_{\mathcal{Q}},$$

where $H|_{\mathcal{Q}}$ and $H_b|_{\mathcal{Q}}$ denote the restrictions of H and H_b to $\mathrm{Rep}(\mathcal{Q}) \subset \mathrm{Rep}(G)$, respectively,

λ is as in 1.2.5,

such that there are an isomorphism of $\otimes$-functors from $\mathrm{Rep}(G_{\mathrm{red}})$ to $\mathbb{R}$HS

$$\nu : (V \mapsto \mathbb{R} \otimes_{\mathbb{Q}} H(V)) \overset{\cong}{\to} (V \mapsto \mathbb{R} \otimes_{\mathbb{Q}} H_b(V))$$

and an isomorphism of $\otimes$-functors from $\mathrm{Rep}(G)$ to $\mathrm{Mod}_{ff}(\mathbb{Q})$

$$\theta : (V \mapsto H(V)_{\mathbb{Q}}) \overset{\cong}{\to} (V \mapsto V) \quad \text{preserving the weight filtrations}$$

which satisfy the following (i)–(iii).

(i) ξ and ν induce the same isomorphism of functors $(V \mapsto \mathbb{R} \otimes_{\mathbb{Q}} H(V)) \overset{\cong}{\to} (V \mapsto \mathbb{R} \otimes_{\mathbb{Q}} H_b(V))$ from $\mathrm{Rep}(\mathcal{Q}_{\mathrm{red}})$ to $\mathbb{R}$HS.

(ii) ξ and θ induce the same isomorphism of functors $(V \mapsto H(V)_{\mathbb{Q}}) \overset{\cong}{\to} (V \mapsto H_b(V)_{\mathbb{Q}})$ from $\mathrm{Rep}(\mathcal{Q})$ to $\mathrm{Mod}_{ff}(\mathbb{Q})$.

(iii) The automorphism $(\mathbf{A}^f_{\mathbb{Q}} \otimes \theta) \circ \tilde{\lambda}$ ($\tilde{\lambda}$ is a representative of λ as in 1.2.5) of the $\otimes$-functor

$$\mathrm{Rep}(G) \to \mathrm{Mod}_{ff}(\mathbf{A}^f_{\mathbb{Q}}) \; ; \; V \mapsto \mathbf{A}^f_{\mathbb{Q}} \otimes_{\mathbb{Q}} V$$

(which is an element of $G(\mathbf{A}^f_{\mathbb{Q}})$ by the theory of Tannakian categories) belongs to $\mathcal{G}(\mathbf{A}^f_{\mathbb{Q}})$.

1.2.8. In 1.2.7, we identify the class of a triple (H, ξ, λ) as above with class$(H', g) \in X_{G,\mathcal{G},H_b,K}(\mathbb{C})$, where $H' = H$ endowed with the identification of $H(V)_{\mathbb{Q}}$ with V via θ, and $g = (\mathbf{A}^f_{\mathbb{Q}} \otimes \theta) \circ \tilde{\lambda} \in \mathcal{G}(\mathbf{A}^f_{\mathbb{Q}})$. Conversely, we identify class$(H', g) \in X_{G,\mathcal{G},H_b,K}(\mathbb{C})$ (class$(H') \in D$, $g \in \mathcal{G}(\mathbf{A}^f_{\mathbb{Q}})$) with the class of the triple (H, ξ, λ), where $H = H'$, $\tilde{\lambda} = g$, and ξ is the evident isomorphism.

In the case $\mathcal{G} = G$, the identification $X_{G,\mathcal{G},H_b,K}(\mathbb{C}) = X_{G,\Upsilon,K}(\mathbb{C})$ corresponds to $(H, \xi, \lambda) \mapsto (H, \lambda)$ (ξ is unique and is the evident isomorphism in this case).

Let $(\mathcal{G}, H_b, K)$ be as in 1.2.3. We define a structure of a complex manifold on $X(\mathbb{C}) = X_{G,\mathcal{G},H_b,K}(\mathbb{C})$ assuming 1.2.9, 1.2.10 (concerning polarization) and assuming 1.2.11 (concerning level structure) below.

1.2.9. We assume we are given $V_0 \in \mathrm{Rep}(G)$, a non-degenerate bilinear form

$$\langle\ ,\ \rangle_{0,w} : \mathrm{gr}^W_w V_0 \times \mathrm{gr}^W_w V_0 \to \mathbb{Q} \cdot (2\pi i)^{-w}$$

for each $w \in \mathbb{Z}$ which is symmetric if w is even and anti-symmetric if w is odd, and a homomorphism

$$\eta : G \to \mathbf{G}_m$$

having the following properties (i)–(iii).

(i) $\langle gx, gy\rangle_{0,w} = \eta(g)^w \langle x, y\rangle_{0,w}$ for any $w \in \mathbb{Z},\ g \in G,\ x, y \in \mathrm{gr}^W_w V_0$.

(ii) The composition $\mathbf{G}_m \xrightarrow{w} G_{\mathrm{red}} \xrightarrow{\eta} \mathbf{G}_m$ is $x \mapsto x^2$.

(iii) The homomorphism $G \to \mathrm{Aut}(V_0) \times \mathbf{G}_m$ is injective, where the part $G \to \mathbf{G}_m$ is η.

1.2.10. Consider the one-dimensional $\mathbb{Q}$-vector space $\mathbb{Q} \cdot (2\pi i)^{-1}$ on which G acts via η. By (ii), $H_b(\mathbb{Q} \cdot (2\pi i)^{-1})$ is a one-dimensional Hodge structure of weight 2 whose underlying $\mathbb{Q}$-vector space is $\mathbb{Q} \cdot (2\pi i)^{-1}$, and hence it is canonically identified with the Hodge structure $\mathbb{Q}(-1)$. The homomorphism $\mathrm{gr}^W_w V_0 \otimes \mathrm{gr}^W_w V_0 \to \mathbb{Q} \cdot (2\pi i)^{-w} = (\mathbb{Q} \cdot (2\pi i)^{-1})^{\otimes w}$ defined by $\langle\ ,\ \rangle_{0,w}$ is a G-homomorphism, and hence induces $\mathrm{gr}^W_w H_b(V_0) \otimes \mathrm{gr}^W_w H_b(V_0) \to \mathbb{Q}(-w)$.

We assume the following (*).

(*) This homomorphism $\mathrm{gr}^W_w H_b(V_0) \otimes \mathrm{gr}^W_w H_b(V_0) \to \mathbb{Q}(-w)$ is a polarization of $\mathrm{gr}^W_w H_b(V_0)$ for any $w \in \mathbb{Z}$.

1.2.11. From now on, we assume that K satisfies the following condition (*) which we call the neat condition.

(*) If $g \in \mathcal{G}(\mathbf{A}_{\mathbb{Q}}^f)$ and $\gamma \in \mathcal{G}(\mathbb{Q}) \cap gKg^{-1} \subset \mathcal{G}(\mathbf{A}_{\mathbb{Q}}^f)$ and if the action of γ on $\mathrm{gr}_w^W V_0$ preserves $\langle\ ,\ \rangle_{0,w}$ for any $w \in \mathbb{Z}$, then for any $V \in \mathrm{Rep}(G)$, the subgroup of $\mathbb{C}^\times$ generated by all eigen values of $\gamma : \mathbb{C} \otimes_{\mathbb{Q}} V \to \mathbb{C} \otimes_{\mathbb{Q}} V$ is torsion free.

Remark 1.2.12. If L is a $\mathbb{Z}$-lattice of V_0 and if $n \geq 3$ is an integer, and if the action of any element g of K on $\mathbf{A}_{\mathbb{Q}}^f \otimes_{\mathbb{Q}} V_0$ satisfies the following (i) and (ii), then K satisfies the neat condition (1.2.11).

(i) $g\hat{L} = \hat{L}$, where $\hat{L} = \hat{\mathbb{Z}} \otimes_{\mathbb{Z}} L$.

(ii) The automorphism of $L/nL = \hat{L}/n\hat{L}$ induced by g is the identity map.

Hence a sufficiently small open compact subgroup K of $\mathcal{G}(\mathbf{A}_{\mathbb{Q}}^f)$ satisfies the neat condition.

Proposition 1.2.13. *(As is said, we assume 1.2.9, 1.2.10, 1.2.11.) The automorphism group of a triple (H, ξ, λ) as in 1.2.7 is trivial.*

Proof. The proof goes in the same way as in the corresponding part of Part I (the proof of Proposition 2.2.7 of Part I). Q.E.D.

By 1.2.13, we will identify a triple (H, ξ, λ) as in 1.2.7 with its class in $X(\mathbb{C})$.

We will sometimes denote $(H, \xi, \lambda) \in X(\mathbb{C})$ also simply as $H \in X(\mathbb{C})$ omitting ξ and λ.

1.2.14. For $(H, \xi, \lambda) \in X(\mathbb{C})$, we have a canonical polarization $\langle\ ,\ \rangle_w$ on $\mathrm{gr}_w^W H(V_0)$ for each $w \in \mathbb{Z}$ defined as follows.

Take θ in 1.2.7. Let $g \in \mathcal{G}(\mathbf{A}_{\mathbb{Q}}^f)$ be the composition $(\mathbf{A}_{\mathbb{Q}}^f \otimes \theta) \circ \tilde{\lambda}$, where $\tilde{\lambda}$ is a representative of λ (1.2.7), and let $c = \prod_p |\eta(g)_p|_p \in \mathbb{Q}_{>0}$ (here p ranges over all prime numbers, $\eta(g)_p$ denotes the p-component of $\eta(g) \in (\mathbf{A}_{\mathbb{Q}}^f)^\times$, and $|\ |_p$ denotes the standard absolute value of $\mathbb{Q}_p$). Then c is independent of the choice of the representative $\tilde{\lambda}$ of λ because $|\eta(k)_p|_p = 1$ for any $k \in K$ and any p.

Let $\mathrm{gr}_w^W H(V_0)_{\mathbb{Q}} \overset{\cong}{\to} \mathrm{gr}_w^W H_b(V_0)_{\mathbb{Q}} = \mathrm{gr}_w^W V_0$ be the isomorphism defined by θ and let $\langle\ ,\ \rangle'_w$ be the bilinear form $\mathrm{gr}_w^W H(V_0)_{\mathbb{Q}} \times \mathrm{gr}_w^W H(V_0)_{\mathbb{Q}} \to \mathbb{Q} \cdot (2\pi i)^{-w}$ corresponding to $\langle\ ,\ \rangle_{0,w}$ via this isomorphism. By the existence of the isomorphism ν (1.2.7), we have either $\langle\ ,\ \rangle'_w$ is a polarization for any w or $(-1)^w \langle\ ,\ \rangle'_w$ is a polarization for any w. We define the canonical polarization $\langle\ ,\ \rangle_w$ on $\mathrm{gr}_w^W H(V_0)$ to be $c^w \langle\ ,\ \rangle'_w$ in the former case and $(-1)^w c^w \langle\ ,\ \rangle'_w$ in the latter case. Then $\langle\ ,\ \rangle_w$ is independent of the choice of θ.

Proposition 1.2.15. *(1) The action of $\mathcal{G}(\mathbb{Q})$ on $D(G, \mathcal{G}, H_b) \times (\mathcal{G}(\mathbf{A}_{\mathbb{Q}}^f)/K)$ is proper. This action is fixed-point free.*

(2) The canonical surjection $D(G, \mathcal{G}, H_b) \times (\mathcal{G}(\mathbf{A}_{\mathbb{Q}}^f)/K) \to X(\mathbb{C})$ is a local homeomorphism (for the quotient topology on $X(\mathbb{C})$).

Proof. (2) follows from (1). (1) is reduced to the fact that for any $g \in \mathcal{G}(\mathbf{A}_{\mathbb{Q}}^f)$, the action of $\mathcal{G}(\mathbb{Q}) \cap gKg^{-1}$ on $D(G, \mathcal{G}, H_b)$ is proper and fixed-point-free. The last thing is proved as follows.

Let $\Phi = ((h(w,r))_{w,r\in\mathbb{Z}}, V_0, W, (\langle\ ,\ \rangle_{0,w})_{w\in\mathbb{Z}})$, where $h(w,r)$ is the dimension of $\mathrm{gr}^r\mathrm{gr}^W_w H_b(V_0)_{\mathbb{C}}$ as a $\mathbb{C}$-vector space, and let $D^{\pm}_{\Phi}$ be the corresponding space $D^{\pm}$ in Part I, Section 2.2. We have an embedding $D(G, \mathcal{G}, H_b) \subset D^{\pm}_{\Phi}$. Let G_{Φ} be the algebraic group over $\mathbb{Q}$ defined as the subgroup of $\mathrm{Aut}_{\mathbb{Q}}(V_0, W) \times \mathbf{G}_m$ consisting of all elements (g, t) such that $\langle gx, gy\rangle_{0,w} = t^w\langle x, y\rangle_{0,w}$ for all $w \in \mathbb{Z}$ and all $x, y \in \mathrm{gr}^W_w$. (This group G_{Φ} was denoted by G in Part I.) We have an embedding $G \overset{\subseteq}{\to} G_{\Phi}$.

By these embeddings $D(G, \mathcal{G}, H_b) \subset D^{\pm}_{\Phi}$ and $G \subset G_{\Phi}$, we are reduced to the fact that the action of $G_{\Phi}(\mathbb{Q}) \cap gKg^{-1}$ on $D^{\pm}_{\Phi}$ is proper and fixed-point-free and this follows from Part I, Section 2.2. Q.E.D.

1.2.16. By 1.2.15 (2), there is a unique structure of a complex analytic manifold on $X_{G,\mathcal{G},H_b,K}(\mathbb{C})$ for which the canonical surjection $D(G, \mathcal{G}, H_b) \times (\mathcal{G}(\mathbf{A}_{\mathbb{Q}}^f)/K) \to X_{G,\mathcal{G},H_b,K}(\mathbb{C})$ is a morphism of complex analytic manifolds and is locally an isomorphism.

Note that $\mathcal{G}_u = \mathcal{G} \cap G_u$ and hence $\mathcal{G}_{\mathrm{red}}$ is the image of $\mathcal{G}$ in G_{red}.

Proposition 1.2.17. *Let K_{red} be the image of K in $\mathcal{G}_{\mathrm{red}}(\mathbf{A}_{\mathbb{Q}}^f)$. Then $(G_{\mathrm{red}}, \mathcal{G}_{\mathrm{red}}, H_b|_{G_{\mathrm{red}}}, K_{\mathrm{red}})$ also satisfies the neat condition.*

Proof. We are reduced to the fact that $\mathcal{G}(\mathbb{Q}) \cap gKg^{-1} \to \mathcal{G}_{\mathrm{red}}(\mathbb{Q}) \cap g_{\mathrm{red}}K_{\mathrm{red}}g_{\mathrm{red}}^{-1}$ is surjective for any $g \in \mathcal{G}(\mathbf{A}_{\mathbb{Q}}^f)$ where g_{red} denotes the image of g in $\mathcal{G}_{\mathrm{red}}(\mathbf{A}_{\mathbb{Q}}^f)$. This last fact is reduced to the following well known fact: For a unipotent algebraic group $\mathcal{U}$ over $\mathbb{Q}$ and for an open subgroup U of $\mathcal{U}(\mathbf{A}_{\mathbb{Q}}^f)$, $\mathcal{U}(\mathbf{A}_{\mathbb{Q}}^f) = \mathcal{U}(\mathbb{Q})U$. Q.E.D.

1.2.18. By 1.2.17, we have the complex analytic manifold $X_{\mathrm{red}}(\mathbb{C}) := X_{G_{\mathrm{red}},\mathcal{G}_{\mathrm{red}},H_b|_{G_{\mathrm{red}}},K_{\mathrm{red}}}(\mathbb{C})$. We have a canonical morphism $X_{G,\mathcal{G},H_b,K}(\mathbb{C}) \to X_{G_{\mathrm{red}},\mathcal{G}_{\mathrm{red}},H_b|_{G_{\mathrm{red}}},K_{\mathrm{red}}}(\mathbb{C})$.

1.2.19. The complex analytic manifold $X(\mathbb{C}) = X_{G,\mathcal{G},H_b,K}(\mathbb{C})$ represents the functor $\mathcal{A} \to$ (Sets) which sends $Y \in \mathcal{A}$ to the set of isomorphism classes of triples $(\mathcal{H}, \xi, \lambda)$, where:

$\mathcal{H}$ is an exact $\otimes$-functors $\mathrm{Rep}(G) \to \mathbb{Q}\mathrm{MHS}(Y)$ (1.1.16),

ξ is an isomorphism of $\otimes$-functors from $\mathrm{Rep}(\mathcal{Q})$ to $\mathbb{Q}\mathrm{MHS}(Y)$

$$\mathcal{H}|_{\mathcal{Q}} \overset{\cong}{\to} H_b|_{\mathcal{Q}},$$

where $\mathcal{H}|_{\mathcal{Q}}$ and $H_b|_{\mathcal{Q}}$ denote the restrictions of $\mathcal{H}$ and H_b to $\mathrm{Rep}(\mathcal{Q}) \subset \mathrm{Rep}(G)$ ($H_b(V)$ for $V \in \mathrm{Rep}(G)$ is defined to be a constant $\mathbb{Q}$MHS $H_b(V)$ on Y), respectively,

λ (called the K-level structure) is a global section of the quotient sheaf $\mathcal{I}/K$, where $\mathcal{I}(U)$ for an open set U of Y is the set of isomorphisms of $\otimes$-functors from $\mathrm{Rep}(G)$ to $\mathrm{Mod}_{ff}(\mathbf{A}^f_{\mathbb{Q}})(U)$

$$(V \mapsto \mathbf{A}^f_{\mathbb{Q}} \otimes_{\mathbb{Q}} V) \overset{\cong}{\to} (V \mapsto \mathbf{A}^f_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathcal{H}_{\mathbb{Q}}|_U),$$

such that there are an isomorphism of $\otimes$-functors from $\mathrm{Rep}(G_{\mathrm{red}})$ to $\mathbb{R}$HS

$$\nu : (V \mapsto \mathbb{R} \otimes_{\mathbb{Q}} \mathcal{H}(V)(y)) \overset{\cong}{\to} (V \mapsto \mathbb{R} \otimes_{\mathbb{Q}} H_b(V))$$

at each $y \in Y$ and an isomorphism of $\otimes$-functors from $\mathrm{Rep}(G)$ to $\mathrm{Mod}_{ff}(\mathbb{Q}, Y)$

$$\theta : (V \mapsto \mathcal{H}(V)_{\mathbb{Q}}) \overset{\cong}{\to} (V \mapsto V) \quad \text{preserving the weight filtrations}$$

locally on Y, which satisfy the following (i)–(iii).

(i) At each $y \in Y$, ξ and ν induce the same isomorphism of functors $(V \mapsto \mathbb{R} \otimes_{\mathbb{Q}} \mathcal{H}(V)(y)) \overset{\cong}{\to} (V \mapsto \mathbb{R} \otimes_{\mathbb{Q}} H_b(V))$ from $\mathrm{Rep}(\mathcal{Q}_{\mathrm{red}})$ to $\mathbb{R}$HS.

(ii) Locally on Y, ξ and θ induce the same isomorphism of functors $(V \mapsto \mathcal{H}(V)_{\mathbb{Q}}) \overset{\cong}{\to} (V \mapsto H_b(V)_{\mathbb{Q}})$ from $\mathrm{Rep}(\mathcal{Q})$ to $\mathrm{Mod}_{ff}(\mathbb{Q}, Y)$.

(iii) Locally on Y, the automorphism $(\mathbf{A}^f_{\mathbb{Q}} \otimes \theta) \circ \tilde{\lambda}$ ($\tilde{\lambda}$ is a representative of λ in $\mathcal{I}$) of the $\otimes$-functor

$$\mathrm{Rep}(G) \to \mathrm{Mod}_{ff}(\mathbf{A}^f_{\mathbb{Q}})\ ;\ V \mapsto \mathbf{A}^f_{\mathbb{Q}} \otimes_{\mathbb{Q}} V$$

belongs to $\mathcal{G}(\mathbf{A}^f_{\mathbb{Q}})$.

1.2.20. In this paper, we will often use the following fact ([30] Thm. 4.14): If $V_1 \in \mathrm{Rep}(G)$ is a faithful representation of G, V_1 generates $\mathrm{Rep}(G)$ as a $\otimes$-category. That is, all finite-dimensional representations of G over $\mathbb{Q}$ can be constructed from V_1 by taking $\otimes$, $\oplus$, the dual, and subquotients.

1.2.21. Taking $Y = X(\mathbb{C})$ in the above, we have the universal object $\mathcal{H}_{X(\mathbb{C})} : \mathrm{Rep}(G) \to \mathbb{Q}\mathrm{MHS}(X(\mathbb{C}))$.

The canonical polarizations on the fibers of $\mathcal{H}_{X(\mathbb{C})}(V_0)$ at each point of $X(\mathbb{C})$ (1.2.14) give a canonical polarization on $\mathcal{H}_{X(\mathbb{C})}$.

For any $V \in \mathrm{Rep}(G)$ and $w \in \mathbb{Z}$, we have a homomorphism $\mathrm{gr}^W_w V \otimes \mathrm{gr}^W_w V \to \mathbb{Q} \cdot (2\pi i)^{-w}$ of representations of G such that $\mathcal{H}(\mathrm{gr}^W_w V) \otimes \mathcal{H}(\mathrm{gr}^W_w V) \to \mathbb{Q} \cdot (2\pi i)^{-w}$ is a polarization of $\mathcal{H}(\mathrm{gr}^W_w V)$. This is true for

$V = V_0$ (the canonical polarization) and is true for general V because $\mathrm{gr}^W(V_0)$ and $\mathbb{Q} \cdot (2\pi i)^{-1}$ generate the $\otimes$-category $\mathrm{Rep}(G_{\mathrm{red}})$ (1.2.20).

1.2.22. Concerning the tangent bundle $T_{X(\mathbb{C})}$ of $X(\mathbb{C})$, we have a canonical isomorphism

$$T_{X(\mathbb{C})} \cong \mathcal{H}_{X(\mathbb{C})}(\mathrm{Lie}(\mathcal{G}))_{\mathcal{O}}/\mathrm{fil}^0,$$

where $\mathrm{Lie}(\mathcal{G})$ is endowed with the adjoint action of G and fil denotes the Hodge filtration.

This follows from 1.2.19 and 1.1.20 (2).

1.2.23. We define the horizontal tangent bundle $T_{X(\mathbb{C}),\mathrm{hor}}$ of $X(\mathbb{C})$ as $\mathrm{fil}^{-1}/\mathrm{fil}^0$, where fil is the Hodge filtration of $\mathcal{H}_{X(\mathbb{C})}(\mathrm{Lie}(\mathcal{G}))_{\mathcal{O}}$.

It is a subbundle of the tangent bundle $T_{X(\mathbb{C})} = \mathcal{H}_{X(\mathbb{C})}(\mathrm{Lie}(\mathcal{G}))_{\mathcal{O}}/\mathrm{fil}^0$.

1.2.24. In 1.2.17, 1.2.18, $T_{X_{\mathrm{red}}(\mathbb{C})}$ and $T_{X_{\mathrm{red}}(\mathbb{C}),\mathrm{hor}}$ are identified with $(\mathrm{gr}^W_0 \mathcal{H}_{X(\mathbb{C})}(\mathrm{Lie}(\mathcal{G}))_{\mathcal{O}})/\mathrm{fil}^0$ and $\mathrm{gr}^{-1}\mathrm{gr}^W_0 \mathcal{H}_{X(\mathbb{C})}(\mathrm{Lie}(\mathcal{G}))_{\mathcal{O}}$, respectively.

1.2.25. Let Y be a complex analytic manifold. We discuss horizontal morphisms from Y to a period domain.

Let $\mathbb{Q}\mathrm{VMHS}(Y)$ be the category of variations of mixed $\mathbb{Q}$-Hodge structure on Y. It is a full subcategory of $\mathbb{Q}\mathrm{MHS}(Y)$. By definition, an object of $\mathbb{Q}\mathrm{MHS}(Y)$ belongs to $\mathbb{Q}\mathrm{VMHS}(Y)$ if and only if it satisfies Griffiths transversality.

A morphism $f : Y \to X(\mathbb{C}) = X_{G,\mathcal{G},H_b,K}(\mathbb{C})$ is said to be *horizontal* if the morphism $T_Y \to T_{X(\mathbb{C})}$ of tangent bundles associated to f factors through the horizontal tangent bundle $T_{X(\mathbb{C}),\mathrm{hor}} \subset T_{X(\mathbb{C})}$. A morphism $Y \to X(\mathbb{C})$ is horizontal if and only if the corresponding object $\mathcal{H} : \mathrm{Rep}(G) \to \mathbb{Q}\mathrm{MHS}(Y)$ satisfies that $\mathcal{H}(V) \in \mathbb{Q}\mathrm{VMHS}(Y)$ for any $V \in \mathrm{Rep}(G)$.

1.2.26. Let Y be a one-dimensional complex analytic manifold (that is, a Riemann surface). Let $\mathbb{Q}\mathrm{VMHS}_{\log}(Y)$ be the full subcategory of $\cup_R \ \mathbb{Q}\mathrm{VMHS}(Y \smallsetminus R)$, where R ranges over all discrete subsets of Y, consisting of objects which satisfy the conditions in Part I, Section 1.6 at any point of R when we replace C there by Y.

We say an element of $\cup_R \mathrm{Mor}_{\mathrm{hor}}(Y \smallsetminus R, X(\mathbb{C}))$, where R ranges over all discrete subsets of Y, is *meromorphic* on Y if the corresponding functor $\mathrm{Rep}(G) \to \cup_R \ \mathbb{Q}\mathrm{VMHS}(Y \smallsetminus R)$ factors through $\mathrm{Rep}(G) \to \mathbb{Q}\mathrm{VMHS}_{\log}(Y)$. Let $\mathcal{M}_{\mathrm{hor}}(Y, X(\mathbb{C}))$ be the subset of $\cup_R \mathrm{Mor}_{\mathrm{hor}}(Y \smallsetminus R, X(\mathbb{C}))$ consisting of all elements which are meromorphic on Y.

1.2.27. For Y as in 1.2.26, we define the subset $\mathcal{M}_{\mathrm{hor,gen}}(Y, X(\mathbb{C}))$ of $\mathcal{M}_{\mathrm{hor}}(Y, X(\mathbb{C}))$ consisting of *generic* elements.

Let $f \in \mathcal{M}_{\mathrm{hor}}(Y, X(\mathbb{C}))$ and let $\mathcal{H} : \mathrm{Rep}(G) \to \mathbb{Q}\mathrm{VMHS}_{\log}(Y)$ be the corresponding exact $\otimes$-functor. Then by our definition, f belongs to $\mathcal{M}_{\mathrm{hor,gen}}(Y, X(\mathbb{C}))$ if and only if there is no algebraic subgroup G' of G such that $\dim(G') < \dim(G)$ and such that $\mathcal{H}$ is isomorphic to the composition $\mathrm{Rep}(G) \to \mathrm{Rep}(G') \overset{\mathcal{H}'}{\to} \mathbb{Q}\mathrm{VMHS}_{\log}(Y)$ for some exact $\otimes$-functor $\mathcal{H}' : \mathrm{Rep}(G') \to \mathbb{Q}\mathrm{VMHS}_{\log}(Y)$

1.3. The set $X(F)$ of motives

In this Section 1.3, we consider a set $X(F) = X_{G,\mathcal{G},M_b,K}(F)$ of G-mixed motives (see 1.3.5 for the definition). We also describe a special case $X_{G,\Upsilon,K}(F)$ (see 1.3.3), which is simpler, as a guide. We define $X_{G,\Upsilon,K}(F)$ (resp. $X_{G,\mathcal{G},M_b,K}(F)$) imitating the presentation 1.2.5 of $X_{G,\Upsilon,K}(\mathbb{C})$ (resp. 1.2.7 of $X_{G,\mathcal{G},H_b,K}(\mathbb{C})$).

1.3.1. As in Part I, we use the formulation of mixed motives due to Jannsen [18]. See Part I, Section 1.1 for a summary of points in [18] which are important in our study.

For a number field F, we denote the category of mixed motives with $\mathbb{Q}$-coefficients over F by $MM(F)$.

1.3.2. Let G and $w : \mathbf{G}_m \to G_{\mathrm{red}}$ be as in 1.1.4. Let $V_0 \in \mathrm{Rep}(G)$, $\langle\ ,\ \rangle_{0,w} : \mathrm{gr}^W_w V_0 \times \mathrm{gr}^W_w V_0 \to \mathbb{Q} \cdot (2\pi i)^{-w}$ for $w \in \mathbb{Z}$, and $\eta : G \to \mathbf{G}_m$ be as in 1.2.9. Let $\mathcal{G}$ be as in 1.1.10. In what follows, when we discuss $X_{G,\Upsilon,K}(F)$, $\mathcal{G}$ denotes G. Let K be an open subgroup of $\mathcal{G}(\mathbf{A}^f_{\mathbb{Q}})$ satisfying the neat condition (1.2.11).

1.3.3. Let Υ be as in 1.1.7.

For a number field $F \subset \mathbb{C}$, let $X_{G,\Upsilon,K}(F)$ be the set of isomorphism classes of pairs (M, λ), where

M is an exact $\otimes$-functor $\mathrm{Rep}(G) \to MM(F)$,

λ (called the K-level structure) is a mod K class of an isomorphism $\tilde{\lambda}$ of $\otimes$-functors from $\mathrm{Rep}(G)$ to $\mathrm{Mod}_{ff}(\mathbf{A}^f_{\mathbb{Q}})$

$$(V \mapsto \mathbf{A}^f_{\mathbb{Q}} \otimes_{\mathbb{Q}} V) \overset{\cong}{\to} (V \mapsto M(V)_{et}) \quad \text{preserving the weight filtrations}$$

satisfying the condition (i) below,

such that there is an isomorphism of $\otimes$-functors from $\mathrm{Rep}(G)$ to $\mathrm{Mod}_{ff}(\mathbb{Q})$

$$\theta : (V \mapsto M(V)_B) \overset{\cong}{\to} (V \mapsto V) \quad \text{preserving the weight filtrations}$$

satisfying the conditions (ii) and (iii) below. Here $(-)_B$ denotes the Betti realization with respect to the embedding $F \subset \mathbb{C}$.

(i) There is a homomorphism $k : \mathrm{Gal}(\bar{F}/F) \to K$ such that $\tilde{\lambda}(k(\sigma)x) = \sigma\tilde{\lambda}(x)$ for any $V \in \mathrm{Rep}(G)$, $x \in \mathbf{A}^f_{\mathbb{Q}} \otimes_{\mathbb{Q}} V$, and $\sigma \in \mathrm{Gal}(\bar{F}/F)$. Here $\bar{F}$ denotes the algebraic closure of F in $\mathbb{C}$.

(ii) The homomorphism $S_{\mathbb{C}/\mathbb{R}} \to G_{\mathrm{red},\mathbb{R}}$ induced by the restrictions of M_H and θ to $\mathrm{Rep}(G_{\mathrm{red}})$ belongs to Υ. Here $(-)_H$ denotes the associated $\mathbb{Q}$-mixed Hodge structure with respect to the embedding $F \subset \mathbb{C}$.

(iii) Consider the object $\mathbb{Q} \cdot (2\pi i)^{-1}$ of $\mathrm{Rep}(G)$ on which G acts via η. Then the isomorphism $M(\mathbb{Q} \cdot (2\pi i)^{-1})_B \cong \mathbb{Q} \cdot (2\pi i)^{-1}$ in $\mathrm{Mod}_{ff}(\mathbb{Q})$ induced by θ comes from an isomorphism

(*) $M(\mathbb{Q} \cdot (2\pi i)^{-1}) \cong \mathbb{Q}(-1)$ in $MM(F)$,

and concerning the morphism $p_w : \mathrm{gr}^W_w M(V_0) \otimes \mathrm{gr}^W_w M(V_0) \to \mathbb{Q}(-w)$ in $MM(F)$ induced by the G-homomorphism $\mathrm{gr}^W_w V_0 \otimes \mathrm{gr}^W_w V_0 \to \mathbb{Q} \cdot (2\pi i)^{-w}$ and the above isomorphism (*), we have either p_w is a polarization of $\mathrm{gr}^W_w M(V_0)$ for any $w \in \mathbb{Z}$ or $(-1)^w p_w$ is a polarization of $\mathrm{gr}^W_w M(V_0)$ for any $w \in \mathbb{Z}$.

1.3.4. To discuss $X_{G,\mathcal{G},M_b,K}(F)$, we assume that we are given a number field $F_0 \subset \mathbb{C}$ and an exact $\otimes$-functor

$$M_b : \mathrm{Rep}(G) \to MM(F_0)$$

and that we are given an isomorphism between $\otimes$-functors from $\mathrm{Rep}(G)$ to $\mathrm{Mod}_{ff}(\mathbb{Q})$

$$(V \mapsto M_b(V)_B) \cong (V \mapsto V) \quad \text{preserving weight filtrations.}$$

Here $(\ )_B$ denotes the Betti realization with respect to the inclusion map $F_0 \overset{\subseteq}{\to} \mathbb{C}$. We regard this isomorphism as an identification.

We further assume the following (i) and (ii).

Consider the action of G on $\mathbb{Q} \cdot (2\pi i)^{-1}$ via η. By the condition (ii) in 1.2.9, we have a unique isomorphism $M_b(\mathbb{Q} \cdot (2\pi i)^{-1})_H \cong \mathbb{Q}(-1)$ of Hodge structures whose underlying isomorphism of $\mathbb{Q}$-vector spaces is the identity map. Here $(\ )_H$ denotes the associated Hodge structure with respect to the inclusion map $F_0 \overset{\subseteq}{\to} \mathbb{C}$. We assume

(i) There is an isomorphism of motives $M_b(\mathbb{Q} \cdot (2\pi i)^{-1}) \cong \mathbb{Q}(-1)$ over F_0 whose underlying isomorphism of Hodge structures is the above one.

Since $\langle\ ,\ \rangle_{0,w}$ gives a G-homomorphism $\mathrm{gr}^W_w V_0 \otimes \mathrm{gr}^W_w V_0 \to \mathbb{Q} \cdot (2\pi i)^{-w} = (\mathbb{Q} \cdot (2\pi i)^{-1})^{\otimes w}$, we have a morphism $\mathrm{gr}^W_w M_b(V_0) \otimes \mathrm{gr}^W_w M_b(V_0) \to \mathbb{Q}(-w)$ of motives over F_0. We assume

(ii) The last morphism is a polarization on $\mathrm{gr}^W_w M_b(V_0)$ for any $w \in \mathbb{Z}$.

We further assume

(iii) The homomorphism $\mathrm{Gal}(\bar{F}/F) \to G(\mathbf{A}^f_{\mathbb{Q}})$ defined by the action of $\mathrm{Gal}(\bar{F}/F)$ on $M_b(V)_{et} = \mathbf{A}^f_{\mathbb{Q}} \otimes_{\mathbb{Q}} V$ ($V \in \mathrm{Rep}(G)$) satisfies $\sigma K \sigma^{-1} = K$ for any $\sigma \in \mathrm{Gal}(\bar{F}/F)$.

1.3.5. Let $\mathcal{G}$ be as in 1.1.10, let F_0 and M_b be as in 1.3.4, and let F be a finite extension of F_0 in $\mathbb{C}$. Let $X(F) = X_{G,\mathcal{G},M_b,K}(F)$ be the set of isomorphism classes of triples (M, ξ, λ), where

M is an exact $\otimes$-functor from $\mathrm{Rep}(G)$ to $MM(F)$,

ξ is an isomorphism $M|_{\mathcal{Q}} \overset{\cong}{\to} M_b|_{\mathcal{Q}}$, where $M|_{\mathcal{Q}}$ (resp. $M_b|_{\mathcal{Q}}$) denotes the restriction of M (resp. M_b) to $\mathrm{Rep}(\mathcal{Q})$, and

λ (called the K-level structure) is a mod K class of an isomorphism of $\otimes$-functors from $\mathrm{Rep}(G)$ to $\mathrm{Mod}_{ff}(\mathbf{A}^f_{\mathbb{Q}})$

$$\tilde{\lambda} : (V \mapsto M_b(V)_{et}) \overset{\cong}{\to} (V \mapsto M(V)_{et}) \quad \text{preserving the weight filtrations}$$

satisfying the condition (i) below,

such that there are an isomorphism of $\otimes$-functors from $\mathrm{Rep}(G_{\mathrm{red}})$ to $\mathbb{R}$HS

$$\nu : (V \mapsto \mathbb{R} \otimes_{\mathbb{Q}} M(V)_H) \overset{\cong}{\to} (V \mapsto \mathbb{R} \otimes_{\mathbb{Q}} M_b(V)_H)$$

and an isomorphism of $\otimes$-functors from $\mathrm{Rep}(G)$ to $\mathrm{Mod}_{ff}(\mathbb{Q})$

$$\theta : (V \mapsto M(V)_B) \overset{\cong}{\to} (V \mapsto V) \quad \text{preserving the weight filtrations}$$

which satisfy the following conditions (ii)–(iv) and the condition (iii) in 1.3.3.

(i) For any $\sigma \in \mathrm{Gal}(\bar{F}/F)$, there is $k(\sigma) \in K$ satisfying $\sigma\tilde{\lambda}(x) = \tilde{\lambda}(k(\sigma)\sigma x)$ for any $V \in \mathrm{Rep}(G)$ and $x \in M_b(V)_{et}$ ($\bar{F}$ denotes the algebraic closure of F in $\mathbb{C}$; here K acts on $M_b(V)_{et}$ because $M_b(V)_{et} = \mathbf{A}^f_{\mathbb{Q}} \otimes_{\mathbb{Q}} M_b(V)_B = \mathbf{A}^f_{\mathbb{Q}} \otimes_{\mathbb{Q}} V$),

(ii) ξ and ν induce the same isomorphism of functors $(V \mapsto \mathbb{R} \otimes_{\mathbb{Q}} M(V)_H) \overset{\cong}{\to} (V \mapsto \mathbb{R} \otimes_{\mathbb{Q}} M_b(V)_H)$ from $\mathrm{Rep}(\mathcal{Q}_{\mathrm{red}})$ to $\mathbb{R}$HS.

(iii) ξ and θ induce the same isomorphism of functors $(V \mapsto H(V)_B) \overset{\cong}{\to} (V \mapsto M_b(V)_B = V)$ from $\mathrm{Rep}(\mathcal{Q})$ to $\mathrm{Mod}_{ff}(\mathbb{Q})$.

(iv) The automorphism $(\mathbf{A}^f_{\mathbb{Q}} \otimes \theta) \circ \tilde{\lambda}$ of the $\otimes$-functor

$$\mathrm{Rep}(G) \to \mathrm{Mod}_{ff}(\mathbf{A}^f_{\mathbb{Q}}) \; ; \; V \mapsto \mathbf{A}^f_{\mathbb{Q}} \otimes_{\mathbb{Q}} V$$

(which is an element of $G(\mathbf{A}^f_{\mathbb{Q}})$ by the theory of Tannakian categories) belongs to $\mathcal{G}(\mathbf{A}^f_{\mathbb{Q}})$.

1.3.6. We have a canonical map $X_{G,\Upsilon,K}(F) \to X_{G,\Upsilon,K}(\mathbb{C})$ (resp. $X(F) = X_{G,\mathcal{G},M_b,K}(F) \to X(\mathbb{C}) = X_{G,\mathcal{G},M_{b,H},K}(\mathbb{C})$). Using 1.2.5 (resp. 1.2.7), it is given by $\mathrm{class}(M,\lambda) \mapsto \mathrm{class}(M_H,\lambda)$ (resp. $\mathrm{class}(M,\xi,\lambda) \mapsto \mathrm{class}(M_H,\xi_H,\lambda)$).

1.3.7. (1) Let $(M_1,\xi_1,\lambda_1) \in X_{G,\mathcal{G},M_b,K}(F)$. Then we have a bijection

$$X_{G,\mathcal{G},M_b,K}(F) \overset{\cong}{\to} X_{G,\mathcal{G},M_1,K'}(F)$$

where to define the last set, we identify $\mathrm{Rep}(G) \to \mathrm{Mod}_{ff}(\mathbb{Q})$; $V \mapsto M_1(V)_B$ with $V \mapsto V$ by fixing $\theta = \theta_1$ for M_1, and we define $K' = g_1Kg_1^{-1}$ where $g_1 = (\mathbf{A}^f_\mathbb{Q} \otimes \theta_1) \circ \tilde{\lambda}_1 \in \mathcal{G}(\mathbf{A}^f_\mathbb{Q})$. The bijection is defined by $\mathrm{class}(M,\xi,\lambda) \mapsto \mathrm{class}(M,\xi_1^{-1} \circ \xi, \lambda \circ \lambda_1^{-1})$ where $\lambda \circ \lambda_1^{-1}$ denotes the mod K' class of $\tilde{\lambda} \circ (\tilde{\lambda}_1)^{-1}$.

Via the maps $X_{G,\mathcal{G},M_b,K}(F) \to X_{G,\mathcal{G},M_{b,H},K}(\mathbb{C})$ and $X_{G,\mathcal{G},M_1,K'}(F) \to X_{G,\mathcal{G},M_{1,H},K'}(\mathbb{C})$ in 1.3.6, this bijection is compatible with the isomorphism

$$X_{G,\mathcal{G},M_{b,H},K}(\mathbb{C}) = \mathcal{G}(\mathbb{Q})\backslash(D(G,\mathcal{G},M_{b,H}) \times (\mathcal{G}(\mathbf{A}^f_\mathbb{Q})/K)) \overset{\cong}{\to}$$

$$X_{G,\mathcal{G},M_{1,H},K'}(\mathbb{C}) = \mathcal{G}(\mathbb{Q})\backslash(D(G,\mathcal{G},M_{1,H}) \times (\mathcal{G}(\mathbf{A}^f_\mathbb{Q})/K')) \;; \mathrm{class}(H,g) \mapsto \mathrm{class}(H,gg_1^{-1}).$$

(2) In 1.3.3, let $F_0 \subset \mathbb{C}$ be a number field and assume that $X_{G,\Upsilon,K}(F_0)$ is not empty. Fix an element $\mathrm{class}(M_1,\lambda_1) \in X_{G,\Upsilon,K}(F_0)$ and identify $\mathrm{Rep}(G) \to \mathrm{Mod}_{ff}(\mathbb{Q})$; $V \mapsto M_1(V)_B$ with $V \mapsto V$ by fixing $\theta = \theta_1$ for M_1. Then we have a bijection

$$X_{G,\Upsilon,K}(F) \overset{\cong}{\to} X_{G,G,M_1,K'}(F) \;;\; \mathrm{class}(M,\lambda) \mapsto \mathrm{class}(M,\xi,\lambda \circ \lambda_1^{-1})$$

where $K' = g_1Kg_1^{-1}$ with $g_1 = (\mathbf{A}^f_\mathbb{Q} \otimes \theta) \circ \tilde{\lambda}_1$, ξ is the evident one, and $\lambda \circ \lambda_1^{-1}$ is the mod K' class of $\tilde{\lambda} \circ (\tilde{\lambda}_1)^{-1}$.

Via the maps $X_{G,\Upsilon,K}(F) \to X_{G,\Upsilon,K}(\mathbb{C})$ and $X_{G,G,M_1,K'}(F) \to X_{G,G,M_{1,H},K'}(\mathbb{C})$ in 1.3.6, this bijection is compatible with the isomorphism

$$X_{G,\Upsilon,K}(\mathbb{C}) = G(\mathbb{Q})\backslash(D(G,\Upsilon) \times (G(\mathbf{A}^f_\mathbb{Q})/K)) \overset{\cong}{\to}$$

$$X_{G,G,M_{1,H},K'}(\mathbb{C}) = G(\mathbb{Q})\backslash(D(G,G,M_{1,H}) \times (G(\mathbf{A}^f_\mathbb{Q})/K')) \;; \mathrm{class}(H,g) \mapsto \mathrm{class}(H,gg_1^{-1}).$$

Proposition 1.3.8. *The automorphism group of a triple (M,ξ,λ) as in 1.3.5 is trivial.*

This follows from the Hodge version 1.2.13.

By 1.3.8, we will identify a triple (M, ξ, λ) as in 1.3.5 with its class in $X(F)$.

We will often denote $(M, \xi, \lambda) \in X(F)$ simply as $M \in X(F)$.

1.3.9. We expect that the canonical map $X(F) \to X(\mathbb{C})$ is always injective.

We expect that if F' is a finite Galois extension of F in $\mathbb{C}$, the map $X(F) \to X(F')$ induces a bijection from $X(F)$ to the $\mathrm{Gal}(F'/F)$-fixed part of $X(F')$.

1.3.10. (1) Note that $M \in X_{G,\Upsilon,K}(F)$ defines a K-conjugacy class of a continuous homomorphism $\mathrm{Gal}(\bar{F}/F) \to K$; $\sigma \mapsto k(\sigma)$, that is, an element of $H^1_{\mathrm{cont}}(\mathrm{Gal}(\bar{F}/F), K)$ where $\mathrm{Gal}(\bar{F}/F)$ acts on K trivially.

(2) Note that $M \in X_{G,\mathcal{G},M_b,K}(F)$ defines the class of a continuous 1-cocycle $\sigma \mapsto k(\sigma)$ in $H^1_{\mathrm{cont}}(\mathrm{Gal}(\bar{F}/F), K)$ where $\sigma \in \mathrm{Gal}(\bar{F}/F)$ acts on K as $k \mapsto \sigma \circ k \circ \sigma^{-1}$ (here σ is regarded as an automorphism of $(V \mapsto M_b(V)_{et} = \mathbf{A}^f_{\mathbb{Q}} \otimes V)$).

1.3.11. Questions.

(1) Is the map $X_{G,\Upsilon,K}(F) \to \mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}(\bar{F}/F), K)$ in 1.3.10 (1) always injective?

(2) Is the map $X_{G,\mathcal{G},M_b,K}(F) \to H^1_{\mathrm{cont}}(\mathrm{Gal}(\bar{F}/F), K)$ in 1.3.10 (2) always injective?

1.3.12. For $(M, \xi, \lambda) \in X(F)$, we define the canonical polarization on $\mathrm{gr}^W_w M(V_0)$ for each $w \in \mathbb{Z}$ in the similar way to the Hodge version 1.2.14.

Take θ in 1.3.5. Let $g \in \mathcal{G}(\mathbf{A}^f_{\mathbb{Q}})$ be the composition $(\mathbf{A}^f_{\mathbb{Q}} \otimes \theta) \circ \tilde{\lambda}$, and let $c = \prod_p |\eta(g)_p|_p \in \mathbb{Q}_{>0}$. Then the canonical polarization is either $c^w p_w$ or $(-c)^w p_w$ where p_w is as in the condition (iii) in 1.3.3.

1.3.13. We define a subset $X_{\mathrm{gen}}(F) = X_{G,\mathcal{G},M_b,K,\mathrm{gen}}(F)$ of $X(F) = X_{G,\mathcal{G},M_b,K}(F)$ consisting of *generic* elements. Here $(M, \xi, \lambda) \in X(F)$ is generic means that

(i) There is no algebraic subgroup G' of G such that $\dim(G') < \dim(G)$ and such that the exact $\otimes$-functor $\mathrm{Rep}(G) \to MM(F)$; $V \mapsto M(V)$ comes from an exact $\otimes$-functor $\mathrm{Rep}(G') \to MM(F)$.

Consider the following conditions (ii)–(iv). We have (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i), and (iv) $\Rightarrow$ (i). In (ii) and (iii), we consider the homomorphism $\mathrm{Gal}(\bar{F}/F) \to G(\mathbf{A}^f_{\mathbb{Q}})$ induced by a representative $\tilde{\lambda}$ of λ.

(ii) The image of $\mathrm{Gal}(\bar{F}/F)$ in $G(\mathbb{Q}_p)$ is open for some prime number p,

(iii) The image of $\mathrm{Gal}(\bar{F}/F)$ in $G(\mathbb{Q}_p)$ is open for any prime number p.

(iv) There is no algebraic subgroup G' of G such that $\dim(G') < \dim(G)$ and such that the exact $\otimes$-functor $\mathrm{Rep}(G) \to \mathbb{Q}\mathrm{MHS}$; $V \mapsto M(V)_H$ comes from an exact $\otimes$-functor $\mathrm{Rep}(G') \to \mathbb{Q}\mathrm{MHS}$.

By the philosophy of Mumford-Tate groups, we expect that these (i)–(iv) are equivalent.

Lemma 1.3.14. *Let $M \in X(F) = X_{G,\mathcal{G},M_b,K}(F)$ and assume that the functor $M : \mathrm{Rep}(G) \to MM(F)$ is fully faithful. Then $M \in X_{\mathrm{gen}}(F)$.*

Proof. Let G' be a linear subgroup of G and assume that M_b is isomorphic to the composition $\mathrm{Rep}(G) \to \mathrm{Rep}(G') \stackrel{a}{\to} MM(F)$ for an exact $\otimes$-functor $a : \mathrm{Rep}(G') \to MM(F_0)$. We prove $G' = G$.

Let $\mathcal{P}$ be the smallest full subcategory of $MM(F)$ which contains the image of a and which is stable under $\otimes$, $\oplus$, the dual, and subquotients. Let P be the Tannakian group of $\mathcal{P}$ associated to the fiber functor $\mathcal{P} \to \mathrm{Mod}_{ff}(\mathbb{Q}) : M \mapsto M_B$. Then a induces a homomorphism $P \to G'$, and the composition $P \to G' \to G$ is faithfully flat because the corresponding functor $M_b : \mathrm{Rep}(G) \to \mathcal{P}$ is fully faithful ([10], Proposition 2.20 (a)). Hence $G' = G$. Q.E.D.

1.3.15. We show that any point of $X_{G,\mathcal{G},M_b,K}(F)$ comes from a point of $X_{G',\mathcal{G}',M'_b,K'}(F)$ which is generic for some algebraic subgroup G' of G and for some $\mathcal{G}'$, M'_b, K'.

Let $(M_1, \xi_1, \lambda_1) \in X_{G,\mathcal{G},M_b,K}(F)$. Let $\mathcal{C}$ be the smallest full subcategory of $MM(F)$ which contains any subquotients of $M_1(V)$ for any $V \in \mathrm{Rep}(G)$. Let G' be the Tannakian group of $\mathcal{C}$ with respect to the fiber functor $\mathcal{C} \to \mathrm{Mod}_{ff}(\mathbb{Q})$; $M \mapsto M_B$. Then $\mathcal{C} \simeq \mathrm{Rep}(G')$. Let M'_b be the composition $\mathrm{Rep}(G') \simeq \mathcal{C} \stackrel{\subsetneq}{\to} MM(F)$. Then the fiber functor $\mathrm{Rep}(G') \to \mathrm{Mod}_{ff}(\mathbb{Q})$; $V \mapsto M'_b(V)_B$ coincides with the evident fiber functor. Fix $\theta_1 = \theta$ of M_1. The composition $\mathrm{Rep}(G) \stackrel{M_1}{\to} \mathcal{C} \simeq \mathrm{Rep}(G') \to \mathrm{Mod}_{ff}(\mathbb{Q})$ is $V \mapsto M_1(V)_B$, and by identifying this fiber functor with the evident fiber functor $\mathrm{Rep}(G) \to \mathrm{Mod}_{ff}(\mathbb{Q})$ by using θ_1, we obtain a homomorphism $G' \to G$ which induces the composition $\mathrm{Rep}(G) \stackrel{M_1}{\to} \mathcal{C} \simeq \mathrm{Rep}(G')$. This homomorphism $G' \to G$ is a closed immersion by the fact that any object of $\mathrm{Rep}(G') \simeq \mathcal{C}$ is isomorphic to a subquotient of an object which comes from $\mathrm{Rep}(G)$, and by [10], Proposition 2.20 (b). Let $\mathcal{G}' = \mathcal{G} \cap G'$, and let $K' = g_1 K g_1^{-1} \cap \mathcal{G}'(\mathbf{A}^f_{\mathbb{Q}})$ where $g_1 = (\mathbf{A}^f_{\mathbb{Q}} \otimes \theta_1) \circ \tilde{\lambda} \in \mathcal{G}(\mathbf{A}^f_{\mathbb{Q}})$. Consider the set $X'(F) = X_{G',\mathcal{G}',M'_b,K'}(F)$.

We have a map

$$X'(F) \to X(F) := X_{G,\mathcal{G},M_b,K}(F) \ ; \ (M,\xi,\lambda) \mapsto (M,\xi_1 \circ \xi, \lambda \circ \lambda_1)$$

which sends $M'_b \in X'(F)$ to $(M_1, \xi_1, \lambda_1) \in X(F)$. By 1.3.14, M'_b is generic in $X'(F)$.

1.4. Examples

We give some examples. In 1.4.4, we explain that $X(\mathbb{C})$ and $X(F)$ in Part I can be regarded as special cases of $X(\mathbb{C})$ and $X(F)$ of this Part II, respectively. In 1.4.7, by using [22], we explain that a higher Albanese manifold ([15]) is an example of $X(\mathbb{C})$, and then in 1.4.8, we give an arithmetic version of a higher Albanese manifold as an example of $X(F)$.

1.4.1. Shimura variety.

Assume G is reductive. If Υ satisfies the conditions of Deligne [8] to define a Shimura variety, $X_{G,\Upsilon,K}(\mathbb{C})$ is the space of $\mathbb{C}$-points of the Shimura variety associated to (G,Υ) of level K ([8]).

If Υ defines a Shimura variety of Hodge type ([29] Section 7), we have a canonical map $X'(F) \to X_{G,\Upsilon,K}(F)$, where X' is the Shimura variety associated to (G,Υ) of level K ([8]) and $X'(F)$ is the set of F-points of X'. This map is defined by sending a class of an abelian variety A to the class of the motive $H^1(A)$ (X' is a moduli space of abelian varieties). We expect that this map $X'(F) \to X_{G,\Upsilon,K}(F)$ is bijective.

1.4.2. The period domain associated to a mixed Hodge structure (this is the mixed Hodge version of the Mumford-Tate domain associated to pure Hodge structures in [12]).

Let H_0 be a $\mathbb{Q}$-mixed Hodge structure and assume that a polarization $p_w : \mathrm{gr}^W_w H_0 \otimes \mathrm{gr}^W_w H_0 \to \mathbb{Q}(-w)$ of $\mathrm{gr}^W_w H_0$ is given for each $w \in \mathbb{Z}$.

Let $\mathcal{C}$ be the smallest full subcategory of $\mathbb{Q}$MHS which contains H_0 and $\mathbb{Q}(-1)$ and is stable under $\otimes$, $\oplus$, the dual, and subquotients. Let G be the Tannakian group of $\mathcal{C}$ associated to the fiber functor $H \mapsto H_{\mathbb{Q}} \ ; \ \mathcal{C} \to \mathrm{Mod}_{ff}(\mathbb{Q})$. It is the linear algebraic group over $\mathbb{Q}$ defined as the automorphism group of this fiber functor. Then this fiber functor induces an equivalence of categories $\mathcal{C} \overset{\cong}{\to} \mathrm{Rep}(G)$. Let H_b be the composite functor $\mathrm{Rep}(G) \overset{\cong}{\to} \mathcal{C} \overset{\subsetneq}{\to} \mathbb{Q}\mathrm{MHS}$.

Let $\mathcal{C}_{\mathrm{red}}$ be the full subcategory of $\mathcal{C}$ consisting of objects which belong to $\mathbb{Q}$HS. Then the Tannakian group of $\mathcal{C}_{\mathrm{red}}$ associated to the fiber functor $H \mapsto H_{\mathbb{Q}}$ is identified with G_{red}. The weight filtrations of objects of $\mathcal{C}_{\mathrm{red}}$ which uniquely split define a homomorphism $w : \mathbf{G}_m \to G_{\mathrm{red}}$ and this homomorphism w satisfies the condition in 1.1.4.

Let $V_0 = H_{0,\mathbb{Q}}$ with the action of G, and for $w \in \mathbb{Z}$, let $\langle\ ,\ \rangle_{0,w} : \mathrm{gr}^W_w V_0 \times \mathrm{gr}^W_w V_0 \to \mathbb{Q}\cdot(2\pi i)^{-w}$ be the pairing induced by the polarization p_w. Let $\eta : G \to \mathbf{G}_m$ be the homomorphism defined by the action of G on $\mathbb{Q}(-1)_{\mathbb{Q}} = \mathbb{Q}\cdot(2\pi i)^{-1}$. Then the conditions in 1.2.9 and 1.2.10 are satisfied.

Let G_Φ be the algebraic subgroup of $\mathrm{Aut}_{\mathbb{Q}}(H_{0,\mathbb{Q}}, W) \times \mathbf{G}_m$ consisting of all (g,t) such that $\langle gx, gy\rangle_{0,w} = t^w\langle x,y\rangle_{0,w}$ for all x, y and w. Let Υ (resp. Υ_Φ) be the $G(\mathbb{R})$ (resp. $G_\Phi(\mathbb{R})$)-conjugacy class of the homomorphism $S_{\mathbb{C}/\mathbb{R}} \to G_{\mathrm{red},\mathbb{R}}$ (resp. $S_{\mathbb{C}/\mathbb{R}} \to G_{\Phi,\mathrm{red},\mathbb{R}}$) associated to $\mathrm{gr}^W H_0$. Then the representation G on $H_{0,\mathbb{Q}}$ induces an injective homomorphism $G \to G_\Phi$ and induces an injective morphism $D(G,\Upsilon) \to D(G_\Phi,\Upsilon)$.

For an algebraic normal subgroup $\mathcal{G}$ of G and for an open compact subgroup K of $\mathcal{G}(\mathbf{A}^f_{\mathbb{Q}})$ satisfying the neat condition (1.2.11), we have the complex analytic manifold $X_{G,\mathcal{G},H_b,K}(\mathbb{C})$.

1.4.3. Motive version of 1.4.2.

Let $F_0 \subset \mathbb{C}$ be a number field, let M_0 be a mixed motive over F_0 with $\mathbb{Q}$-coefficients, and assume that a polarization $p_w : \mathrm{gr}^W_w M_0 \otimes \mathrm{gr}^W_w M_0 \to \mathbb{Q}(-w)$ of $\mathrm{gr}^W_w M_0$ is given for each $w \in \mathbb{Z}$.

Let $\mathcal{C}$ be the smallest full subcategory of $MM(F_0)$ which contains M_0 and $\mathbb{Q}(-1)$ and is stable under $\otimes$, $\oplus$, the dual, and subquotients. Let G be the Tannakian group of $\mathcal{C}$ associated to the fiber functor $M \mapsto M_B$; $\mathcal{C} \to \mathrm{Mod}_{ff}(\mathbb{Q})$. Then this fiber functor induces an equivalence of categories $\mathcal{C} \overset{\simeq}{\to} \mathrm{Rep}(G)$. Let M_b be the composite functor $\mathrm{Rep}(G) \overset{\simeq}{\to} \mathcal{C} \hookrightarrow MM(F_0)$.

Let $\mathcal{C}_{\mathrm{red}}$ be the full subcategory of $\mathcal{C}$ consisting of objects which are direct sums of pure objects. Then the Tannakian group of $\mathcal{C}_{\mathrm{red}}$ associated to the fiber functor $M \mapsto M_B$ is identified with G_{red}. The weight filtrations of objects of $\mathcal{C}_{\mathrm{red}}$ which uniquely split define a homomorphism $w : \mathbf{G}_m \to G_{\mathrm{red}}$ and this homomorphism w satisfies the condition in 1.1.4.

Let $V_0 = M_{0,B}$ with the action of G, and for $w \in \mathbb{Z}$, let $\langle\ ,\ \rangle_{0,w} : \mathrm{gr}^W_w V_0 \times \mathrm{gr}^W_w V_0 \to \mathbb{Q}\cdot(2\pi i)^{-w}$ be the pairing induced by the polarization p_w. Let $\eta : G \to \mathbf{G}_m$ be the homomorphism defined by the action of G on $\mathbb{Q}(-1)_B = \mathbb{Q}\cdot(2\pi i)^{-1}$. Then the conditions in 1.2.9, 1.2.10, and 1.3.4 are satisfied.

For an algebraic normal subgroup $\mathcal{G}$ of G, for an open compact subgroup K of $\mathcal{G}(\mathbf{A}^f_{\mathbb{Q}})$ satisfying the neat condition (1.2.11), and for a finite extension F of F_0 in $\mathbb{C}$, we have the set $X_{G,\mathcal{G},M_b,K}(F)$.

1.4.4. Relation to Part I.

We show that the period domain $X(\mathbb{C})$ and the set of motives $X(F)$ in Part I is regarded as an example of $X_{G,\Upsilon,K}(\mathbb{C})$ and $X_{G,\Upsilon,K}(F)$ of this Part II, respectively.

As in Section 2.2 of Part I, let $\Phi = ((h(w,r))_{w,r\in\mathbb{Z}}, H_{0,\mathbb{Q}}, W, (\langle\ ,\ \rangle_{0,w})_{w\in\mathbb{Z}})$, where $h(w,r) \in \mathbb{Z}_{\geq 0}$ satisfying $h(w,r) = 0$ for almost all (w,r) and $h(w,r) = h(w,w-r)$ for all w,r, $H_{0,\mathbb{Q}}$ is a $\mathbb{Q}$-vector space of dimension $\sum_{w,r} h(w,r)$, W is an increasing filtration on $H_{0,\mathbb{Q}}$ such that $\dim_{\mathbb{Q}}(\mathrm{gr}^W_w) = \sum_r h(w,r)$ for all w, and $\langle\ ,\ \rangle_{0,w}$ is a non-degenerate $\mathbb{Q}$-bilinear form $\mathrm{gr}^W_w H_{0,\mathbb{Q}} \times \mathrm{gr}^W_w H_{0,\mathbb{Q}} \to \mathbb{Q}\cdot(2\pi i)^{-w}$ for each $w \in \mathbb{Z}$ which is symmetric if w is even and is anti-symmetric if w is odd.

As in Part I, Section 2.2, define the linear algebraic group G over $\mathbb{Q}$ as

$$G = \{(g,t) \in \mathrm{Aut}(H_{0,\mathbb{Q}}, W) \times \mathbf{G}_m \mid \langle gx, gy\rangle_{0,w} = t^w\langle x,y\rangle_{0,w} \quad \text{for any } w \in \mathbb{Z}, x,y \in \mathrm{gr}^W_w H_{0,\mathbb{Q}}\}$$

As in Part I, Section 2.2, let $D^{\pm}$ be the set of decreasing filtrations fil on $H_{0,\mathbb{C}}$ such that either $(\mathrm{gr}^W_w H_{0,\mathbb{Q}}, \mathrm{gr}^W_w \mathrm{fil}, \langle\ ,\ \rangle_{0,w})$ is a polarized Hodge structure of weight w for any $w \in \mathbb{Z}$ or $(\mathrm{gr}^W_w H_{0,\mathbb{Q}}, \mathrm{gr}^W_w \mathrm{fil}, (-1)^w\langle\ ,\ \rangle_{0,w})$ is a polarized Hodge structure of weight w for any w. Let $X_{\Phi,K}(\mathbb{C})$ (resp. $X_{\Phi,K}(F)$) be $X(\mathbb{C})$ (resp. $X(F)$) in Part I, Section 2.2.

Since $G(\mathbb{R})G_u(\mathbb{C})$ acts transitively on $D^{\pm}$ (Part I, Section 2.2), $\mathrm{gr}^W_w \mathrm{fil}$ ($w \in \mathbb{Z}$) for fil $\in D^{\pm}$ give a $G_{\mathrm{red}}(\mathbb{R})$-conjugacy class Υ of $S_{\mathbb{C}/\mathbb{R}} \to G_{\mathrm{red},\mathbb{R}}$.

We show

$$D^{\pm} = D(G,\Upsilon), \quad X_{\Phi,K}(\mathbb{C}) = X_{G,\Upsilon,K}(\mathbb{C}), \quad X_{\Phi,K}(F) = X_{G,\Upsilon,K}(F).$$

We have a canonical map $D(G,\Upsilon) \to D^{\pm}$; $H \mapsto H(H_{0,\mathbb{Q}})$. Since $G(\mathbb{R})G_u(\mathbb{C})$ acts on $D(G,\Upsilon)$ and $D^{\pm}$ transitively and since this map is compatible with these actions, this map is surjective. The injectivity follows from the fact that $G \to \mathrm{Aut}(H_{0,\mathbb{Q}}) \times \mathbf{G}_m$ is injective and from 1.2.20.

Hence the space $X_{\Phi,K}(\mathbb{C}) = G(\mathbb{Q})\backslash(D^{\pm} \times (G(\mathbf{A}^f_{\mathbb{Q}})/K))$ is identified with $X_{G,\Upsilon,K}(\mathbb{C}) = G(\mathbb{Q})\backslash(D(G,\Upsilon) \times (G(\mathbf{A}^f_{\mathbb{Q}})/K))$.

We have a canonical map

$$X_{G,\Upsilon,K}(F) \to X_{\Phi,K}(F)\ ;\ \mathrm{class}(M,\lambda) \mapsto \mathrm{class}(M', \lambda_1, \theta_1, \lambda_2, \theta_2)$$

where $M' = M(H_{0,\mathbb{Q}})$. $\lambda_1 = \tilde{\lambda}_{H_{0,\mathbb{Q}}}$. $\theta_1 = \theta_{H_{0,\mathbb{Q}}}$. $\lambda_2 = \theta \circ \tilde{\lambda}$. $\theta_2 = 1$. The converse map

$$X_{\Phi,K}(F) \to X_{G,\Upsilon,K}(F)\ ; \mathrm{class}(M, \lambda_1, \lambda_2, \theta_1, \theta_2) \mapsto (\tilde{M}, \lambda)$$

is defined as follows. Let $\mathcal{C}$ be the smallest full subcategory of $MM(F)$ which contains M and $\mathbb{Q}(-1)$ and which is stable under $\otimes$, $\oplus$, the dual, and subquotients, and let G' be the Tannakian group of $\mathcal{C}$ with respect to the fiber functor $(-)_B$. Then $\mathcal{C} \simeq \mathrm{Rep}(G')$. We have a canonical injective homomorphism $G' \to G$. Let $\tilde{M}$ be the composite functor $\mathrm{Rep}(G) \to \mathrm{Rep}(G') \simeq \mathcal{C} \subset MM(F)$. Then for $V \in \mathrm{Rep}(G)$, $\tilde{M}(V)_B$ is identified with V. We define λ by $\tilde{\lambda} = (\theta_1\lambda_1, \theta_2\lambda_2) \in G(\mathbf{A}^f_{\mathbb{Q}})$.

1.4.5. Mixed Hodge structures with fixed pure graded quotients.

Let Φ, G, $D^{\pm}$, and Υ be as in 1.4.4. Fix $H_0 \in D^{\pm}$ and assume that H_0 is endowed with a $\mathbb{Z}$-structure $H_{0,\mathbb{Z}} \subset H_{0,\mathbb{Q}}$. Let

$$\mathcal{G} := G_u = \mathrm{Ker}(G \to \mathrm{Aut}(\mathrm{gr}^W H_{0,\mathbb{Q}}) \times \mathbf{G}_m)$$

and let

$$K = \{g \in \mathrm{Aut}_{\hat{\mathbb{Z}}}(\hat{\mathbb{Z}} \otimes_{\mathbb{Z}} H_{0,\mathbb{Z}}, W) \mid \mathrm{gr}^W(g) = 1\} \subset \mathcal{G}(\mathbf{A}^f_{\mathbb{Q}}).$$

Let $H_b := H_0 \in D(G, \Upsilon) = D^{\pm}$.

As a subset of $D(G, \Upsilon)$, $D(G, \mathcal{G}, H_b)$ is identified with the set of decreasing filtrations fil on $H_{0,\mathbb{C}}$ such that $\mathrm{gr}^W \mathrm{fil} = \mathrm{gr}^W H_{0,\mathbb{C}}$. This induces a bijection between $X_{G,\mathcal{G},H_b,K}(\mathbb{C})$ and the set of all isomorphism classes of a $\mathbb{Z}$MHS H such that $\mathrm{gr}^W H = \mathrm{gr}^W H_0$.

1.4.6. Mixed motives with fixed pure graded quotients.

Let $F_0 \subset \mathbb{C}$ be a number field, and let M_0 be a mixed motive over F_0 with $\mathbb{Q}$-coefficients and with a polarization p_w on $\mathrm{gr}^W_w M_0$ for each $w \in \mathbb{Z}$. Assume M_0 is endowed with a structure of a mixed motive with $\mathbb{Z}$-coefficients, that is, a $\mathbb{Z}$-structure $M_{0,B,\mathbb{Z}}$ of $M_{0,B}$ such that $\hat{\mathbb{Z}} \otimes_{\mathbb{Z}} M_{0,B,\mathbb{Z}}$ is stable in $M_{et} = \hat{\mathbb{Z}} \otimes_{\mathbb{Z}} M_{0,B}$ under the action of $\mathrm{Gal}(\bar{F}_0/F_0)$. We show that as an example of $X(F)$, we have the set of isomorphism classes of mixed motives M over F with $\mathbb{Z}$-coefficients such that $\mathrm{gr}^W M = \mathrm{gr}^W M_0$.

Let H_0 be the Hodge realization of M_0, and consider Φ and $D^{\pm} \ni H_0$ associated to H_0 (1.4.4). Let G, $\mathcal{G} = G_u$ and K be as in 1.4.5. We have $M_b : \mathrm{Rep}(G) \to MM(F_0)$ as follows. Let $\mathcal{C}$ be the smallest full subcategory of $MM(F_0)$ which contains M_0 and $\mathbb{Q}(-1)$ and is stable under $\otimes$, $\oplus$, the dual, and subquotients. Let G_M be the Tannakian group of $\mathcal{C}$ associated to the fiber functor $M \mapsto M_B$; $\mathcal{C} \to \mathrm{Mod}_{ff}(\mathbb{Q})$. Then the action of G_M on $M_{0,B} = H_{0,\mathbb{Q}}$ induces a homomorphism $G_M \to G$. Let M_b be the composite functor $\mathrm{Rep}(G) \to \mathrm{Rep}(G_M) \simeq \mathcal{C} \hookrightarrow MM(F_0)$.

Then $X_{G,\mathcal{G},M_b,K}(F)$ is identified with the set E of isomorphism classes of mixed motives M over F with $\mathbb{Z}$-coefficients such that $\mathrm{gr}^W M = \mathrm{gr}^W M_0$. In fact, the map $X_{G,\mathcal{G},M_b,K}(F) \to E$ is given by $M \mapsto M(H_{0,\mathbb{Q}})$.

The converse map $E \to X_{G,\mathcal{G},M_b,K}(F)$; $M \mapsto (\tilde{M}, \xi, \lambda)$ is obtained as follows. Let $\mathcal{C}'$ be the smallest full subcategory of $MM(F)$ which contains M and $\mathbb{Q}(-1)$ and which is stable under $\otimes$, $\oplus$, the dual, and subquotients. Let G' be the Tannakian group of $\mathcal{C}'$ with respect to the fiber functor $M' \mapsto M'_B$. Choose an isomorphism of $\mathbb{Z}$-modules $M_{B,\mathbb{Z}} \cong M_{0,B,\mathbb{Z}} = H_{0,\mathbb{Z}}$ which is compatible with the weight filtrations and which induces the identity map on gr^W. Then the induced action of G' on $H_{0,\mathbb{Q}}$ induces a homomorphism $G' \to G$. We have the composite functor $\tilde{M} : \mathrm{Rep}(G) \to \mathrm{Rep}(G') \simeq \mathcal{C}' \subset MM(F)$. The identification $\mathrm{gr}^W M = \mathrm{gr}^W M_0$ gives ξ and the isomorphism $M_{B,\mathbb{Z}} \cong M_{0,B,\mathbb{Z}} = H_{0,\mathbb{Z}}$ gives λ.

1.4.7. Higher Albanese manifolds.

Here by using [22], we show that the higher Albanese manifold of Hain [15] is regarded as an example of $X(\mathbb{C})$. We also correct a mistake in [22].

Let Z be a connected smooth quasi-projective algebraic variety over $\mathbb{C}$. Fix $b \in Z$. Let J be the augmentation ideal $\mathrm{Ker}(\mathbb{Q}[\pi_1(Z,b)] \to \mathbb{Q})$ of the group ring $\mathbb{Q}[\pi_1(Z,b)]$. Fix $n \geq 0$, and let $\Gamma = \Gamma_n$ be the image of $\pi_1(X,b) \to \mathbb{Q}[\pi_1(Z,b)]/J^{n+1}$. Then Γ is a finitely generated torsion-free nilpotent group.

Let $\mathcal{G}$ be the unipotent algebraic group over $\mathbb{Q}$ whose Lie algebra is defined as follows. Let I be the augmentation ideal $\mathrm{Ker}(\mathbb{Q}[\Gamma] \to \mathbb{Q})$ of $\mathbb{Q}[\Gamma]$. Then $\mathrm{Lie}(\mathcal{G})$ is the $\mathbb{Q}$-subspace of $\mathbb{Q}[\Gamma]^\wedge := \varprojlim_r \mathbb{Q}[\Gamma]/I^r$ generated by all $\log(\gamma)$ $(\gamma \in \Gamma)$.

We have

$$\mathrm{Lie}(\mathcal{G}) = \{h \in \mathbb{Q}[\Gamma]^\wedge \mid \Delta(h) = h \otimes 1 + 1 \otimes h\},$$

$$\mathcal{G}(R) = \{g \in (R[\Gamma]^\wedge)^\times \mid \Delta(g) = g \otimes g\}$$

for any commutative ring R over $\mathbb{Q}$, where $\Delta : R[\Gamma]^\wedge \to R[\Gamma \times \Gamma]^\wedge$ is the ring homomorphism induced by the ring homomorphism $R[\Gamma] \to R[\Gamma \times \Gamma]$; $\gamma \mapsto \gamma \otimes \gamma$ $(\gamma \in \Gamma)$. The Lie product of $\mathrm{Lie}(\mathcal{G})$ is defined by $[x, y] = xy - yx$. We have $\Gamma \subset \mathcal{G}(\mathbb{Q})$.

By the work [16] of Hain-Zucker, $\mathrm{Lie}(\mathcal{G})$ is regarded as a polarizable mixed $\mathbb{Q}$-Hodge structure. The Lie product $\mathrm{Lie}(\mathcal{G}) \otimes \mathrm{Lie}(\mathcal{G}) \to \mathrm{Lie}(\mathcal{G})$ is a homomorphism of mixed Hodge structures.

The n-th higher Albanese manifold $\mathrm{Alb}_{Z,n}(\mathbb{C})$ of Z is as follows. Let $\mathrm{fil}^0\mathcal{G}(\mathbb{C})$ be the algebraic subgroup of $\mathcal{G}(\mathbb{C})$ over $\mathbb{C}$ corresponding to the Lie subalgebra $\mathrm{fil}^0\mathrm{Lie}(\mathcal{G})_{\mathbb{C}}$ (fil^0 here denotes the Hodge filtration) of $\mathrm{Lie}(\mathcal{G})_{\mathbb{C}}$. Then

$$\mathrm{Alb}_{Z,n}(\mathbb{C}) := \Gamma\backslash\mathcal{G}(\mathbb{C})/\mathrm{fil}^0\mathcal{G}(\mathbb{C}).$$

This is a complex analytic manifold but usually it is not an algebraic variety.

In [22], we took as Γ any quotient group of $\pi_1(Z,b)$ which is nilpotent and torsion-free (we did not assume $\Gamma = \Gamma_n$). This was a mistake because for such general Γ, the Lie algebra $\mathrm{Lie}(\mathcal{G})$ need not have a mixed Hodge structure. The authors of [22] correct this mistake by adding in 5.1.1 of [22] the assumption that $\mathrm{Lie}(\mathcal{G})$ has a mixed Hodge structure which is a quotient of the case $\Gamma = \Gamma_n$ for some n. With this correction, all arguments and results of [22] work. (This correction will be written also in the joint paper [21] Part V.)

By the work [22], the higher Albanese manifold $\mathrm{Alb}_{Z,n}(\mathbb{C})$ is interpreted as an example of $X(\mathbb{C})$ as follows.

Let $\mathcal{C}$ be a full subcategory of $\mathbb{Q}$MHS which contains the mixed Hodge structures $\mathrm{Lie}(\mathcal{G})$ and $\mathbb{Q}(-1)$, which is stable under $\otimes$, $\oplus$, the dual, and subquotients, and which is of finite type as a Tannakian category.

We define categories $\mathcal{C}_Z$ and $\mathcal{C}'_Z$.

Let $\mathcal{C}_Z$ be the category of variations of mixed $\mathbb{Q}$-Hodge structure $\mathcal{H}$ on Z satisfying the following conditions (i)–(iv).

(i) All graded quotients $\mathrm{gr}^W_w\mathcal{H}$ for the weight filtration are constant mixed Hodge strutures.

(ii) The monodromy actions of $\pi_1(Z,b)$ on the fiber $\mathcal{H}_{\mathbb{Q},b}$ at b factors through the projection $\pi_1(Z,b) \to \Gamma$.

(iii) The fiber $\mathcal{H}(b)$ of $\mathcal{H}$ at b belongs to $\mathcal{C}$.

(iv) $\mathcal{H}$ is good at the boundary of Z in the sense of [16].

Let $\mathcal{C}'_Z$ be the category of $h \in \mathcal{C}$ which is endowed with a morphism $\mathrm{Lie}(\mathcal{G}) \otimes h \to h$ in $\mathbb{Q}$MHS which is a Lie action of $\mathrm{Lie}(\mathcal{G})$ on h.

By [16], we have an equivalence of categories

$$\mathcal{C}_Z \overset{\simeq}{\to} \mathcal{C}'_Z$$

which sends $\mathcal{H} \in \mathcal{C}_Z$ to the fiber $\mathcal{H}(b)$ of $\mathcal{H}$ at b with the action of $\mathrm{Lie}(\mathcal{G})$ induced by the monodromy action of Γ on $\mathcal{H}_{\mathbb{Q},b}$.

Let $\mathcal{Q}_{\mathcal{C}}$ be the Tannakian group of $\mathcal{C}$ associated to the fiber functor $\mathcal{C} \to \mathrm{Mod}_{ff}(\mathbb{Q})$; $H \mapsto H_{\mathbb{Q}}$. By the mixed $\mathbb{Q}$-Hodge structure on $\mathrm{Lie}(\mathcal{G})$, we have an action of $\mathcal{Q}_{\mathcal{C}}$ on the Lie algebra $\mathrm{Lie}(\mathcal{G})$. This induces an action of $\mathcal{Q}_{\mathcal{C}}$ on the algebraic group $\mathcal{G}$. Let $G_{\mathcal{C}}$ be the semi-direct product of $\mathcal{G}$ and $\mathcal{Q}_{\mathcal{C}}$ in which $\mathcal{G}$ is a normal subgroup of $G_{\mathcal{C}}$ and in which the inner-automorphism action of $\mathcal{Q}_{\mathcal{C}}$ on $\mathcal{G}$ is the action which we just defined. Then we have equivalences of categories

$$\mathcal{C} \simeq \mathrm{Rep}(\mathcal{Q}_{\mathcal{C}}), \quad \mathcal{C}'_Z \simeq \mathrm{Rep}(G_{\mathcal{C}}),$$

where the second equivalence is because a linear representation of $\mathcal{G}$ and a representation of the Lie algebra $\mathrm{Lie}(\mathcal{G})$ are equivalent. The functor $\mathrm{Rep}(\mathcal{Q}_{\mathcal{C}}) \to \mathrm{Rep}(G_{\mathcal{C}})$ induced by $G_{\mathcal{C}} \to \mathcal{Q}_{\mathcal{C}}$ corresponds to the functor $\mathcal{C} \to \mathcal{C}'_Z$ to give the trivial action of $\mathrm{Lie}(\mathcal{G})$. The functor $\mathrm{Rep}(G_{\mathcal{C}}) \to \mathrm{Rep}(\mathcal{Q}_{\mathcal{C}})$ induced by the inclusion map $\mathcal{Q}_{\mathcal{C}} \to G_{\mathcal{C}}$ corresponds to the functor $\mathcal{C}'_Z \to \mathcal{C}$ to forget the action of $\mathrm{Lie}(\mathcal{G})$. Let $H_b : \mathrm{Rep}(G_{\mathcal{C}}) \to \mathbb{Q}\mathrm{MHS}$ be the composition $\mathrm{Rep}(G_{\mathcal{C}}) \to \mathrm{Rep}(\mathcal{Q}_{\mathcal{C}}) \simeq \mathcal{C} \overset{\subseteq}{\to} \mathbb{Q}\mathrm{MHS}$. Then the action of $\mathcal{G}(\mathbb{C})$ on $D = D(G_{\mathcal{C}}, \mathcal{G}, H_b)$ is transitive and $\mathrm{fil}^0\mathcal{G}(\mathbb{C})$ coincides with the isotropy group of $\mathrm{class}(H_b) \in D$ in $\mathcal{G}(\mathbb{C})$. Hence we have $D \cong \mathcal{G}(\mathbb{C})/\mathrm{fil}^0\mathcal{G}(\mathbb{C})$.

Let K be the profinite completion of Γ, which is naturally regarded as an open compact subgroup of $\mathcal{G}(\mathbf{A}^f_{\mathbb{Q}})$. Then $\mathcal{G}(\mathbb{Q})/\Gamma \overset{\cong}{\to} \mathcal{G}(\mathbf{A}^f_{\mathbb{Q}})/K$. Hence

$$\begin{aligned} X_{G_{\mathcal{C}},\mathcal{G},H_b,K}(\mathbb{C}) = \mathcal{G}(\mathbb{Q})\backslash(D \times (\mathcal{G}(\mathbf{A}^f_{\mathbb{Q}})/K)) &= \mathcal{G}(\mathbb{Q})\backslash(D \times (\mathcal{G}(\mathbb{Q})/\Gamma)) \\ &= \Gamma\backslash D = \mathrm{Alb}_{Z,n}(\mathbb{C}). \end{aligned}$$

By this, $\mathrm{Alb}_{Z,n}(\mathbb{C})$ is identified with $X_{G_{\mathcal{C}},\mathcal{G},H_b,K}(\mathbb{C})$.

The higher Albanese map $Z(\mathbb{C}) \to \mathrm{Alb}_{Z,n}(\mathbb{C})$ of Hain is interpreted as follows. Let $X_{Z,\mathcal{C}}(\mathbb{C})$ be the set of isomorphism classes of triples (H, ξ, λ), where H is an exact $\otimes$-functor $\mathcal{C}_Z \to \mathbb{Q}\mathrm{MHS}$, ξ is an isomorphism of $\otimes$-functors from $\mathbb{Q}\mathrm{MHS}$ to $\mathbb{Q}\mathrm{MHS}$

$$(h \mapsto H(h_Z)) \overset{\cong}{\to} (h \mapsto h),$$

where h_Z denotes the constant varitation of mixed $\mathbb{Q}$-Hodge structure on Z associated to h, and λ is a mod Γ class of an isomorphism of $\otimes$-functors from $\mathcal{C}_Z$ to $\mathrm{Mod}_{ff}(\mathbb{Q})$

$$(\mathcal{H} \mapsto \mathcal{H}(b)_{\mathbb{Q}}) \overset{\cong}{\to} (\mathcal{H} \mapsto H(\mathcal{H})_{\mathbb{Q}}) \quad \text{preserving the weight filtrations}$$

such that the mod K class of the isomorphism of $\otimes$-functors from $\mathbb{Q}\mathrm{MHS}$ to $\mathrm{Mod}_{ff}(\mathbb{Q})$

$$(h \mapsto h_{\mathbb{Q}}) \overset{\cong}{\to} (h \mapsto H(h_Z)_{\mathbb{Q}})$$

induced by ξ^{-1} coincides with that induced by λ.

The commutative diagram of categories

$$\begin{array}{ccccc} \mathcal{C} & = & \mathcal{C} & \simeq & \mathrm{Rep}(\mathcal{Q}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{C}_Z & \overset{\simeq}{\to} & \mathcal{C}'_Z & \simeq & \mathrm{Rep}(G_{\mathcal{C}}), \end{array}$$

where the vertical arrows are the pullback functors, induces a bijection

$$X_{G_{\mathcal{C}},\mathcal{G},H_b,K}(\mathbb{C}) \stackrel{\cong}{\to} X_{Z,\mathcal{C}}(\mathbb{C}).$$

The higher Albanese map $Z(\mathbb{C}) \to \mathrm{Alb}_{Z,n}(\mathbb{C})$ of Hain is interpreted as the map $Z(\mathbb{C}) \to X_{Z,\mathcal{C}}(\mathbb{C})$, which sends $s \in Z$ to roughly speaking, the functor to take the fiber at s. Precisely speaking, it sends s to the class of (H, ξ, λ), where H is the exact $\otimes$-functor $\mathcal{C}_Z \to \mathbb{Q}\mathrm{MHS}$; $\mathcal{H} \mapsto \mathcal{H}(s)$, ξ is the evident identification $h(s) = h$ ($h \in \mathbb{Q}\mathrm{MHS}$), and λ is the mod Γ class of $\mathcal{H}(s)_{\mathbb{Q}} \cong \mathcal{H}(b)_{\mathbb{Q}}$ of the local system $\mathcal{H}_{\mathbb{Q}}$.

1.4.8. Motive version of 1.4.7.

We consider the arithmetic version of the higher Albanese manifold, which is obtained as $X(F)$ and relate it to the Selmer variety of Kim [25] which is also an arithmetic version of a higher Albanese manifold.

Let $F_0 \subset \mathbb{C}$ be a number field and let Z be a geometrically connected smooth quasi-projective algebraic variety over F_0. Assume we are given $b \in Z(F_0)$. Let $n \geq 0$, and define $\mathcal{G}$ and K as in 1.4.7 by taking $Z(\mathbb{C})$ as Z in 1.4.7.

By Deligne-Goncharov [9], 3.12, we have a mixed motive $\mathrm{Lie}(\mathcal{G})_{\mathrm{mot}}$ with $\mathbb{Q}$-coefficients over F_0 whose Hodge realization is the mixed Hodge structure $\mathrm{Lie}(\mathcal{G})$ in 1.4.7. We have a morphism $\mathrm{Lie}(\mathcal{G})_{\mathrm{mot}} \otimes \mathrm{Lie}(\mathcal{G})_{\mathrm{mot}} \to \mathrm{Lie}(\mathcal{G})_{\mathrm{mot}}$ of $MM(F_0)$ which induces the Lie product $\mathrm{Lie}(\mathcal{G}) \otimes \mathrm{Lie}(\mathcal{G}) \to \mathrm{Lie}(\mathcal{G})$.

We define a set $\mathrm{Alb}_{Z,n}(F)$ for a finite extension F of F_0 in $\mathbb{C}$.

Let $\mathcal{C}$ be a full subcategory of $MM(F_0)$, which contains $\mathrm{Lie}(\mathcal{G})_{\mathrm{mot}}$ and $\mathbb{Q}(-1)$ and which is stable under $\otimes$, $\oplus$, the dual, and subquotients, and which is of finite type as a Tannakian category.

Let $\mathcal{Q}_{\mathcal{C}}$ be the Tannakian group of $\mathcal{C}$ with respect to the fiber functor $\mathcal{C} \to \mathrm{Mod}_{ff}(\mathbb{Q})$; $M \mapsto M_B$. Then $\mathcal{C} \simeq \mathrm{Rep}(\mathcal{Q}_{\mathcal{C}})$. Since $\mathrm{Lie}(\mathcal{G})_{\mathrm{mot}} \in \mathcal{C}$, $\mathcal{Q}_{\mathcal{C}}$ acts on $\mathrm{Lie}(\mathcal{G})_{\mathrm{mot},B} = \mathrm{Lie}(\mathcal{G})$ preserving the Lie product. Hence $\mathcal{Q}$ acts on the algebraic group $\mathcal{G}$. Let $G_{\mathcal{C}}$ be the semi-direct product of $\mathcal{G}$ and $\mathcal{Q}_{\mathcal{C}}$ in which $\mathcal{G}$ is a normal subgroup and in which the inner-automorphism action of $\mathcal{Q}_{\mathcal{C}}$ on $\mathcal{G}$ is the one just defined. Let M_b : $\mathrm{Rep}(G_{\mathcal{C}}) \to MM(F_0)$ be the composition $\mathrm{Rep}(G_{\mathcal{C}}) \to \mathrm{Rep}(\mathcal{Q}_{\mathcal{C}}) \simeq \mathcal{C} \stackrel{\subsetneq}{\to} MM(F_0)$. Define

$$\mathrm{Alb}_{Z,n,\mathcal{C}}(F) := X_{G_{\mathcal{C}},\mathcal{G},M_b,K}(F).$$

We have a canonical map $\mathrm{Alb}_{Z,n,\mathcal{C}}(F) \to \mathrm{Alb}_{Z,n}(\mathbb{C})$ defined as follows. Let $\mathcal{C}'$ be the Tannakian subcategory of $\mathbb{Q}\mathrm{MHS}$ generated by Hodge realizations of objects of $\mathcal{C}$. Let $G_{\mathcal{C}'} \supset \mathcal{G}$ be the $G_{\mathcal{C}'}$ in 1.4.7 associated to $\mathcal{C}'$. Then we have a canonical homomorphism $G_{\mathcal{C}'} \to$

$G_{\mathcal{C}}$ which induces the identity map of $\mathcal{G}$. Denote by H_b the functor $\mathrm{Rep}(G_{\mathcal{C}}) \to \mathbb{Q}\mathrm{MHS}$ induced by M_b, and denote by H'_b the canonical functor $\mathrm{Rep}(G_{\mathcal{C}'}) \to \mathbb{Q}\mathrm{MHS}$. Then we can show that the canonical map $X_{G_{\mathcal{C}'},\mathcal{G},H'_b,K}(\mathbb{C}) \to X_{G_{\mathcal{C}},\mathcal{G},H_b,K}(\mathbb{C})$ is an isomorphism. Hence we have a map $\mathrm{Alb}_{Z,n,\mathcal{C}}(F) = X_{G_{\mathcal{C}},\mathcal{G},M_b,K}(F) \to X_{G_{\mathcal{C}},\mathcal{G},H_b,K}(\mathbb{C}) \cong X_{G_{\mathcal{C}'},\mathcal{G},H'_b,K}(\mathbb{C}) = \mathrm{Alb}_{Z,n}(\mathbb{C})$.

We have a canonical map from $\mathrm{Alb}_{Z,n,\mathcal{C}}(F)$ to the Selmer variety of M. Kim by 1.3.10 (2).

The author can not show that $\mathrm{Alb}_{Z,n,\mathcal{C}}(F)$ is independent of $\mathcal{C}$. We define $\mathrm{Alb}_{Z,n}(F)$ as the inverse limit of $\mathrm{Alb}_{Z,n,\mathcal{C}}(F)$ for all $\mathcal{C}$.

Concerning the higher Albanese map from $Z(F)$, we may define the motive version of $\mathcal{C}_Z$ of 1.4.7 and a motive version $X_{Z,\mathcal{C}}(F)$ of $X_{Z,\mathcal{C}}(\mathbb{C})$ of 1.4.7 and define a map $Z(F) \to X_{Z,\mathcal{C}}(F)$ as in 1.4.7. But the author can not show that the canonical map $\mathrm{Alb}_{Z,n,\mathcal{C}}(F) \to X_{Z,\mathcal{C}}(F)$ is bijective. We do not discuss these points in this paper.

§2. Curvature forms and Hodge theory

2.1. Reviews on curvature forms of line bundles

2.1.1. Let Y be a complex analytic manifold and let L be a line bundle on Y. A metric $|\ |_L$ on L is said to be C^∞ if $|e|_L$ is a C^∞ function for a local basis e of L. A C^∞ metric on L exists if Y is paracompact.

2.1.2. Let Y be as in 2.1.1 and let L be a line bundle on Y endowed with a C^∞ metric $|\ |_L$. Then the curvature form $\kappa(L)$ of L is a C^∞ $(1,1)$-form on Y defined by

$$\kappa(L) = \kappa(L, |\ |_L) := \partial\bar{\partial}\log(|e|_L^2) = \partial\bar{\partial}\log((e,e)_L)$$

where e is a local basis of L and $(\ ,\ )_L$ is the Hermitian form on L corresponding to $|\ |_L$. (Recall that if Y is n-dimensional and $(z_j)_{1\le j\le n}$ is a local coordinate of Y, for a C^∞ function g on Y, $\partial\bar{\partial}g = \sum_{j,k}(\frac{\partial}{\partial z_j}\frac{\partial}{\partial \bar{z}_k}g)dz_j \wedge d\bar{z}_k$.)

This $\kappa(L)$ does not depend on the local choice of a basis e and hence defined globally.

2.1.3. Let Y and L be as in 2.1.2 and assume that Y is a connected compact Riemann surface. Then the theory of Chern forms and Chern classes shows

$$\frac{1}{2\pi i}\int_Y \kappa(L) = \deg(L).$$

2.1.4. Let Y be as in 2.1.1 and let T_Y be the tangent bundle of Y.

Then there is a canonical one-to-one correspondence between C^∞ Hermitian forms on T_Y and purely imaginary C^∞ $(1,1)$-forms on Y (C^∞ $(1,1)$-forms ω such that $\bar{\omega} = -\omega$). Using a local coordinate $(z_j)_j$ of Y, a $(1,1)$-form $\sum_{1\le j,k\le n} f_{j,k} d\bar{z}_k \wedge dz_j$ with C^∞ functions $f_{j,k}$ is purely imaginary if and only if $\bar{f}_{j,k} = f_{k,j}$. If it is purely imaginary, it corresponds to the Hermitian form $(\partial/\partial z_j, \partial/\partial z_k) \mapsto f_{j,k}$ on T_Y.

The curvature form $\kappa(L)$ of a line bundle L on Y with a C^∞ metric is a purely imaginary C^∞ $(1,1)$-form and hence corresponds to a C^∞ Hermitiain form on T_Y. Locally, this Hermitian form is given as

$$(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}) \mapsto -\frac{\partial}{\partial z_j}\frac{\partial}{\partial \bar{z}_k} \log((e,e)_L) = -\frac{\partial}{\partial \bar{z}_k}\frac{\partial}{\partial z_j} \log((e,e)_L).$$

We say $\kappa(L)$ is positive definite if the corresponding Hermitian form on T_Y is positive definite. By a theorem of Kodaira, a line bundle L on Y is ample if and only if L has a C^∞ metric whose curvature form $\kappa(L)$ is positive definite.

Example. For $Y = \mathbf{P}^1(\mathbb{C})$, the ample line bundle $L = \mathcal{O}_Y(\infty)$ has the C^∞ metric such that $|1|_L = (1 + z\bar{z})^{-1/2}$ where z is the canonical coordinate of $\mathbb{C} \subset \mathbf{P}^1(\mathbb{C})$. Its curvature form $\kappa(L) = (1 + z\bar{z})^{-2} d\bar{z} \wedge dz$ and the associated Hermitian form $(\partial/\partial z, \partial/\partial z) \mapsto (1 + z\bar{z})^{-2}$ on T_Y is positive definite.

2.2. Review on the result of Griffiths on curvature forms

2.2.1. Let Y be a complex analytic manifold. Let $\mathcal{H}$ be a variation of polarized $\mathbb{Q}$-Hodge structure of weight w on Y. Let $r \in \mathbb{Z}$, and consider the line bundle $\det(\mathrm{fil}^r \mathcal{H}_{\mathcal{O}})$ with the Hodge metric. (Here fil is the Hodge filtration.) The result of Griffiths on the curvature form $\kappa(\det(\mathrm{fil}^r \mathcal{H}_{\mathcal{O}}))$ is as follows.

Let $r \in \mathbb{Z}$. We have a homomorphism

$$h_r : T_Y \to \mathrm{Hom}_{\mathcal{O}_Y}(\mathrm{gr}^r \mathcal{H}_{\mathcal{O}}, \mathrm{gr}^{r-1} \mathcal{H}_{\mathcal{O}})$$

which sends $\alpha \in T_Y$ to the composition $\mathrm{gr}^r \mathcal{H}_{\mathcal{O}} \xrightarrow{\nabla} \mathrm{gr}^{r-1} \mathcal{H}_{\mathcal{O}} \otimes_{\mathcal{O}_Y} \Omega^1_Y \xrightarrow{\alpha} \mathrm{gr}^{r-1} \mathcal{H}_{\mathcal{O}}$. Here $\mathrm{gr}^r = \mathrm{fil}^r/\mathrm{fil}^{r+1}$, the first arrow is induced by the connection $\mathcal{H}_{\mathcal{O}} = \mathcal{O}_Y \otimes_{\mathbb{Q}} \mathcal{H}_{\mathbb{Q}} \to \mathcal{H}_{\mathcal{O}} \otimes_{\mathcal{O}_Y} \Omega^1_Y = \Omega^1_Y \otimes_{\mathbb{Q}} \mathcal{H}_{\mathbb{Q}}$ which kills $\mathcal{H}_{\mathbb{Q}}$, and the second arrow is by the duality between T_Y and Ω^1_Y.

Theorem (Griffiths ([14])): The Hermitian form $\kappa(\det(\mathrm{fil}^r \mathcal{H}_{\mathcal{O}}))$ on T_Y is described as $(\alpha, \beta) \mapsto (h_r(\alpha), h_r(\beta))$ where the last $(\ ,\ )$ denotes the Hodge metric of $(\mathrm{gr}^r \mathcal{H}_{\mathcal{O}})^* \otimes \mathrm{gr}^{r-1} \mathcal{H}_{\mathcal{O}}$.

2.2.2. As is explained in Part I, 2.4.3, the curvature form $\kappa(\det(\mathrm{fil}^r\mathcal{H}_{\mathcal{O}}))$ is independent of the polarization. This can be seen also as follows.

Let

$$A_{\mathbb{R},Y} \quad (\text{resp. } A_{\mathbb{C},Y})$$

be the sheaf of $\mathbb{R}$ (resp. $\mathbb{C}$) valued C^∞ functions on Y.

Let λ_r be the composition of the bijections

$$A_{\mathbb{C},Y} \otimes_{\mathcal{O}_Y} \mathrm{gr}^r\mathcal{H}_{\mathcal{O}} \cong (A_{\mathbb{C},Y} \otimes_{\mathcal{O}_Y} \mathcal{H}_{\mathcal{O}})^{r,w-r} \to (A_{\mathbb{C},Y} \otimes_{\mathcal{O}_Y} \mathcal{H}_{\mathcal{O}})^{w-r,r} \cong A_{\mathbb{C},Y} \otimes_{\mathcal{O}_Y} \mathrm{gr}^{w-r}\mathcal{H}_{\mathcal{O}}$$

where $(-)^{p,q}$ denotes the (p,q)-Hodge component and the middle bijection is induced by the complex conjugation $A_{\mathbb{C},Y} \otimes_{\mathbb{Q}} \mathcal{H}_{\mathbb{Q}} \to A_{\mathbb{C},Y} \otimes_{\mathbb{Q}} \mathcal{H}_{\mathbb{Q}}$; $a \otimes b \mapsto \bar{a} \otimes b$.

Let

$$h_r^* : T_Y \to A_{\mathbb{C},Y} \otimes_{\mathcal{O}_Y} \mathcal{H}om_{\mathcal{O}_Y}(\mathrm{gr}^{r-1}\mathcal{H}_{\mathcal{O}}, \mathrm{gr}^r\mathcal{H}_{\mathcal{O}})$$

be the map which sends $\alpha \in T_Y$ to the following composite map:

$$A_{\mathbb{C},Y} \otimes_{\mathcal{O}_Y} \mathrm{gr}^{r-1}\mathcal{H}_{\mathcal{O}} \overset{\lambda_{r-1}}{\to} A_{\mathbb{C},Y} \otimes_{\mathcal{O}_Y} \mathrm{gr}^{w-r+1}\mathcal{H}_{\mathcal{O}} \overset{h_{w-r+1}(\alpha)}{\longrightarrow} A_{\mathbb{C},Y} \otimes_{\mathcal{O}_Y} \mathrm{gr}^{w-r}\mathcal{H}_{\mathcal{O}} \overset{\lambda_{w-r}}{\to} A_{\mathbb{C},Y} \otimes_{\mathcal{O}} \mathrm{gr}^r\mathcal{H}_{\mathcal{O}}.$$

Then

$$(h_r(\alpha), h_r(\beta)) = \langle h_r(\alpha), h_r^*(\beta)\rangle$$

where $\langle\ ,\ \rangle$ denotes the duality $(\mathrm{gr}^r\mathcal{H}_{\mathcal{O}})^* \otimes \mathrm{gr}^{r-1}\mathcal{H}_{\mathcal{O}} \times (\mathrm{gr}^{r-1}\mathcal{H}_{\mathcal{O}})^* \otimes \mathrm{gr}^r\mathcal{H}_{\mathcal{O}} \to \mathcal{O}_Y$. The right hand side of this equation is defined without using the polarization.

2.2.3. Consider the period domain $X(\mathbb{C}) = X_{G,\mathcal{G},H_b,K}(\mathbb{C})$ with G reductive.

Let $\mathcal{H}_{X(\mathbb{C})}$ be universal object on $X(\mathbb{C})$, let $V \in \mathrm{Rep}(G)$, and let $r \in \mathbb{Z}$. Consider the restriction $\kappa(\det(\mathrm{fil}^r\mathcal{H}_{X(\mathbb{C})}(V)_{\mathcal{O}}))_{\mathrm{hor}}$ to $T_{X(\mathbb{C}),\mathrm{hor}}$ of the curvature form $\kappa(\det(\mathrm{fil}^r\mathcal{H}_{X(\mathbb{C})}(V)_{\mathcal{O}}))$ of the line bundle $\det(\mathrm{fil}^r\mathcal{H}_{X(\mathbb{C})}(V)_{\mathcal{O}})$ with the Hodge metric of a polarization (1.2.21) (the curvature form is independent of the choice of the polarization). Recall that $\mathcal{H}_{X(\mathbb{C})}(V)$ need not be a variation of Hodge structure (it need not satisfy the Griffiths transversality). Recall that $T_{X(\mathbb{C}),\mathrm{hor}} = \mathrm{gr}^{-1}\mathcal{H}_{X(\mathbb{C})}(\mathrm{Lie}(\mathcal{G}))_{\mathcal{O}}$. The action of G on V induces a Lie action $\mathrm{Lie}(\mathcal{G}) \otimes V \to V$ of $\mathrm{Lie}(\mathcal{G})$ and hence a homomorphism $\mathrm{gr}^{-1}\mathcal{H}_{X(\mathbb{C})}(\mathrm{Lie}(\mathcal{G}))_{\mathcal{O}} \otimes \mathrm{gr}^r\mathcal{H}_{X(\mathbb{C})}(V)_{\mathcal{O}} \to \mathrm{gr}^{r-1}\mathcal{H}_{X(\mathbb{C})}(V)_{\mathcal{O}}$. Let

$$h_r : T_{X(\mathbb{C}),\mathrm{hor}} = \mathrm{gr}^{-1}\mathcal{H}_{X(\mathbb{C})}(\mathrm{Lie}(\mathcal{G}))_{\mathcal{O}} \to (\mathrm{gr}^r\mathcal{H}_{X(\mathbb{C})}(V)_{\mathcal{O}})^* \otimes \mathrm{gr}^{r-1}\mathcal{H}_{X(\mathbb{C})}(V)_{\mathcal{O}}$$

be the induced homomorphism. Just as in 2.2.2, we have a homomorphism

$$h_r^*: T_{X(\mathbb{C}),\mathrm{hor}} \to A_{\mathbb{C},X(\mathbb{C})} \otimes_{\mathcal{O}_{X(\mathbb{C})}} (\mathrm{gr}^{r-1}\mathcal{H}_{X(\mathbb{C})}(V)_{\mathcal{O}})^* \otimes \mathrm{gr}^r\mathcal{H}_{X(\mathbb{C})}(V)_{\mathcal{O}}$$

.

Theorem 2.2.3.1. The Hermitian form $\kappa(\det(\mathrm{fil}^r\mathcal{H}_{X(\mathbb{C})}(V)_{\mathcal{O}}))_{\mathrm{hor}}$ on $T_{X(\mathbb{C}),\mathrm{hor}}$ coincides with $(\alpha,\beta) \mapsto (h_r(\alpha), h_r(\beta)) = \langle h_r(\alpha), h_r^*(\beta)\rangle$ where the second $(\ ,\ )$ is the Hodge metric of $(\mathrm{gr}^r\mathcal{H}_{X(\mathbb{C})}(V)_{\mathcal{O}})^* \otimes \mathrm{gr}^{r-1}\mathcal{H}_{X(\mathbb{C})}(V)_{\mathcal{O}}$ and $\langle\ ,\ \rangle$ is the paring

$$(\mathrm{gr}^r\mathcal{H}_{X(\mathbb{C})}(V)_{\mathcal{O}})^*\otimes\mathrm{gr}^{r-1}\mathcal{H}_{X(\mathbb{C})}(V)_{\mathcal{O}}\times(\mathrm{gr}^{r-1}\mathcal{H}_{X(\mathbb{C})}(V)_{\mathcal{O}})^*\otimes\mathrm{gr}^r\mathcal{H}_{X(\mathbb{C})}(V)_{\mathcal{O}} \to \mathcal{O}_{X(\mathbb{C})}$$

.

This is proved in the same way as the theorem of Griffiths (2.2.1), and is also reduced to that theorem by the following fact. For any $x \in X(\mathbb{C})$ and $v \in T_{x,X(\mathbb{C}),\mathrm{hor}}$, there are a one-dimensional complex analytic space Y, $y \in Y$, and a horizontal morphism $f: Y \to X(\mathbb{C})$ such that $f(y) = x$ and v is in the image of $T_{y,Y} \to T_{x,X(\mathbb{C}),\mathrm{hor}}$.

2.2.4. The pullback of $\kappa(\det(\mathrm{fil}^r\mathcal{H}_{X(\mathbb{C})}(V)_{\mathcal{O}}))_{\mathrm{hor}}$ to $T_{D(G,\mathcal{G},H_b),\mathrm{hor}}$ is invariant under the action of $\mathcal{G}(\mathbb{R})$. Recall that by our assumption G is reductive, $D(G,\mathcal{G},H_b)$ is a finite disjoint union of $\mathcal{G}(\mathbb{R})$-orbits (1.1.21).

2.3. $X(\mathbb{C})$ is like a hyperbolic space in the case G is reductive

2.3.1. A complex analytic manifold Y is said to be Brody hyperbolic if any morphism $\mathbb{C} \to Y$ from the complex plane $\mathbb{C}$ is a constant map. If Y is hyperbolic in the sense of Kobayashi, then Y is Brody hyperbolic. The converse is true in the case Y is compact ([4]).

Kobayashi conjectures ([26] page 370) that if Y is a compact hyperbolic complex manifold, the line bundle $\det(\Omega^1_Y)$ (the canonical bundle) of Y is ample. The author thinks that the experts believe the following non-compact version of this conjecture is true:

Conjecture 2.3.1.1. If Y is Brody hyperbolic and if $Y = \bar{Y} \smallsetminus D$ for a compact complex manifold $\bar{Y}$ and for a normal crossing divisor D on Y, then the line bundle $\det(\Omega^1_{\bar{Y}}(\log D))$ (the log canonical bundle) on $\bar{Y}$ is ample.

The result 2.3.4 below shows that in the case G is reductive, the period domains $Y = D(G,\mathcal{G},H_b)$ and $Y = X_{G,\mathcal{G},H_b,K}(\mathbb{C})$ are "like Brody hyperbolic", which means that any horizontal morphisms from $\mathbb{C}$ to Y are constant (notice that the condition "horizontal" is put on morphisms). This is reduced to the case Y is the classical period domain of Griffiths [13] classifying polarized Hodge structures, and the author believes that this case of 2.3.4 is well known to the experts. Another result 2.3.3, which is deduced from 2.2.3.1, shows that if G is reductive, $X_{G,\mathcal{G},H_b,K}(\mathbb{C})$ satisfies something like the ampleness condition in the above Conjecture 2.3.1.1 (see 3.8.14).

2.3.2. Let $\kappa_{X(\mathbb{C}),\spadesuit} = \kappa(\det(\Omega^1_{X(\mathbb{C})}))_{\mathrm{hor}}$ be the Hermitian form on $T_{X(\mathbb{C}),\mathrm{hor}}$ with respect to the Hodge metric defined by the identification $\Omega^1_{X(\mathbb{C})}$ as the dual of $\mathcal{H}_{X(\mathbb{C})}(\mathrm{Lie}(\mathcal{G}))_{\mathcal{O}}/\mathrm{fil}^0$ and by polarizations (1.2.21) on $\mathrm{gr}^W_w\mathcal{H}_{X(\mathbb{C})}(\mathrm{Lie}(\mathcal{G}))$ for $w \in \mathbb{Z}$. Here the curvature form is independent of the choices of polarizations (2.2.2).

This $\kappa_{X(\mathbb{C}),\spadesuit}$ is regarded as an Hermitian form on the quotient $T_{X_{\mathrm{red}}(\mathbb{C}),\mathrm{hor}}$ of $T_{X(\mathbb{C}),\mathrm{hor}}$.

Theorem 2.3.3. *Assume G is reductive. Then $\kappa_{X(\mathbb{C}),\spadesuit}$ is positive definite as an Hermitian form on $T_{X(\mathbb{C}),\mathrm{hor}}$.*

Proof. We have

$$\kappa(\det(\Omega^1_{X(\mathbb{C})})) = -\kappa(\det(\mathcal{H}_{X(\mathbb{C})}(\mathrm{Lie}(\mathcal{G}))_{\mathcal{O}}/\mathrm{fil}^0))$$

$$= \kappa(\det(\mathrm{fil}^0\mathcal{H}_{X(\mathbb{C})}(\mathrm{Lie}(\mathcal{G}))_{\mathcal{O}})) - \kappa(\det(\mathcal{H}_{X(\mathbb{C})}(\mathrm{Lie}(\mathcal{G}))_{\mathcal{O}}))$$

and $\kappa(\det(\mathcal{H}_{X(\mathbb{C})}(\mathrm{Lie}(\mathcal{G}))_{\mathcal{O}})) = \kappa(\mathcal{O}_{X(\mathbb{C})} \otimes_{\mathbb{Q}} \det_{\mathbb{Q}}(\mathcal{H}_{X(\mathbb{C})}(\mathrm{Lie}(\mathcal{G}))_{\mathbb{Q}})) = 0$. Hence

$$\kappa(\det(\Omega^1_{X(\mathbb{C})}))_{\mathrm{hor}} = \kappa(\det(\mathrm{fil}^0\mathcal{H}_{X(\mathbb{C})}(\mathrm{Lie}(\mathcal{G}))_{\mathcal{O}}))_{\mathrm{hor}}.$$

By Theorem 2.2.3.1, the Hermitian form $\kappa(\det(\mathrm{fil}^0\mathcal{H}_{X(\mathbb{C})}(\mathrm{Lie}(\mathcal{G}))_{\mathcal{O}}))_{\mathrm{hor}}$ on $T_{X(\mathbb{C}),\mathrm{hor}} = \mathrm{gr}^{-1}\mathcal{H}_{X(\mathbb{C})}(\mathrm{Lie}(\mathcal{G}))_{\mathcal{O}}$ is the pullback of a positive definite Hermitian form on $\mathcal{H}om(\mathrm{gr}^0\mathcal{H}_{X(\mathbb{C})}(\mathrm{Lie}(\mathcal{G}))_{\mathcal{O}}, \mathrm{gr}^{-1}\mathcal{H}_{X(\mathbb{C})}(\mathrm{Lie}(\mathcal{G}))_{\mathcal{O}})$ by the map

$$\mathrm{gr}^{-1}\mathcal{H}_{X(\mathbb{C})}(\mathrm{Lie}(\mathcal{G}))_{\mathcal{O}} \to \mathcal{H}om(\mathrm{gr}^0\mathcal{H}_{X(\mathbb{C})}(\mathrm{Lie}(\mathcal{G}))_{\mathcal{O}}, \mathrm{gr}^{-1}\mathcal{H}_{X(\mathbb{C})}(\mathrm{Lie}(\mathcal{G}))_{\mathcal{O}}).$$

Hence it is sufficient to prove that the last map is injective. Since G is reductive, there is an isomorphism of Lie algebras $\mathrm{Lie}(G) \cong \mathrm{Lie}(\mathcal{G}) \times \mathrm{Lie}(\mathcal{Q})$ which is compatible with the inclusion map $\mathrm{Lie}(\mathcal{G}) \to \mathrm{Lie}(G)$. Hence it is sufficient to prove that the map

$$\mathrm{gr}^{-1}\mathcal{H}_{X(\mathbb{C})}(\mathrm{Lie}(G))_{\mathcal{O}} \to \mathcal{H}om(\mathrm{gr}^0\mathcal{H}_{X(\mathbb{C})}(\mathrm{Lie}(G))_{\mathcal{O}}, \mathrm{gr}^{-1}\mathcal{H}_{X(\mathbb{C})}(\mathrm{Lie}(G))_{\mathcal{O}})$$

is injective. Let $S_{\mathbb{C}/\mathbb{R}} \to G_{\mathbb{R}}$ be a homomorphism associated to $\mathcal{H}$, let $h : \mathbb{C} = \mathrm{Lie}(S_{\mathbb{C}/\mathbb{R}}) \to \mathrm{Lie}(G)_{\mathbb{R}}$ be the induced homomorphism of Lie algebras, and consider $h(i) \in \mathrm{Lie}(G)_{\mathbb{R}}$. We have $\mathrm{Lie}(G)_{\mathbb{R}} = \oplus_{r\in\mathbb{Z}}\mathrm{Lie}(G)^{(r)}_{\mathbb{R}}$

where $\mathrm{Lie}(G)_{\mathbb{R}}^{(r)} = \{x \in \mathrm{Lie}(G)_{\mathbb{R}} \mid [h(i), x] = rx\}$. It is sufficient to prove that

$$\mathrm{Lie}(G)_{\mathbb{R}}^{(-1)} \to \mathrm{Hom}_{\mathbb{R}}(\mathrm{Lie}(G)_{\mathbb{R}}^{(0)}, \mathrm{Lie}(G)_{\mathbb{R}}^{(-1)}) \; ; \; x \mapsto (y \mapsto [x, y])$$

is injective. But this is seen by $h(i) \in \mathrm{Lie}(G)_{\mathbb{R}}^{(0)}$ and by the fact any $x \in \mathrm{Lie}(G)_{\mathbb{R}}^{(-1)}$ satisfies $[x, h(i)] = x$. Q.E.D.

Proposition 2.3.4. *Assume G is reductive. Then any horizontal holomorphic map $f : \mathbb{C} \to X(\mathbb{C})$ is a constant map.*

Proof. By Schmid nilpotent orbit theorem [33], f gives a nilpotent orbit at $\infty \in \mathbf{P}^1(\mathbb{C}) \supset \mathbb{C}$. But since $\pi_1(\mathbb{C}) = \{1\}$, this nilpotent orbit has no local monodromy and hence no degeneration. Hence f extends to a horizontal morphism $f : \mathbf{P}^1(\mathbb{C}) \to X(\mathbb{C})$. But as is well known, a variation of pure Hodge structure $\mathcal{H}$ on $\mathbf{P}^1(\mathbb{C})$ is constant. (This is proved by using the fact that for any $r \in \mathbb{Z}$, $\deg(\mathrm{fil}^r \mathcal{H}_{\mathcal{O}}) \geq 0$ (a consequence of the theorem of Griffiths in 2.2.1) and hence $\dim(H^0(\mathbf{P}^1(\mathbb{C}), \mathrm{fil}^r \mathcal{H}_{\mathcal{O}})) \geq \mathrm{rank}(\mathrm{fil}^r \mathcal{H}_{\mathcal{O}})$ by Riemann-Roch.) Hence f is constant. Q.E.D.

2.4. Height pairings in Hodge theory and curvature forms

2.4.1. Let Y be a complex analytic manifold. Assume we are given a variation of $\mathbb{Z}$-Hodge structure $\mathcal{H}_0$ of weight -1 on Y and variations of mixed $\mathbb{Z}$-Hodge structure $\mathcal{H}_1$ and $\mathcal{H}_2$ on Y with exact sequences

$$\text{(i)} \quad 0 \to \mathcal{H}_0 \to \mathcal{H}_1 \to \mathbb{Z} \to 0 \quad \text{and} \quad \text{(ii)} \; 0 \to \mathcal{H}_0^*(1) \to \mathcal{H}_2 \to \mathbb{Z} \to 0$$

where $\mathcal{H}_0^*$ denotes the $\mathbb{Z}$-dual of $\mathcal{H}_0$. Note that the exact sequence (ii) corresponds to an exact sequence (iii) $0 \to \mathbb{Z}(1) \to \mathcal{H}_2^*(1) \to \mathcal{H}_0 \to 0$.

Then we have a line bundle on Y with a C^∞ metric as follows (see for example [5]).

Locally on Y, we have a variation $\mathcal{H}$ of mixed Hodge structure on Y which satisfies $\mathrm{gr}_{-1}^W \mathcal{H} = \mathcal{H}_0$, $\mathcal{H}/W_{-2}\mathcal{H} = \mathcal{H}_1$ (with the exact sequence (i)), $W_{-1}\mathcal{H} = \mathcal{H}_2^*(1)$ (with the exact sequence (iii)).

The isomorphism classes of such $\mathcal{H}$ form a $\mathbf{G}_m$-torsor on Y (defined globally on Y). The action of $\mathbf{G}_m$ is as follows. A local section s of $\mathbf{G}_m$ sends the class of $\mathcal{H}$ to the Baer sum of the following two extensions of $\mathcal{H}_1$ by $\mathbb{Z}(1)$. One is $\mathcal{H}$, and the other is the extension of $\mathcal{H}_1$ by $\mathbb{Z}(1)$ which is obtained from the extension of $\mathbb{Z}$ by $\mathbb{Z}(1)$ corresponding to s and from the projection $\mathcal{H}_1 \to \mathbb{Z}$.

Hence we obtain a line bundle $L(\mathcal{H}_1, \mathcal{H}_2)$ on Y associated to this $\mathbf{G}_m$-torsor. This line bundle has the C^∞ metric $\mathrm{class}(\mathcal{H}) \mapsto |\mathcal{H}|$ characterized by the property that locally there is a lifting v of $1 \in \mathrm{gr}_0^W \mathcal{H}_{\mathbb{R}}$ in $A_{\mathbb{R},Y} \otimes_{\mathbb{R}} \mathcal{H}_{\mathbb{R}}$ such that $v + \log(|\mathcal{H}|) \in A_{\mathbb{C},Y} \otimes_{\mathcal{O}_Y} \mathrm{fil}^0 \mathcal{H}_{\mathcal{O}}$ in $A_{\mathbb{C},Y} \otimes_{\mathcal{O}_Y} \mathcal{H}_{\mathcal{O}} =$

$A_{\mathbb{C},Y} \otimes_{\mathbb{R}} \mathcal{H}_{\mathbb{R}}$. Here $\log(|\mathcal{H}|) \in A_{\mathbb{R},Y}$ is regarded as a local section of $A_{\mathbb{C},Y} \otimes_{\mathbb{R}} W_{-2}\mathcal{H}_{\mathbb{R}}$ via the isomorphism $A_{\mathbb{C},Y} \cong A_{\mathbb{C},Y} \otimes_{\mathbb{R}} W_{-2}\mathcal{H}_{\mathcal{O}}$ which is induced by $\mathbb{Z} \cdot 2\pi i \cong W_{-2}\mathcal{H}_{\mathbb{Z}}$. (Hence $\log(|\mathcal{H}|)$ is regarded as a local section of $i \cdot A_{\mathbb{R},Y} \otimes_{\mathbb{R}} W_{-2}\mathcal{H}_{\mathbb{R}}$.)

In this Section 2.4, we compute the curvature form of $L(\mathcal{H}_1, \mathcal{H}_2)$.

As in [5], this line bundle $L(\mathcal{H}_1, \mathcal{H}_2)$ with metric is related to the theory of height pairing (cf. 3.4.5).

2.4.2. (1) We have $\mathcal{O}_Y$-homomorphisms

$$h_{\mathcal{H}_1} : T_Y \to A_{\mathbb{C},Y} \otimes_{\mathcal{O}_Y} \mathrm{gr}^{-1}\mathcal{H}_{0,\mathcal{O}}, \quad h_{\mathcal{H}_2} : T_Y \to A_{\mathbb{C},Y} \otimes_{\mathcal{O}_Y} \mathrm{gr}^{-1}\mathcal{H}_0^*(1)_{\mathcal{O}}$$

defined as follows. For $\alpha \in T_Y$, consider the composition $\mathrm{gr}^0\mathcal{H}_{1,\mathcal{O}} \overset{\nabla}{\to} \mathrm{gr}^{-1}\mathcal{H}_{1,\mathcal{O}} \otimes_{\mathcal{O}_Y} \Omega^1_Y \overset{\alpha}{\to} \mathrm{gr}^{-1}\mathcal{H}_{1,\mathcal{O}} = \mathrm{gr}^{-1}\mathcal{H}_{0,\mathcal{O}}$. Define $h_{\mathcal{H}_1}(\alpha)$ as the image of $1 \in \mathbb{R}$ under $\mathbb{R} \to A_{\mathbb{C},Y} \otimes_{\mathcal{O}_Y} \mathrm{gr}^0\mathcal{H}_{1,\mathcal{O}} \overset{\alpha}{\to} A_{\mathbb{C},Y} \otimes_{\mathcal{O}_Y} \mathrm{gr}^{-1}\mathcal{H}_{0,\mathcal{O}}$ where the first arrow is induced by the inverse of the isomorphism $A_{\mathbb{C},Y} \otimes_{\mathcal{O}_Y} \mathrm{fil}^0\mathcal{H}_{1,\mathcal{O}} \cap A_{\mathbb{R},Y} \otimes_{\mathbb{R}} \mathcal{H}_{1,\mathbb{R}} \overset{\cong}{\to} A_{\mathbb{R},Y} \otimes_{\mathbb{R}} \mathrm{gr}^W_0 \mathcal{H}_{1,\mathbb{R}}$. We define $h_{\mathcal{H}_2}$ by replacing $0 \to \mathcal{H}_0 \to \mathcal{H}_1 \to \mathbb{Z} \to 0$ by $0 \to \mathcal{H}_0^*(1) \to \mathcal{H}_2 \to \mathbb{Z} \to 0$ in the definition of $h_{\mathcal{H}_1}$.

(2) We define

$$h^*_{\mathcal{H}_1} : T_Y \to A_{\mathbb{C},Y} \otimes_{\mathcal{O}_Y} \mathrm{gr}^0\mathcal{H}_{0,\mathcal{O}}, \quad h^*_{\mathcal{H}_2} : T_Y \to A_{\mathbb{C},Y} \otimes_{\mathcal{O}_Y} \mathrm{gr}^0\mathcal{H}_0^*(1)_{\mathcal{O}}$$

by $h^*_{\mathcal{H}_i} = \lambda_{-1} \circ h_{\mathcal{H}_i}$ $(i = 1, 2)$, where λ_{-1} in this definition of $h^*_{\mathcal{H}_1}$ (resp. $h^*_{\mathcal{H}_2}$) is the map in 2.2.2 for $\mathcal{H}_0$ (resp. $\mathcal{H}_0^*(1)$).

Theorem 2.4.3. *The Hermitian form $\kappa(L(\mathcal{H}_1, \mathcal{H}_2))$ on T_Y coincides with*

$$(\alpha, \beta) \mapsto -\langle h_{\mathcal{H}_1}(\alpha), h^*_{\mathcal{H}_2}(\beta)\rangle + \langle h^*_{\mathcal{H}_1}(\beta), h_{\mathcal{H}_2}(\alpha)\rangle,$$

where the pairing $\langle\ \rangle$ is defined by the canonical pairing $A_{\mathbb{C},Y} \otimes_{\mathcal{O}_Y} \mathcal{H}_{0,\mathcal{O}} \times A_{\mathbb{C},Y} \otimes_{\mathcal{O}_Y} \mathcal{H}_0^*(1)_{\mathcal{O}} \to A_{\mathbb{C},Y}$.

2.4.4. The proof of 2.4.3 is the reduction to the case of a period domain.

Fix a free $\mathbb{Z}$-module $H_{0,\mathbb{Z}}$ of finite rank, and let $H_{1,\mathbb{Z}} = \mathbb{Z}e_1 \oplus H_{0,\mathbb{Z}}$ and $H_{2,\mathbb{Z}} = \mathbb{Z}e_2 \oplus H^*_{0,\mathbb{Z}}(1)$ be free $\mathbb{Z}$-modules of rank $\mathrm{rank}_{\mathbb{Z}}(H_{0,\mathbb{Z}}) + 1$. Here $H^*_{0,\mathbb{Z}}(1) = \mathrm{Hom}_{\mathbb{Z}}(H_{0,\mathbb{Z}}, \mathbb{Z} \cdot 2\pi i)$. Define an increasing filtrations on $H_{1,\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} H_{1,\mathbb{Z}}$ and on $H_{2,\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} H_{2,\mathbb{Z}}$ by

$$W_{-2} = 0 \subset W_{-1} = H_{0,\mathbb{Q}} \subset W_0 = H_{1,\mathbb{Q}},$$
$$W_{-2} = 0 \subset W_{-1} = H^*_{0,\mathbb{Q}}(1) \subset W_0 = H_{2,\mathbb{Q}}.$$

Fix integers $h(r) \in \mathbb{Z}_{\geq 0}$ for $r \in \mathbb{Z}$ such that $\sum_r h(r) = \dim_{\mathbb{Q}} H_{0,\mathbb{Q}}$ and $h(-1-r) = h(r)$ for all r. Let D be the set of all pairs $(\mathrm{fil}_{(1)}, \mathrm{fil}_{(2)})$, where $\mathrm{fil}_{(i)}$ is a decreasing filtration on $H_{i,\mathbb{C}}$ satisfying the following conditions (i)–(iii).

(i) The restriction of $\mathrm{fil}_{(2)}$ to $H^*_{0,\mathbb{Z}}(1)_{\mathbb{C}}$ coincides with the filtration induced by the restriction of $\mathrm{fil}_{(1)}$ to $H_{0,\mathbb{C}}$.
(ii) For $i = 1, 2$, $(H_{i,\mathbb{Z}}, W, \mathrm{fil}_{(i)})$ is a mixed Hodge structures.
(iii) $\dim \mathrm{gr}^r W_{-1}\mathrm{fil}_{(i)} = h(r)$ for all r $(i = 1, 2)$.

Then D is naturally regarded as a complex analytic manifold. On D, we have the universal objects $\mathcal{H}_{0,D}$, $\mathcal{H}_{1,D}$, $\mathcal{H}_{2,D}$ of $\mathbb{Q}$MHS(D) (1.1.16) (but these need not belong to $\mathbb{Q}$VMHS(D) (1.2.25)) with exact sequences $0 \to \mathcal{H}_{0,D} \to \mathcal{H}_{1,D} \to \mathbb{Z} \to 0$ and $0 \to \mathcal{H}^*_{0,D}(1) \to \mathcal{H}_{2,D} \to \mathbb{Z} \to 0$. By the method of 2.4.1, we have a line bundle $L(\mathcal{H}_{1,D}, \mathcal{H}_{2,D})$ on D with a C^∞ metric.

The tangent bundle of D is identified with $\mathcal{E}/\mathrm{fil}^0$, where $\mathcal{E}$ is a part of $\mathcal{E}nd_{\mathcal{O}_D}(\mathcal{H}_{1,D,\mathcal{O}}, W) \times \mathcal{E}nd_{\mathcal{O}_D}(\mathcal{H}_{2,D,\mathcal{O}}, W)$ consisting of all pairs (a, b) such that the restriction of a to $\mathcal{H}_{0,D,\mathcal{O}}$ and the restriction of b to $\mathcal{H}^*_{0,D}(1)_{\mathcal{O}}$ are induced from the other, and fil is the Hodge filtration.

The horizontal tangent bundle $T_{D,\mathrm{hor}}$ of D is defined to be $\mathrm{fil}^{-1}/\mathrm{fil}^0$ of $\mathcal{E}$.

Let $\kappa(L(\mathcal{H}_{1,D}, \mathcal{H}_{2,D}))$ be the curvature form of the metric of $L(\mathcal{H}_{1,D}, \mathcal{H}_{2,D})$ and let $\kappa(L(\mathcal{H}_{1,D}, \mathcal{H}_{2,D}))_{\mathrm{hor}}$ be its restriction to $T_{D,\mathrm{hor}}$.

By the method of 2.4.2, we have

$$h_{\mathcal{H}_{1,D}} : T_{D,\mathrm{hor}} \to A_{\mathbb{C},D} \otimes_{\mathcal{O}_Y} \mathrm{gr}^{-1}\mathcal{H}_{0,D,\mathcal{O}},$$

$$h_{\mathcal{H}_{2,D}} : T_{D,\mathrm{hor}} \to A_{\mathbb{C},D} \otimes_{\mathcal{O}_D} \mathrm{gr}^{-1}\mathcal{H}^*_{0,D}(1)_{\mathcal{O}},$$

$$h^*_{\mathcal{H}_{1,D}} : T_{D,\mathrm{hor}} \to A_{\mathbb{C},D} \otimes_{\mathcal{O}_Y} \mathrm{gr}^{0}\mathcal{H}_{0,D,\mathcal{O}},$$

$$h^*_{\mathcal{H}_{2,D}} : T_{D,\mathrm{hor}} \to A_{\mathbb{C},D} \otimes_{\mathcal{O}_D} \mathrm{gr}^{0}\mathcal{H}^*_{0,D}(1)_{\mathcal{O}}.$$

Proposition 2.4.5. *The Hermitian form* $\kappa(L(\mathcal{H}_{1,D}, \mathcal{H}_{2,D}))_{\mathrm{hor}}$ *on* $T_{D,\mathrm{hor}}$ *coincides with*

$$(\alpha, \beta) \mapsto -\langle h_{\mathcal{H}_{1,D}}(\alpha), h^*_{\mathcal{H}_{2,D}}(\beta)\rangle + \langle h^*_{\mathcal{H}_{1,D}}(\beta), h_{\mathcal{H}_{2,D}}(\alpha)\rangle.$$

2.4.6. Theorem 2.4.3 is reduced to 2.4.5 as follows. We may assume that Y is connected. Take $b \in Y$ and define the above period domain D by taking the stalks $\mathcal{H}_{i,\mathbb{Z},b}$ at b as $H_{i,\mathbb{Z}}$ $(i = 0, 1, 2)$ and by taking the Hodge numbers of $\mathcal{H}_0$ as $h(r)$. Then we have the horizontal morphism $\tilde{Y} \to D$ (the period map) from the universal covering $\tilde{Y}$ of Y to

D associated to $(\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2)$, and the pullback of $\kappa(L(\mathcal{H}_1, \mathcal{H}_2))$ on $\tilde{Y}$ coincides with the pullback of $\kappa(L(\mathcal{H}_{1,D}, \mathcal{H}_{2,D}))_{\mathrm{hor}}$. Hence 2.4.3 follows from 2.4.5.

2.4.7. The proof of 2.4.5 occupies 2.4.7–2.4.18. For the proof of 2.4.5, we use several period domains related to D.

We first define the period domain $\tilde{D}$ and describe the $\mathbf{G}_m$-torsor on D associated to $L(\mathcal{H}_{1,D}, \mathcal{H}_{2,D})$ explicitly. Let $H_{0,\mathbb{Z}}$ be as in 2.4.4 and consider a free $\mathbb{Z}$-module $H_{\mathbb{Z}} := \mathbb{Z}e \oplus H_{0,\mathbb{Z}} \oplus \mathbb{Z}e'$ of rank $\mathrm{rank}_{\mathbb{Z}}(H_{0,\mathbb{Z}}) + 2$. Define the increasing filtration W on $H_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} H_{0,\mathbb{Z}}$ by

$$W_{-3} = 0 \subset W_{-2} = \mathbb{Q}e' \subset W_{-1} = H_{0,\mathbb{Q}} \oplus \mathbb{Q}e' \subset W_0 = H_{0,\mathbb{Q}}.$$

Let $\tilde{D}$ be the set of all decreasing filtrations φ on $H_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{Z}} H_{\mathbb{Z}}$ such that $(H_{\mathbb{Z}}, W, \varphi)$ is a mixed Hodge structure and such that $\dim \mathrm{gr}^r \mathrm{gr}^W_{-1} \varphi = h(r)$ for all r. Then $\tilde{D}$ is naturally regarded as a complex analytic manifold.

We have a surjective morphism

$$\tilde{D} \to D\ ;\ \varphi \mapsto (\mathrm{fil}_{(1)}, \mathrm{fil}_{(2)}).$$

Here $\mathrm{fil}_{(1)}$ is the filtration induced on $H_{\mathbb{C}}/W_{-2}H_{\mathbb{C}}$ by φ where we identify e with e_1 and $\mathrm{fil}_{(2)}$ is the filtration induced by φ by identifying $H^*_{2,\mathbb{Z}}(1)_{\mathbb{C}}$ with $W_{-1}H_{\mathbb{C}}$ where we identify e' with $e_2^* \otimes 2\pi i$. Consider the action of the additive group $\mathbb{C}$ on $H_{\mathbb{C}}$ as follows: $z \in \mathbb{C}$ sends e to $e + z \cdot (2\pi i)^{-1} e'$ and fix all elements of $W_{-1}H_{\mathbb{C}}$. This action induces an action of $\mathbb{C}$ on $\tilde{D}$ and induces an isomorphism $\mathbb{C} \backslash \tilde{D} \overset{\cong}{\to} D$ and $\tilde{D}$ is a $\mathbb{C}$-torsor over D. Via $\mathbb{C}/\mathbb{Z}(1) \cong \mathbb{C}^\times\ ;\ z \mapsto \exp(z)$ ($\mathbb{Z}(1) = \mathbb{Z} \cdot 2\pi i$), $\mathbb{Z}(1) \backslash \tilde{D}$ becomes a $\mathbb{C}^\times$-torsor over D.

This $\mathbb{Z}(1) \backslash \tilde{D} \to D$ is the $\mathbf{G}_m$-torsor associated to the line bundle $L(\mathcal{H}_{1,D}, \mathcal{H}_{2,D})$ on D. Its metic is $\tilde{D} \ni \varphi \mapsto |\varphi|$ where $|\varphi| \in \mathbb{R}_{>0}$ is characterized by the following property. There is an element $v \in H_{0,\mathbb{R}}$ such that $e + v + \log(|\varphi|) \cdot (2\pi i)^{-1} e' \in \varphi^0$.

2.4.8. We define period domains D', $D^{(i)}$ ($i = 1, 2$), and $D(s)$. Let the notation be as in 2.4.4.

Let D' be the set of all decreasing filtrations φ on $H_{0,\mathbb{C}}$ such that $(H_{0,\mathbb{Z}}, \varphi)$ is a Hodge structure of weight -1 and $\dim_{\mathbb{C}}(\mathrm{gr}^r \varphi) = h(r)$ for all r. We have a projection $p : D \to D'\ ;\ (\mathrm{fil}_{(1)}, \mathrm{fil}_{(2)}) \mapsto \varphi$ where φ is the restriction of $\mathrm{fil}_{(1)}$ to $H_{0,\mathbb{C}}$. For $s \in D'$, let $D(s) \subset D$ be the inverse image of s in D.

For $i = 1, 2$, let $D^{(i)}$ be the set of all decreasing filtrations φ on $H_{i,\mathbb{C}}$ such that $(H_{i,\mathbb{Z}}, W, \varphi)$ is a mixed Hodge structure and $\dim_{\mathbb{C}}(\mathrm{gr}^r \mathrm{gr}^W_{-1} \varphi) = h(r)$ for all r. Let $p_i : D \to D^{(i)}$ be the map $(\mathrm{fil}_{(1)}, \mathrm{fil}_{(2)}) \to \mathrm{fil}_{(i)}$.

For $x = (\mathrm{fil}_{(1)}(x), \mathrm{fil}_{(2)}(x)) \in D$, we have a map $f_{x,i} : D^{(i)} \to D$; $\varphi \mapsto (\mathrm{fil}_{(1)}, \mathrm{fil}_{(2)})$ defined as follows. Let $S^{(i)}$ be the one-dimensional $\mathbb{R}$-linear space $\mathrm{fil}_{(i)}(x)^0 \cap H_{i,\mathbb{R}}$. Then for $\varphi \in D^{(i)}$, $\mathrm{fil}_{(i)} = \varphi$ and $\mathrm{fil}_{(3-i)}$ is characterized by the property that $\mathrm{fil}^0_{(3-i)} = \varphi^0 \cap W_{-1}H_{3-i,\mathbb{C}} \oplus \mathbb{C} \otimes_{\mathbb{R}} S^{(3-i)}$. The composition $D^{(i)} \overset{f_{x,i}}{\to} D \overset{p_i}{\to} D^{(i)}$ is the identity map, and we have $f_{x,i}(p_i(x)) = x$.

Let $g_{x,i} : D' \to D^{(i)}$ be the following map $\varphi' \mapsto \varphi$. φ is characterized by the properties that $W_{-1}\varphi = \varphi'$ and $\varphi^0 = (\varphi')^0 \oplus \mathbb{C} \otimes_{\mathbb{R}} S^{(i)}$. Then $g_{x,i}$ sends $p(x) \in D'$ to $p_i(x)$. We have $f_{x,1} \circ g_{x,1} = f_{x,2} \circ g_{x,2} : D' \to D$. We denote this map $D' \to D$ by h_x. The composition $D' \overset{h_x}{\to} D \overset{p}{\to} D'$ is the identity map and $h_x(p(x)) = x$.

We consider the tangent bundles and horizontal tangent bundles of these period domains.

We have $T_{D'} = (\mathcal{O}_{D'} \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(H_{0,\mathbb{C}}))/\mathrm{fil}^0$ where fil is the Hodge filtration. Define $T_{D',\mathrm{hor}} = \mathrm{fil}^{-1}/\mathrm{fil}^0 \subset T_{D'}$. We have $T_{D^{(i)}} = (\mathcal{O}_{D^{(i)}} \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(H_{i,\mathbb{C}}))/\mathrm{fil}^0$ where fil is the Hodge filtration. Define $T_{D^{(i)},\mathrm{hor}} = \mathrm{fil}^{-1}/\mathrm{fil}^0 \subset T_{D^{(i)}}$. We have $T_{D(s)} = (W_{-1}$ part of the pullback of T_D to $D(s))$. Let $T_{D(s),\mathrm{hor}}$ be the W_{-1} part of the pullback of $W_{-1}T_{D,\mathrm{hor}}$ to $D(s)$.

For $x \in D$ with $s := p(x) \in D'$, by the embedding $D(s) \subset D$ and the embeddings $D' \subset D^{(i)} \subset D$ induced by $g_{x,i}$ and $f_{x,i}$, regard $T_{x,D(s),\mathrm{hor}} \subset T_{x,D,\mathrm{hor}}$ and $T_{s,D',\mathrm{hor}} \subset T_{x,D^{(i)},\mathrm{hor}} \subset T_{x,D,\mathrm{hor}}$. We have

$$T_{x,D,\mathrm{hor}} = T_{s,D',\mathrm{hor}} \oplus T_{x,D(s),\mathrm{hor}}, \quad T_{x,D,\mathrm{hor}} = T_{x,D^{(1)},\mathrm{hor}} + T_{x,D^{(2)},\mathrm{hor}}.$$

Hence

$$T_{x,D,\mathrm{hor}} \otimes T_{x,D,\mathrm{hor}} = (\sum_{i=1}^{2} T_{x,D^{(i)},\mathrm{hor}} \otimes T_{x,D^{(i)},\mathrm{hor}}) + T_{x,D(s),\mathrm{hor}} \otimes T_{x,D(s),\mathrm{hor}}.$$

Hence for the proof of 2.4.5, it is sufficient to prove the pullbacks of $\kappa(L(\mathcal{H}_{1,D}, \mathcal{H}_{2,D}))$ to $T_{D^{(i)},\mathrm{hor}}$ ($i = 1, 2$, under $f_{x,i}$) and to $T_{D(s),\mathrm{hor}}$ are described as in 2.4.5.

2.4.9. We consider first $D^{(i)}$. As is easily seen, under the map $f_{x,i} : D^{(i)} \to D$, the pullback of $L(\mathcal{H}_{1,D}, \mathcal{H}_{2,D})$ with the metric is isomorphic to the trivial line bundle $\mathcal{O}_{D^{(i)}}$ with the standard metric. Hence its curvature form is zero. On the other hand, the pullbacks of the map $h_{\mathcal{H}_1}$ and $h^*_{\mathcal{H}_1}$ on $T_{D^{(2)},\mathrm{hor}}$ under $f_{x,2}$ are the zero maps and the pullbacks of the maps $h_{\mathcal{H}_2}$ and $h^*_{\mathcal{H}_2}$ on $T_{D^{(1)},\mathrm{hor}}$ under $f_{x,1}$ is the zero map.

2.4.10. Fix $s \in D'$. Let $\tilde{D}(s)$ be the inverse image of $D(s)$ in $\tilde{D}$. Let U be a $\mathbb{C}$-subspace of $H_{0,\mathbb{C}}$ such that $H_{0,\mathbb{C}} = \mathrm{fil}(s)^0 \oplus U$ and let U' be a $\mathbb{C}$-subspace of $H^*_{0,\mathbb{Z}}(1)_\mathbb{C}$ such that $H^*_{0,\mathbb{Z}}(1)_\mathbb{C} = (\mathrm{fil}(s))^*(1)^0 \oplus U'$. Then we have isomorphisms

$$U \times U' \overset{\cong}{\to} D(s)\ ;\ (z, w) \mapsto (\mathrm{fil}(z), \mathrm{fil}(w)),$$

$$U \times U' \times \mathbb{C} \overset{\cong}{\to} \tilde{D}(s)\ ;\ (z, w, u) \mapsto \varphi(z, w, u).$$

Here $\mathrm{fil}(z)$ on $H_{1,\mathbb{C}}$ extends $\mathrm{fil}(s)$ on $H_{0,\mathbb{C}}$ by $e+z \in \mathrm{fil}(z)^0$, $\mathrm{fil}(w)$ on $H_{2,\mathbb{C}}$ extends $\mathrm{fil}(s)^*(1)$ on $H^*_{0,\mathbb{Z}}(1)_\mathbb{C}$ by $e+w \in \mathrm{fil}(w)^0$, and $\varphi(z, w, u)$ extends the filtration $\mathrm{fil}(w)^*(1)$ on $W_{-1}H_\mathbb{C}$ by $e+z+u\cdot(2\pi i)^{-1}e' \in \varphi(z, w, u)^0$.

These isomorphisms are compatible with the projection $\tilde{D} \to D$.

2.4.11. By using $H_{0,\mathbb{C}} = \mathrm{fil}(s)^0 \oplus H_{0,\mathbb{R}}$, write $z = z_1 + z_2$ ($z \in H_{0,\mathbb{C}}$, $z_1 \in \mathrm{fil}(s)^0$, $z_2 \in H_{0,\mathbb{R}}$). Similarly, by using $H^*_{0,\mathbb{Z}}(1)_\mathbb{C} = \mathrm{fil}(s)^*(1)^0 \oplus H^*_{0,\mathbb{Z}}(1)_\mathbb{R}$, write $w = w_1 + w_2$ ($w \in H^*_{0,\mathbb{Z}}(1)_\mathbb{C}$, $w_1 \in \mathrm{fil}(s)^*(1)^0$, $w_2 \in H^*_{0,\mathbb{Z}}(1)_\mathbb{R}$).

We also use the notation

$$z = \mathrm{Re}(z) + i\mathrm{Im}(z) \quad \text{for } z \in H_{0,\mathbb{C}}, \text{where } \mathrm{Re}(z), \mathrm{Im}(z) \in H_{0,\mathbb{R}}.$$

Let $\langle\ ,\ \rangle : H_{0,\mathbb{R}} \times H^*_{0,\mathbb{Z}}(1)_\mathbb{R} \to \mathbb{R}(1)$ be the canonical pairing. We denote the induced $\mathbb{C}$-linear pairing $H_{0,\mathbb{C}} \times H^*_{0,\mathbb{Z}}(1)_\mathbb{C} \to \mathbb{C}$ also by $\langle\ ,\ \rangle$.

Lemma 2.4.12. $\log(|\varphi(z, w, u)|) = i\langle \mathrm{Im}(z), w_2\rangle + \mathrm{Re}(u)$.
(Note that $\langle \mathrm{Im}(z), w_2\rangle \in \mathbb{R}(1)$ *and hence* $i\langle \mathrm{Im}(z), w_2\rangle \in \mathbb{R}$.*)*

Proof. $\varphi(z, w, u)^0$ is generated by $e + z + u \cdot (2\pi i)^{-1}e'$ and $\lambda - \langle\lambda, w\rangle \cdot (2\pi i)^{-1}e'$ ($\lambda \in \mathrm{fil}(s)^0$). In particular, $z_1 - \langle z_1, w\rangle \cdot (2\pi i)^{-1}e' \in \varphi(z, w, u)^0$. Hence $e + z_2 + (\langle z_1, w\rangle + u) \cdot (2\pi i)^{-1}e' \in \varphi(z, w, u)^0$. This shows $\log(|\varphi|) = \mathrm{Re}(\langle z_1, w\rangle) + \mathrm{Re}(u)$. We have $\langle z_1, w_1\rangle = 0$ because $\langle \mathrm{fil}(s)^0, (\mathrm{fil}(s)^*(1))^0\rangle = 0$. Hence

$$\mathrm{Re}(\langle z_1, w\rangle) = \mathrm{Re}(\langle z_1, w_2\rangle) = i\langle \mathrm{Im}(z_1), w_2\rangle = i\langle \mathrm{Im}(z), w_2\rangle.$$

Q.E.D.

2.4.13. On $\tilde{D}(s)$, we have $\partial\bar{\partial}\mathrm{Im}(u) = 0$. Hence $2\partial\bar{\partial}\ \log(|\varphi|)$ descends to $D(s)$ and this is the pullback $\kappa(L(\mathcal{H}_{1,D(s)}, \mathcal{H}_{2,D(s)}))$ of $\kappa(L(\mathcal{H}_{1,D}, \mathcal{H}_{2,D}))$ to $D(s)$ where $\mathcal{H}_{i,D(s)}$ ($i = 1, 2$) denotes the pullback of $\mathcal{H}_{i,D}$ to $D(s)$. It is equal to $2i\partial\bar{\partial}\langle \mathrm{Im}(z), w_2\rangle$.

Lemma 2.4.14. *On* $D(s)$*, we have* $\partial\bar{\partial}z_1 = \partial\bar{\partial}z_2 = 0$. $\partial\bar{\partial}w_1 = \partial\bar{\partial}w_2 = 0$.

Proof. Write the $\mathbb{R}$-linear map $\mathrm{Im} : \mathrm{fil}^0 H_{0,\mathbb{C}} \to H_{0,\mathbb{R}}$ by ℓ. Then ℓ is a bijection. Since $z_1 = \ell^{-1}\mathrm{Im}(z_1) = \ell^{-1}\mathrm{Im}(z)$, we have $\partial\bar{\partial}z_1 = \ell^{-1}(\partial\bar{\partial}\mathrm{Im}(z)) = 0$. The statement for z_2 follows from this by $\bar{\partial}z = 0$. Results for w_i are proved similarly. Q.E.D.

2.4.15. By 2.4.12 and 2.4.14, we have $2\partial\bar{\partial}\log(|\varphi|) = \langle\partial z, \bar{\partial} w_2\rangle + \langle\bar{\partial}\bar{z}, \partial w_2\rangle$ in $A^2_{\mathbb{C},D(s)}$. Here $A^p_{\mathbb{C},D(s)}$ denotes the sheaf of complex valued C^∞ p-forms on $D(s)$, and $\langle\ ,\ \rangle$ denotes the pairing $(A^1_{\mathbb{C},D(s)} \otimes_{\mathbb{R}} H_{0,\mathbb{R}}) \times (A^1_{\mathbb{C},D(s)} \otimes_{\mathbb{R}} H^*_{0,\mathbb{Z}}(1)_{\mathbb{R}}) \to A^2_{\mathbb{C},D(s)}$; $(\omega\otimes h, \omega'\otimes h') \mapsto (\omega\wedge\omega')\otimes\langle h, h'\rangle$.

2.4.16. Let Ω^1_{hor} be the quotient of $\Omega^1_{D(s)}$ corresponding to the subbundle $T_{D(s),\mathrm{hor}}$ of $T_{D(s)}$ by duality. Let A^1_{hor} be the quotient $A_{\mathbb{C},D(s)}\otimes_{\mathcal{O}_{D(s)}}\Omega^1_{\mathrm{hor}}\oplus(A_{\mathbb{C},D(s)}\otimes_{\mathcal{O}_{D(s)}}\Omega^1_{\mathrm{hor}})^-$ of $A^1_{\mathbb{C},D(s)} = A_{\mathbb{C},D(s)}\otimes_{\mathcal{O}_{D(s)}} \Omega^1_{D(s)}\oplus(A_{\mathbb{C},D(s)}\otimes_{\mathcal{O}_{D(s)}}\Omega^1_{D(s)})^-$ where $(\ )^-$ denotes the complex conjugate. Consider

$$\nabla_{\mathrm{hor}}: \mathcal{H}_{1,D(s),\mathcal{O}} = \mathcal{O}_{D(s)}\otimes_{\mathbb{R}} H_{1,\mathbb{R}} \to \mathcal{H}_{1,D(s),\mathcal{O}}\otimes_{\mathcal{O}_{D(s)}}\Omega^1_{\mathrm{hor}} = \Omega^1_{\mathrm{hor}}\otimes_{\mathbb{R}} H_{1,\mathbb{R}}$$

which satisfies the Griffiths transversality. This induces

$$\nabla_{\mathrm{hor}} = (\partial, \bar{\partial}) : A_{\mathbb{C},D(s)}\otimes_{\mathbb{R}}\mathcal{H}_{1,D(s),\mathbb{R}} \to A^1_{\mathrm{hor}}\otimes_{\mathbb{R}}\mathcal{H}_{0,D(s),\mathbb{R}}$$

$$= ((A_{\mathbb{C},D(s)}\otimes_{\mathcal{O}_{D(s)}}\Omega^1_{\mathrm{hor}})\oplus(A_{\mathbb{C},D(s)}\otimes_{\mathcal{O}_{D(s)}}\Omega^1_{\mathrm{hor}})^-)\otimes_{\mathbb{R}} H_{0,\mathbb{R}}.$$

In the rest of the proof of 2.4.5 below, ∇_{hor}, ∂, $\bar{\partial}$ ($\nabla = \partial + \bar{\partial}$) are considered by using Ω^1_{hor} and A^1_{hor}, not using $\Omega^1_{D(s)}$ and $A^1_{\mathbb{C},D(s)}$.

Lemma 2.4.17. *On $D(s)$, we have:*
(1) $\partial z \in \mathrm{fil}^{-1}\mathcal{H}_{0,D(s),\mathcal{O}}\otimes_{\mathcal{O}_{D(s)}}\Omega^1_{\mathrm{hor}}$.
(2) $\nabla_{\mathrm{hor}} z_1 \in \mathrm{fil}^0\mathcal{H}_{0,D(s),\mathcal{O}}\otimes_{\mathcal{O}_{D(s)}} A^1_{\mathrm{hor}}$.
(3) $\partial z_2 \in \mathrm{fil}^{-1}\mathcal{H}_{0,D(s),\mathcal{O}}\otimes_{\mathcal{O}_{D(s)}} A^1_{\mathrm{hor}}$.
(4) $\bar{\partial} z_2 \in \mathrm{fil}^0\mathcal{H}_{0,D(s),\mathcal{O}}\otimes_{\mathcal{O}_{D(s)}} A^1_{\mathrm{hor}}$.
(5) The images of ∂z and ∂z_2 in $\mathrm{gr}^{-1}\mathcal{H}_{0,D(s),\mathcal{O}}\otimes_{\mathcal{O}_D(s)} A^1_{\mathrm{hor}}$ coincide.
We have similar results for w, w_1, w_2.

Proof. Since $e + z \in \mathrm{fil}^0\mathcal{H}_{1,D(s),\mathcal{O}}$ and $\nabla_{\mathrm{hor}}(e+z) = \nabla z$, we have (1) by Griffiths transversality for ∇_{hor}. (2) follows from the fact that the Hodge filtration of $\mathcal{H}_{0,D(s)}$ is constant. By (1) and (2), we have (3). (4) follows from (2) and $\bar{\partial} z = 0$. (5) follows from (2). Q.E.D.

2.4.18. By 2.4.15 and by (4) and (5) of 2.4.17,

$$2\partial\bar{\partial}\log(|\varphi|) = \langle(\partial z_2)^{-1,0}, (\bar{\partial} w_2)^{0,-1}\rangle + \langle(\bar{\partial} z_2)^{0,-1}, (\partial w_2)^{-1,0}\rangle.$$

Here $(\ )^{p,q}$ denotes the (p,q)-Hodge component. Since w_2 (resp. z_2) is real, $(\bar{\partial} w_2)^{0,-1}$ (resp. $(\bar{\partial} z_2)^{0,-1}$) is the complex conjugate of $(\partial w_2)^{-1,0}$ (resp. $(\partial z_2)^{-1,0}$). Hence by 2.1.4, the right hand side of the above formula is a purely imaginary $(1,1)$-form on $D(s)$ corresponding to the restriction of the Hermitian form $(\alpha,\beta) \mapsto -\langle h_{\mathcal{H}_{1,D}}(\alpha), h^*_{\mathcal{H}_{2,D}}(\beta)\rangle +$

$\langle h^*_{\mathcal{H}_{1,D}}(\beta), h_{\mathcal{H}_{2,D}}(\alpha)\rangle$. The left hand side is the restriction of the curvature form of $L(\mathcal{H}_{1,D}, \mathcal{H}_{2,D})$ to $T_{D(s),\mathrm{hor}}$.

This completes the proof of 2.4.5.

2.4.19. We have the evident variant of 2.4.3 for a variation of $\mathbb{Q}$-Hodge structure $\mathcal{H}_0$ of weight -1 and for variations of $\mathbb{Q}$-mixed Hodge structures $\mathcal{H}_1$, $\mathcal{H}_2$ with exact sequences $0 \to \mathcal{H}_0 \to \mathcal{H}_1 \to \mathbb{Q} \to 0$ and $0 \to \mathcal{H}_0^*(1) \to \mathcal{H}_2 \to \mathbb{Q} \to 0$. The maps $h_{\mathcal{H}_i}$ and $h^*_{\mathcal{H}_i}$ $(i = 1, 2)$ are defined in the same way. We have the associated $\mathbf{G}_m \otimes \mathbb{Q}$-torsor which has a C^∞ metric and its curvature form. This variant states that the formula in 2.4.3 is true in this generalized situation.

This variant can be proved by the following simple reduction to 2.4.3: Locally, such $\mathcal{H}_0$ comes from a $\mathbb{Z}$-Hodge structure $\tilde{\mathcal{H}}_0$. For some $n \geq 1$, the exact sequence $0 \to \mathcal{H}_0 \to \mathcal{H}'_1 \to \mathbb{Q} \to 0$ with $\mathcal{H}'_1$ the fiber product of $\mathcal{H}_1 \to \mathbb{Q} \overset{n}{\leftarrow} \mathbb{Q}$, and the exact sequence $0 \to \mathcal{H}_0^*(1) \to \mathcal{H}'_2 \to \mathbb{Q} \to 0$ with $\mathcal{H}'_2$ the fiber product of $\mathcal{H}_2 \to \mathbb{Q} \overset{n}{\leftarrow} \mathbb{Q}$, come from exact sequences of $\mathbb{Z}$-mixed Hodge structures $0 \to \tilde{\mathcal{H}}_0 \to \tilde{\mathcal{H}}'_1 \to \mathbb{Z} \to 0$ and $0 \to \tilde{\mathcal{H}}_0^*(1) \to \tilde{\mathcal{H}}'_2 \to \mathbb{Z} \to 0$, respectively. The maps $h_{\mathcal{H}'_i}$ and $h^*_{\mathcal{H}'_i}$ of $\mathcal{H}'_i$ are equal to n times the maps $h_{\mathcal{H}_i}$ and $h^*_{\mathcal{H}_i}$ of $\mathcal{H}_i$ $(i = 1, 2)$, respectively, and the curvature form associated to $(\mathcal{H}'_1, \mathcal{H}'_2)$ is n^2 times the curvature form associated to $(\mathcal{H}_1, \mathcal{H}_2)$. Hence this variant is reduced to 2.4.3 for $(\tilde{\mathcal{H}}'_1, \tilde{\mathcal{H}}'_2)$.

Remark 2.4.20. Let H be a $\mathbb{Q}$-Hodge structure of weight w and let $\langle\ ,\ \rangle : H_{\mathbb{Q}} \times H_{\mathbb{Q}} \to \mathbb{Q} \cdot (2\pi i)^{-w}$ be a polarization. Then we have the associated Hodge metric $(\ ,\ )$ on $H_{\mathbb{C}}$ which is a positive definite Hermitian form. Though it is not stated in Part I, the author used the following definition of $(\ ,\ )$ in Part I:

(1) $(x, y) := (-1)^p \langle x, \bar{y}\rangle \quad$ for $x \in H^{p,q}_{\mathbb{C}}, y \in H_{\mathbb{C}}$,

where $H^{p,q}_{\mathbb{C}}$ $(p+q = w)$ is the (p, q)-Hodge component of $H_{\mathbb{C}}$. The author now finds that the definition

(2) $(x, y) := (2\pi i)^w i^{p-q}\langle x, \bar{y}\rangle = (2\pi)^w(-1)^p\langle x, \bar{y}\rangle \quad$ for $x \in H^{p,q}_{\mathbb{C}}, y \in H_{\mathbb{C}}$

is used in some literatures and (2) may be the standard definition. To have the compatibility with Part I, we continue to use the definition (1) in this Part II.

Proposition 2.4.21. *In the $\mathbb{Q}$-Hodge version 2.4.19 of 2.4.3, assume that $\mathcal{H}_0$ is endowed with a polarization. Then the Hermitian form on T_Y corresponding to the curvature form of $L(\mathcal{H}_1, \mathcal{H}_2)$ is $(\alpha, \beta) \mapsto (h_{\mathcal{H}_1}(\alpha), h_{\mathcal{H}_2}(\beta)) + (h_{\mathcal{H}_2}(\alpha), h_{\mathcal{H}_1}(\beta))$, where $(\ ,\ )$ is the Hodge metric on $\mathrm{gr}^{-1}\mathcal{H}_{0,\mathcal{O}}$.*

Proof. This follows from 2.4.3 by 2.4.20. Q.E.D.

2.4.22. Consider our period domain $X(\mathbb{C})$. We define an Hermitian form $\kappa_{X(\mathbb{C}),\diamondsuit,w,1}$ on $T_{X(\mathbb{C}),\mathrm{hor}}$.

Consider the variation of $\mathbb{Q}$-MHS

$$\mathcal{H}_0 = (\mathrm{gr}^W_w \mathcal{H}_{X(\mathbb{C})}(V_0))^* \otimes \mathrm{gr}^W_{w-1}\mathcal{H}_{X(\mathbb{C})}(V_0)$$

on $X(\mathbb{C})$ of weight -1. The exact sequence

$$0 \to \mathrm{gr}^W_{w-1}\mathcal{H}_{X(\mathbb{C})}(V_0) \to W_w\mathcal{H}_{X(\mathbb{C})}(V_0)/W_{w-2}\mathcal{H}_{X\mathbb{C})}(V_0) \to \mathrm{gr}^W_w\mathcal{H}_{X(\mathbb{C})}(V_0) \to 0$$

gives an exact sequence $0 \to \mathcal{H}_0 \to \mathcal{H}_1 \to \mathbb{Q} \to 0$ in $\mathbb{Q}\mathrm{MHS}(X(\mathbb{C}))$. By the isomorphism $\mathcal{H}_0 \cong \mathcal{H}_0^*(1)$ obtained by the canonical polarizations (1.2.21) of $\mathrm{gr}^W_w\mathcal{H}_{X(\mathbb{C})}(V_0)$ and $\mathrm{gr}^W_{w-1}\mathcal{H}_{X(\mathbb{C})}(V_0)$, we have an exact sequence $0 \to \mathcal{H}_0^*(1) \to \mathcal{H}_1 \to \mathbb{Q} \to 0$.

By 2.4.19, we obtain a $\mathbf{G}_m \otimes \mathbb{Q}$-torsor $L(\mathcal{H}_1, \mathcal{H}_1)$ with C^∞ metric. We define

$$\kappa_{X(\mathbb{C}),\diamondsuit,w,1} := \kappa(L(\mathcal{H}_1,\mathcal{H}_1))_{\mathrm{hor}},$$

the restriction of the curvature form of $L(\mathcal{H}_1,\mathcal{H}_1)$ to $T_{X(\mathbb{C}),\mathrm{hor}}$.

To describe $\kappa_{X(\mathbb{C}),\diamondsuit,w,1}$, let

$$h_{\mathcal{H}_1} : T_{X(\mathbb{C}),\mathrm{hor}} \to A_{\mathbb{C},X(\mathbb{C})} \otimes_{\mathcal{O}_{X(\mathbb{C})}} \mathrm{gr}^{-1}\mathcal{H}_{0,\mathcal{O}}$$

be the map which sends $\alpha \in T_{X(\mathbb{C}),\mathrm{hor}}$ to the image of $1 \in \mathbb{R}$ under

$$\begin{aligned}\mathbb{R} &\to A_{\mathbb{C},X(\mathbb{C})} \otimes_{\mathcal{O}_{X(\mathbb{C})}} \mathrm{gr}^0\mathcal{H}_{1,\mathcal{O}} \\ &\stackrel{\alpha}{\to} A_{\mathbb{C},X(\mathbb{C})} \otimes_{\mathcal{O}_{X(\mathbb{C})}} \mathrm{gr}^{-1}\mathcal{H}_{1,\mathcal{O}} = A_{\mathbb{C},X(\mathbb{C})} \otimes_{\mathcal{O}_{X(\mathbb{C})}} \mathrm{gr}^{-1}\mathcal{H}_{0,\mathcal{O}}.\end{aligned}$$

Here the first arrow comes from the inverse of the isomorphism

$$A_{\mathbb{C},X(\mathbb{C})} \otimes_{\mathcal{O}_{X(\mathbb{C})}} \mathrm{fil}^0\mathcal{H}_{1,\mathcal{O}} \cap A_{\mathbb{R},X(\mathbb{C})} \otimes_{\mathbb{R}} \mathcal{H}_{1,\mathbb{R}} \stackrel{\cong}{\to} A_{\mathbb{R},X(\mathbb{C})} \otimes_{\mathbb{R}} \mathrm{gr}^W_0\mathcal{H}_{1,\mathbb{R}}.$$

Proposition 2.4.23. *(1) The Hermitian form* $\kappa_{X(\mathbb{C}),\diamondsuit,w,1}$ *coincides with* $(\alpha,\beta) \mapsto 2(h_{\mathcal{H}_1}(\alpha), h_{\mathcal{H}_1}(\beta))$ *where the last* (,) *is the Hodge metric on* $\mathrm{gr}^{-1}\mathcal{H}_{0,\mathcal{O}}$.

(2) This Hermitian form $\kappa_{X(\mathbb{C}),\diamondsuit,w,1}$ *on* $T_{X(\mathbb{C}),\mathrm{hor}}$ *is semi-positive definite. It comes from an Hermitian form on the quotient* $\mathrm{gr}^{-1}(W_0/W_{-2}$ *of* $\mathcal{H}_{X(\mathbb{C})}(\mathrm{Lie}(\mathcal{G}))_{\mathcal{O}})$ *of* $T_{X(\mathbb{C}),\mathrm{hor}} = \mathrm{gr}^{-1}\mathcal{H}_{X(\mathbb{C})}(\mathrm{Lie}(\mathcal{G}))_{\mathcal{O}}$. *The Hermitian form* $\sum_{w\in\mathbb{Z}} \kappa_{X(\mathbb{C}),\diamondsuit,w,1}$ *on* $\mathrm{gr}^{-1}\mathrm{gr}^W_{-1}\mathcal{H}_{X(\mathbb{C})}(\mathrm{Lie}(\mathcal{G}))_{\mathcal{O}}$ *is positive definite.*

Proof. (1) follows from 2.4.21 by using horizontal morphisms $Y \to X(\mathbb{C})$ from one-dimensional complex analytic manifolds Y (see the proof of 2.2.3.1 in 2.2.3).

(2) follows from (1).

Q.E.D.

§3. Height functions

3.1. The setting

3.1.1. We consider the following three situations (I), (II), (III).

In Situation (I), we consider a number field $F_0 \subset \mathbb{C}$.

In Situation (II), we consider a connected one-dimensional complex analytic manifold B endowed with a finite flat morphism $\Pi : B \to \mathbb{C}$.

In Situation (III), we consider a connected projective smooth curve C over $\mathbb{C}$.

3.1.2. Let G, $w : \mathbf{G}_m \to G_{\mathrm{red}}$ be as in 1.1.4. Let $\mathcal{G}$ be a normal algebraic subgroup of G. Let $\mathcal{Q} = G/\mathcal{G}$.

In Situation (I), we assume that we are given $M_b : \mathrm{Rep}(G) \to MM(F_0)$ as in 1.3.4 and we assume 1.3.4.

In Situations (II) and (III), we assume we are given $H_b : \mathrm{Rep}(G) \to \mathbb{Q}\mathrm{MHS}$ as in 1.1.10 and we assume 1.2.9 and 1.2.10.

Let K be an open compact subgroup of $\mathcal{G}(\mathbf{A}^f_K)$ which satisfies the neat condition (1.2.11). In Situation (I), let $X(F) = X_{G,\mathcal{G},M_b,K}(F)$. In Situation (II) (resp. (III)), let $X(\mathbb{C}) = X_{G,\mathcal{G},H_b,K}(\mathbb{C})$ and let $\mathcal{M}_{\mathrm{hor}}(B, X(\mathbb{C}))$ (resp. $\mathcal{M}_{\mathrm{hor}}(C, X(\mathbb{C}))$) be as in 1.2.26.

3.1.3. Let

$$\Lambda = ((V_i, w(i), s(i), c(i))_{1 \le i \le m}, (t(w,d))_{w,d \in \mathbb{Z}, d \ge 1}),$$

where $m \ge 0$, $V_i \in \mathrm{Rep}(G)$, $w(i), s(i) \in \mathbb{Z}$, $c(i) \in \mathbb{R}$, and $t(w,d) \in \mathbb{R}$.

3.1.4. The organization of Section 3 is as follows.

In Section 3.2, we define the height function

$$H_\Lambda : X(F) \to \mathbb{R}_{>0}$$

in Situation (I) for a finite extension F of F_0 in $\mathbb{C}$ fixing somethings and assuming somethings, and we also define the height function

$$h_\Lambda : \mathcal{M}_{\mathrm{hor}}(C, X(\mathbb{C})) \to \mathbb{R}$$

in Situation (III).

Sections 3.3 and 3.4 are preparations for Section 3.5. In Section 3.3, we review height functions in the usual Nevanlinna theory. In Section 3.4, we give results on the degeneration of Hodge structures which we use in Section 3.5. In Section 3.5, we define the height function

$$T_{f,\Lambda}(r) \in \mathbb{R} \ \ (r \in \mathbb{R}_{\ge 1}) \quad \text{for} \ \ f \in \mathcal{M}_{\mathrm{hor}}(B, X(\mathbb{C}))$$

in Situation (II).

In Section 3.6, we consider special cases $h_{\spadesuit}$, $H_{\spadesuit}$, $T_{f,\spadesuit}(r)$ of h_Λ, H_Λ, $T_{f,\Lambda}(r)$, respectively, and then define a height functions $N_{f,\heartsuit}(r)$ for Situation (II) and $h_\heartsuit : \mathcal{M}_{\rm hor}(C, X(\mathbb{C})) \to \mathbb{R}$ in Situation (III), and we give some comments on height functions.

In Section 3.7, we explain that the height functions of the different situations (I), (II), (III) are connected via asymptotic behaviors.

In Section 3.8, we describe the relations of these height functions to the geometry of a toroidal partial compactification $\bar{X}(\mathbb{C})$ of $X(\mathbb{C})$.

3.1.5. Situation (II) and Situation (III) are directly related when $B = C \smallsetminus R$ with B as in Situation (II), C as in Situation (III), and R a finite subset of C. In this case, by the "Great Picard Theorem" applied to small neighborhoods of points of R in C, $\Pi : B \to \mathbb{C}$ extends to a morphism $C \to \mathbf{P}^1(\mathbb{C})$ for which R is the inverse image of $\infty \in \mathbf{P}^1(\mathbb{C})$.

3.2. Height functions $h_\Lambda(\mathcal{H})$ and $H_\Lambda(M)$

3.2.1. Assume we are in Situation (III).

We define the height function $h_\Lambda : \mathcal{M}_{\rm hor}(C, X(\mathbb{C})) \to \mathbb{R}$ as

$$h_\Lambda(\mathcal{H}) := \sum_{i=1}^{n} c(i)\deg(\mathrm{gr}^{s(i)}\mathrm{gr}^W_{w(i)}\mathcal{H}(V_i)) + \sum_{w\in\mathbb{Z}, d\geq 1} t(w,d)h_{\diamondsuit,w,d}(\mathcal{H}(V_0)),$$

where $h_{\diamondsuit,w,d}(\mathcal{H}(V_0))$ is defined as in Part I, Section 1.6 by using the canonical polarization 1.2.14.

3.2.2. Let F be a finite extension of F_0 in $\mathbb{C}$.

To define the height function $H_\Lambda : X(F) \to \mathbb{R}_{>0}$, we fix a $\mathbb{Z}$-lattice $V_{i,\mathbb{Z}}$ in V_i $(1 \leq i \leq m)$ such that $\hat{\mathbb{Z}} \otimes_{\mathbb{Z}} V_{i,\mathbb{Z}}$ is stable under the action of K.

3.2.3. Define

$$H_\Lambda(M) = \prod_i H_{s(i)}(\mathrm{gr}^W_{w(i)}M(V_i))^{c(i)} \cdot \prod_{w\in\mathbb{Z}, d\geq 1} H_{\diamondsuit,w,d}(M(V_0))^{t(w,d)}$$

using Part I.

Here in the definition of $H_{\diamondsuit,w,d}(M(V_0))$, we assume the conjectures in Part I, Section 1.7, which we assumed to define $H_{\diamondsuit,w,d}$ there. We use the canonical polarization on $\mathrm{gr}^W_w M(V_0)$ (1.3.12) for the definition of $H_{\diamondsuit,w,d}$.

3.3. Reviews on height functions in Nevanlinna theory

See [28] and [36] for example.

3.3.1. Let Y be a compact complex analytic manifold.

Let E be a divisor on Y. A Weil function of E is an $\mathbb{R}$-valued continuous function on $Y \smallsetminus \mathrm{Supp}(E)$ which is written locally on Y as

$$W = -\log(|g|) + \text{a continuous function},$$

where g is a meromorphic function (found locally) such that $\mathrm{div}(g) = -E$.

A Weil function of E exists.

3.3.2. Let B be a connected one-dimensional complex analytic manifold endowed with a finite flat morphism $\Pi : B \to \mathbb{C}$. Let $f : B \to Y$ be a holomorphic map, let E be a divisor on Y, and assume $f(B)$ is not contained in $\mathrm{Supp}(E)$. Assume that a Weil function W of E is given. Then the height function $T_{f,E}(r)$ ($r \in \mathbb{R}_{\geq 0}$) for f and E in Nevanlinna theory with respect to W is defined by

$$T_{f,E}(r) := m_{f,E}(r) + N_{f,E}(r)$$

where $m_{f,E}(r)$ and $N_{f,E}(r)$ are defined as follows. First,

$$m_{f,E}(r) := \frac{1}{2\pi}\int_0^{2\pi} (\Pi_* f^* W)(re^{2\pi i\theta})d\theta.$$

Here Π_* is the trace map associated to Π. Writing $f^*E = \sum_{x\in B} n(x)x$ ($n(x) \in \mathbb{Z}$, note that this can be an infinite sum), let

$$N_{f,E}(r) := \sum_{x\in B, 0<|\Pi(x)|<r} n(x)\log(r/|\Pi(x)|) + \sum_{x\in B, \Pi(x)=0} n(x)\log(r).$$

If W' is another Weil function of E, then $W' - W$ is an $\mathbb{R}$-valued continuous function on the compact space Y, and hence there is a constant $c \in \mathbb{R}_{\geq 0}$ such that $|W - W'| \leq c$. Hence in this case, if we denote by $m'_{f,E}$ and $T'_{f,E}$ the $m_{f,E}$ and $T_{f,E}$ defined using W' in place of W, respectively, we have

$$|m_{f,E}(r) - m'_{f,E}(r)| \leq c, \quad |T_{f,E}(r) - T'_{f,E}(r)| \leq c.$$

3.3.3. The Nevanllina height functions can be defined also by using the curvature forms of line bundles with C^∞ metrics, as below. In our Hodge-Nevanlinna theory, we will follow this formulation.

3.3.4. Let L be a line bundle on Y. Then a C^∞ metric on L exists.

Let B be as in 3.3.2 and let $f : B \to Y$ be a holomorphic map. Take a C^∞ metric $|\ |_L$ on L and define the height function of (f, L) by

$$T_{f,L}(r) = T_{f,L,|\ |_L}(r) := \frac{1}{2\pi i}\int_0^r \Big(\int_{B(t)} f^*\kappa(L)\Big)\frac{dt}{t} \in \mathbb{R} \quad \text{for } r \in \mathbb{R}_{\geq 0}$$

where $B(t) = \{x \in B \mid |\Pi(x)| < t\}$ and $\kappa(L)$ is the curvature form of $(L, |\ |_L)$.

3.3.5. Let the notation be as in 3.3.2. Assume $L = \mathcal{O}_Y(E)$ for a divisor E. Then $W := -\log(|1|_L)$ is a Weil function of E. The height functions in 3.3.2 and 3.3.4 (the former is defined by using this W and the latter is defined by using $|\ |_L$) are related as

$$T_{f,L}(r) = T_{f,E}(r) - c$$

where c is the constant $\lim_{z\to 0}(\Pi_* f^* W - n\log(|z|))$ with n the coefficient of the divisor $-\Pi_* f^* E$ on $\mathbb{C}$ at $z = 0$.

This formula follows from Stokes' formula.

3.4. Some results on degeneration

Our definitions of the height functions in Situation (II) are similar to those in Section 3.3, but we have to take care of degenerations. Here we give preparations for it.

3.4.1. Concerning 3.4.1–3.4.3, see [32].

Let Y be a one-dimensional complex manifold, let R be a discrete subset of Y, let L be a line bundle on Y, and let $|\ |$ be a C^∞ metric on the restriction of L to $Y \smallsetminus R$. We say the metric $|\ |$ is good at R if the following condition (*) is satisfied at each $x \in R$. Let z be a local coordinate function on Y at x such that $z(x) = 0$.

(*) There are an open neighborhood U of x on which $|z| < 1$, a basis e of L on U, and constants $C_1, C_2, C_3, C_4, c \in \mathbb{R}_{>0}$ such that we have the following (i)–(iii) on $U \cap (Y \smallsetminus R)$.

(i) $C_1 \log(1/|z|)^{-c} \le |e| \le C_2 \log(1/|z|)^c$.

(ii) $|z\frac{\partial}{\partial z} \log|e|| \le C_3 \log(1/|z|)^{-1}$.

(iii) $|z^2 \frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} \log(|e|)| \le C_4 \log(1/|z|)^{-2}$.

3.4.2. In 3.4.1, consider the case Y is a connected projective smooth curve C over $\mathbb{C}$. Assume that the metric $|\ |$ is good at R. Let $\kappa(L)$ be the associated curvature form on $C \smallsetminus R$. Then by [31], $\kappa(L)$ is integrable on C, and $(2\pi i)^{-1} \int_C \kappa(L) = \deg(L)$.

3.4.3. Let Y and R be as in 3.4.1, and let $\mathcal{H} \in \mathbb{Q}\mathrm{VMHS}_{\log}(Y) \cap \mathbb{Q}\mathrm{VMHS}(Y \smallsetminus R)$. Assume $\mathcal{H}$ is pure and polarizable. Then the Hodge metric on $\det(\mathrm{gr}^r \mathcal{H}_{\mathcal{O}})$ given by a polarization of $\mathcal{H}$ is good at R by [32].

3.4.4. Let Y and R be as in 3.4.1. Assume we are given $\mathcal{H}_0$, $\mathcal{H}_1$, $\mathcal{H}_2$ on $Y \smallsetminus R$ as in Section 2.4 (we replace Y there by $Y \smallsetminus R$ here). Assume $\mathcal{H}_0$ is polarizable and assume that $\mathcal{H}_i \in \mathbb{Q}\mathrm{VMHS}_{\log}(Y)$ for $i = 0, 1, 2$. Then by [5], the $\mathbf{G}_m \otimes \mathbb{Q}$-torsor $L(\mathcal{H}_1, \mathcal{H}_2)$ on $Y \smallsetminus R$ (Section 2.4) extends uniquely to a $\mathbf{G}_m \otimes \mathbb{Q}$-torsor on Y and its metric on $Y \smallsetminus R$ is good at R.

3.4.5. In 3.4.4, assume that Y is a connected projective smooth curve over $\mathbb{C}$. Then by [5], $\deg(L(\mathcal{H}_1, \mathcal{H}_2)) \in \mathbb{Q}$ is equal to the height paring $\mathrm{ht}(\mathcal{H}_1, \mathcal{H}_2)$. By 3.4.2 and 3.4.4, this $\mathrm{ht}(\mathcal{H}_1, \mathcal{H}_2)$ coincides with $\frac{1}{2\pi i} \int_C \kappa(L(\mathcal{H}_1, \mathcal{H}_2))$.

Hence Prop. 2.4.21 gives an explanation of the positive definite property of the height pairing $\mathrm{ht}(\mathcal{H}_1, \mathcal{H}_2)$.

3.4.6. Recall that for an $\mathbb{R}$-mixed Hodge structure H and for $w \in \mathbb{Z}$ and $d \geq 2$, we have an $\mathbb{R}$-linear map $\delta_{w,d} : \mathrm{gr}^W_w H_{\mathbb{R}} \to \mathrm{gr}^W_{w-d} H_{\mathbb{R}}$ (Part I, Section 1.6). We give a property of $\delta_{w,d}$ in degeneration.

Let $\Delta = \{q \in \mathbb{C} \mid |q| < 1\}$ and let $\Delta^* = \Delta \smallsetminus \{0\}$. Then the upper half plane $\mathfrak{H}$ is the universal covering of Δ^* with the covering map $x + iy \mapsto q = e^{2\pi i(x+iy)}$. Endow Δ with the log structure associated to $\{0\} \subset \Delta$ and let $\Delta^{\log}$ be the space over Δ in [20] (see also [23], [21]). We have $\Delta^{\log} = |\Delta| \times \mathbf{S}^1$ where $|\Delta| := \{r \in \mathbb{R} \mid 0 \leq r < 1\}$ and $\mathbf{S}^1 := \{u \in \mathbb{C}^\times \mid |u| = 1\}$, and the map $\Delta^{\log} \to \Delta$ is given by $(r, u) \mapsto ru$. Let $\tilde{\Delta}^{\log} = |\Delta| \times \mathbb{R}$ be the universal covering of $\Delta^{\log}$ with the covering map $\tilde{\Delta}^{\log} \to \Delta^{\log}$ induced by $\mathbb{R} \to \mathbf{S}^1$; $x \mapsto e^{2\pi i x}$. Then $\mathfrak{H}$ is identified with an open set of $\tilde{\Delta}^{\log}$ via the embedding $x+iy \mapsto (|q|, x) = (e^{-2\pi y}, x)$.

Assume we are given $\mathcal{H} \in \mathbb{Q}\mathrm{VMHS}_{\log}(\Delta) \cap \mathbb{Q}\mathrm{VMHS}(\Delta^*)$ and assume that $\mathrm{gr}^W_w \mathcal{H}$ are polarized for all $w \in \mathbb{Z}$. We consider the behavior of $(\delta_{w,d}(\mathcal{H}(q)), \delta_{w,d}(\mathcal{H}(q)))$ when $q \in \Delta^*$ converges to 0, where $\mathcal{H}(q)$ denotes the fiber of $\mathcal{H}$ at q and $(\ ,\ )$ denotes the Hodge metric of $(\mathrm{gr}^W_w \mathcal{H})^* \otimes \mathrm{gr}^W_{w-d} \mathcal{H}$.

Fix an element $\tilde{0}$ of $\tilde{\Delta}^{\log}$ lying over $0 \in \Delta$. The local system $\mathcal{H}_{\mathbb{Q}}$ on Δ^* extends uniquely to a local system on $\Delta^{\log}$ which we denote also by $\mathcal{H}_{\mathbb{Q}}$. Let $H_{0,\mathbb{Q}}$ be the stalk of $\mathcal{H}_{\mathbb{Q}}$ at the image of $\tilde{0}$ in $\Delta^{\log}$. Then $H_{0,\mathbb{Q}}$ has the weight filtration and the polarization $\mathrm{gr}^W_w \mathcal{H}_{\mathbb{Q}} \times \mathrm{gr}^W_w \mathcal{H}_{\mathbb{Q}} \to \mathbb{Q} \cdot (2\pi i)^{-w}$ induces a $\mathbb{Q}$-bilinear form $\langle\ ,\ \rangle_w$ on $\mathrm{gr}^W_w H_{0,\mathbb{Q}}$. Let $h(w, r)$ be the rank of $\mathrm{gr}^r \mathrm{gr}^W_w \mathcal{H}_{\mathcal{O}}$.

Let D be the set of all descending filtrations fil on $H_{0,\mathbb{C}}$ such that $(H_{0,\mathbb{Q}}, W, (\langle\ ,\ \rangle_w)_w, \mathrm{fil})$ is a mixed Hodge structure with polarized pure graded quotients and such that $\dim_{\mathbb{C}}(\mathrm{gr}^r \mathrm{gr}^W_w \mathrm{fil}) = h(w, r)$ for all w, r.

We have the period map $p : \mathfrak{H} \to D$ associated to $\mathcal{H}$. The theory of associated SL(2)-orbit in [21] Part II, Part III gives the following $((\rho_w)_{w \in \mathbb{Z}}, s, \mathbf{r})$. ρ_w for each $w \in \mathbb{Z}$ is a homomorphism $\mathrm{SL}(2)_{\mathbb{R}} \to \mathrm{Aut}_{\mathbb{R}}(\mathrm{gr}^W_w H_{0,\mathbb{R}}, \langle\ ,\ \rangle_w)$ of algebraic groups over $\mathbb{R}$. The canonical $\mathbb{R}$-splitting of W ([21] Part I, §4, Part II, §1.2) associated to $p(\tilde{q})$ converges when $\tilde{q} \in \mathfrak{H}$ converges to $\tilde{0}$. We denote the limit splitting of W over $\mathbb{R}$ by s. Let $\tau : \mathbf{G}_{m,\mathbb{R}} \to \mathrm{Aut}_{\mathbb{R}}(H_{0,\mathbb{R}}, W)$ be the homomorphism which corresponds via s to the direct sum of $t^w\rho_w \begin{pmatrix} 1/t & 0 \\ 0 & t \end{pmatrix}$ on $\mathrm{gr}^W_w H_{0,\mathbb{R}}$ ([21] Part II, 2.3.5). $\tau(y^{1/2})p(\tilde{q}) \in D$ converges in D when $\tilde{q} \in \mathfrak{H}$ converges to $\tilde{0}$. We denote the limit by $\mathbf{r} \in D$. (This $\mathbf{r}$ is written as $\exp(iN)\hat{F}$ in [21] Part IV, Theorem 6.2.4.)

Proposition 3.4.7. *When $q \in \Delta$ converges to* 0, $y^{-d}(\delta_{w,d}(\mathcal{H}(q)), \delta_{w,d}\mathcal{H}(q)))$ *converges to* $(\delta_{w,d}(\mathbf{r}), \delta_{w,d}(\mathbf{r}))$.

Here (,) is the Hodge metric.

Proof. We have

$$y^{-d}(\delta_{w,d}(\mathcal{H}(q)), \delta_{w,d}(\mathcal{H}(q))) = (\tau(y^{1/2})\delta_{w,d}(p(\tilde{q})), \tau(y^{1/2})\delta_{w,d}(p(\tilde{q})))$$

$$= (\delta_{w,d}(\tau(y^{1/2})p(\tilde{q})), \delta_{w,d}(\tau(y^{1/2})p(\tilde{q})) \to (\delta_{w,d}(\mathbf{r}), \delta_{w,d}(\mathbf{r}))$$

($\tilde{q} \in \mathfrak{H}$ denotes a lifting of q). Q.E.D.

3.5. Height functions $T_{f,\Lambda}(r)$

3.5.1. Assume we are in Situation (II). Let $f \in \mathcal{M}_{\mathrm{hor}}(B, X(\mathbb{C}))$. We denote f also by $\mathcal{H}$ when we regard it as the exact $\otimes$-functor $\mathrm{Rep}(G) \to \mathbb{Q}\mathrm{VMHS}_{\log}(B)$ (1.2.26).

3.5.2. Let $f \in \mathcal{M}_{\mathrm{hor}}(B, X(\mathbb{C}))$. For $r \in \mathbb{R}_{\geq 1}$, we define

$$T_{f,\Lambda}(r) := T_{f,\Lambda,\mathrm{red}}(r) + \sum_{w,d} t(w,d) T_{f,\diamondsuit,w,d}(r),$$

where each term of the right hand side is defined in 3.5.3, 3.5.4, 3.5.5 below.

Note that we consider $r \in \mathbb{R}_{\geq 1}$ here, not $r \in \mathbb{R}_{\geq 0}$ as in the classical Nevanlinna theory. This is to avoid the divergence in the definition of the height function which could arise from the singularity of the involved variation of Hodge structure. See 3.5.3 below. The author is very thankful to the referee who pointed out this problem of divergence which was overlooked by the author in an earlier version of the manuscript.

In Nevanlinna theory and in our Hodge-Nevanlinna theory, we are interested in the behavior of the height function when $r \to \infty$. So, it is harmless to consider only $r \geq 1$.

3.5.3. We define the first term $T_{f,\Lambda,\mathrm{red}}(r)$ in the definition of $T_{f,\Lambda}(r)$ in 3.5.2.

Let

$$\kappa_{X(\mathbb{C}),\Lambda,\mathrm{red}} := \sum_i c(i)\kappa(\det(\mathrm{gr}^{s(i)}\mathrm{gr}^W_{w(i)}\mathcal{H}_{X(\mathbb{C})}(V_i)_{\mathcal{O}}))_{\mathrm{hor}}$$

where $\kappa(\det(\mathrm{gr}^{s(i)}\mathrm{gr}^W_{w(i)}\mathcal{H}_{X(\mathbb{C})}(V_i)_{\mathcal{O}}))_{\mathrm{hor}}$ is the restriction to $T_{X(\mathbb{C}),\mathrm{hor}}$ of the curvature form of the line bundle $\det(\mathrm{gr}^{s(i)}\mathrm{gr}^W_{w(i)}\mathcal{H}_{X(\mathbb{C})}(V_i)_{\mathcal{O}})$ with the Hodge metric. The Hodge metric is defined by a polarization (1.2.21) but the curvature form is independent of the choice of the polarization. This $\kappa_{X(\mathbb{C}),\Lambda,\mathrm{red}}$ comes from an Hermitian form on $T_{X_{\mathrm{red}}(\mathbb{C}),\mathrm{hor}}$ via $T_{X(\mathbb{C}),\mathrm{hor}} \to T_{X_{\mathrm{red}}(\mathbb{C}),\mathrm{hor}}$.

Define

$$T_{f,\Lambda,\mathrm{red}}(r) := \frac{1}{2\pi i}\int_1^r (\int_{B(t)} \kappa_{f,\Lambda,\mathrm{red}})\frac{dt}{t} \quad \text{with } \kappa_{f,\Lambda,\mathrm{red}} := f^*\kappa_{X(\mathbb{C}),\Lambda,\mathrm{red}}.$$

Here we have to be careful that not as in 3.3.4, the above differential 2-form $\kappa_{f,\Lambda,\mathrm{red}}$ is C^∞ on $U = B \smallsetminus R$, where R is the set of points at which $\mathcal{H}(V_i)$ for some i has singularity and it is a discrete subset of B, and $\kappa_{f,\Lambda,\mathrm{red}}$ may have singularities at R. But the integration on the right hand side in the definition of $T_{f,\Lambda,\mathrm{red}}(r)$ converges by 3.4.3. Here we take $\int_1^r$ assuming $r \geq 1$, and do not take $\int_0^r$ as in the classical Nevanlinna theory. If we take $\int_0^r$, the integration would diverge at $t = 0$ due to the singularity at $\Pi^{-1}(0)$.

3.5.4. We define $T_{f,\diamondsuit,w,1}(r)$.

Let $\kappa_{X(\mathbb{C}),\diamondsuit,w,1}$ be the Hermitian form on $T_{X(\mathbb{C}),\mathrm{hor}}$ defined in 2.4.22. Define

$$T_{f,\diamondsuit,w,1}(r) := \frac{1}{2\pi i}\int_1^r (\int_{B(t)} \kappa_{f,\diamondsuit,w,1})\frac{dt}{t} \quad \text{with } \kappa_{f,\diamondsuit,w,1} := f^*\kappa_{X(\mathbb{C}),\diamondsuit,w,1}.$$

The curvature form $\kappa_{f,\diamondsuit,w,1}$ may have singularities but the integration on the right hand side of the definition of $T_{f,\diamondsuit,w,1}(r)$ converges by 3.4.4.

3.5.5. Assume $d \geq 2$. We define

$$T_{f,\diamondsuit,w,d}(r) := m_{f,\diamondsuit,w,d}(r) + N_{f,\diamondsuit,w,d}(r),$$

where:

$$m_{f,\diamondsuit,w,d}(r) := \frac{1}{2\pi}\int_0^{2\pi} g(re^{\theta i})d\theta \quad \text{with } g = \Pi_*(\langle \delta_{w,d}, \delta_{w,d}\rangle^{1/d}),$$

where $\delta_{w,d}$ is that of $\mathcal{H}(V_0)$ in Part I, Section 1.6, Π_* is the trace map for $\Pi : B \to \mathbb{C}$,

$$N_{f,\diamondsuit,w,d}(r) := \sum_{x\in B, 0<|\Pi(x)|<r} \langle N_{x,w,d}, N_{x,w,d}\rangle^{1/d}_{N_{x,0}} \cdot \log(r/|\Pi(x)|) + \sum_{x\in B, \Pi(x)=0} \langle N_{x,w,d}, N_{x,w,d}\rangle^{1/d}_{N_{x,0}} \cdot \log(r),$$

where $N_{x,w,d}$ and $N_{x,0}$ are those of $\mathcal{H}(V_0)$ in Part I, Section 1.6.

The function $\delta_{w,d}$ may have singularities but the integral $m_{f,\diamondsuit,w,d}(r)$ converges by 3.4.7.

3.6. ♠-height functions, ♡-height functions, complements on height functions

We consider height functions $H_\spadesuit$ in Situation (I), $T_{f,\spadesuit}(r)$ in Situation (II), and $h_\spadesuit$ in situation (III). These are special cases of H_Λ, $T_{f,\Lambda}(r)$, h_Λ, respectively. We define also height functions $N_{f,\heartsuit}(r)$ in Situation (II), $h_\heartsuit$ in Situation (III). We give complements to height functions.

3.6.1. The following height functions $h_\spadesuit(\mathcal{H}), H_\spadesuit(M), T_{f,\spadesuit}(r)$, which are special cases of the height functions $h_\Lambda(\mathcal{H})$, $H_\Lambda(M)$, $T_{f,\Lambda}(r)$, will be important in Section 4:

$$h_\spadesuit(\mathcal{H}) := -\deg(\mathcal{H}(\mathrm{Lie}(\mathcal{G}))_{\mathcal{O}}/\mathrm{fil}^0) \quad \text{in Situation (III)},$$

$$H_\spadesuit(M) := \prod_{r<0} H_r(M(\mathrm{Lie}(\mathcal{G})))^{-1} \quad \text{in Situation (I)},$$

$$T_{f,\spadesuit}(r) := \frac{1}{2\pi i}\int_1^r (\int_{B(t)} f^*\kappa_{X(\mathbb{C}),\spadesuit})\frac{dt}{t} \quad \text{in Situation (II)}.$$

Here G acts on $\mathrm{Lie}(\mathcal{G})$ by the adjoint action. See 2.3.2 for the definition of $\kappa_{X(\mathbb{C}),\spadesuit}$.

This height function is the special case $\Lambda = \spadesuit$ of the height functions in Section 3.2 and 3.5, where $\Lambda = \spadesuit$ means that Λ is such that $(w(i), s(i))$ $(1 \le i \le m)$ are all different pairs such that $\mathrm{gr}^{s(i)}\mathrm{gr}^W_{w(i)}H_b(\mathrm{Lie}(\mathcal{G})) \neq 0$ and $s(i) < 0$, $V_i = \mathrm{Lie}(\mathcal{G})$ for $1 \le i \le m$, $c(i) = -1$ for all i, and $t(w,d) = 0$ for all w, d.

3.6.2. For a one-dimensional complex analytic manifold Y, for $\mathcal{H} \in \mathcal{M}_{\mathrm{hor}}(Y, X(\mathbb{C}))$, and for $x \in Y$, we define $e(x) = h_{\heartsuit,x}(\mathcal{H}) \in \mathbb{Z}_{\ge 0}$ as follows.

Let $N'_x : \mathcal{H}(V)_{\mathbb{Q},x} \to \mathcal{H}(V)_{\mathbb{Q},x}$ be the local monodromy operator at x. It is the logarithm of the action of the canonical generator of the local monodromy group at x. Here $\mathcal{H}(V)_{\mathbb{Q},x}$ denotes the stalk of $\mathcal{H}(V)_{\mathbb{Q}}$ at a point $\neq x$ of Y which is near to x. By the theory of Tannakian categories, N'_x for all $V \in \mathrm{Rep}(G)$ come from $N'_x \in \mathcal{H}(\mathrm{Lie}(\mathcal{G}))_{\mathbb{Q},x}$.

By level structure, we have $N'_x \in \mathbf{A}^f_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathrm{Lie}(\mathcal{G})$ mod the adjoint action of K on $\mathbf{A}^f_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathrm{Lie}(\mathcal{G})$. We define $e(x) = 0$ if $N'_x = 0$. If $N'_x \neq 0$, $e(x)$ is defined by $\mathbb{Z}e(x)^{-1} = \{b \in \mathbb{Q} \mid \exp(bN'_x) \in K\}$.

3.6.3. In Situation (III), for $\mathcal{H} \in \mathcal{M}_{\rm hor}(C, X(\mathbb{C}))$, define

$$h_{\heartsuit}(\mathcal{H}) := \sum_{x\in C} h_{\heartsuit,x}(\mathcal{H}) \in \mathbb{Z}_{\geq 0}.$$

3.6.4. We have $h_{\heartsuit}(\mathcal{H}) \geq \sharp(\Sigma(C,\mathcal{H}))$, where $\Sigma(C,\mathcal{H})$ denotes the set of points of C at which $\mathcal{H}$ has singularity.

3.6.5. In Situation (II), for $f \in \mathcal{M}_{\rm hor}(B, X(\mathbb{C}))$, define

$$N_{f,\heartsuit}(r) := \sum_{x\in B, 0<|\Pi(x)|<r} h_{\heartsuit,x}(\mathcal{H}) \log(r/|\Pi(x)|)) + \sum_{x\in B, \Pi(x)=0} h_{\heartsuit,x}(\mathcal{H}) \log(r).$$

3.6.6. Let

$$N^{(1)}_{f,\heartsuit}(r) := \sum_{x\in\Sigma(B,\mathcal{H}), 0<|\Pi(x)|<r} \log(r/|\Pi(x)|) + \sum_{x\in\Sigma(B,\mathcal{H}), \Pi(x)=0} \log(r).$$

Here $\Sigma(B,\mathcal{H})$ is the set of points of B at which $\mathcal{H}$ has singularity.

We have $N_{f,\heartsuit}(r) \geq N^{(1)}_{f,\heartsuit}(r)$.

3.6.7. We give comments on height functions.

(1) The bijections in 1.3.7 (1) do not change the height functions in (I), (II), (III).

(2) In this Section 3, we considered the height functions on $X_{G,\mathcal{G},M_b,K}(F)$ but not yet on $X_{G,\Upsilon,K}(F)$. We define height functions on $X_{G,\Upsilon,K}(F)$ by using the bijection $X_{G,\Upsilon,K}(F) \cong X_{G,G,M_1,K'}(F)$ in 1.3.7 (2) by choosing $M_1 \in X_{G,\Upsilon,K}(F)$, and using the height functions on the latter. Then they are independent of the choice of M_1.

Proposition 3.6.8. *Assume G is reductive and assume $\kappa_{X(\mathbb{C}),\Lambda,\rm red}$ (3.5.3) is positive semi-definite. Then*

(1) In Situation (II), $T_{f,\Lambda}(r) \geq 0$ for any $f \in \mathcal{M}_{\rm hor}(B, X(\mathbb{C}))$ and any $r \in \mathbb{R}_{\geq 1}$.

(2) In Situation (III), $h_\Lambda(\mathcal{H}) \geq 0$ for any $\mathcal{H} \in \mathcal{M}_{\rm hor}(C, X(\mathbb{C}))$.

Proof. (1) is evident. (2) follows from $\frac{1}{2\pi i}\int_C f^*\kappa_{X(\mathbb{C}),\Lambda,\rm red} = h_\Lambda(\mathcal{H})$ (3.4.2, 3.4.3) where $f = \mathcal{H}$. Q.E.D.

3.6.9. We say Λ is ample if the Hermitian form $\kappa_{X(\mathbb{C}),\Lambda,\rm red}$ on $T_{X_{\rm red}(\mathbb{C}),\rm hor}$ (3.5.3) is positive definite and $t(w,d) > 0$ for any w, d.

For example, by the theorem of Griffiths in 2.2.1, Λ is ample if $V_i = V_0$ for all i, $c(i) = s(i)$ for all i, $\{(w(i), s(i)) \mid i \in \mathbb{Z}\}$ covers all pairs (w,s) such that $\mathrm{gr}^s\mathrm{gr}^W_w V_0 \neq 0$, and $t(w,d) > 0$ for all w, d.

Proposition 3.6.10. *Assume we are in Situation (II). Assume Λ is ample and G is reductive.*

Let $f \in \mathcal{M}_{\mathrm{hor}}(B, X(\mathbb{C}))$ and assume that $T_{f,\Lambda}(r) = o(\log(r))$. Then f is constant.

Proof. If f is not constant, the Hermitian form $\kappa_{f,\Lambda,\mathrm{red}}$ is positive definite and hence there are constants $b \geq 1$ and $c > 0$ such that $\frac{1}{2\pi i}\int_{B(t)} \kappa_{f,\Lambda,\mathrm{red}} \geq c$ for any $t \geq b$. For any $r \geq b$, we have $T_{f,\Lambda}(r) \geq \int_b^r c dt/t = c(\log(r) - \log(b))$. Q.E.D.

The author expects that the assumption G is reductive in 3.6.10 is unnecessary.

Proposition 3.6.11. *Assume we are in Situation (III). Assume Λ is ample and let $\mathcal{H} \in \mathcal{M}_{\mathrm{hor}}(C, X(\mathbb{C}))$. Then $h_\Lambda(\mathcal{H}) \geq 0$. If $h_\Lambda(\mathcal{H}) = 0$, then $\mathcal{H}$ is constant.*

Proof. This is proved in the same way as Part I, Proposition 1.6.16. Q.E.D.

The proof of the following Proposition is easy.

Proposition 3.6.12. *(1) Assume we are in Situation (II) and let B' be a connected smooth curve over $\mathbb{C}$ endowed with a finite flat morphism $B' \to B$ of degree $[B' : B]$. Let $f \in \mathcal{M}_{\mathrm{hor}}(B, X(\mathbb{C}))$ and let $f' \in \mathcal{M}_{\mathrm{hor}}(B', X(\mathbb{C}))$ be the composition of f and $B' \to B$. Then*

$$T_{f',\Lambda}(r) = [B' : B] \cdot T_{f,\Lambda}(r), \quad N_{f',\heartsuit}(r) = [B' : B] \cdot N_{f,\heartsuit}(r).$$

(2) Assume we are in Situation (III) and let C' be a proper smooth curve over $\mathbb{C}$ endowed with a finite flat morphism $C' \to C$ of degree $[C' : C]$. Let $\mathcal{H} \in \mathcal{M}_{\mathrm{hor}}(C, X(\mathbb{C}))$ and let $\mathcal{H}' \in \mathcal{M}_{\mathrm{hor}}(C', X(\mathbb{C}))$ be the pull back of $\mathcal{H}$. Then

$$h_\Lambda(\mathcal{H}') = [C' : C] \cdot h_\Lambda(\mathcal{H}), \quad h_\heartsuit(\mathcal{H}') = [C' : C] \cdot h_\heartsuit(\mathcal{H}).$$

Remark 3.6.13. In Part I ([19]) Section 2.3, we defined the height function $H_{\heartsuit,v}(M)$, which is the motive version of the above $h_{\heartsuit,x}(\mathcal{H})$, for a mixed motive M over a number field F and for a non-archimedean place v of F, and defined the height function $H_{\heartsuit,S} := \prod_{v\notin S} H_{\heartsuit,v}$ for a finite set S of places of F containing all archimedean places of F. The author found that the definitions of these $H_{\heartsuit,v}$ and $H_{\heartsuit,S}$ were not the good ones. Hence we do not discuss these height functions in this Part II. He hopes that in Part III (a sequel of this paper), he gives the good definitions together with the definitions of $H_{\heartsuit,v}$ for archimedean places v.

3.7. Asymptotic behaviors

In the following 3.7.1 and 3.7.2, we describe relations between the height functions in Situation (II) and the height functions in Situation (III). At the end of this Section 3.7, we briefly describe the relations between the height functions in Situation (I) and the height functions in (II) or (III), but the details of them will be given elsewhere.

Proposition 3.7.1. *Assume $B = C \smallsetminus R$ for a finite subset R of C (that is, we are in the situation of 3.1.5). Let $\mathcal{H} \in \mathcal{M}_{\mathrm{hor}}(C, X(\mathbb{C}))$ and let $f \in \mathcal{M}_{\mathrm{hor}}(B, X(\mathbb{C}))$ be the restriction of $\mathcal{H}$ to B.*

(1) $T_{f,\Lambda}(r)/\log(r)$ converges when $r \to \infty$.

(2) If either one of the following conditions (i)–(iii) is satisfied, $T_{f,\Lambda}(r)/\log(r)$ converges to $h_\Lambda(\mathcal{H})$.

(i) G is reductive.

(ii) $t(w, d) = 0$ for any $d \geq 2$.

(iii) $\mathcal{H}$ has no degeneration at R.

Proof. We prove (2) first. Case (i) follows from Case (ii). Assume (ii). Then $T_{f,\Lambda}(r) = \frac{1}{2\pi i}\int_1^r(\int_{B(t)}\kappa)dt/t$ with $\kappa := \kappa_{f,\Lambda,\mathrm{red}} + \sum_w t(w,1)\kappa_{f,\diamondsuit,w,1}$. Let $A(t) = \frac{1}{2\pi i}\int_{B(t)}\kappa$. Then when $t \to \infty$, $A(t)$ converges to $A := (2\pi i)^{-1}\int_C \kappa = h_\Lambda(\mathcal{H})$. Take $\epsilon > 0$. Then there is $r_0 \geq 1$ such that $|A(r) - A| \leq \epsilon$ if $r \geq r_0$. Let $c := \int_1^{r_0} A(t)dt/t$. We have $T_{f,\Lambda}(r) = c + \int_{r_0}^r A(t)dt/t = c + \int_{r_0}^r (A + u(t))dt/t$ with $|u(t)| \leq \epsilon$, and hence $|T_{f,\Lambda}(r) - c - A\log(r/r_0)| \leq \epsilon\int_{r_0}^r dt/t = \epsilon\log(r/r_0)$. Hence if r is sufficiently large, we have $|T_{f,\Lambda}(r)/\log(r) - A| \leq 2\epsilon$.

Assume (iii). In this case, $m_{f,\diamondsuit,w,d}$ is bounded when $r \to \infty$ and

$$T_{f,\diamondsuit,w,d}(r)/\log(r) \to \sum_{x\in B}\langle N_{x,w,d}, N_{x,w,d}\rangle_{N_{x,0}}^{1/d} = h_{\diamondsuit,w,d}(\mathcal{H}).$$

(1) follows from 3.4.7 and the case (ii) of (2).

Q.E.D.

Proposition 3.7.2. *Let the assumption be as in 3.7.1. Let $\mathcal{H} \in \mathcal{M}_{\mathrm{hor}}(C, X(\mathbb{C}))$.*

(1) $\lim_{r\to\infty} N_{f,\heartsuit}(r)/\log(r) = \sum_{x\in B} h_{\heartsuit,x}(\mathcal{H})$.

(2) $\lim_{r\to\infty} N_{f,\heartsuit}^{(1)}(r)/\log(r) = \sharp\{x \in B \mid \mathcal{H}$ has singularity at $x\}$.

Proof. This is clear. Q.E.D.

Remark 3.7.3. (1) In 3.7.2, if $\mathcal{H}$ has no degeneration at R, the rhs becomes $h_\heartsuit(\mathcal{H})$ in (1) and $\sharp\{x \in C \mid \mathcal{H}$ has singularity at $x\}$ in (2).

(2) 3.7.1 and 3.7.2 tell that the study of height functions in Situation (III) is essentially reduced to that in Situation (II).

3.7.4. In the above, we considered relations between Situation (II) and Situation (III).

We have relations between Situation (I) and Situation (III), and also relations between Situation (I) and Situation (II). We give only rough stories here. We hope to discuss more precise things elsewhere.

Let C_0 be a projective smooth curve over a number field F_0. We can define the set $X(F_0(C_0))$, where $F_0(C_0)$ denotes the function field of C_0, in the similar way as the definition of $X(F)$. For $M \in X(F_0(C_0))$, for a finite extension F of F_0 in $\mathbb{C}$ and for $x \in C_0(F)$ at which M does not have singularity, we can define the specialization $M(x) \in X(F)$ of M at x.

Concerning the relation between (I) and (III), we can show that

$$\log(H_\Lambda(M(x)))/\log(H(x)) \text{ in (I)} \quad \text{and} \quad h_\Lambda(M_H) \text{ in (III)}$$

are closely related. Here M_H is the Hodge realization of M on $C := C_0(\mathbb{C})$ and $H(x)$ is the height of x as an element of $C_0(F)$ defined by a $\mathbf{G}_m \otimes \mathbb{Q}$-torsor on C_0 of degree 1. In some cases, $h_\Lambda(M_H)$ is the limit of $\log(H_\Lambda(M(x)))/\log(H(x))$ when F and x move.

Concerning the relation between (I) and (II), we can show that for an affine dense open subset B_0 of C_0, if we denote by f the element of $\mathcal{M}_{\text{hor}}(B, X(\mathbb{C}))$ $(B := B_0(\mathbb{C}))$ induced by M,

$$\log(H_\Lambda(M(x))) \text{ in (I)} \quad \text{and} \quad T_{f,\Lambda}(H(x)) \text{ in (II)}$$

are closely related.

3.8. Toroidal partial compactifications and height functions

Here we describe the toroidal partial compactification $\bar{X}(\mathbb{C})$ of $X(\mathbb{C})$ and its relation to height functions.

This $\bar{X}(\mathbb{C})$ is a generalization of $\bar{X}(\mathbb{C})$ in Part I. It is given by the work of Kerr and Pearlstein [24] in the case G is reductive and $\mathcal{G} = G$. For general G, it is described in [22] briefly in the case $\mathcal{G} = G$. The details of the general case will be given in a forth-coming paper [21], Part V.

In the rest of this paper, this Section 3.8 serves to make our philosophy clearer, and we do not use the contents of this Section 3.8 to prove a result except that 3.8.10 is used in Remark 4.2.3.9 to prove a result there.

3.8.1. Let Σ be the set of cones in $\text{Lie}(\mathcal{G})_\mathbb{R}$ of the form $\mathbb{R}_{\geq 0}N$ for some element N of $\text{Lie}(\mathcal{G})$ satisfying the following conditions (a) and (b) for any $V \in \text{Rep}(G)$.

(a) The image of N under $\mathrm{Lie}(G) \to \mathrm{End}(V)$ is a nilpotent operator on V.

(b) There is a relative monodromy filtration of $N : V \to V$ with respect to the weight filtration $W_\bullet V$.

If $V_1 \in \mathrm{Rep}(G)$ is a faithful representation, (a) and (b) are satisfied for any $V \in \mathrm{Rep}(G)$ if they are satisfied by $V = V_1$ (1.2.20).

3.8.2. Let D_Σ be the set of pairs (σ, Z), where $\sigma \in \Sigma$ and Z is a non-empty subset of $\check{D}(G, \Upsilon)$ satisfying the following (i)–(iv).

(i) Write $\sigma = \mathbb{R}_{\geq 0} N$ with $N \in \mathrm{Lie}(\mathcal{G})$. Then Z is an $\exp(\mathbb{C}N)$-orbit in $\check{D}(G, \Upsilon)$.

(ii) The image of Z in $\check{D}(\mathcal{Q}, \Upsilon_{\mathcal{Q}})$ is $\mathrm{class}(H_{b,\mathcal{Q}}) \in D(\mathcal{Q}, \Upsilon_{\mathcal{Q}})$,

(iii) If $H \in Z$, N belongs to $\mathrm{fil}^{-1} H(\mathrm{Lie}(\mathcal{G}))_{\mathbb{C}}$, where G acts on $\mathrm{Lie}(\mathcal{G})$ by the adjoint action and fil^{-1} is the Hodge filtration.

(iv) Let $H \in Z$. Then $\exp(zN)H \in D(G, \Upsilon)$ if $z \in \mathbb{C}$ and $\mathrm{Im}(z)$ is sufficiently large.

Let $\bar{X}(\mathbb{C}) := \mathcal{G}(\mathbb{Q}) \backslash (D_\Sigma \times (\mathcal{G}(\mathbf{A}^f_{\mathbb{Q}})/K))$

3.8.3. $\bar{X}(\mathbb{C})$ has a structure of a logarithmic manifold which extends the complex analytic structure of $X(\mathbb{C})$. (Logarithmic manifold is a generalization of complex analytic manifold ([23], 3.5.7). It is like a complex manifold with slits.)

3.8.4. For a one-dimensional complex analytic manifold Y, for a discrete subset R of Y, and for a horizontal holomorphic map $f : Y \smallsetminus R \to X(\mathbb{C})$, the following (i)–(iii) are equivalent. (i) f is meromorphic on Y (1.2.26). (ii) f extends to a morphism $Y \to \bar{X}(\mathbb{C})$ of locally ringed spaces over $\mathbb{C}$. (iii) f extends to a morphism $Y \to \bar{X}(\mathbb{C})$ of logarithmic manifolds. If these equivalent conditions are satisfied, the extensions of f in (ii) and (iii) are unique.

3.8.5. We describe the relations of $h_\spadesuit$ and $h_\heartsuit$ to the extended period domain $\bar{X}(\mathbb{C})$.

Around here, Y is a one-dimensional complex analytic manifold and $f \in \mathcal{M}_{\mathrm{hor}}(Y, X(\mathbb{C}))$ and let $\mathcal{H}$ be the corresponding exact $\otimes$-functor $\mathrm{Rep}(G) \to \mathbb{Q}\mathrm{VMHS}_{\log}(Y)$. We denote the morphism $Y \to \bar{X}(\mathbb{C})$ which extends f by the same letter f.

3.8.6. For $V \in \mathrm{Rep}(G)$, we have a universal log mixed Hodge structure $\mathcal{H}_{\bar{X}(\mathbb{C})}(V)$ on $\bar{X}(\mathbb{C})$.

We have

$$\mathcal{H}(V) = f^* \mathcal{H}_{\bar{X}(\mathbb{C})}(V).$$

3.8.7. We have the sheaf $\Omega^1_{\bar{X}(\mathbb{C})}(\log)$ of differential forms on $\bar{X}(\mathbb{C})$ with log poles outside $X(\mathbb{C})$ and the sheaf $\Omega^1_{\bar{X}(\mathbb{C})} \subset \Omega^1_{\bar{X}(\mathbb{C})}(\log)$ of differential forms on $\bar{X}(\mathbb{C})$. These are vector bundles on $\bar{X}(\mathbb{C})$.

3.8.8. Let $I_{\bar{X}(\mathbb{C})}$ be the invertible ideal of $\mathcal{O}_{\bar{X}(\mathbb{C})}$ which is locally generated by generators of log structures. For $x \in \bar{X}(\mathbb{C})$, the stalk I_x coincides with $\mathcal{O}_{\bar{X}(\mathbb{C}),x}$ if and only if $x \in X(\mathbb{C})$.

We have an exact sequence

$$0 \to \Omega^1_{\bar{X}(\mathbb{C})} \to \Omega^1_{\bar{X}(\mathbb{C})}(\log) \to \mathcal{O}_{\bar{X}(\mathbb{C})}/I_{\bar{X}(\mathbb{C})} \to 0,$$

where the third arrow sends $d\log(q)$ for a local generator q of the log structure to 1.

3.8.9. The vector bundle $\mathcal{H}_{\bar{X}(\mathbb{C})}(\mathrm{Lie}(\mathcal{G}))_{\mathcal{O}}/\mathrm{fil}^0$ on $\bar{X}(\mathbb{C})$ is canonically isomorphic to the logarithmic tangent bundle of $\bar{X}(\mathbb{C})$, that is, it is canonically isomorphic to the dual vector bundle of $\Omega^1_{\bar{X}(\mathbb{C})}(\log)$.

3.8.10. In Situation (III), for $Y = C$, by 3.8.6 and 3.8.9, we have

$$h_{\spadesuit}(\mathcal{H}) = \deg(f^*\Omega^1_{\bar{X}(\mathbb{C})}(\log)).$$

3.8.11. In Situation (II) (resp. (III)), for $Y = B$ (resp. $Y = C$) and for $x \in Y$, $h_{\heartsuit,x}(\mathcal{H}) = e(x)$, where $e(x)$ is the integer ≥ 0 such that $f^*I_{\bar{X}(\mathbb{C})} = m_{Y,x}^{e(x)}$ in $\mathcal{O}_{Y,x}$.

3.8.12. Assume $Y = C$.

By 3.8.11, we have

$$h_{\heartsuit}(\mathcal{H}) = -\deg(f^*I_{\bar{X}(\mathbb{C})}).$$

By this and by 3.8.10, we have

$$h_{\spadesuit}(\mathcal{H}) - h_{\heartsuit}(\mathcal{H}) = \deg(f^*\Omega^1_{\bar{X}(\mathbb{C})}).$$

3.8.13. In the comparison of (I) and (1) in Section 0, $\bar{X}(\mathbb{C}) \supset X(\mathbb{C})$ is like $\bar{V} \supset V$ such that $\bar{V}$ is smooth and $E := \bar{V} \smallsetminus V$ is a divisor on $\bar{V}$ with normal crossings. Basing on 3.8.10 and the analogy between (I) and (III), we think that the height function $H_{\spadesuit}$ for (I) is similar to the height function H_{K+E} for (1), where K is the canonical divisor of $\bar{V}$.

3.8.14. It can be shown that the Hodge metric on $\det(\Omega^1_{X(\mathbb{C})})$ (defined by a polarization on $\mathcal{H}_{X(\mathbb{C})}(\mathrm{Lie}(\mathcal{G}))$ (1.2.21) and the duality between $\Omega^1_{X(\mathbb{C})}$ and $T_{X(\mathbb{C})} = \mathcal{H}_{X(\mathbb{C})}(\mathrm{Lie}(\mathcal{G}))_{\mathcal{O}}/\mathrm{fil}^0$) extends to a metric on $\det(\Omega^1_{\bar{X}(\mathbb{C})}(\log))$ with at worst log singularity. Hence the positivity of the curvature form $\kappa(\det(\Omega^1_{X(\mathbb{C})}))_{\mathrm{hor}}$ (2.3.3) tells that $\det(\Omega^1_{\bar{X}(\mathbb{C})}(\log))$ is something like an ample line bundle.

§4. Speculations

We extend speculations in Part I to the setting of this Part II.
Let the setting be as in 3.1.1–3.1.2.

4.1. Speculations on positivity

4.1.1. Question 1. Assume G is reductive.
For Λ as in 3.1.3, are the following (i)–(iv) equivalent?

(i) The Hermitian form $\kappa_{X(\mathbb{C}),\Lambda,\mathrm{red}}$ on $T_{X(\mathbb{C}),\mathrm{hor}}$ is positive semi-definite.

(ii) In Situation (III), we have $h_\Lambda(\mathcal{H}) \geq 0$ for any C and for any $\mathcal{H} \in \mathcal{M}_{\mathrm{hor}}(C, X(\mathbb{C}))$.

(iii) In Situation (II), we have $T_{f,\Lambda}(r) \geq 0$ for any B, any $f \in \mathcal{M}_{\mathrm{hor}}(B, X(\mathbb{C}))$ and any $r \in \mathbb{R}_{\geq 1}$.

(iv) In Situation (I), there is $c \in \mathbb{R}_{>0}$ such that for any number field $F \subset \mathbb{C}$ such that $F_0 \subset F$ and for any $M \in X(F)$, we have $H_\Lambda(M) \geq c^{[F:F_0]}$.

Remark 4.1.1.1. We have (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii). In fact, (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) follow from 3.6.8, and (iii) $\Rightarrow$ (ii) follows from 3.7.1.

4.1.2. Question 2. Assume G is reductive.
For Λ as in 3.1.3, are the following (i)–(iv) equivalent?

(i) Λ is ample (3.6.9).

(ii) In Situation (III), if $\mathcal{H} \in \mathcal{M}_{\mathrm{hor}}(C, X(\mathbb{C}))$ and if $h_\Lambda(\mathcal{H}) \leq 0$, then $\mathcal{H}$ is constant.

(iii) In Situation (II), if $f \in \mathcal{M}_{\mathrm{hor}}(B, X(\mathbb{C}))$ and if there is $c \in \mathbb{R}$ such that $T_{f,\Lambda}(r) \leq c$ for any $r \in \mathbb{R}_{\geq 1}$, then f is constant.

(iv) In Situation (I), for any $d \geq 1$ and $c \in \mathbb{R}_{>0}$, there are finitely many pairs (F_i, M_i) $(1 \leq i \leq n)$, where F_i are finite extensions of F_0 in $\mathbb{C}$ and $M_i \in X(F_i)$, satisfying the following condition. If $F \subset \mathbb{C}$ is a finite extension of F_0 and $M \in X(F)$ and if $[F : F_0] \leq d$ and $H_\Lambda(M) \leq c$, there is i $(1 \leq i \leq n)$ such that $F_i \subset F$ and such that M is the image of M_i under $X(F_i) \to X(F)$.

Remark 4.1.2.1. We have (i) $\Rightarrow$ (ii) by 3.6.11, (iii) $\Rightarrow$ (ii) by 3.7.1, and (i) $\Rightarrow$ (iii) by 3.6.10.

Now we do not assume G is reductive.

4.1.3. Question 3. Assume Λ is ample (3.6.9). Are the following (1) and (2) true?

(1) In Situation (I), for any finite extension F of F_0 in $\mathbb{C}$ and for any $c \in \mathbb{R}_{>0}$, the set $\{M \in X(F) \mid H_\Lambda(M) \leq c\}$ is finite.

(2) (A stronger form of (1).) Fix $d \geq 1$ and $c \in \mathbb{R}_{>0}$. Then there is a finite number of pairs pairs (F_i, M_i) $(1 \leq i \leq n)$, where F_i are finite extensions of F_0 in $\mathbb{C}$ and $M_i \in X(F_i)$ satisfying the following condition. If F is a finite extension of F_0 in $\mathbb{C}$ such that $[F : F_0] \leq d$ and if $M_i \in X(F)$ and $H_\Lambda(M) \leq c$, then there is i $(1 \leq i \leq n)$ such $F_i \subset F$ and such that M is the image of M_i under $X(F_i) \to X(F)$.

Remark 4.1.3.1. Question 3 asks whether the motive version of the finiteness theorem of Northcott on the usual height is true.

Remark 4.1.3.2. It seems that through the analogies between Situation (I) and Situation (II) (resp. Situation (I) and Situation (III)), the result 3.6.10 (resp. 3.6.11) supports the answer Yes to Question 3.

4.2. Speculations on Vojta conjectures

4.2.1. In Situation (I), for a finite extension F of F_0 and for a finite set S_0 of places of F_0 containing all archimedean places of F_0, let $D_{S_0}(F/F_0)$ be the norm of the component of the different ideal of F/F_0 outside S_0.

In Situation (II), we consider the Nevanlinna analogue of it

$$N_{\mathrm{Ram}(\Pi)}(r) := \sum_{x \in B, 0 < |\Pi(x)| < r} (e_x(\Pi) - 1) \log(r/|\Pi(x)|) + \sum_{x \in B, \Pi(x) = 0} (e_x(\Pi) - 1) \log(r),$$

where $e_x(\Pi)$ is the ramification index of B over $\mathbb{C}$ at x.

4.2.2. In Situation (I), for a finite extension F of F_0 in $\mathbb{C}$, for $M \in X(F)$, and for a non-archimedean place v of F, we say M is of good reduction at v if $M(V)$ is of good reduction at v for any $V \in \mathrm{Rep}(G)$. We say M is of bad reduction at v if it is not of good reduction at v.

For each $M \in X(F)$, there are only finitely many non-archimedean places v at which M is of bad reduction. This is because for a faithful representation $V_1 \in \mathrm{Rep}(G)$, by 1.2.20, M is good reduction at v if and only if the mixed motive $M(V_1)$ is of good reduction at v.

4.2.3. Question 4 (1). Assume we are in Situation (I). Assume Λ is ample (3.6.9). Is the following statement true?

Fix a finite set of places S_0 of F_0 which contains all archimedean places of F_0. Then there is a constant $c \in \mathbb{R}_{>0}$ such that

$$\Big(\prod_{v \in \Sigma_S(M)} \sharp(\mathbb{F}_v)\Big) \cdot D_{S_0}(F/F_0) \geq c^{[F:F_0]} \cdot H_{\spadesuit}(M) H_\Lambda(M)^{-1}$$

for any finite extension F of F_0 in $\mathbb{C}$ and for any $M \in X_{\mathrm{gen}}(F)$. Here S denotes the set of all places of F lying over S_0, $\Sigma_S(M)$ denotes the set of places of F which do not belong to S and at which M has bad reduction, and $\mathbb{F}_v$ denotes the residue field of v.

Remark 4.2.3.1. If we follow the analogy with conjectures in [36], the reader may think that $\epsilon > 0$ should be fixed and $H_\Lambda(M)$ should be replaced by $H_\Lambda(M)^\epsilon$. However, we have $H_\Lambda(M)^\epsilon = H_{\Lambda'}(M)$ where Λ' is obtained from Λ by replacing $c(i)$ with $\epsilon c(i)$ and replacing $t(w,d)$ with $\epsilon t(w,d)$ (then Λ' is also ample). Hence ϵ can be deleted.

Remark 4.2.3.2. In the case G is reductive, basing on 2.3.3, we have a variant of Question 4 (1) replacing $H_\spadesuit(M)H_\Lambda(M)^{-1}$ by $H_\spadesuit(M)^{1-\epsilon}$ for a fixed $\epsilon > 0$.

Question 4 (2). Assume we are in Situation (II). Let Λ be ample (3.6.9). Fix B. Is the following statement true?

If $f \in \mathcal{M}_{\mathrm{hor,gen}}(B, X(\mathbb{C}))$ and $c \in \mathbb{R}$, we have

$$N^{(1)}_{f,\heartsuit}(r) + N_{\mathrm{Ram}(\Pi)}(r) \geq_{\mathrm{exc}} T_{f,\spadesuit}(r) - T_{f,\Lambda}(r) + c$$

where $\geq_{\mathrm{exc}}$ means that the inequality $\geq$ holds for any r outside some subset of $\mathbb{R}_{\geq 1}$ of finite Lebesgue measure. Here $N^{(1)}_{f,\heartsuit}(r)$ is as in 3.6.6.

Remark 4.2.3.3. Question 4 (2) treats the Hodge-Nevanlinna version of the conjecture of Griffiths in the usual Nevanlinna theory treated in [36] Section 14, Section 26.

Remark 4.2.3.4. There is a variant of Question 4 (2) in which we replace $N^{(1)}_{f,\heartsuit}(r)$ by $N_{f,\heartsuit}(r)$.

Remark 4.2.3.5. If G is reductive, by 2.3.3, Question 4 (2) becomes equivalent to the question in which we replace $T_{f,\spadesuit}(r) - T_{f,\Lambda}(r)$ in the above by $(1-\epsilon)T_{f,\spadesuit}(r)$ $(\epsilon > 0)$.

Question 4 (3). Assume we are in Situation (III). Let Λ be ample (3.6.9). Fix C. Is the following statement true?

There is $c \in \mathbb{R}$ such that $\sharp(\Sigma(\mathcal{H})) \geq h_\spadesuit(\mathcal{H}) + c$ for any $\mathcal{H} \in \mathcal{M}_{\mathrm{hor,gen}}(C, X(\mathbb{C}))$.

Here $\Sigma(\mathcal{H})$ denotes the set of points in C at which $\mathcal{H}$ degenerates.

Remark 4.2.3.6. In Situation (III), various results are known concerning the comparison of $\sharp(\Sigma(\mathcal{H}))$ and kinds of heights of $\mathcal{H}$ for a variation of Hodge structure $\mathcal{H}$ on C with log degeneration. For example, see Proposition 2.1 of [34].

Remark 4.2.3.7. If the answer to Question 4 (2) is Yes, the answer to Question 4 (3) is Yes. This follows from 3.7.1 and 3.7.2.

Remark 4.2.3.8. If $X(\mathbb{C})$ is of dimension 1, Question 4 (3) has an affirmative answer. In this case, if f is not constant, then for $R := \Sigma(\mathcal{H})$, we have a canonical injective map $f^*(\Omega^1_{\bar{X}(\mathbb{C})}(\log)) \to \Omega^1_C(\log R)$ and hence the comparison of degrees gives $h_{\spadesuit}(\mathcal{H}) \leq \deg(\Omega^1_C(\log R)) = 2g_C - 2 + \sharp(R)$.

Remark 4.2.3.9. There is a variant of Question 4 (3) in which we replace $\sharp(\Sigma(\mathcal{H}))$ by $h_{\heartsuit}(\mathcal{H})$.

4.2.4. In this Question 5, we assume G is reductive.

Question 5 (1). Are the following (i) and (ii) equivalent?

(i) $X(F)$ is finite for any finite extension F of F_0 in $\mathbb{C}$.
(ii) Any $f \in \mathcal{M}_{\text{hor}}(\mathbb{C}, X(\mathbb{C}))$ is constant.

Question 5 (2). Are the following (i)–(iii) equivalent?

(i) $X_{\text{gen}}(F)$ is finite for any finite extension F of F_0 in $\mathbb{C}$ in Situation (I).

(ii) Any $f \in \mathcal{M}_{\text{hor,gen}}(\mathbb{C}, X(\mathbb{C}))$ is constant.

(iii) There is an ample Λ (3.6.9) such that $h_{\spadesuit}(\mathcal{H}) - h_{\heartsuit}(\mathcal{H}) \geq h_{\Lambda}(\mathcal{H})$ in Situation (III) for any C and for any $\mathcal{H} \in \mathcal{M}_{\text{hor,gen}}(C, X(\mathbb{C}))$.

Question 5 (3). Is the following true?

Let F be a finite extension of F_0 in $\mathbb{C}$ and let S be a finite set of places of F containing all archimedean places of F. Then there are only finitely many $M \in X(F)$ which are of good reduction outside S.

Remark 4.2.4.1. The answer Yes to Question 5 (3) is a generalization of Shafarevich conjecture for abelian varieties proved by Faltings ([11]) to G-motives.

Remark 4.2.4.2. The last section of Koshikawa [27] contains an affirmative result on the Shafarevich conjecture for motives assuming the finiteness of the number of motives with bounded heights and assuming the semi-simplicity of the category of pure motives.

Renark 4.2.4.3. Via the analogy in 4.2.5 below, Proposition 2.3.4 supports the answer Yes to Question 5 (3).

Remark 4.2.5. The following conjecture of Bombieri-Lang-Vojta is well known in Diophantine geometry.

Conjecture 4.2.5.1. Let $F_0 \subset \mathbb{C}$ be a number field, let V be a scheme of finite type over O_{F_0}, and let V' be a closed subscheme of V. Then the following conditions are equivalent.

(i) The image of any morphism $\mathbb{C} \to V(\mathbb{C})$ of complex analytic spaces is contained in $V'(\mathbb{C})$.

(ii) For any finite extension F of F_0 and for any finite set S of places of F containing all archimedean places of F, $V(O_S) \smallsetminus V'(O_S)$ is finite.

Question 5 is based on the analogy with this conjecture.

In Question 5 (1), we consider an arithmetic version $\bar{X}$ of $\bar{X}(\mathbb{C})$ (if it exists) in place of V and we consider $\bar{X} \smallsetminus X$ in place of V'.

In Question 5 (2), we consider $\bar{X}$ in place of V.

In Question 5 (3), we consider X in place of V taking the empty set as V'.

The following Question 6 (1)–(3) present inequalities which are similar to those in Question 4, but the difference is that the generic parts ($X_{\rm gen}(F)$ etc.) are considered in Question 4 whereas the total space ($X(F)$ etc.) are considered in Question 6.

4.2.6. Question 6.

(1) Assume we are in Situation (I) and assume G is reductive. Is there an ample Λ (3.6.9) satisfying the following? Fix a finite set of places S_0 of F_0 which contains all archimedean places of F_0. Then there exists $c \in \mathbb{R}_{>0}$ such that

$$(\prod_{v\in\Sigma_S(M)} \sharp(\mathbb{F}_v)) \cdot D_{S_0}(F/F_0) \geq c^{[F:F_0]} \cdot H_\Lambda(M)$$

for any finite extension F of F_0 in $\mathbb{C}$ and for any $M \in X(F)$.

Here S and $\Sigma_S(M)$ are as in Question 4 (1).

(2) Assume we are in Situation (II) and assume G is reductive. Fix B. Is there an ample Λ satisfying the following? If $f \in \mathcal{M}_{\rm hor}(B, X(\mathbb{C}))$ and $c \in \mathbb{R}$, we have

$$N^{(1)}_{f,\heartsuit}(r) + N_{\mathrm{Ram}(\Pi)}(r) \geq_{\rm exc} T_{f,\Lambda}(r) + c.$$

(3) Assume we are in Situation (III) and assume G is reductive. Fix C. Are there an ample Λ and $c \in \mathbb{R}$ such that

$$\sharp(\Sigma(\mathcal{H})) \geq h_\Lambda(\mathcal{H}) + c$$

for any $\mathcal{H} \in \mathcal{M}_{\rm hor}(C, X(\mathbb{C}))$? Here $\Sigma(\mathcal{H})$ denotes the set of points in C at which $\mathcal{H}$ degenerates.

Remark 4.2.6.1. If the answer to Question 6 (2) is Yes, then the answer to Question 6 (3) is Yes. This follows from 3.7.1 and 3.7.2.

Remark 4.2.6.2. If the answer to Question 6 (1) is Yes and if (i) $\Rightarrow$ (iv) in Question 2 is true, we have the answer Yes to Question 5 (3).

4.3. Speculations on the number of motives of bounded height

4.3.1. Fix an ample Λ (3.6.9). Consider Situation (I). We fix a finite extension F of F_0 in $\mathbb{C}$.

For $t \in \mathbb{R}_{>0}$, define

$$N(H_\Lambda, t) := \sharp\{M \in X(F) \mid H_\Lambda(M) \leq t\},$$
$$N_{\rm gen}(H_\Lambda, t) := \sharp\{M \in X_{\rm gen}(F) \mid H_\Lambda(M) \leq t\}.$$

We expect that these numbers are finite (Question 3 in Section 4.1).

4.3.2. Question 7. Do we have

$$N(H_\Lambda, t) = ct^a \log(t)^b(1 + o(1)), \quad N_{\rm gen}(H_\Lambda, t) = c't^{a'} \log(t)^{b'}(1 + o(1))$$

for some constants $a, a' \in \mathbb{R}_{\geq 0}$, $b, b' \in \frac{1}{2}\mathbb{Z}$, $c, c' \in \mathbb{R}_{\geq 0}$? Do we have $a, a' \in \mathbb{Q}$ in the case $c(i), t(w, d) \in \mathbb{Q}$ in the definition of Λ?

4.3.3. Define $\alpha \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ to be the inf of all $s \in \mathbb{R}_{\geq 0}$ satisfying the following condition (i).

(i) For any connected projective smooth curve C over $\mathbb{C}$ and for any $\mathcal{H} \in \mathcal{M}_{\rm hor}(C, X(\mathbb{C}))$, we have

$$sh_\Lambda(\mathcal{H}) + h_\spadesuit(\mathcal{H}) - h_\heartsuit(\mathcal{H}) \geq 0.$$

(α is defined to be ∞ if such s does not exist).

Remark 4.3.4. The inequality $sh_\Lambda(\mathcal{H}) + h_\spadesuit(\mathcal{H}) \geq h_\heartsuit(\mathcal{H})$ can be written also as $\deg(f^*\Omega^1_{\bar{X}(\mathbb{C})}) + sh_\Lambda(\mathcal{H}) \geq 0$ ($f = \mathcal{H} \in \mathcal{M}_{\rm hor}(C, X(\mathbb{C}))$) by using the partial toroidal compactification $\bar{X}(\mathbb{C})$ of $X(\mathbb{C})$.

The next question arises following the analogy with the conjecture [1] of Batyrev-Manin.

4.3.5. Question 8. Assume that $X_{\rm gen}(F)$ is not empty. Do we have

$$\lim_{t\to\infty} \frac{\log(N_{\rm gen}(H_\Lambda, t))}{\log(t)} = \alpha < \infty?$$

4.3.6. Example. In Example 1.4.6, consider the case $F_0 = \mathbb{Q}$ and $M_0 = \mathbb{Z} \oplus \mathbb{Z}(1)^n$ for some $n \geq 1$.

We have $X(\mathbb{C}) = \mathrm{Ext}^1_{\mathbb{Q}\mathrm{MHS}}(\mathbb{Z}, \mathbb{Z}(1)^n) = (\mathbb{C}^\times)^n$. $X(F)=$ $\mathrm{Ext}^1_{MM(F,\mathbb{Z})}(\mathbb{Z}, \mathbb{Z}(1)^n)$, where $MM(F, \mathbb{Z})$ is the category of mixed motives with $\mathbb{Z}$-coefficients over F. Let $X'(F) = (F^\times)^n$. We have a canonical map $X'(F) \to X(F)$ and a philosophy of mixed motives suggests that this map should be bijective. We will compute the number of motives of bounded height replacing $X(F)$ by $X'(F)$ and replacing $X_{\mathrm{gen}}(F)$ by the inverse image $X'_{\mathrm{gen}}(F)$ of $X_{\mathrm{gen}}(F)$ in $X'(F)$. Then $(a_i)_{1\le i\le n} \in X'(F) = (F^\times)^n$ belongs to $X'_{\mathrm{gen}}(F)$ if and only if $a_1, \dots, a_n$ are linearly independent in the $\mathbb{Q}$-vector space $F^\times \otimes \mathbb{Q}$. For $a \in (F^\times)^n$, let $M_a \in X(F)$ be its image.

We take Λ such that $t(0,2) = 1$. We have

$$H_\Lambda(M_a) = \exp(\sum_v (\sum_{i=1}^n \log(|a_i|_v)^2)^{1/2}),$$

where v ranges over all places of F. (This is proved as Part I, 1.7.8 which treats the case $n = 1$.)

We prove

$$\lim_{t\to\infty} \frac{\log(N_{\mathrm{gen}}(H_\Lambda, t))}{\log(t)} = 1 = \alpha.$$

First we prove $\alpha = 1$. If $\mathcal{H} \in \mathcal{M}_{\mathrm{hor}}(C, X(\mathbb{C}))$ corresponds to the extension $0 \to \mathbb{Z}(1)^n \to \mathcal{H} \to \mathbb{Z} \to 0$ defined by a family $(f_i)_{1\le i\le n}$ of non-zero elements f_i of the function field of C, then

$$h_\Lambda(\mathcal{H}) = \sum_{x\in C} (\sum_{i=1}^n \mathrm{ord}_x(f_i)^2)^{1/2},$$

$$h_\spadesuit(\mathcal{H}) = 0,$$

$$h_\heartsuit(\mathcal{H}) = \sum_{x\in C} \mathrm{GCD}\{|\mathrm{ord}_x(f_i)| \mid 1 \le i \le n\},$$

where GCD is the greatest common divisor. Hence $sh_\Lambda(\mathcal{H}) + h_\spadesuit(\mathcal{H}) \ge h_\heartsuit(\mathcal{H})$ means $sh_\Lambda(\mathcal{H}) \ge h_\heartsuit(\mathcal{H})$. We have clearly $h_\Lambda(\mathcal{H}) \ge h_\heartsuit(\mathcal{H})$. On the other hand, if we take $f_i = 1$ for $2 \le i \le n$, we have $h_\Lambda(\mathcal{H}) = h_\heartsuit(\mathcal{H}) = \sum_{x\in C} |\mathrm{ord}_x(f_1)|$. These prove $\alpha = 1$.

Next we prove $\lim_{t\to\infty} \log(N_{\mathrm{gen}}(H_\Lambda, t))/\log(t) = 1$.

Let $V := \mathbf{G}_m^n$, $\bar{V}_1 = (\mathbf{P}^1)^n$, and let D_1 be the divisor $\bar{V}_1 \smallsetminus V$ with simple normal crossings. Let $\bar{V}_2$ be the blowing-up of $\bar{V}_1$ along all intersections of two irreducible components of D_1 (i.e. the bowing-up of $\bar{V}_1$ by the product ideal of $\mathcal{O}_{\bar{V}_1}$ of the ideals which define intersections of two irreducible components of D_1). Let D_2 be the divisor $\bar{V}_2 \smallsetminus V$ on $\bar{V}_2$ with simple normal crossings. Then for $i = 1, 2$, the height function $H_{(i)}$ associated to the divisor D_i on $\bar{V}_i$ is described as follows. For

$a = (a_i)_{1 \le i \le n} \in (F^\times)^n$,

$$H_{(1)}(a) = \exp(\sum_v \sum_{i=1}^n |\log(|a_i|_v)|),$$
$$H_{(2)}(a) = \exp(\sum_v \max\{|\log(|a_i|_v)| \mid 1 \le i \le n\})$$

where v ranges over all places of F. Since D_i is rationally equivalent to $-K_i$ where K_i is the canonical divisor of $\bar{V}_i$, the work [2] shows that $\lim_{t\to\infty} \log(N(H_{(i)}, t))/\log(t) = 1$ for $i = 1, 2$ by [2]. Since $H_{(1)}(a) \ge H_\Lambda(M_a) \ge H_{(2)}(a)$, this shows $\lim_{t\to\infty} \log(N(H_\Lambda, t))/\log(t) = 1$. We can obtain $\lim_{t\to\infty} \log(N_{\mathrm{gen}}(H_\Lambda, t))/\log(t) = 1$ from it easily.

4.3.7. The following Question 9 is a refined version of Question 8. In Question 9, we consider the type of monodromy.

Consider the quotient set $\Sigma/\sim$ of Σ (3.8.1) where $\sim$ is the following equivalence relation. For $N, N' \in \mathrm{Lie}(\mathcal{G})$, $\mathbb{R}_{\ge 0}N \sim \mathbb{R}_{\ge 0}N'$ if and only if there are $(g, t) \in \mathcal{G}(\mathbb{C})$ and $c \in \mathbb{C}^\times$ such that $N' = c\mathrm{Ad}(g)(N)$ in $\mathrm{Lie}(\mathcal{G})_{\mathbb{C}}$. Let Σ' be a subset of $\Sigma/\sim$ which contains the class of the cone $\{0\}$.

Let S be a finite set of places of F which contains all archimedean places of F and all non-archimedean places of F at which M_b is of bad reduction. We define the height function $h_{\heartsuit,\Sigma'}(\mathcal{H})$ and the counting function $N_{\mathrm{gen},S,\Sigma'}(H, t)$, which take the shapes of the monodromy operators into account.

For a connected projective smooth curve C and for $\mathcal{H} \in \mathcal{M}_{\mathrm{hor}}(C, X(\mathbb{C}))$, define $h_{\heartsuit,\Sigma'}(\mathcal{H}) = \sum_x h_{\heartsuit,x}(\mathcal{H}) \in \mathbb{Z}_{\ge 0}$ where x ranges over all points of C which satisfy the following condition (i).

(i) The image of x under the composite map

$$C \xrightarrow{\mathcal{H}} \bar{X}(\mathbb{C}) \to \mathcal{G}(\mathbb{Q})\backslash(\Sigma \times (\mathcal{G}(\mathbf{A}^f_{\mathbb{Q}})/K)) \to \mathcal{G}(\mathbb{Q})\backslash\Sigma \to \Sigma/\sim$$

belongs to Σ'. Here $\mathcal{G}(\mathbb{Q})$ acts on Σ by conjugation.

Let $N_{\mathrm{gen},S,\Sigma'}(H_\Lambda, t)$ be the number of $M \in X_{\mathrm{gen}}(F)$ which appear in the definition of $N_{\mathrm{gen}}(H_\Lambda, t)$ and which satisfy the following condition (ii) at each place $v \notin S$ of F.

(ii) There is an element $\mathrm{class}(\mathbb{R}_{\ge 0}N)$ (with $N \in \mathrm{Lie}(\mathcal{G})$ nilpotent) of Σ' which satisfies the following: Let $p = \mathrm{char}(\mathbb{F}_v)$. For any prime number $\ell \ne p$, there are $t, c \in \bar{\mathbb{Q}}_\ell^\times$ and an isomorphism $\bar{\mathbb{Q}}_\ell \otimes_{\mathbb{Q}_\ell} M_{et,\mathbb{Q}_\ell} \cong H_{0,\bar{\mathbb{Q}}_\ell}$ of $\bar{\mathbb{Q}}_\ell$-vector spaces which preserves W and via which $\langle\ ,\ \rangle_w$ on $\bar{\mathbb{Q}}_\ell \otimes_{\mathbb{Q}_\ell} \mathrm{gr}^W_w M_{et,\mathbb{Q}_\ell}$ corresponds to $t^w\langle\ ,\ \rangle_{0,w}$ on $\mathrm{gr}^W_w H_{0,\bar{\mathbb{Q}}_\ell}$ for any $w \in \mathbb{Z}$

and the local monodromy operator N'_v corresponds to cN. Furthermore, there are $t, c \in \bar{F}_v^\times$ and an isomorphism $\bar{F}_v \otimes_{F_{v,0,\mathrm{ur}}} D_{\mathrm{pst}}(F_v, M_{et,\mathbb{Q}_p}) \cong H_{0,\bar{F}_v}$ of $\bar{F}_v$-vector spaces which preserves W and via which $\langle\ ,\ \rangle_w$ on $\bar{F}_v \otimes_{F_{v,0,\mathrm{ur}}} \mathrm{gr}^W_w D_{\mathrm{pst}}(F_v, M_{et,\mathbb{Q}_p})$ corresponds to $t^w\langle\ ,\ \rangle_{0,w}$ on $\mathrm{gr}^W_w H_{0,\bar{F}_v}$ for any $w \in \mathbb{Z}$ and the local monodromy operator N'_v corresponds to cN.

4.3.8. The above $h_{\heartsuit,\Sigma'}$ can be expressed in the form

$$h_{\heartsuit,\Sigma'}(\mathcal{H}) = \sum_{x\in C} e(x), \quad (f^*I_{\Sigma'})_x = m_{C,x}^{e(x)}$$

where $I_{\Sigma'}$ is the invertible ideal of $\mathcal{O}_{\bar{X}(\mathbb{C})}$ which coincides with $I_{\bar{X}(\mathbb{C})}$ in 3.8.8 at points of $\bar{X}(\mathbb{C})$ whose classes in $\Sigma/\sim$ belong to Σ' and coincides with $\mathcal{O}_{\bar{X}(\mathbb{C})}$ at the other points of $\bar{X}(\mathbb{C})$.

4.3.9. Let the notation be as in 4.3.7. Let F be a number field. Let S be a finite set of places of F containing all archimedean places. Define the modified version

$$\alpha_{\Sigma'} \quad \text{of} \quad \alpha \text{ in 4.3.3}$$

by replacing

$$sh(\mathcal{H}) + h_\spadesuit(\mathcal{H}) - h_\heartsuit(\mathcal{H}) \geq 0 \quad \text{by} \quad sh(\mathcal{H}) + h_\spadesuit(\mathcal{H}) - h_{\heartsuit,\Sigma'}(\mathcal{H}) \geq 0.$$

Question 9. Assume $X_{\mathrm{gen}}(F) \neq \emptyset$. Do we have

$$\lim_{t\to\infty} \frac{\log(N_{\mathrm{gen},S,\Sigma'}(H_\Lambda, t))}{\log(t)} = \alpha_{\Sigma'}?$$

4.3.10. Iwasawa's class number formula ([17], page 212, Theorem 4) describes the asymptotic behavior of the class numbers of the fields $\mathbb{Q}(\zeta_{p^n})$ when n moves, where ζ_{p^n} denotes a primitive p^n-th root of 1. We wonder whether there is a similar formula for the asymptotic behavior of the orders of $\{M \in X(F_0(\zeta_{p^n})) \mid H_\Lambda(M) \leq t^{d(n)}\}$ when n moves (t is fixed, $d(n) := [F_0(\zeta_{p^n}) : F_0]$). As is described in Part I, 2.6.15, the study of the number of motives of bounded height is related to Tamagawa number conjecture (class number formula) for motives. Hence this behavior of the numbers of motives when number fields change should be regarded as a problem of Iwasawa's class number formula for motives.

Correction to Part I. In Part I ([19]), page 471, line 5, the name of Tung Nguyen is written as N. Tung, not as T. Nguyen. The author is very sorry to him for this.

Complement to Part I. The work of K. Česnavičius and T. Koshikawa quoted in Part I, 1.3.8 (2) is described in their paper [7].

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