Iwasawa Theory for Modular Forms

Xin Wan

Abstract.

In this paper we give an overview of some aspects of Iwasawa theory for modular forms. We start with the classical formulation in terms of p-adic L-functions in the ordinary case and the $\pm$-formulation for supersingular elliptic curves. Then we discuss some recent progresses in the proof of the corresponding Iwasawa main conjectures formulated by Kato (Conjecture 4.1), which relates the index of his zeta element to the characteristic ideal of the *strict* Selmer groups. [1]

§1. Introduction

An important problem in number theory is to study relations between analytic L-functions and arithmetic objects. A first example of this flavor is the class number formula which relates the values of Dedekind ζ-functions of number fields at $s = 1$ to its class number. Let p be a prime. In the 1950's, Iwasawa initiated the study of such relation for cyclotomic $\mathbb{Z}_p$-towers of field extensions, which resulted in the asymptotic formula for the p-part of class numbers of such tower of field extensions. Beyond the case for number fields (and Hecke characters), another important example for this relations is the Birch-Swinnerton-Dyer conjecture for elliptic curves E defined over $\mathbb{Q}$. According to the Shimura-Taniyama-Weil conjecture proved by Wiles [37] and Breuil-Conrad-Diamond-Taylor [3], its associated L-function $L(E, s)$ is an entire function for s. The L-series has a functional equation symmetric with respect to the point $s = 1$. The BSD conjecture states that

- The vanishing order $r = r_{\mathrm{an}}$ of $L(E, s)$ at $s = 1$ is equal to the rank of the Mordell-Weil group $E(\mathbb{Q})$. This is called the *rank part* of the conjecture.

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- We have the following expression for the leading coefficient of $L(E,s)$ at 1
$$\frac{L^{(r)}(E,1)}{r!\Omega_E R_E} = \frac{\sharp \text{Ш}_{E,\mathbb{Q}} \prod_\ell c_\ell}{(\sharp E(\mathbb{Q})_{\mathrm{tor}})^2}.$$
Here Ω_E is the period, R_E is the regulator, $\text{Ш}_{E,\mathbb{Q}}$ is the Shafarevich-Tate group and c_ℓ is the Tamagawa number of E at ℓ. This is called the refined BSD formula.

In order to study the refined BSD formula, one first studies if the p-parts of both hand sides are equal. It is Mazur who first realized that the idea from Iwasawa theory can be applied to elliptic curves, getting results for the p-part of its BSD formula. In fact the Tamagawa numbers above come out naturally in Mazur's control theorems for Selmer groups (we refer to [8] for a detailed presentation of such theory). Later on, the formulation of Iwasawa theory has been greatly generalized by Greenberg [7] and others to certain "motives" (or automorphic forms which are algebraic in the sense of Clozel). In this paper we are going to give a survey on the case of elliptic modular forms, including elliptic curves as a special case.

§2. Modular Forms, Galois Representations and Assumptions

Fix a prime p. Let $f = \sum_{m=1}^{\infty} a_m q^m$ be a normalized cuspidal eigenform for $\mathrm{GL}_2/\mathbb{Q}$, of weight $k \geq 2$ and conductor N. For simplicity we assume that f has trivial character throughout this paper. By work of Shimura, Langlands, Deligne, etc, one can associate a Galois representation
$$\rho_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(L)$$
for some finite extension L of $\mathbb{Q}_p$. It is well known that one can find a Galois stable lattice in the representation space for ρ_f over the integer ring $\mathcal{O}_L$ of L, from which we can talk about the residual representation $\bar{\rho}_f$.

- Assume throughout this paper that this residual representation is absolutely irreducible.

In this case the choice of the Galois stable lattice is unique up to multiplying by a nonzero number. In fact such assumption is also used in several places in the argument (e.g. the lattice construction, and also Kato's Euler system argument). Write T_f for this lattice and let $V_f := T_f \otimes_{\mathcal{O}_L} L$. It is characterized by requiring that for all primes ℓ not dividing Np,
$$\mathrm{Tr}(\mathrm{Frob}_\ell|_{V_f}) = a_\ell.$$

Moreover the representation of $G_{\mathbb{Q}_p}$ on V_f is de Rham (potentially semi-stable) in the sense of p-adic Hodge theory. On the other hand, standard theory for modular forms provides an L-series to f, which we denote as $L(f,s)$. This is an entire function on the complex plane $\mathbb{C}$. More generally for each Hecke character χ of $\mathbb{Q}^\times\backslash\mathbb{A}_\mathbb{Q}^\times$ of finite order, there is a twisted L-series $L(f,\chi,s)$ (entire functions) of f by χ. The special values of $L(f,\chi,s)$ at $s=1,\cdots,k-1$ are called critical values (following Deligne). Critical L-values are those related to Selmer groups by the Bloch-Kato conjecture [5] which we recall momentarily.

- From now on throughout this paper we assume p is an odd prime for simplicity. The main reason is right now the prime 2 is still not accessible in the automorphic form arguments. The Euler system argument does not need to exclude the prime 2 nevertheless.
- We also always assume that k is even. The reason is we need the central critical point for the L-function for f is an integer (again for the automorphic arguments, we need some vanishing of anticyclotomic μ-invariant results which require this).

§3. Iwasawa Theory for Ordinary Modular Forms

As we are mainly interested in applications to central L-values, we are going to consider even weight k and look at the Galois representations $T_f(-\frac{k-2}{2})$ (the $-\frac{k-2}{2}$-th Tate twist) and $V_f(-\frac{k-2}{2})$. Let $\mathbb{Q}_\infty$ be the cyclotomic $\mathbb{Z}_p$ extension of $\mathbb{Q}$. Write $\Gamma=\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})\simeq\mathbb{Z}_p$. We define the Iwasawa algebra over $\mathcal{O}_L$ to be $\Lambda=\Lambda_{\mathcal{O}_L}=\mathcal{O}_L[[\Gamma]]$. We say the modular form f is ordinary at p if a_p is a p-adic unit. In this case we have a nice description of the representation of $G_{\mathbb{Q}_p}$ on T_f: there is a rank one $\mathcal{O}_L$ submodule T_f^+ (direct summand) of T_f stable under the action of $G_{\mathbb{Q}_p}$, and such that the action of $G_{\mathbb{Q}_p}$ on the quotient $T_f^-:=T_f/T_f^+$ is unramified. In this section we focus on the good ordinary case, i.e. f is ordinary and $p\nmid N$. In order to state the Iwasawa main conjecture for f, we define objects on arithmetic side and analytic side.

Arithmetic Side

Write Σ for the set of primes dividing pN. For any finite sub-extension $\mathbb{Q}_n/\mathbb{Q}$ of $\mathbb{Q}_\infty$, we define the Selmer group for f over $\mathbb{Q}_n$ as follows. For $v\nmid p$ we define the finite part of local cohomology

$$H^1_f(\mathbb{Q}_{n,v},V_f(-\frac{k-2}{2})):=\ker\{H^1(\mathbb{Q}_{n,v},V_f(-\frac{k-2}{2}))\to H^1(I_{n,v},V_f(-\frac{k-2}{2}))\}$$

($I_{n,v}$ is the inertial group of $\mathbb{Q}_{n,v}$) and define $H^1_f(\mathbb{Q}_{n,v}, V_f(-\frac{k-2}{2})/T_f(-\frac{k-2}{2}))$ as the image of

$$H^1_f(\mathbb{Q}_{n,v}, V_f(-\frac{k-2}{2})).$$

Let

$$\mathrm{Sel}(\mathbb{Q}_n, V_f(-\frac{k-2}{2})/T_f(-\frac{k-2}{2})) := \ker\{H^1(\mathbb{Q}_n^\Sigma, V_f(-\frac{k-2}{2})/T_f(-\frac{k-2}{2}))$$
$$\to \prod_{v\nmid p} \frac{H^1(\mathbb{Q}_{n,v}, V_f(-\frac{k-2}{2})/T_f(-\frac{k-2}{2}))}{H^1_f(\mathbb{Q}_{n,v}, V_f(-\frac{k-2}{2})/T_f(-\frac{k-2}{2}))} \times \frac{H^1(\mathbb{Q}_{n,p}, V_f(-\frac{k-2}{2})/T_f(-\frac{k-2}{2}))}{H^1(\mathbb{Q}_{n,p}, V_f^+(-\frac{k-2}{2})/T_f^+(-\frac{k-2}{2}))}\}.$$

We also define

$$\mathrm{Sel}(\mathbb{Q}_\infty, V_f(-\frac{k-2}{2})/T_f(-\frac{k-2}{2})) := \varinjlim_n \mathrm{Sel}(\mathbb{Q}_n, V_f(-\frac{k-2}{2})/T_f(-\frac{k-2}{2})).$$

We define the dual Selmer group

$$X_{f,\mathbb{Q}_\infty} := \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Sel}(\mathbb{Q}_\infty, V_f(-\frac{k-2}{2})/T_f(-\frac{k-2}{2})), \mathbb{Q}_p/\mathbb{Z}_p).$$

This is equipped with an action of Γ and thus a Λ-module structure, which can be easily proved to be finitely generated. This $X_{f,\mathbb{Q}_\infty}$ is the main arithmetic object to study. We need one more definition.

Definition 3.1. *Suppose A is a Noetherian normal domain and X is a finitely generated A module. We define the characteristic ideal of X as A module by*

$$\mathrm{char}_A(X) = \{x \in A | \mathrm{length}_{A_P} X_P \leq \mathrm{ord}_P x, \textit{for any height one prime } P \textit{ of } A.\}.$$

We also define it to be 0 if X is not a torsion A module.

Analytic Side

Now we write α, β for the roots of the Hecke polynomial $X^2 - a_pX + p^{k-1} = 0$. Since we assume a_p is a p-adic unit, there is a unique such root which is a p-adic unit. Without loss of generality we write α for this root. Let γ be the topological generator of Γ which corresponds to $(1+p)$ under the local reciprocity law in class field theory at p. We say a $\bar{\mathbb{Q}}_p$-point ϕ in $\mathrm{Spec}\,\Lambda$ is arithmetic if $\phi(\gamma) = \zeta(1+p)^m$ for $0 \leq m \leq k-2$ for some p-power root of unity ζ. We write χ_ϕ for the Hecke character of Γ corresponding to ϕ. By work of Amice-Vélu [1] and Vishik [32], there is a p-adic L-function $\mathcal{L}_{f,\mathbb{Q}} \in \Lambda$ with the following interpolation property

$$\phi(\mathcal{L}_{f,\mathbb{Q}}) = \frac{m!(p^{t'_\phi}N)^{m+1}L(f,\omega^{-m}\chi_\phi^{-1}, m+1)}{(-2\pi i)^{m+1}\Omega_f^{(-1)^m}G(\omega^{-m}\chi_\phi^{-1})}e_p(\phi)$$

for each arithmetic point ϕ. Here $e_p(\phi)$ is some local normalization factor at p (see [28, 3.4.4] for details. In fact there is an error for power of $2\pi i$ there. See also the formula in [28, Section 12.3]. The formula in [28, 3.4.4] follows [18] but the definition for $\Omega_f^\pm$ differs from [18] by $2\pi i$). We assume the ζ corresponding to ϕ is a primitive $p^{t_\phi-1}$-th root of unity, and the t'_ϕ is defined as 0 if $t_\phi = 1$ and $p-1|m$, and is t_ϕ otherwise. The ω is the Teichmuller character, G is the Gauss sum, $\Omega_f^\pm$ are period factors associated to f. These periods are defined up to multiplying by a p-adic unit (see [28, 3.3.3] for a precise definition).

Now we are ready to formulate the Iwasawa main conjecture for f.

Conjecture 3.2. *Assume $\bar{\rho}_f$ is absolutely irreducible. The Λ-module $X_{f,\mathbb{Q}_\infty}$ is torsion, and its characteristic ideal is the principal ideal generated by $\mathcal{L}_{f,\mathbb{Q}}$.*

This formulation is due to Greenberg in a series of papers (e.g. [7]). In the elliptic curve case the conjecture was already made by Mazur-Swinnerton-Dyer in 1974 [19, Conjecture 3]. The conjecture was greatly influenced by early work of Mazur on control theorems for Selmer groups of elliptic curves (see [8]). We will see in Remark 4.3 that this is a special case of Kato's more general formulation in the ordinary case. Now the conjecture is already a theorem, due to work of Kato and Skinner-Urban [28].

Theorem 3.3. *Suppose the weight k of f is congruent to 2 modulo $(p-1)$, and that $\bar{T}_f$ is an absolutely irreducible representation of $G_\mathbb{Q}$. Suppose moreover that there is a prime $\ell||N$, such that $\bar{T}_f|_{G_{\mathbb{Q}_\ell}}$ is ramified. Then the Iwasawa main conjecture is true.*

Note that the assumption excluded the CM cases, which are treated by results of Karl Rubin [23]. Note also that as noted in [29], the assumption of Kato that the image of $G_\mathbb{Q}$ contains $\mathrm{SL}_2(\mathbb{Z}_p)$ can be slightly weakened by only requiring the mod p irreducibility of the Galois representation, and the existence of the ℓ as above.

The proof consists of two parts. To prove the upper bound for Selmer groups, Kato [11] constructed an Euler system coming from the Siegel units on modular curves. An Euler system is, roughly speaking a set of Galois cohomology classes satisfying the norm relation and some local conditions. We refer to [24] for the precise definition and methods of Euler systems. Thaine [31] made an annihilator of the class groups of real abelian fields over $\mathbb{Q}$, using cyclotomic units via an argument which was later refined and generalized by many people including Rubin and Kolyvagin. It was Kolyvagin who made the machinery "Euler systems" and used it to study Heegner points and proved the rank part of the BSD conjecture when the analytic rank is 0 or 1. He also suggested to use this machinery to study Iwasawa theory. Karl Rubin studied it systematically, axiomatized the argument in [24], and exploited it in the case of elliptic units to prove Iwasawa main conjectures for quadratic imaginary fields [23]. (Strictly speaking the Heegner point case of Kolyvagin is not an Euler system in the sense of Rubin, but an "anticyclotomic Euler system in his terminology.)

To prove the lower bound, Skinner-Urban studied the congruences between families $\mathbf{E}_{Kling}$ of Klingen Eisenstein series and cusp forms on the rank four unitary group $\mathrm{U}(2,2)$, and constructed "sufficiently many" elements in the Selmer group. By doing this, it is necessary to study two and three variable main conjectures over the quadratic imaginary field $\mathcal{K}$ that is used to define the unitary group $\mathrm{U}(2,2)$. The key part of the argument is to find out a suitable Fourier coefficient of $\mathbf{E}_{Kling}$ and show that it is co-prime to the p-adic L-function under study. For this purpose, they used the pullback formula of Piatetski-Shapiro-Rallis and Shimura to realize $\mathbf{E}_{Kling}$ as the pullback of a family of (simpler) Siegel type Eisenstein series $\mathbf{E}_{Sieg}$ on $\mathrm{U}(3,3)$ under the pullback

$$\mathrm{U}(2,2)\times\mathrm{U}(1,1)\hookrightarrow\mathrm{U}(3,3).$$

So it is reduced to computing the Fourier-Jacobi coefficients for $\mathbf{E}_{Sieg}$, which turns out to be a finite sum of products of Eisenstein series and theta functions on $\mathrm{U}(1,1)$. We refer to [36] for a concise introduction to Skinner-Urban's work. After this, a Galois representation theoretic argument called "lattice construction" (generalizations of the so-called Ribet's lemma by Wiles, and further generalized later on by E. Urban [28, Section 4]. This construction is also explored in Bellaiche-Chenevier [2].) gives the Selmer elements needed.

§4. Kato's Formulation for Main Conjecture

Now we briefly discuss Kato's formulation of Iwasawa main conjecture for modular forms in [11] using his zeta elements. In this section for simplicity we exclude the situation described in [11, (12.5.1)]. In particular it is satisfied if f is "potentially of good reduction" at p. This formulation does not involve p-adic L-functions and does not require any ordinarity on the form.

Strict Selmer Groups

We define

$$H^1_{\mathrm{Iw}}(\mathbb{Q}^\Sigma, T_f(-\frac{k-2}{2})) := \varprojlim_n H^1(\mathbb{Q}^\Sigma_n, T_f(-\frac{k-2}{2}))$$

where $\mathbb{Q}_n$ is running over all intermediate field extensions between $\mathbb{Q}_\infty$ and $\mathbb{Q}$. Kato proved that it is a torsion-free rank one module over Λ, and defined a zeta element z_{Kato} in $H^1_{\mathrm{Iw}}(\mathbb{Q}^\Sigma, T_f(-\frac{k-2}{2}))\otimes\mathbb{Q}_p$. In fact what Kato constructed is a *canonical* morphism

$$V_f^\pm \to H^1_{\mathrm{Iw}}(\mathbb{Q}^\Sigma, T_f(-\frac{k-2}{2}))\otimes\mathbb{Q}_p, \gamma\mapsto z_\gamma,$$

where $\pm$ denotes the parts where the complex conjugation acts by ±1. The z_{Kato} is the image of a generator of T_f^+ under the above map (determined up to a p-adic unit).

On the arithmetic side, we define the Selmer group

$$\mathrm{Sel}_{\mathrm{str},\mathbb{Q}_n}(f) := \ker\{H^1(\mathbb{Q}_n^\Sigma, V_f(-\frac{k-2}{2})/T_f(-\frac{k-2}{2}))$$
$$\to \prod_{v\nmid p}\frac{H^1(\mathbb{Q}_{n,v}, V_f(-\frac{k-2}{2})/T_f(-\frac{k-2}{2}))}{H^1_f(\mathbb{Q}_{n,v}, V_f(-\frac{k-2}{2})/T_f(-\frac{k-2}{2}))}\times H^1(\mathbb{Q}_{n,p}, V_f(-\frac{k-2}{2})/T_f(-\frac{k-2}{2}))\}.$$

Here strict means the local image at p is required to be 0.

$$\mathrm{Sel}_{\mathrm{str},\mathbb{Q}_\infty}(f) := \varinjlim \mathrm{Sel}_{\mathrm{str},\mathbb{Q}_n}(f)$$

and

$$X_{\mathrm{str}} := \mathrm{Sel}_{\mathrm{str},\mathbb{Q}_\infty}(f)^*.$$

Here $*$ means Pontryagin dual. Kato formulated the Iwasawa main conjecture as

Conjecture 4.1. *The X_{str} is torsion over Λ. If the image of ρ_f contains $\mathrm{SL}_2(\mathbb{Z}_p)$, then the z_{Kato} is in $H^1_{\mathrm{Iw}}(\mathbb{Q}^\Sigma, T_f(-\frac{k-2}{2}))$, and*

$$\mathrm{char}_\Lambda X_{\mathrm{str}} = \mathrm{char}_\Lambda(\frac{H^1_{\mathrm{Iw}}(\mathbb{Q}^\Sigma, T_f(-\frac{k-2}{2}))}{\Lambda z_{\mathrm{Kato}}}).$$

Without the assumption that the image of ρ_f contains $\mathrm{SL}_2(\mathbb{Z}_p)$, then the related statement is true after tensoring with $\mathbb{Q}_p$.

In fact Kato constructed zeta elements for various levels, so that these classes form an Euler system and can be used to bound the strict Selmer groups. The z_{Kato} above is actually the zeta element of level 1. We will see in a moment that in the ordinary case, Kato's formulation of the Iwasawa main conjecture is the same as the one given in Section 3. Kato proved the following

Theorem 4.2. *(Kato) The X_{str} is a torsion module over Λ, and*

$$\mathrm{char}_{(\Lambda\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)}(X_{\mathrm{str}}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p) \supseteq \mathrm{char}_{(\Lambda\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)}\Big(\frac{H^1_{\mathrm{Iw}}(\mathbb{Q}^\Sigma, T_f(-\frac{k-2}{2}))\otimes_{\mathbb{Z}_p}\mathbb{Q}_p}{(\Lambda\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)z_{\mathrm{Kato}}}\Big).$$

If moreover the image of ρ_f contains some conjugation of $\mathrm{SL}_2(\mathbb{Z}_p)$ then

$$\mathrm{char}_\Lambda X_{\mathrm{str}} \supseteq \mathrm{char}_\Lambda\Big(\frac{H^1_{\mathrm{Iw}}(\mathbb{Q}^\Sigma, T_f(-\frac{k-2}{2}))}{\Lambda z_{\mathrm{Kato}}}\Big).$$

Remark 4.3. *Now we briefly explain the equivalence between Kato's formulation and the formulation using p-adic L-functions of the Iwasawa main conjecture in the ordinary case (this is explained in detail in* [11, Section 16]*). By the Poitou-Tate exact sequence we have the following*

$$0\to H^1_{\mathrm{Iw}}(\mathbb{Q}^\Sigma, T_f(-\frac{k-2}{2}))\to \frac{H^1_{\mathrm{Iw}}(\mathbb{Q}_p, T_f(-\frac{k-2}{2}))}{H^1_{\mathrm{Iw}}(\mathbb{Q}_p, T_f^+(-\frac{k-2}{2}))}\to X_{f,\mathbb{Q}_\infty}\to X_{\mathrm{str},\mathbb{Q}_\infty}\to 0.$$

On the other hand there is a pseudo-isomorphism from $H^1_{\mathrm{Iw}}(\mathbb{Q}_p, T_f^+(-\frac{k-2}{2}))$ to $\mathcal{O}_L[[T]]$, mapping z_{Kato} to the p-adic L-function for f. Then the equivalence follows from the multiplicativity of characteristic ideals in exact sequences.

§5. Iwasawa Main Conjecture for Supersingular Elliptic Curves

We note in the above section that Kato's result does not make any assumptions on ordinarity. So the next natural question to ask is: can we get the lower bound for Selmer groups in the non-ordinary case as well?

Recall in the proof in the ordinary case, Skinner-Urban used the formulation using p-adic L-functions. In the non-ordinary case, there are two problems to face. (1) The p-adic L-function constructed using similar interpolation formula is not in the Iwasawa algebra. (2) The Λ-module of dual Selmer groups defined by taking limits of Bloch-Kato Selmer groups, is not necessarily torsion. So these suggests that some modification is needed.

In the special case of supersingular elliptic curves E, if $a_p=0$, there is a nice $\pm$-theory on both analytic and arithmetic sides. Write α and $-\alpha$ for the roots of the Hecke polynomial $X^2+p=0$. On the analytic side, R. Pollack [20] constructed a pair of $\pm$ p-adic L-function, which we briefly summarize. Let $\mathcal{L}_{E,\alpha}$ and $\mathcal{L}_{E,-\alpha}$ be the p-adic L-functions associated to the roots α and $-\alpha$ respectively. These are not in the Iwasawa algebra Λ. Instead they are rigid analytic functions on the unit disc with certain growth condition. We introduce two additional half log functions $\log_p^{\pm}$ as follows:

$$\log_p^+(1+X) = \frac{1}{p}\prod_m \Phi_{2m}(1+X),\ \log_p^-(1+X) = \frac{1}{p}\prod_m \Phi_{2m-1}(1+X).$$

Here Φ_m denotes the p^m-th cyclotomic polynomial. It follows from the interpolation formulas that $\mathcal{L}_{E,\alpha} + \mathcal{L}_{E,-\alpha}$ is 0 at all zero points of $\log_p^+$, and $\mathcal{L}_{E,\alpha} - \mathcal{L}_{E,-\alpha}$ is 0 at all zero points of $\log_p^-$. Based on this observation Pollack proved the following theorem.

Theorem 5.1. *There are elements $\mathcal{L}_E^{\pm} \in \Lambda$ such that*

$$\mathcal{L}_{E,\alpha} = \mathcal{L}_E^- \log_p^+(1+X) + \mathcal{L}_E^+ \log_p^-(1+X)\alpha,$$

$$\mathcal{L}_{E,-\alpha} = \mathcal{L}_E^- \log_p^+(1+X) - \mathcal{L}_E^+ \log_p^-(1+X)\alpha.$$

Later on Kobayashi [14] found analogues of Pollack's $\pm$ p-adic L-functions on the arithmetic side. Kobayashi put more strict local Selmer conditions at p, and defined the $\pm$ dual Selmer groups which we denote as $X_E^{\pm}$. Kobayashi made the following conjecture.

Conjecture 5.2. *The characteristic ideals of $X_E^{\pm}$ are generated by $\mathcal{L}_E^{\pm}$, respectively.*

Kobayashi also proved that the characteristic ideal of $X_E^{\pm}$ above contain the principal ideal $(\mathcal{L}_E^{\pm})$, using Kato's theorem above. Kobayashi also proved that the $\pm$ main conjecture are both equivalent to Kato's main conjecture in this case.

For CM elliptic curves, Pollack and Rubin [22] proved the lower bounds for Selmer groups as well, completing the full main conjecture. For general elliptic curves, we proved the following theorem.

Theorem 5.3. *Assume $a_p = 0$. Suppose there is at least one prime ℓ dividing N exactly once, such that the representation $E[p]$ of $G_{\mathbb{Q}_\ell}$ is ramified. Then the $\pm$ main conjecture above is true.*

Note that our theorem is slightly more general than in that we do not need N to be square-free, since we removed such assumptions for the Greenberg type main conjecture in [35]. Note also that we do not assume the image of $G_{\mathbb{Q}}$ contains $\mathrm{SL}_2(\mathbb{Z}_p)$. The reason is observed by Skinner that to make the Euler system argument work, we only need to assume the weaker assumption that there is an ℓ as in the theorem above, and that $E[p]$ is an absolutely irreducible $G_{\mathbb{Q}}$-representation (which is automatic since E has supersingular reduction at p). We have the following corollary.

Corollary 5.4. *Assumptions are the same as in Theorem 5.3. If $r_{\mathrm{an}} = 0$ or 1, then the p-part of the refined BSD formula for E is true.*

The rank 0 case follows from a control theorem for Selmer groups. In the rank 1 case this follows from an old result (unpublished) of Perrin-Riou on non-degeneracy of p-adic height pairing and also explained in [15, Introduction and Section 4]. The key is the non-vanishing of the p-adic height pairing in the supersingular case.

The proof of Theorem 5.3 is a little convoluted. We first need some backgrounds on Greenberg's work on Iwasawa theory. At the moment suppose T is a $\mathbb{Z}_p$-Galois representation of $G_{\mathbb{Q}}$ and $V := T \otimes \mathbb{Q}_p$. Suppose V is geometric (i.e. de Rham at p and unramified almost everywhere). Then we have the Hodge-Tate decomposition

$$V|_{G_{\mathbb{Q}_p}} \otimes \mathbb{C}_p = \oplus_i \mathbb{C}_p(i)^{h_i}$$

where $\mathbb{C}_p(i)$ is the i-th Tate twist and h_i is the multiplicity. Let d be the dimension of T and let $d^{\pm}$ be the dimensions of the subspaces whose eigenvalues of the complex conjugation c is ± 1. We assume

- $d^+ = \sum_{i>0} h_i$.

This is put by Greenberg as a p-adic version of the assumption that $L(T, 0)$ (in favorable situations when this makes sense) is critical in the sense of Deligne. Assume moreover the following Panchishkin condition at p

- There is a d^+-dimensional $\mathbb{Q}_p$-subspace V^+ of V which is stable under the action of the decomposition group $G_{\mathbb{Q}_p}$ at p such that $V^+ \otimes \mathbb{C}_p = \oplus_{i>0} \mathbb{C}_p(i)^{h_i}$.

Write $T^+ := V^+ \cap T$. Under this Panchishkin condition Greenberg defined the following local Selmer condition

$$H^1_f(\mathbb{Q}_p, V/T) = \mathrm{Ker}\{H^1(\mathbb{Q}_p, V/T) \to H^1(\mathbb{Q}_p, \frac{V/T}{V^+/T^+})\}.$$

In other words under the Panchishkin condition the local Selmer condition above is very analogous to the ordinary case, thus making the corresponding Iwasawa main conjecture (when an appropriate p-adic L-function is available) accessible to proof (especially the "lattice construction" discussed in [28, Chapter 4]). The following example is crucial for this paper.

Example 5.5. *Let f be a cuspidal eigenform of weight k and g be a CM form of weight k' with respect to a quadratic imaginary field $\mathcal{K}$ such that p splits. Then g is ordinary at p by definition. Assume $k + k'$ is*

an odd number. We consider critical values for Rankin-Selberg products $L(f, g, i)$ (which means $L(\rho_f \otimes \rho_g(-i), 0)$ if we write ρ_f and ρ_g for the corresponding Galois representations). We consider two possibilities:
1. If $k > k'$, then the Panchishkin's condition is true if f is ordinary;
2. If $k' > k$, then the Panchishkin's condition is always true, regardless of whether f is ordinary or not. This can be seen as follows: we have $d^{\pm} = 2$, ρ_f and ρ_g have Hodge-Tate weights $(0, k-1)$ and $(0, k'-1)$ respectively. The L-values are critical when $k-1 \leq i \leq k'-1$. So for those i above $\rho_f \otimes \rho_g(-i)$ has two positive Hodge-Tate weights. On the other hand ρ_g as a $G_{\mathbb{Q}_p}$-representation is the direct sum of two characters. Thus the Panchishkin condition is easily seen.

We pick an auxiliary quadratic imaginary field $\mathcal{K}$ above. Let $\mathcal{K}_\infty$ be the unique $\mathbb{Z}_p^2$-extension of $\mathcal{K}$ with Galois group denoted as $\Gamma_\mathcal{K}$. Write $\Lambda_\mathcal{K} = \mathbb{Z}_p[[\Gamma_\mathcal{K}]]$. Let $\Gamma_\mathcal{K}^\pm$ be the rank one submodules of $\Gamma_\mathcal{K}$ such that the complex conjugation acts by ± 1. We are going to use the case 2 in the above example when f is the weight two cuspidal eigenform associated to E, and g varying in the Hida family $\mathbf{g}$ corresponding to characters of $\Gamma_\mathcal{K}$. We vary the data in a two-variable family: one corresponds to Hida family $\mathbf{g}$, one corresponds to twisting by cyclotomic characters (i.e. considering tower of field extensions $\mathbb{Q}_\infty/\mathbb{Q}$). The corresponding Greenberg type p-adic L-function has been constructed by Hida using Rankin-Selberg method (which we denote as $\mathcal{L}^{\mathrm{Gr}}_{\mathbf{g}\otimes f}$). We also denote $X^{\mathrm{Gr}}_{\mathbf{g}\otimes f}$ for the two-variable dual Selmer group defined by Greenberg. We proved the following theorem (see [34] and [33]).

Theorem 5.6. *Assume the residual Galois representation associated to f is absolutely irreducible over $G_\mathcal{K}$, and that there is at least one prime ℓ dividing N exactly once. We have the characteristic ideal of $X^{\mathrm{Gr}}_{\mathbf{g}\otimes f}$ as $\Lambda_\mathcal{K}$-module is contained in $(\mathcal{L}^{\mathrm{Gr}}_{\mathbf{g}\otimes f})$, up to height one primes of $\mathbb{Z}_p[[\Gamma_\mathcal{K}^+]]$.*

The proof of this theorem used Eisenstein congruences on the unitary group $\mathrm{U}(3,1)$, extending Skinner-Urban's strategy for the group $\mathrm{U}(2,2)$. Unlike Skinner-Urban's case, there is only Fourier-Jacobi expansion (instead of Fourier expansion) on $\mathrm{U}(3,1)$, making the argument more complicated. In particular we need to carefully make the local constructions and analyze the p-adic properties. This is the most technical part of our argument. We refer to the introduction of [33] for a detailed account of this argument. We also mention that this theorem has other important consequences. For example it is the key ingredient for Skinner's proof [26] of the converse of the theorem of Gross-Zagier and Kolyvagin, and the proof for the p-part of the refined BSD formulas for elliptic curves of analytic rank one, by Jetchev-Skinner-Wan [10].

Go back to the proof of the $\pm$ main conjectures. B. D. Kim [12] is able to upgrade the $\pm$ theory into a two-variable setting over $\Lambda_{\mathcal{K}}$. In fact since there are two primes of $\mathcal{K}$ above p, there are four possibilities in Kim's theory: $++, +-, -+, --$. Now we explain how we prove the $\pm$ main conjecture. We consider the $++$ p-adic L-function $\mathcal{L}^{++}_{E,\mathcal{K}}$ above. The key input comes from the explicit reciprocity law for Beilinson-Flach elements, studied by Kings-Loeffler-Zerbes [16], [13]. We first constructed the $\pm$ Beilinson-Flach element $\mathrm{BF}^{\pm}$, imitating Pollack's construction of the $\pm$ p-adic L-functions. Next we further develop some $\pm$ local theory and construct $\pm$ Coleman maps and Log maps, which are denoted as $\mathrm{Col}^{\pm}$ and $\mathrm{Log}^{\pm}$. Writing $p = v_0\bar{v}_0$ for the decomposition of p in $\mathcal{K}$. The work of Kings-Loeffler-Zerbes tells us that the images of BF^{+} under $\mathrm{Col}^{+}_{\bar{v}_0}$ and $\mathrm{Log}^{+}_{v_0}$ are essentially given by $\mathcal{L}^{++}_{E,\mathcal{K}}$ and $\mathcal{L}^{\mathrm{Gr}}_{\mathbf{g}\otimes f}$, respectively. Then by an argument using Poitou-Tate exact sequence, we see that the one side divisibility of the Greenberg main conjecture is equivalent to the one side divisibility of the $++$ main conjecture. Thus we get the desired divisibility for the $++$-main conjecture from Theorem 5.6. (Here we need some μ-invariant argument to deal with height one primes of $\mathbb{Z}_p[[\Gamma^{+}_{\mathcal{K}}]]$.) Finally we specialize this two-variable one divisibility to the cyclotomic line, and get the $+$ main conjecture of Kobayashi.

Note that we have assumed $a_p = 0$ in all above discussion. This is automatic when $p > 3$. When $a_p \neq 0$ and $p = 3$, F. Sprung developed a more complicated $\flat/\sharp$ theory in the place of $\pm$ theory here. Following the main line of our argument above, using more technical algebra, Sprung is able to prove in [30] the $\flat/\sharp$-main conjecture, which are again, equivalent to Kato's main conjecture.

§6. Iwasawa Main Conjecture for Modular Forms

In fact we are also able to prove Kato's main conjecture for modular forms of general weight k, provided $p \nmid N$. More precisely we have the following theorem [35].

Theorem 6.1. *Assume $2|k$, $p \nmid N$, f has trivial character, and that $\bar{T}_f|_{\mathbb{Q}(\zeta_p)}$ is absolutely irreducible. Suppose $\bar{T}_f|_{G_{\mathbb{Q}_p}}$ is absolutely irreducible. Assume moreover that the p-component of the automorphic representation π_f is a principal series representation with distinct Satake parameters. If there is an $\ell||N$ such that π_ℓ is the Steinberg representation twisted by $\chi_{\mathrm{ur}}^{\frac{k}{2}}$ for χ_{ur} being the unramified character sending p to $(-1)^{\frac{k}{2}}p^{\frac{k}{2}-1}$, then*

(1) *We have*

$$\mathrm{char}_{\Lambda[1/p]}(X_{\mathrm{str}}) = \mathrm{char}_{\Lambda[1/p]}(\frac{H^1_{\mathrm{Iw}}(\mathbb{Q}^\Sigma, T_f(-\frac{k-2}{2}))}{\Lambda z_{\mathrm{Kato}}}).$$

(2) *If moreover the image of the Galois representation* ρ_f *contains* $\mathrm{SL}_2(\mathbb{Z}_p)$. *Then*

$$\mathrm{char}_{\Lambda}(X_{\mathrm{str}}) = \mathrm{char}_{\Lambda}(\frac{H^1_{\mathrm{Iw}}(\mathbb{Q}^\Sigma, T_f(-\frac{k-2}{2}))}{\Lambda z_{\mathrm{Kato}}}).$$

The proof of this theorem goes mainly in 3 steps:

- Prove the Greenberg-type main conjecture (generalization of Theorem 5.6) for Rankin-Selberg product of f with CM forms of higher weight. Here the main difficulty is when $k > 2$, we are forced to consider forms on $\mathrm{U}(3,1)$ of vector-valued weights. Then the Archimedean computation for Fourier-Jacobi coefficient is very complicated. Our idea is to fix the Archimedean weight and vary the nebentypus at p to obtain the two-variable family. Then we use Ikeda's theory to show that the Fourier-Jacobi coefficients for Siegel Eisenstein series are finite sums of products of Eisenstein series and theta functions. From this we can use a less explicit argument to factor out an Archimedean integral which is non-zero and fixed throughout the family. This is enough for our purpose.
- In general we do not have a nice integral local theory at p as in the ordinary or $\pm$ case. So we first use the trianguline Iwasawa theory to prove the main conjecture after inverting p. This is a finite slope analogue of the ordinary Iwasawa theory, devloped by J. Pottharst [21]. By trianguline we mean the (φ, Γ)-module for $V_f|_{G_{\mathbb{Q}_p}}$ is upper triangular. (In the ordinary case, we have a much stronger property that the representation T_f of $G_{\mathbb{Q}_p}$ is upper triangular.)
- Finally we treat powers of p. Using some control theorem of Selmer groups, it actually suffices to show that at some arithmetic point where the value for the L-function of f over $\mathcal{K}$ is non-zero, the size of the Selmer groups is precisely as predicted. We do so by looking at one line in the two-variable family. It corresponds to twisting by Hecke characters of $\mathcal{K}^\times\backslash\mathbb{A}_{\mathcal{K}}^\times$ which is unramified at v_0 but ramified at $\bar{v}_0$. Therefore along this line the local Iwasawa theory at v_0 is easy to understand. We pick up such a line on which the Greenberg type p-adic L-function

is not identically 0, and do the Poitou-Tate exact sequence argument: we compute the relations between the index of the Beilinson-Flach element, or the Greenberg p-adic L-function, with their corresponding Selmer groups along this line chosen. Then we specialize to the point on the cyclotomic line, and compare with the special value of the L-function for f over $\mathcal{K}$ at *this* point. Note that in this argument we do not need any local Iwasawa theory at $\bar{v}_0$.

This theorem can be viewed as a generalization of Theorem 5.3, however, it does not completely cover the latter due to different assumptions.

It is also possible to deduce the p-part of the Tamagawa number formula if $L(f,k/2)\neq 0$, provided $k>p$. Write T for $T_f(-\frac{k-2}{2})$. The notations in the following corollary are slightly different from [35].

Corollary 6.2. *Assumptions are as in part two of Theorem 6.1. Assume moreover that the weight k is in the Fontaine-Laffaille range $k<p$. If $L(f,\frac{k}{2})\neq 0$, then the full Iwasawa main conjecture for T is true, and we have*

$$\prod_{l|pN} c_l(T)\cdot \mathrm{char}_{\mathcal{O}_L}(\mathrm{Sel}_{p^\infty}(T)) = \mathrm{char}_{\mathcal{O}_L}\left(\frac{\mathcal{O}_L}{\left(\frac{L(f,\frac{k}{2})}{(2\pi i)^{\frac{k}{2}}\Omega_f^{(-1)^{\frac{k}{2}-1}}}\right)\mathcal{O}_L}\right).$$

The reason for putting this low weight assumption is, in general we do not know how to have a nice purely local definition for the local Tamagawa numbers at p. If $k<p$ this is given by the Fontaine-Laffaille functor as in Bloch-Kato [5]. We note that there is recent work of Bhatt-Morrow-Scholze [4] on integral p-adic Hodge theory giving a purely local functor from lattices in crystalline p-adic $G_{\mathbb{Q}_p}$-representations V to lattices in $D_{\mathrm{cris}}(V)$ giving by $\mathrm{BK}(T)\otimes_{\mathfrak{S}}W(\mathbb{F})$. Here BK means Breuil-Kisin module functor over some coefficient ring $\mathfrak{S}$ (we refer to *loc.cit.* for precise definitions). The $W(\mathbb{F})$ is the Witt vector over some finite field $\mathbb{F}$ of characteristic p. Moreover they proved cohomology comparison result for trivial coefficient sheaf. More precisely, they have the following theorem

Theorem 6.3. *(Bhatt-Morrow-Scholze) Let X be a proper smooth variety defined over $\mathcal{O}_L$. Suppose*

$$H^i_{\mathrm{crys}}(X_{\mathbb{F}}/W(\mathbb{F}))_{\mathrm{tor}}=0,\ H^{i+1}_{\mathrm{crys}}(X_{\mathbb{F}}/W(\mathbb{F}))_{\mathrm{tor}}=0.$$

Then there is a canonical isomorphism

$$\mathrm{BK}(H^i_{\mathrm{et}}(X,\mathbb{Z}_p))\otimes_{\mathfrak{S}} W(\mathbb{F})\simeq H^i_{\mathrm{crys}}(X_{\mathbb{F}}/W(\mathbb{F})).$$

Here the assumption on the torsion-freeness of the crystalline cohomology of the special fibre is crucial. Morrow mentioned that there is work in progress to generalize the above theorem to general coefficient sheaves. In our setting, we can consider the construction by Tony Scholl [25] on the motive for modular forms. Let Y and X be the modular curve $Y_0(N)$ and the compactified modular curve $X_0(N)$, respectively, and E_{univ} the universal Abelian variety over Y. Let $Z=X\backslash Y$ be the set of cusps. Write $\mathcal{E}$ for the first relative de Rham cohomology of E_{univ} over $Y_0(N)$. Let $\Omega^1_X(\log Z)$ be the sheaf of differentials with log poles at cusps. Let $\omega=e^*\Omega^1_{E_{\mathrm{univ}}/X}$. Then for an appropriate extension of $\mathcal{E}$ from Y to X (which we still denote as $\mathcal{E}$), there is a short exact sequence

$$0\to\omega\to\mathcal{E}\to\omega^{-1}\to 0.$$

Let $\mathcal{E}_{k-2}=\mathrm{Sym}^{k-2}\mathcal{E}$. There is also a differential operator (we refer to [25] for details)

$$\nabla_{k-2}:\mathcal{E}_{k-2}\to\mathcal{E}_{k-2}\otimes\Omega^1_X(Z).$$

Tony Scholl defined a complex of coherent sheaves $\Omega^\bullet(\mathcal{E}_{k-2})$ on X with two non-zero terms in degrees 0 and 1, and studied its hyper-cohomology. From the proof of [25, Theorem 2.7], there is a filtration F^i on $\mathcal{E}_{k-2}$ whose graded pieces Gr^i are given by ω^{2i-k+2}, $0\le i\le k-2$, satisfying Griffiths transversality for ∇_{k-2}. One thus defines a filtered complex structure on $\Omega^\bullet(\mathcal{E}_k)$. Scholl worked out the graded pieces for this complex as follows:

$$\mathrm{Gr}^0(\Omega^\bullet(\mathcal{E}))=[\omega^{-(k-2)}\to 0],$$
$$\mathrm{Gr}^i(\Omega^\bullet(\mathcal{E}))=[\omega^{2i-(k-2)}\to\omega^{2i-k}\otimes\Omega^1_X(\log Z)],\ 1\le i\le k-2,$$
$$\mathrm{Gr}^{k-1}(\Omega^\bullet(\mathcal{E}))=[0\to\omega^{k-2}\otimes\Omega^1_X].$$

Consider the spectral sequence for hyper-cohomology of filtered complexes. From this description, if k is not in the Fontaine-Laffaille range, there might indeed be torsion for the integral de Rham cohomology (even after localizing at non-Eisenstein maximal ideals of Hecke algebras), whose effects to the comparison theorem is still not clear. On the other hand, those torsion may also contribute to the difference between the integral de Rham cohomology (the hyper-cohomology of $\Omega^\bullet(\mathcal{E}_k)$) and the space $H^0(X_0(N),\omega_k)$, by the above spectral sequence. We hope experts in p-adic Hodge theory can provide a nice formulation for a Tamagawa number conjecture for general weight forms, compatible with Kato's main conjecture we proved.

§7. Ramified Cases

So far we have assumed the automorphic representation π_f is unramified at p. Now we discuss cases when π_f is ramified at p. If f has weight two and the automorphic representation for f is special at p, then the main conjecture is proved by Skinner in [27] (in this case the f is ordinary at p). It seems not hard to adapt our argument above in the general weight special at p case, where f has finite slope. (But the details need to be checked.)

The strategy we summarized above does not apply to cases with certain bad ramification at p, especially when the automorphic representation of f is supercuspidal at p. (Kato's Euler system method works well also in these cases to get the upper bounds for Selmer groups.) The main difficulty comes from the explicit reciprocity law for Beilinson-Flach elements. In the case when $p \nmid N$, Kings-Loeffler-Zerbes first studied the *explicit* interpolation maps for various Perrin-Riou regulator maps over families, and used analytic continuation to prove the explicit reciprocity law from the information at a Zariski dense set of points where they get the formulas from geometry. In the ramified cases, however, it seems impossible to understand the interpolation formulas for Perrin-Riou maps.

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Morningside Center of Mathematics, Academy of Mathematics and Systems Science, CAS, No. 55 Zhongguancun East Road, Beijing, China, 100190
E-mail address: `xwan@math.ac.cn`