On primitive p-adic Rankin-Selberg L-functions

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Abstract.

In this note, we revisit Hida's construction of p-adic Rankin-Selberg L-functions by incorporating Jacquet's approach to automorphic L-functions on $\mathrm{GL}(2)\times\mathrm{GL}(2)$. This allows us to give a construction of *primitive* three variable p-adic Rankin-Selberg L-functions associated with a pair of two primitive Hida families in full generality and prove the functional equation of this p-adic Rankin-Selberg L-function.

§1. Introduction

The theory of p-adic Rankin-Selberg L-functions for Hida families of elliptic modular forms has been developed extensively by Hida in [Hid85] and [Hid88a] and presents a landmark in the search of p-adic L-functions for motives. The p-adic L-functions constructed by Hida are in general imprimitive in the sense that they interpolate the critical values of automorphic Rankin-Selberg L-function with local L-factors at ramified places removed. The primitive p-adic Rankin-Selberg L-functions were constructed in [Hid09, Theorem 3.3] under certain local assumptions. The aim of this note is to go through Hida's construction of p-adic Rankin-Selberg L-functions with some new ingredients from Jacquet's representation theoretic approach to automorphic L-functions on $\mathrm{GL}(2)\times\mathrm{GL}(2)$ in [Jac72]. As a result, we obtain the *primitive* p-adic Rankin-Selberg L-functions in great generality and deduce the interpolation formula in the form conjectured by Coates and Perrin-Riou [CPR89], [Coa89] (See Remark 1.1(1) for the precise meaning). We hope that brining in representation theory to Hida's work mentioned above and the primitive p-adic L-functions can be useful in some applications, for example, the precise formulation of three variable Iwasawa-Greenberg

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main conjecture for Rakin-Selberg convolutions. In order to give a precise statement of the main formula, we begin with some notation from Hida theory for elliptic modular forms and technical items such as the modified Euler factors at the archimedean place and the place p as well as the canonical periods of primitive Hida families. To begin with, let $p > 3$ be a prime. Let $\mathcal{O}$ be a valuation ring finite flat over $\mathbf{Z}_p$. Let $\mathbf{I}$ be a normal domain finite flat over the Iwasawa algebra $\Lambda = \mathcal{O}[\![\Gamma]\!]$ of the topological group $\Gamma = 1 + p\mathbf{Z}_p$.

1.1. Galois representations attached to Hida families

For a primitive cuspidal Hida family $\mathcal{F} = \sum_{n\geq 1} \mathbf{a}(n, \mathcal{F})q^n \in \mathbf{I}[\![q]\!]$ of tame conductor $N_{\mathcal{F}}$, we let $\rho_{\mathcal{F}} : G_{\mathbf{Q}} = \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\mathrm{Frac}\,\mathbf{I})$ be the associated big Galois representation such that $\mathrm{Tr}\,\rho_{\mathcal{F}}(\mathrm{Frob}_\ell) = \mathbf{a}(\ell, \mathcal{F})$ for primes $\ell \nmid N_{\mathcal{F}}$, where Frob_ℓ is the geometric Frobenius at ℓ and let $V_{\mathcal{F}}$ denote the natural realization of $\rho_{\mathcal{F}}$ inside the étale cohomology groups of modular curves. Thus, $V_{\mathcal{F}}$ is a lattice in $(\mathrm{Frac}\,\mathbf{I})^2$ with the continuous Galois action via $\rho_{\mathcal{F}}$, and the $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$-invariant subspace $\mathrm{Fil}^0 V_{\mathcal{F}} := V_{\mathcal{F}}^{I_p}$ fixed by the inertia group I_p at p is free of rank one over $\mathbf{I}$ ([Oht00, Corollary, page 558])). We recall the specialization of $V_{\mathcal{F}}$ at arithmetic points. A point $Q \in \mathrm{Spec}\,\mathbf{I}(\overline{\mathbf{Q}}_p)$ is called an arithmetic point of weight k and finite part ϵ if $Q|_\Gamma : \Gamma \to \Lambda^\times \xrightarrow{Q} \overline{\mathbf{Q}}_p^\times$ is given by $Q(x) = x^k\epsilon(x)$ for some integer $k \geq 2$ and a finite order character $\epsilon : \Gamma \to \overline{\mathbf{Q}}_p^\times$. For an arithmetic point Q, denote by k_Q the weight of Q and ϵ_Q the finite part of Q. Let $\mathfrak{X}_{\mathbf{I}}^+$ be the set of arithmetic points of $\mathbf{I}$. For each arithmetic point $Q \in \mathfrak{X}_{\mathbf{I}}^+$, the specialization $V_{\mathcal{F}_Q} := V_{\mathcal{F}} \otimes_{\mathbf{I},Q} \overline{\mathbf{Q}}_p$ is the geometric p-adic Galois representation associated with the eigenform $\mathcal{F}_Q$ constructed by Shimura and Deligne.

1.2. Rankin-Selberg L-functions

Let $\boldsymbol{\varepsilon}_{\mathrm{cyc}} : G_{\mathbf{Q}} \to \mathbf{Z}_p^\times$ be the p-adic cyclotomic character. Let $\mathbf{Q}_\infty/\mathbf{Q}$ be the cyclotomic $\mathbf{Z}_p$-extension and let $\langle\boldsymbol{\varepsilon}_{\mathrm{cyc}}\rangle_\Lambda : G_{\mathbf{Q}} \to \mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q}) \overset{\boldsymbol{\varepsilon}_{\mathrm{cyc}}}{\simeq} 1 + p\mathbf{Z}_p \hookrightarrow \Lambda^\times$ be the universal cyclotomic character. Let

$$\mathcal{R} := \mathbf{I}\widehat{\otimes}_{\mathcal{O}}\mathbf{I}[\![\Gamma]\!]$$

be a finite extension of the three-variable Iwasawa algebra. Let

$$\boldsymbol{F} = (\boldsymbol{f}, \boldsymbol{g})$$

be the pair of two primitive Hida families of tame conductor (N_1, N_2) and nebentypus (ψ_1, ψ_2) with coefficients in $\mathbf{I}$. Let $\boldsymbol{\omega} : (\mathbf{Z}/p\mathbf{Z})^\times \to \mu_{p-1}$

be the Teichmüller character. For each integer $0 \le a < p-1$, we consider the big Galois representation $\rho_{\mathbf{V}} : G_{\mathbf{Q}} \to \mathrm{GL}_4(\mathrm{Frac}\,\mathcal{R})$ realized on the lattice

$$\mathbf{V} = V_{\boldsymbol{f}} \widehat{\otimes}_{\mathcal{O}} V_{\boldsymbol{g}} \widehat{\otimes}_{\mathcal{O}} \langle \boldsymbol{\varepsilon}_{\mathrm{cyc}} \rangle_{\Lambda}\, \boldsymbol{\omega}^a \boldsymbol{\varepsilon}_{\mathrm{cyc}}^{-1}.$$

Let $\mathfrak{X}_{\mathcal{R}}^{\boldsymbol{f}} \subset \mathrm{Spec}\,\mathcal{R}(\overline{\mathbf{Q}}_p)$ be the $\boldsymbol{f}$-dominated weight space of arithmetic points of $\mathcal{R}$ given by

$$\mathfrak{X}_{\mathcal{R}}^{\boldsymbol{f}} := \left\{ \underline{Q} = (Q_1, Q_2, P) \in \mathfrak{X}_{\mathbf{I}}^{+} \times \mathfrak{X}_{\mathbf{I}}^{+} \times \mathfrak{X}_{\Lambda} \mid k_{Q_2} < k_P \le k_{Q_1} \right\}.$$

For each arithmetic point $\underline{Q} = (Q_1, Q_2, P) \in \mathfrak{X}_{\mathcal{R}}^{\boldsymbol{f}}$, the specialization

$$\mathbf{V}_{\underline{Q}} = V_{\boldsymbol{f}_{Q_1}} \otimes V_{\boldsymbol{g}_{Q_2}} \otimes \epsilon_P \boldsymbol{\omega}^{a-k_P} \boldsymbol{\varepsilon}_{\mathrm{cyc}}^{k_P-1}$$

is a p-adic geometric Galois representation of pure weight $w_{\underline{Q}} := k_{Q_1} + k_{Q_2} - 2k_P$. Next we briefly recall the complex L-function associated with the specialization $\mathbf{V}_{\underline{Q}}$. For each place ℓ, denote by $W_{\mathbf{Q}_\ell}$ the Weil-Deligne group of $\mathbf{Q}_\ell$. To the geometric p-adic Galois representation $\mathbf{V}_{\underline{Q}}$, we can associate the Weil-Deligne representation $\mathrm{WD}_\ell(\mathbf{V}_{\underline{Q}})$ of $W_{\mathbf{Q}_\ell}$ over $\overline{\mathbf{Q}}_p$ (See [Tat79, (4.2.1)] for $\ell \neq p$ and [Fon94, (4.2.3)] for $\ell = p$). Fixing an isomorphism $\iota_p : \overline{\mathbf{Q}}_p \simeq \mathbf{C}$ once and for all, we define the complex L-function of $\mathbf{V}_{\underline{Q}}$ by the Euler product

$$L(\mathbf{V}_{\underline{Q}}, s) = \prod_{\ell < \infty} L_\ell(\mathbf{V}_{\underline{Q}}, s)$$

of the local L-factors $L_\ell(\mathbf{V}_{\underline{Q}}, s)$ attached to $\mathrm{WD}_\ell(\mathbf{V}_{\underline{Q}}) \otimes_{\overline{\mathbf{Q}}_p, \iota_p} \mathbf{C}$ ([Del79, (1.2.2)], [Tay04, page 85]). According to the recipe in [Del79, page 329], the Gamma factor $\Gamma_{\mathbf{V}_{\underline{Q}}}(s)$ iof $\mathbf{V}_{\underline{Q}}$ is defined by

(1.1)
$$\Gamma_{\mathbf{V}_{\underline{Q}}}(s) := \Gamma_{\mathbf{C}}(s + k_P - 1)\Gamma_{\mathbf{C}}(s + k_P - k_{Q_2}) \quad (\Gamma_{\mathbf{C}}(s) = 2(2\pi)^{-s}\Gamma(s)).$$

On the other hand, denote by $\pi_{\boldsymbol{f}_{Q_1}} = \otimes_v \pi_{\boldsymbol{f}_{Q_1}, v}$ (resp. $\pi_{\boldsymbol{g}_{Q_1}}$) the irreducible unitary cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A})$ associated with $\boldsymbol{f}_{Q_1}$ (resp. $\boldsymbol{g}_{Q_2}$). In terms of automorphic L-functions, by [Jac72, Corollary 19.16] we have

$$\Gamma_{\mathbf{V}_{\underline{Q}}}(s) \cdot L(\mathbf{V}_{\underline{Q}}, s) = L(s + \frac{2k_P - k_{Q_1} - k_{Q_2}}{2}, \pi_{\boldsymbol{f}_{Q_1}} \times \pi_{\boldsymbol{g}_{Q_2}} \otimes \epsilon_P \boldsymbol{\omega}^{-k_P}),$$

where $L(s, \pi_{\boldsymbol{f}_{Q_1}} \times \pi_{\boldsymbol{g}_{Q_2}} \otimes \epsilon_P \boldsymbol{\omega}^{-k_P})$ is the Rankin-Selberg automorphic L-function on $\mathrm{GL}_2(\mathbf{A}) \times \mathrm{GL}_2(\mathbf{A})$.

1.3. The modified Euler factors at p and ∞

Let $G_{\mathbf{Q}_p}$ be the decomposition group at p. We consider the following rank two $G_{\mathbf{Q}_p}$-invariant subspaces of $\mathbf{V}_{\underline{Q}}$:

$$\mathrm{Fil}^+\mathbf{V} := \mathrm{Fil}^0 V_{\boldsymbol{f}} \otimes V_{\boldsymbol{g}} \otimes \boldsymbol{\omega}^a \left\langle \boldsymbol{\varepsilon}_{\mathrm{cyc}} \right\rangle_\Lambda \boldsymbol{\varepsilon}_{\mathrm{cyc}}^{-1}. \tag{1.2}$$

The pair $(\mathrm{Fil}^+\mathbf{V}, \mathfrak{X}_{\mathcal{R}}^{\boldsymbol{f}})$ satisfies the *Panchishkin condition* in [Gre94, page 217]) in the sense that for each arithmetic point $\underline{Q} \in \mathfrak{X}_{\mathcal{R}}^{\bullet}$, the Hodge-Tate numbers of $\mathrm{Fil}^+\mathbf{V}_{\underline{Q}}$ are all positive, while the Hodge-Tate numbers of $\mathbf{V}_{\underline{Q}}/\mathrm{Fil}^+\mathbf{V}_{\underline{Q}}$ are all non-positive.[1] Define the modified p-Euler factor by

$$\mathcal{E}_p(\mathrm{Fil}^+\mathbf{V}_{\underline{Q}}) := \frac{L_p(\mathrm{Fil}^+\mathbf{V}_{\underline{Q}},0)}{\varepsilon(\mathrm{WD}_p(\mathrm{Fil}^+\mathbf{V}_{\underline{Q}}))\cdot L_p((\mathrm{Fil}^+\mathbf{V}_{\underline{Q}})^\vee,1)} \cdot \frac{1}{L_p(\mathbf{V}_{\underline{Q}},0)}. \tag{1.3}$$

Here $(?)^\vee$ means the dual representation. We note that this modified p-Euler factor is precisely the ratio between the factor $\mathcal{L}_p^{(\rho)}(\mathbf{V}_{\underline{Q}})$ in [Coa89, page 109, (18)] and the local L-factor $L_p(\mathbf{V}_{\underline{Q}},0)$.

In the conjectural interpolation formula of p-adic L-functions for motives, we also need the modified Euler factor $\mathcal{E}_\infty(\mathbf{V}_{\underline{Q}})$ at the archimedean place as observed by Deligne. In our case, this Euler factor is given by

$$\mathcal{E}_\infty(\mathbf{V}_{\underline{Q}}) = (\sqrt{-1})^{1+k_{Q_2}-2k_P}.$$

This factor is the ratio between the factor $\mathcal{L}_\infty^{(\sqrt{-1})}(\mathbf{V}_{\underline{Q}})$ and the Gamma factor $\Gamma_{\mathbf{V}_{\underline{Q}}}(0)$ in [Coa89, page 103 (4)].

1.4. Hida's canonical periods

We review Hida's canonical period of an $\mathbf{I}$-adic primitive cuspidal Hida family $\mathcal{F}$ of tame conductor $N_{\mathcal{F}}$. Let $\mathfrak{m}_{\mathbf{I}}$ be the maximal ideal of $\mathbf{I}$. For a subset Σ of the support of $N_{\mathcal{F}}$, we consider the following

Hypothesis (CR)**.** The residual Galois representation

$$\bar{\rho}_{\mathcal{F}} := \rho_{\mathcal{F}} \,(\mathrm{mod}\ \mathfrak{m}_{\mathbf{I}}) : G_{\mathbf{Q}} \to \mathrm{GL}_2(\bar{\mathbb{F}}_p)$$

is absolutely irreducible and p-distinguished.

We say $\bar{\rho}_{\mathcal{F}}$ is *p-distinguished* if the semi-simplication of the restriction of the residual Galois representation $\rho_{\mathcal{F}}\,(\mathrm{mod}\ \mathfrak{m}_{\mathbf{I}})$ to the decomposition group at p is a sum of two characters $\chi_{\mathcal{F}}^+ \oplus \chi_{\mathcal{F}}^-$ with $\chi_{\mathcal{F}}^+ \not\equiv$

[1] The Hodge-Tate number of $\mathbf{Q}_p(1)$ is one in our convention.

$\chi_{\mathcal{F}}^{-} \pmod{\mathfrak{m}_{\mathbf{I}}}$. Suppose that $\mathcal{F}$ satisfies (CR). The local component of the universal cuspidal ordinary Hecke algebra corresponding to $\mathcal{F}$ is known to be Gorenstein by [MW86, Prop.2, §9] and [Wil95, Corollary 2, page 482], and with this Gorenstein property, Hida proved in [Hid88b, Theorem 0.1] that the congruence module for $\mathcal{F}$ is isomorphic to $\mathbf{I}/(\eta_{\mathcal{F}})$ for some non-zero element $\eta_{\mathcal{F}} \in \mathbf{I}$. Moreover, for any arithmetic point $Q \in \mathfrak{X}_{\mathbf{I}}^{+}$, the specialization $\eta_{\mathcal{F}_Q} = \iota_p(Q(\eta_{\mathcal{F}}))$ generates the congruence ideal of $\mathcal{F}_Q$. We denote by $\mathcal{F}_Q^{\circ}$ the normalized newform of weight k_Q, conductor $N_Q = N_{\mathcal{F}} p^{n_Q}$ with nebentypus χ_Q corresponding to $\mathcal{F}_Q$. There is a unique decomposition $\chi_Q = \chi'_Q \chi_{Q,(p)}$, where χ'_Q and $\chi_{Q,(p)}$ are Dirichlet characters modulo $N_{\mathcal{F}}$ and p^{n_Q} respectively. Let $\alpha_Q = \mathbf{a}(p, \mathcal{F}_Q)$. Define the modified Euler factor $\mathcal{E}_p(\mathcal{F}_Q, \mathrm{Ad})$ for adjoint motive of $\mathcal{F}_Q$ by

(1.4)

$$\mathcal{E}_p(\mathcal{F}_Q, \mathrm{Ad}) = \alpha_Q^{-2n_Q}$$
$$\times \begin{cases} (1-\alpha_Q^{-2}\chi_Q(p)p^{k_Q-1})(1-\alpha_Q^{-2}\chi_Q(p)p^{k_Q-2}) & \text{if } n_Q = 0, \\ -1 & \text{if } n_Q = 1, \chi_{Q,(p)} = 1, \\ \mathfrak{g}(\chi_{Q,(p)})\chi_{Q,(p)}(-1) & \text{if } n_Q > 0,\ \chi_{Q,(p)} \neq 1. \end{cases}$$

Here $\mathfrak{g}(\chi_{Q,(p)})$ is the usual Gauss sum. Fixing a choice of the generator $\eta_{\mathcal{F}}$ and letting $\|\mathcal{F}_Q^{\circ}\|^2_{\Gamma_0(N_Q)}$ be the usual Petersson norm of $\mathcal{F}_Q^{\circ}$, we define the *canonical period* $\Omega_{\mathcal{F}_Q}$ of $\mathcal{F}$ at Q by

$$\Omega_{\mathcal{F}_Q} := (-2\sqrt{-1})^{k_Q} \|\mathcal{F}_Q^{\circ}\|^2_{\Gamma_0(N_Q)} \cdot \frac{\mathcal{E}_p(\mathcal{F}_Q, \mathrm{Ad})}{\eta_{\mathcal{F}_Q}} \in \mathbf{C}^{\times}. \tag{1.5}$$

By [Hid16, Corollary 6.24, Theorem 6.28], one can show that for each arithmetic point Q, up to a p-adic unit, the period $\Omega_{\mathcal{F}_Q}$ is equal to the product of the plus/minus canonical period $\Omega(+\,;\mathcal{F}_Q^{\circ})\Omega(-\,;\mathcal{F}_Q^{\circ})$ introduced in [Hid94, page 488].

1.5. Statement of the interpolation formula

Now we give the statement of the main formula. Let $(f, g) = (\boldsymbol{f}_{Q_1}, \boldsymbol{g}_{Q_2})$ for some arithmetic specilization. Let Σ_{exc} be the finite set of primes ℓ such that (i) $\pi_{f,\ell}$ and $\pi_{g,\ell}$ are supercuspidal, and (ii) $\pi_{f,\ell} \simeq \pi_{f,\ell} \otimes \tau_{\mathbf{Q}_{\ell^2}} \simeq \pi_{g,\ell}^{\vee} \otimes \sigma$, where $\tau_{\mathbf{Q}_{\ell^2}}$ is the unramified quadratic character of $\mathbf{Q}_\ell^{\times}$ and σ is some unramified character σ of $\mathbf{Q}_\ell^{\times}$. Note that this set Σ_{exc} does not depend on the choice of arithmetic specializations.

Theorem A. *Suppose that $\boldsymbol{f}$ satisfies (CR). For the fixed generator $\eta_{\boldsymbol{f}}$ of the congruence ideal of $\boldsymbol{f}$, there exists a unique element $\mathcal{L}^{\boldsymbol{f}}_{\boldsymbol{F},a} \in \mathcal{R}$*

such that for every $\underline{Q} = (Q_1, Q_2, P) \in \mathfrak{X}_{\mathcal{R}}^{\boldsymbol{f}}$ *in the unbalanced range dominated by* $\boldsymbol{f}$*, we have the following interpolation formula*

$$\begin{aligned}&\mathcal{L}_{\boldsymbol{F},a}^{\boldsymbol{f}}(\underline{Q})\\ =&\Gamma_{\mathbf{V}_{\underline{Q}}}(0) \cdot \frac{L(\mathbf{V}_{\underline{Q}}, 0)}{\Omega_{\boldsymbol{f}_{Q_1}}} \cdot (\sqrt{-1})^{1+k_{Q_2}-2k_P} \mathcal{E}_p(\mathrm{Fil}_{\boldsymbol{f}}^{+} \mathbf{V}_{\underline{Q}}) \prod_{\ell \in \Sigma_{\mathrm{exc}}} (1 + \ell^{-1}).\end{aligned}$$

Remark 1.1.

(1) We call $\mathcal{L}_{\boldsymbol{F},a}^{\boldsymbol{f}}$ the primitive p-adic Rankin-Selberg L-function for $\boldsymbol{F}$ with the branch character $\boldsymbol{\omega}^a$. The shape of the interpolation formula exactly complies with the form described in [Coa89, Principal Conjecture] in the sense that it has the correct modified Euler factors at p and ∞. Note that $(\sqrt{-1})^{1+k_{Q_2}-2k_P}$ is the modified Euler factor at the archimedean place. However, due to the multiplication by $\eta_{\boldsymbol{f}}$ a generator of the congruence ideal, the period $\Omega_{\boldsymbol{f}_{\mathbf{Q}}}$ we use here may not agree with the period in [Coa89, Principal Conjecture] up to $\mathbf{Q}^\times$. The conjectural form of the interpolation of p-adic L-functions proposed by Coates and Perrin-Riou is in particular useful in the comparison among different constructions of a p-adic L-function.

(2) The p-adic L-function $D(P, Q, R)$ constructed by Hida in [Hid88a, Theorem 5.1d] in general interpolates critical values of *imprimitive* Rankin-Selberg L-functions. Therefore, after making a suitable change of variables, we should be able to see that $D(P, Q, R)$ is the product of $\mathcal{L}_{\boldsymbol{F},a}^{\boldsymbol{f}}$ and local L-factors at places dividing $\mathrm{lcm}(N_1, N_2)$. We do not verify this here.

(3) As an immediate consequence of the above explicit interpolation formulae combined with the functional equation of the automorphic L-functions for $\mathrm{GL}_2(\mathbf{A}) \times \mathrm{GL}_2(\mathbf{A})$, we obtain the functional equation of the primitive p-adic Rankin-Selberg L-functions. For the precise statement, see Corollary 7.2.

Needless to say, the idea of the construction of $\mathcal{L}_{\boldsymbol{F},a}^{\boldsymbol{f}}$ is entirely due to Hida, which we recall briefly as follows. Roughly speaking, one begins with a three variable p-adic family of Eisenstein series $\boldsymbol{E}_{\psi_1,\psi_2,a}$ of tame level $N := \mathrm{lcm}(N_1, N_2)$. Let $\breve{\boldsymbol{f}}$ be the primitive Hida family associated with $\boldsymbol{f}$ twisted by $\psi_1^{-1}\psi_{1,(p)}$, where $\psi_{1,(p)}$ is the p-primary part of ψ. Viewing $\boldsymbol{E}_{\psi_1,\psi_2,a}$ as a q-expansion with coefficient in $\mathcal{R}$, we define

$\mathscr{L}^{\boldsymbol{f}}_{\boldsymbol{F},a} \in \mathcal{R}$ by

$$\mathscr{L}^{\boldsymbol{f}}_{\boldsymbol{F},a} := \text{ the first Fourier coefficient of } \eta_{\boldsymbol{f}} \cdot 1_{\breve{\boldsymbol{f}}} \operatorname{Tr}_{N/N_1} e(\boldsymbol{g}\boldsymbol{E}_{\psi_1,\psi_2,a}),$$

where e is Hida's ordinary projector, $\operatorname{Tr}_{N/N_1}$ is the trace map from the space of ordinary $\mathcal{R}$-adic modular forms of tame level N to that of tame level N_1 and $1_{\breve{\boldsymbol{f}}}$ is the idempotent in the universal $\mathbf{I}$-adic cuspidal Hecke algebra of tame level N_1. The standard Rankin-Selberg method shows that the specialization of $\mathcal{L}^{\boldsymbol{f}}_{\boldsymbol{F},a}$ at (Q_1, Q_2, P) is a product of the value in the right hand side of the equation in Theorem A and certain local fudge factors Ψ^*_ℓ at some bad primes $\ell \mid N$. In order to get the primitive p-adic L-function, one has to choose $\boldsymbol{E}_{\psi_1,\psi_2,a}$ carefully so that these fudge factors Ψ^*_ℓ are essentially 1. It seems we do not have a simple construction of such a nice Eisenstein series in the most general situation. Nonetheless, we can construct such kind of Eisenstein series easily and show that $\Psi^*_\ell = 1$ with small effort whenever $\boldsymbol{F} = (\boldsymbol{f}, \boldsymbol{g})$ satisfies certain minimal hypothesis (See the hypothesis (M) in §6.2), which practically requires $\boldsymbol{F}$ have the minimal conductor among (prime-to-p) Dirichlet twists. We now take a suitable twist $\boldsymbol{F}' = (\boldsymbol{f} \otimes \lambda, \boldsymbol{g} \otimes \lambda^{-1})$ so that $\boldsymbol{F}'$ is minimal. On the other hand, we have shown the right hand side of Theorem A is invariant under (prime-to-p) Dirichlet twists (i.e. Ω_f "=" $\Omega_{f\otimes\lambda}$) in [Hsi17, Prop. 7.5], so the desired primitive L-function can be defined by

$$\mathcal{L}^{\boldsymbol{f}}_{\boldsymbol{F},a} := \mathscr{L}^{\boldsymbol{f}\otimes\lambda}_{\boldsymbol{F}',a}.$$

This idea was already employed in [Hsi17].

This paper is organized as follows. In §2, we review some standard facts and the notation in modular forms and automorphic forms as well as their well-known connection, and in §3, we recall some ingredients in Hida theory for ordinary Λ-adic forms, in particular, the congruence ideal associated with a primitive Hida family. In §4, we give the construction of the three p-adic family of Eisenstein series following the method of Godement-Jacquet in [Jac72, §19]. In §5, we recall Hida's p-adic Rankin-Selberg method, following the exposition in his blue book [Hid93, Chapter 10] but in the language of automorphic representation theory. We explain the construction of $\mathscr{L}^{\boldsymbol{f}}_{\boldsymbol{F},a}$, and in Proposition 5.3, we express the interpolation of $\mathscr{L}^{\boldsymbol{f}}_{\boldsymbol{F},a}$ at arithmetic points as a product of critical Rankin-Selberg L-values and local zeta integrals $\Psi^{\rm ord}_p(s)$ (modified Euler factor at p) and $\Psi^*_\ell(s)$ (fudge factors). In §6, we evaluate these local zeta integrals explicitly. Finally, in §7, we construct the primitive p-adic L-function and prove the interpolation formula Theorem 7.1.

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§2. Classical modular forms and automorphic forms

In this section, we recall basic definitions and standard facts about classical elliptic modular forms and automorphic forms on $\mathrm{GL}_2(\mathbf{A})$, following the notation in [Hsi17, §2] which we reproduce here for the reader's convenience. The main purpose of this section is to set up the notation and introduce some Hecke operators on the space of automorphic forms which will be frequently used in the construction of p-adic L-functions.

2.1. Notation

Let $\mathbf{A}$ be the ring of adeles of $\mathbf{Q}$. If v is a place of $\mathbf{Q}$, let $\mathbf{Q}_v$ be the completion of $\mathbf{Q}$ with respect to v, and for $a \in \mathbf{A}^\times$, let $a_v \in \mathbf{Q}_v^\times$ be the v-component of a. Denote $|\ |_{\mathbf{Q}_v}$ the absolute value of $\mathbf{Q}_v$ normalized so that $|\ |_{\mathbf{Q}_v}$ is the usual absolute value of $\mathbf{R}$ if $v = \infty$ and $|\ell|_{\mathbf{Q}_\ell} = \ell^{-1}$ if $v = \ell$ is finite. For a prime ℓ, let $\mathrm{ord}_\ell : \mathbf{Q}_\ell \to \mathbf{Z}$ be the valuation normalized so that $\mathrm{ord}_\ell(\ell) = 1$. We shall regard $\mathbf{Q}_\ell$ and $\mathbf{Q}_\ell^\times$ as subgroups of $\mathbf{A}$ and $\mathbf{A}^\times$ in a natural way. Let $|\ |_\mathbf{A}$ be the absolute value on $\mathbf{A}^\times$ given by $|a|_\mathbf{A} = \prod_v |a_v|_{\mathbf{Q}_v}$. Let $\zeta_v(s)$ be the usual local zeta function of $\mathbf{Q}_v$. Namely,

$$\zeta_\infty(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right), \ \zeta_\ell(s) = (1 - \ell^{-s})^{-1}.$$

Define the global zeta function $\zeta_\mathbf{Q}(s)$ of $\mathbf{Q}$ by $\zeta_\mathbf{Q}(s) = \prod_v \zeta_v(s)$.

Let $\psi_\mathbf{Q} : \mathbf{A}/\mathbf{Q} \to \mathbf{C}^\times$ be the additive character with the archimedean component $\psi_\mathbf{R}(x) = e^{2\pi\sqrt{-1}x}$ and let $\psi_{\mathbf{Q}_\ell} : \mathbf{Q}_\ell \to \mathbf{C}^\times$ be the local component of $\psi_\mathbf{Q}$ at ℓ.

If R is a commutative ring and $G = \mathrm{GL}_2(R)$, we denote by ρ the right translation of G on the space of $\mathbf{C}$-valued functions on G: $\rho(g)f(g') = f(g'g)$ and by $\mathbf{1} : G \to \mathbf{C}$ the constant function $\mathbf{1}(g) = 1$. For a function $f : G \to \mathbf{C}$ and a character $\chi : R^\times \to \mathbf{C}^\times$, let $f \otimes \chi : G \to \mathbf{C}$ denote the function $f \otimes \chi(g) = f(g)\chi(\det(g))$.

In the algebraic group GL_2, let B be the Borel subgroup consisting of upper triangular matrices and N be its unipotent radical.

2.2. Hecke characters and Dirichlet characters

If $\omega : \mathbf{Q}^\times\backslash\mathbf{A}^\times \to \overline{\mathbf{Q}}^\times$ is a finite order Hecke character, we denote by $\omega_\ell : \mathbf{Q}_\ell \to \mathbf{C}^\times$ the local component of ω at ℓ. For every Dirichlet character χ, we denote by $\mathfrak{c}(\chi)$ the conductor of χ. Let $\chi_{\mathbf{A}}$ be the *adelization* of χ, the unique finite order Hecke character $\chi_{\mathbf{A}} = \prod \chi_\ell : \mathbf{Q}^\times\backslash\mathbf{A}^\times/\mathbf{R}_+(1+\mathfrak{c}(\chi)\widehat{\mathbf{Z}})^\times \to \mathbf{C}^\times$ of conductor $\mathfrak{c}(\chi)$ such that for any prime $\ell \nmid \mathfrak{c}(\chi)$,

$$\chi_\ell(\ell) = \chi(\ell)^{-1}.$$

For every prime ℓ, write $\mathfrak{c}(\chi) = \ell^e C'$ with $\ell \nmid C'$. Then we can decompose $\chi = \chi_{(\ell)}\chi^{(\ell)}$ into a product of two Dirichlet characters $\chi_{(\ell)}$ and $\chi^{(\ell)}$ of conductors ℓ^e and N' respectively. We call $\chi_{(\ell)}$ the ℓ-primary component of χ. The ℓ-primary component of a finite order Hecke character can be defined likewise.

Throughout this paper, we often identify Dirichlet characters with their adelization whenever no confusion arises.

2.3. Classical modular forms

Let $C^\infty(\mathfrak{H})$ be the space of $\mathbf{C}$-valued smooth functions on the upper half complex plane $\mathfrak{H}$. Let k be any integer. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbf{R})$ act on $z \in \mathfrak{H}$ by $\gamma(z) = \frac{az+b}{cz+d}$, and for $f = f(z) \in C^\infty(\mathfrak{H})$, define

$$f|_k\gamma(z) := f(\gamma(z))(cz+d)^{-k}(\det\gamma)^{\frac{k}{2}}.$$

Recall that the Maass-Shimura differential operators δ_k and ε on $C^\infty(\mathfrak{H})$ are given by

$$\delta_k = \frac{1}{2\pi\sqrt{-1}}(\frac{\partial}{\partial z} + \frac{k}{2\sqrt{-1}y}) \text{ and } \varepsilon = -\frac{1}{2\pi\sqrt{-1}}y^2\frac{\partial}{\partial \bar{z}} \quad (y = \mathrm{Im}(z))$$

(cf. [Hid93, (1a, 1b) page 310]). Let N be a positive integer and $\chi : (\mathbf{Z}/N\mathbf{Z})^\times \to \mathbf{C}^\times$ be a Dirichlet character modulo N. For a non-negative integer m, denote by $\mathcal{N}_k^{[m]}(N,\chi)$ the space of nearly holomorphic modular forms of weight k, level N and character χ, consisting of slowly increasing functions $f \in C^\infty(\mathfrak{H})$ such that $\varepsilon^{m+1}f = 0$ and

$$f|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \chi(d)f \quad \text{for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$$

(cf. [Hid93, page 314]). Let $\mathcal{N}_k(N,\chi) = \bigcup_{r=0}^\infty \mathcal{N}_k^{[m]}(N,\chi)$ (cf. [Hid93, (1a), page 310]). By definition, $\mathcal{N}_k^{[0]}(N,\chi) = \mathcal{M}_k(N,\chi)$ is the space of classical holomorphic modular forms of weight k, level N and character

χ. Denote by $\mathcal{S}_k(N,\chi)$ the space of cusp forms in $\mathcal{M}_k(N,\chi)$. Let $\delta_k^m = \delta_{k+2m-2}\cdots\delta_{k+2}\delta_k$. If $f \in \mathcal{N}_k(N,\chi)$ is a nearly holomorphic modular form of weight k, then $\delta_k^m f \in \mathcal{N}_{k+2m}(N,\chi)$ has weight $k+2m$ ([Hid93, page 312]). For a positive integer d, define

$$V_d f(z) = d\cdot f(dz); \quad \mathbf{U}_d f(z) = \frac{1}{d}\sum_{j=0}^{d-1} f(\frac{z+j}{d}).$$

Recall that the classical Hecke operators T_ℓ for primes $\ell \nmid N$ are given by

$$T_\ell f = \mathbf{U}_\ell f + \chi(\ell)\ell^{k-2}V_\ell f.$$

We say $f \in \mathcal{N}_k(N,\chi)$ is a *Hecke eigenform* if f is an eigenfunction of the all Hecke operators T_ℓ for $\ell \nmid N$ and the operators $\mathbf{U}_\ell$ for $\ell \mid N$.

2.4. Automorphic forms on $\mathrm{GL}_2(\mathbf{A})$

For a positive integer N, define open-compact subgroups of $\mathrm{GL}_2(\widehat{\mathbf{Z}})$ by

$$U_0(N) = \left\{ g \in \mathrm{GL}_2(\widehat{\mathbf{Z}}) \;\middle|\; g \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} (\mathrm{mod}\ N\widehat{\mathbf{Z}}) \right\},$$
$$U_1(N) = \left\{ g \in U_0(N) \;\middle|\; g \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} (\mathrm{mod}\ N\widehat{\mathbf{Z}}) \right\}.$$

Let $\omega : \mathbf{Q}^\times\backslash\mathbf{A}^\times \to \mathbf{C}^\times$ be a finite order Hecke character of level N. We extend ω to a character of $U_0(N)$ defined by $\omega\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \prod_{\ell\mid N}\omega_\ell(d_\ell)$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_0(N)$, where $\omega_\ell : \mathbf{Q}_\ell^\times \to \mathbf{C}^\times$ is the ℓ-component of ω. Denote by $\mathcal{A}(\omega)$ the space of automorphic forms on $\mathrm{GL}_2(\mathbf{A})$ with central character ω. For any integer k, let $\mathcal{A}_k(N,\omega) \subset \mathcal{A}(\omega)$ be the space of automorphic forms on $\mathrm{GL}_2(\mathbf{A})$ of weight k, level N and character ω. In other words, $\mathcal{A}_k(N,\omega)$ consists of automorphic forms $\varphi : \mathrm{GL}_2(\mathbf{A}) \to \mathbf{C}$ such that

$$\varphi(\alpha g\kappa_\theta u_{\mathrm{f}}) = \varphi(g)e^{\sqrt{-1}k\theta}\omega(u_{\mathrm{f}})$$
$$(\alpha \in \mathrm{GL}_2(\mathbf{Q}), \kappa_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, u_{\mathrm{f}} \in U_0(N)).$$

Let $\mathcal{A}_k^0(N,\omega)$ be the space of cusp forms in $\mathcal{A}_k(N,\omega)$.

Next we introduce important local Hecke operators on automorphic forms. At the archimedean place, let $V_\pm : \mathcal{A}_k(N,\omega) \to \mathcal{A}_{k\pm 2}(N,\omega)$

be the normalized weight raising/lowering operator in [JL70, page 165] given by

(2.1)
$$V_{\pm} = \frac{1}{(-8\pi)}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes 1 \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \sqrt{-1}\right) \in \mathrm{Lie}(\mathrm{GL}_2(\mathbf{R})) \otimes_{\mathbf{R}} \mathbf{C}.$$

Define the operator $\mathbf{U}_\ell$ acting on $\varphi \in \mathcal{A}_k(N,\omega)$ by

$$\mathbf{U}_\ell\varphi = \sum_{x \in \mathbf{Z}_\ell/\ell\mathbf{Z}_\ell} \rho\left(\begin{pmatrix} \varpi_\ell & x \\ 0 & 1 \end{pmatrix}\right)\varphi,$$

and the level-raising operator $V_\ell : \mathcal{A}_k(N,\omega) \to \mathcal{A}_k(N\ell,\omega)$ at a finite prime ℓ by

$$V_\ell\varphi(g) := \rho\left(\begin{pmatrix} \varpi_\ell^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right)\varphi.$$

Note that $\mathbf{U}_\ell V_\ell \varphi = \ell\varphi$ and that if $\ell \mid N$, then $\mathbf{U}_\ell \in \mathrm{End}_{\mathbf{C}}\mathcal{A}_k(N,\omega)$. For each prime $\ell \nmid N$, let $T_\ell \in \mathrm{End}_{\mathbf{C}}\mathcal{A}_k(N,\omega)$ be the usual Hecke operator defined by

$$T_\ell = \mathbf{U}_\ell + \omega_\ell(\ell)V_\ell.$$

Let $\mathcal{A}^0(\omega)$ be the space of cusp forms in $\mathcal{A}(\omega)$ and let $\mathcal{A}^0_k(N,\omega) = \mathcal{A}^0(\omega) \cap \mathcal{A}_k(N,\omega)$. Define the the $\mathrm{GL}_2(\mathbf{A})$-equivariant pairing $\langle\, , \rangle$: $\mathcal{A}^0_{-k}(N,\omega) \otimes \mathcal{A}_k(N,\omega^{-1}) \to \mathbf{C}$ by

$$\langle \varphi, \varphi' \rangle = \int_{\mathbf{A}^\times \mathrm{GL}_2(\mathbf{Q}) \backslash \mathrm{GL}_2(\mathbf{A})} \varphi(g)\varphi'(g)\mathrm{d}^\tau g, \tag{2.2}$$

where $\mathrm{d}^\tau g$ is the Tamagawa measure of $\mathrm{PGL}_2(\mathbf{A})$. Note that we have $\langle T_\ell\varphi, \varphi' \rangle = \langle \varphi, T_\ell\varphi' \rangle$ for the Hecke operator T_ℓ with $\ell \nmid N$.

2.5.

With every nearly holomorphic modular form $f \in \mathcal{N}_k(N,\chi)$, we associate a unique automorphic form $\Phi(f) \in \mathcal{A}_k(N,\chi_{\mathbf{A}}^{-1})$ defined by the formula

$$\Phi(f)(\alpha g_\infty u) := (f|_k g_\infty)(\sqrt{-1}) \cdot \chi_{\mathbf{A}}^{-1}(u) \tag{2.3}$$

for $\alpha \in \mathrm{GL}_2(\mathbf{Q})$, $g_\infty \in \mathrm{GL}_2^+(\mathbf{R})$ and $u \in U_0(N)$ (cf. [Cas73, §3]). Conversely, we can recover the form f from $\Phi(f)$ by

$$f(x + \sqrt{-1}y) = y^{-k/2}\Phi(f)\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right). \tag{2.4}$$

We call $\Phi(f)$ the *adelic lift* of f.

The weight raising/lowering operators are the adelic avatar of the Maass-Shimura differential operators δ_k^m and ε on the space of automorphic forms. A direct computation shows that the map Φ from the space of modular forms to the space of automorphic forms is *equivariant* for the Hecke action in the sense that

$$\Phi(\delta_k^m f) = V_+^m \Phi(f), \quad \Phi(\varepsilon f) = V_- \Phi(f), \tag{2.5}$$

and for a finite prime ℓ

$$\Phi(T_\ell f) = \ell^{k/2-1} T_\ell \Phi(f), \quad \Phi(\mathbf{U}_\ell f) = \ell^{k/2-1} \mathbf{U}_\ell \Phi(f). \tag{2.6}$$

In particular, f is holomorphic if and only if $V_- \Phi(f) = 0$.

2.6. Preliminaries on irreducible representations of $\mathrm{GL}_2(\mathbf{Q}_v)$

2.6.1. *Measures* We shall normalize the Haar measures on $\mathbf{Q}_v$ and $\mathbf{Q}_v^\times$ as follows. If $v = \infty$, $\mathrm{d}x$ denotes the usual Lebesgue measure on $\mathbf{R}$ and the measure $\mathrm{d}^\times x$ on $\mathbf{R}^\times$ is $|x|_{\mathbf{R}}^{-1}\mathrm{d}x$. If $v = \ell$ is a finite prime, denote by $\mathrm{d}x$ the Haar measure on $\mathbf{Q}_\ell$ with $\mathrm{vol}(\mathbf{Z}_\ell, \mathrm{d}x) = 1$ and by $\mathrm{d}^\times x$ the Haar measure on $\mathbf{Q}_\ell^\times$ with $\mathrm{vol}(\mathbf{Z}_\ell^\times, \mathrm{d}^\times x) = 1$. Define the compact subgroup $\mathbf{K}_v$ of $\mathrm{GL}_2(\mathbf{Q}_v)$ by $\mathbf{K}_v = \mathrm{O}(2, \mathbf{R})$ if $v = \infty$ and $\mathbf{K}_v = \mathrm{GL}_2(\mathbf{Z}_v)$ if v is finite. Let $\mathrm{d}u_v$ be the Haar measure on $\mathbf{K}_v$ so that $\mathrm{vol}(\mathbf{K}_v, \mathrm{d}u_v) = 1$. Let $\mathrm{d}g_v$ be the Haar measure on $\mathrm{PGL}_2(\mathbf{Q}_v)$ given by $\mathrm{d}g_v = |y_v|_{\mathbf{Q}_v}^{-1} \mathrm{d}x_v \mathrm{d}^\times y_v \mathrm{d}u_v$ for $g_v = \begin{pmatrix} y_v & x_v \\ 0 & 1 \end{pmatrix} u_v$ with $y_v \in \mathbf{Q}_v^\times$, $x_v \in \mathbf{Q}_v$ and $u_v \in \mathbf{K}_v$.

2.6.2. *Representations* Denote by $\chi \boxplus \upsilon$ the irreducible principal series representation of $\mathrm{GL}_2(\mathbf{Q}_v)$ attached to two characters $\chi, \upsilon : \mathbf{Q}_v^\times \to \mathbf{C}^\times$ such that $\chi\upsilon^{-1} \neq |\ |_{\mathbf{Q}_v}^{\pm}$. If $v = \infty$ is the archimedean place and $k \geq 1$ is an integer, denote by $\mathcal{D}_0(k)$ the discrete series of lowest weight k if $k \geq 2$ or the limit of discrete series if $k = 1$ with central character sgn^k (the k-the power of the sign function). If v is finite, denote by St the Steinberg representation and by χSt the special representation $\mathrm{St} \otimes \chi \circ \det$.

2.6.3. *L-functions and ε-factors* For a character $\chi : \mathbf{Q}_v^\times \to \mathbf{C}^\times$, let $L(s, \chi)$ be the complex L-function and $\varepsilon(s, \chi) := \varepsilon(s, \chi, \psi_{\mathbf{Q}_v})$ be the ε-factor (cf. [Sch02, Section 1.1]). Define the γ-factor

$$\gamma(s, \chi) := \varepsilon(s, \chi) \cdot \frac{L(1 - s, \chi^{-1})}{L(s, \chi)}. \tag{2.7}$$

If v is a finite prime, denote by $c(\chi)$ the exponent of the conductor of χ, If π is an irreducible admissible generic representation of $\mathrm{GL}_2(\mathbf{Q}_v)$, denote by $L(s,\pi)$ the L-function and by $\varepsilon(s,\pi) := \varepsilon(s,\pi,\psi_{\mathbf{Q}_v})$ the ε-factor defined in [JL70, Theorem 2.18]. Let $\pi^\vee$ be the contragredient representation of π. Define the gamma factor

$$\gamma(s,\pi) = \varepsilon(s,\pi) \cdot \frac{L(1-s,\pi^\vee)}{L(s,\pi)}.$$

If v is a finite prime, we let $c(\pi)$ be the exponent of the conductor of π.

2.6.4. *Whittaker models and the normalized Whittaker newforms* Every admissible irreducible infinite dimensional representation π of $\mathrm{GL}_2(\mathbf{Q}_v)$ admits a realization of the Whttaker model $\mathcal{W}(\pi) = \mathcal{W}(\pi, \psi_{\mathbf{Q}_v})$ associated with the additive character $\psi_{\mathbf{Q}_v}$. Recall that $\mathcal{W}(\pi)$ is a subspace of smooth functions $W : \mathrm{GL}_2(\mathbf{Q}_v) \to \mathbf{C}$ such that

- $W\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) = \psi_{\mathbf{Q}_v}(x) W(g)$ for all $x \in \mathbf{Q}_v$,
- if $v = \infty$ is the archimedean place, there exists an integer M such that

$$W\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) = O(|a|_{\mathbf{R}}^M) \text{ as } |a|_{\mathbf{R}} \to \infty.$$

The group $\mathrm{GL}_2(\mathbf{Q}_v)$ (or the Hecke algebra of $\mathrm{GL}_2(\mathbf{Q}_v)$) acts on $\mathcal{W}(\pi)$ via the right translation ρ. We introduce the (normalized) *local Whittaker newform* W_π in $\mathcal{W}(\pi)$ in the following cases. If $v = \infty$ and $\pi = \mathcal{D}_0(k)$, then the local Whittaker newform $W_\pi \in \mathcal{W}(\pi)$ is defined by

$$\begin{aligned} & W_\pi\left(z\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right) \\ =& \mathbb{I}_{\mathbf{R}_+}(y) \cdot y^{k/2} e^{-2\pi y} \cdot \mathrm{sgn}(z)^k \psi_{\mathbf{R}}(x) e^{\sqrt{-1}k\theta} \\ & (y, z \in \mathbf{R}^\times,\ x, \theta \in \mathbf{R}). \end{aligned} \tag{2.8}$$

Here $\mathbb{I}_{\mathbf{R}_+}(a)$ denotes the characteristic function of the set of positive real numbers. If $v = \ell$ is a finite prime, then the (normalized) local Whittaker newform W_π is the unique function in $\mathcal{W}(\pi)^{\mathrm{new}}$ such that $W_\pi(1) = 1$. The explicit formula for $W_\pi(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix})$ is well-known (See [Sch02, page 21] or [Sah16, Section 2.2] for example).

2.7. p-stabilized newforms

Let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A})$ and let $\mathcal{A}(\pi)$ be the π-isotypic part in the space of automorphic forms on

$\mathrm{GL}_2(\mathbf{A})$. For $\varphi \in \mathcal{A}(\pi)$, the Whittaker function of φ (with respect to the additive character $\psi_{\mathbf{Q}} : \mathbf{A}/\mathbf{Q} \to \mathbf{C}^\times$) is given by

$$W_\varphi(g) = \int_{\mathbf{A}/\mathbf{Q}} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \psi_{\mathbf{Q}}(-x)\mathrm{d}x \quad (g \in \mathrm{GL}_2(\mathbf{A})), \tag{2.9}$$

where $\mathrm{d}x$ is the Haar measure with $\mathrm{vol}(\mathbf{A}/\mathbf{Q}, \mathrm{d}x) = 1$. We have the Fourier expansion:

$$\varphi(g) = \sum_{\alpha \in \mathbf{Q}^\times} W_\varphi\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g\right)$$

(cf. [Bum98, Theorem 3.5.5]). Let $f(q) = \sum_n \mathbf{a}(n, f) q^n \in \mathcal{S}_k(N, \chi)$ be a normalized Hecke eigenform. Denote by $\pi_f = \otimes'_v \pi_{f,v}$ the cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A})$ generated by the adelic lift $\Phi(f)$ of f. Then π_f is irreducible and unitary with the central character χ^{-1}. If f is newform, then the conductor of π_f is N, the adelic lift $\Phi(f)$ is the normalized new vector in $\mathcal{A}(\pi_f)$ and the Mellin transform

$$Z(s, \Phi(f)) = \int_{\mathbf{A}^\times/\mathbf{Q}^\times} \Phi(f)\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) |y|_{\mathbf{A}}^{s-1/2} \mathrm{d}^\times y = L(s, \pi_f)$$

is the automorphic L-function of π_f. Here $\mathrm{d}^\times y$ is the product measure $\prod_v \mathrm{d}^\times y_v$.

Definition 2.1 (p-stabilized newform)**.** Let p be a prime and fix an isomorphism $\iota_p : \mathbf{C} \simeq \overline{\mathbf{Q}}_p$. We say that a normalized Hecke eigenform $f = \sum_n \mathbf{a}(n, f) q^n \in \mathcal{S}_k(Np, \chi)$ is a (ordinary) *p-stabilized newform* (with respoect to ι_p) if f is a new outside p and the eigenvalue of $\mathbf{U}_p$, i.e. the p-th Fourier coefficient $\iota_p(\mathbf{a}(p, f))$, is a p-adic unit. The prime-to-p part of the conductor of f is called *the tame conductor* of f.

By the multiplicity one for new and ordinary vectors, the Whittaker function of $\Phi(f)$ is a product of local Whittaker functions in $\mathcal{W}(\pi_{f,v})$. To be precise,

$$W_{\Phi(f)}(g) = W^{\mathrm{ord}}_{\pi_{f,p}}(g_v) \prod_{v \neq p} W_{\pi_{f,v}}(g_v) \quad (g = (g_v) \in \mathrm{GL}_2(\mathbf{A})).$$

Here $W_{\pi_{f,v}}$ is the normalized local Whittaker newform of $\pi_{f,v}$ and $W^{\mathrm{ord}}_{\pi_{f,p}}$ is the ordinary Whittaker function characterized by

$$W^{\mathrm{ord}}_{\pi_{f,p}}(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}) = \alpha_f |\cdot|^{\frac{1}{2}}_{\mathbf{Q}_p}(a) \cdot \mathbb{I}_{\mathbf{Z}_p}(a) \text{ for } a \in \mathbf{Q}_p^\times, \tag{2.10}$$

where $\alpha_f : \mathbf{Q}_p^\times \to \mathbf{C}^\times$ is the unramified character with $\alpha_f(p) = \mathbf{a}(p, f) \cdot p^{(1-k)/2}$ (See [Hsi17, Corollary 2.3, Remark 2.5]).

§3. Λ-adic modular forms and Hida families

3.1. Ordinary Λ-adic modular forms

Let $p > 2$ be a prime and let $\mathcal{O}$ be the ring of integers of a finite extension of $\mathbf{Q}_p$. Let $\mathbf{I}$ be a normal domain finite flat over $\Lambda = \mathcal{O}[\![1+p\mathbf{Z}_p]\!]$. A point $Q \in \mathrm{Spec}\mathbf{I}(\overline{\mathbf{Q}}_p)$, a ring homomorphism $Q : \mathbf{I} \to \overline{\mathbf{Q}}_p$, is said to be locally algebraic if $Q|_{1+p\mathbf{Z}_p}$ is a locally algebraic character in the sense that $Q(z) = z^{k_Q}\epsilon_Q(z)$ with k_Q an integer and $\epsilon_Q(z) \in \mu_{p^\infty}$. We shall call k_Q the *weight* of Q and ϵ_Q the *finite part* of Q. Let $\mathfrak{X}_{\mathbf{I}}$ be the set of locally algebraic points $Q \in \mathrm{Spec}\mathbf{I}(\overline{\mathbf{Q}}_p)$. A point $Q \in \mathfrak{X}_{\mathbf{I}}$ is called *arithmetic* if the weight $k_Q \geq 2$ and let $\mathfrak{X}_{\mathbf{I}}^+$ be the set of arithmetic points. Let $\wp_Q = \mathrm{Ker}Q$ be the prime ideal of $\mathbf{I}$ corresponding to Q and $\mathcal{O}(Q)$ be the image of $\mathbf{I}$ under Q.

Fix an isomorphism $\iota_p : \mathbf{C}_p \simeq \mathbf{C}$ once and for all. Denote by $\boldsymbol{\omega} : (\mathbf{Z}/p\mathbf{Z})^\times \to \mu_{p-1}$ the p-adic Teichmüller character. Let N be a positive integer prime to p and let $\chi : (\mathbf{Z}/Np\mathbf{Z})^\times \to \mathcal{O}^\times$ be a Dirichlet character modulo Np with value in $\mathcal{O}$. Denote by $\mathbf{S}(N, \chi, \mathbf{I})$ the space of $\mathbf{I}$-adic cusp forms of tame level N and (even) branch character χ, consisting of formal power series $\boldsymbol{f}(q) = \sum_{n\geq 1} \mathbf{a}(n, \boldsymbol{f})q^n \in \mathbf{I}[\![q]\!]$ with the following property: there exists an integer $a_{\boldsymbol{f}}$ such that for arithemtic points $Q \in \mathfrak{X}_{\mathbf{I}}^+$ with $k_Q \geq a_{\boldsymbol{f}}$, the specialization $\boldsymbol{f}_Q(q)$ is the q-expansion of a cusp form $\boldsymbol{f}_Q \in \mathcal{S}_{k_Q}(Np^e, \chi\boldsymbol{\omega}^{-k_Q}\epsilon_Q)$. We call the character χ is the *branch character* of $\boldsymbol{f}$.

The space $\mathbf{S}(N, \chi, \mathbf{I})$ is equipped with the action of the usual Hecke operators T_ℓ for $\ell \nmid Np$ as in [Wil88, page 537] and the operators $\mathbf{U}_\ell$ for $\ell \mid pN$ given by $\mathbf{U}_\ell(\sum_n \mathbf{a}(n, \boldsymbol{f})q^n) = \sum_n \mathbf{a}(n\ell, \boldsymbol{f})q^n$. Recall that Hida's ordinary projector e is defined by

$$e := \lim_{n\to\infty} \mathbf{U}_p^{n!}.$$

This ordinary projector e has a well-defined action on the space of classical modular forms preserving the cuspidal part as well as on the space $\mathbf{S}(N, \chi, \mathbf{I})$ of $\mathbf{I}$-adic cusp forms (cf. [Wil88, page 537 and Proposition 1.2.1]). The space $e\mathbf{S}(N, \chi, \mathbf{I})$ is called the space of ordinary $\mathbf{I}$-adic forms defined over $\mathbf{I}$. A key result in Hida's theory of ordinary $\mathbf{I}$-adic cusp forms is that if $\boldsymbol{f} \in e\mathbf{S}(N, \chi, \mathbf{I})$, then for *every* arithmetic points $Q \in \mathfrak{X}_{\mathbf{I}}$, we have $\boldsymbol{f}_Q \in e\mathcal{S}_{k_Q}(Np^e, \chi\boldsymbol{\omega}^{-k_Q}\epsilon_Q)$. We say $\boldsymbol{f} \in e\mathbf{S}(N, \chi, \mathbf{I})$ is a *primitive Hida family* if for every arithmetic points $Q \in \mathfrak{X}_{\mathbf{I}}$, $\boldsymbol{f}_Q$ is a p-stabilized cuspidal newform of tame conductor N. Let $\mathfrak{X}_{\mathbf{I}}^{\mathrm{cls}}$ be the set of classical points (for $\boldsymbol{f}$) given by the subset of $Q \in \mathfrak{X}_{\mathbf{I}}$ such that $k_Q \geq 1$ and $\boldsymbol{f}_Q$ is the q-expansion of a classical modular form. Note that $\mathfrak{X}_{\mathbf{I}}^{\mathrm{cls}}$ contains the set of arithmetic points $\mathfrak{X}_{\mathbf{I}}^{+}$ but may be strictly larger than $\mathfrak{X}_{\mathbf{I}}^{+}$ as we allow the possibility of the points of weight one.

3.2. Galois representation attached to Hida families

Let $\langle\cdot\rangle : \mathbf{Z}_p^{\times} \to 1 + p\mathbf{Z}_p$ be character defined by $\langle x\rangle = x\boldsymbol{\omega}^{-1}(x)$ and write $z \mapsto [z]_{\Lambda}$ for the inclusion of group-like elements $1 + p\mathbf{Z}_p \to \mathcal{O}[\![1 + p\mathbf{Z}_p]\!]^{\times} = \Lambda^{\times}$. For $z \in \mathbf{Z}_p^{\times}$, denote by $\langle z\rangle_{\mathbf{I}} \in \mathbf{I}^{\times}$ the image of $[\langle z\rangle]_{\Lambda}$ in $\mathbf{I}$ under the structure morphism $\Lambda \to \mathbf{I}$. By definition, $Q(\langle z\rangle_{\mathbf{I}}) = Q|_{1+p\mathbf{Z}_p}(\langle z\rangle)$ for $Q \in \mathfrak{X}_{\mathbf{I}}$. Let $\varepsilon_{\rm cyc} : G_{\mathbf{Q}} \to \mathbf{Z}_p^{\times}$ be the p-adic cyclotomic character and let $\langle\varepsilon_{\rm cyc}\rangle_{\mathbf{I}} : G_{\mathbf{Q}} \to \mathbf{I}^{\times}$ be the character $\langle\varepsilon_{\rm cyc}\rangle_{\mathbf{I}}(\sigma) = \langle\varepsilon_{\rm cyc}(\sigma)\rangle_{\mathbf{I}}$. For each Dirichlet chatacter χ, we define $\chi_{\mathbf{I}} : G_{\mathbf{Q}} \to \mathbf{I}^{\times}$ by $\chi_{\mathbf{I}} := \boldsymbol{\sigma}_{\chi}\langle\varepsilon_{\rm cyc}\rangle^{-2}\langle\varepsilon_{\rm cyc}\rangle_{\mathbf{I}}$, where $\boldsymbol{\sigma}_{\chi}$ is the Galois character which sends the geometric Frobenious element ${\rm Frob}_{\ell}$ at ℓ to $\chi(\ell)^{-1}$.

If $\boldsymbol{f} \in e\mathbf{S}(N,\chi,\mathbf{I})$ is primitive Hida family of tame conductor N, we let $\rho_{\boldsymbol{f}} : G_{\mathbf{Q}} \to {\rm GL}_2({\rm Frac}\mathbf{I})$ be the $\mathbf{I}$-adic Galois representation attached to $\boldsymbol{f}$ which is unramified outside pN and characterized by

$$\mathrm{Tr}(\rho_{\boldsymbol{f}}({\rm Frob}_{\ell})) = \mathbf{a}(\ell,\boldsymbol{f}); \quad \det\rho_{\boldsymbol{f}}({\rm Frob}_{\ell}) = \chi(\ell)\langle\ell\rangle_{\mathbf{I}}\ell^{-1} \quad (\ell \nmid pN).$$

Note that $\det\rho_{\boldsymbol{f}} = \chi_{\mathbf{I}}^{-1}\cdot\varepsilon_{\rm cyc}^{-1}$. The description of the restriction of $\rho_{\boldsymbol{f}}$ to the local decomposition group $G_{\mathbf{Q}_{\ell}}$ is well-understood. For $\ell = p$, according to [Wil88, Theorem 2.2.1], we have

$$\rho_{\boldsymbol{f}}|_{G_{\mathbf{Q}_p}} \sim \begin{pmatrix} \alpha_p & * \\ 0 & \alpha_p^{-1}\chi_{\mathbf{I}}^{-1}\varepsilon_{\rm cyc}^{-1}\end{pmatrix},$$

where $\alpha_p : G_{\mathbf{Q}_p} \to \mathbf{I}^{\times}$ is the unramified character with $\alpha_p({\rm Frob}_p) = \mathbf{a}(p,\boldsymbol{f})$.[2]

3.3. Hecke algebras and congruence numbers

If N is a positive and χ is a Dirichlet character modulo N, we let $\mathbb{T}_k(N,\chi)$ be the $\mathcal{O}$-subalgebra in ${\rm End}_{\mathbf{C}}e\mathcal{S}_k(N,\chi)$ generated over $\mathcal{O}$ by the Hecke operators T_{ℓ} for $\ell \nmid Np$ and the operators $\mathbf{U}_{\ell}$ for $\ell \mid Np$. Suppose that N is *prime to p*. Let $\Delta = (\mathbf{Z}/Np\mathbf{Z})^{\times}$ and $\widehat{\Delta}$ be the group of Dirichlet characters modulo Np. Enlarging $\mathcal{O}$ if necessary, we assume that every $\chi \in \widehat{\Delta}$ takes value in $\mathcal{O}^{\times}$. We are going to consider the Hecke algebra $\mathbf{T}(N,\mathbf{I})$ acting on the space of ordinary Λ-adic cusp forms of tame level $\Gamma_1(N)$ defined by

$$\mathbf{S}(N,\mathbf{I})^{\rm ord} := \bigoplus_{\chi\in\widehat{\Delta}} e\mathbf{S}(N,\chi,\mathbf{I}).$$

[2]Our representation $\rho_{\boldsymbol{f}}$ is the dual of $\rho_{\mathscr{F}}$ considered in [Wil88].

In addition to the action of Hecke operators, denote by σ_d the usual diamond operator for $d \in \Delta$ acting on $\mathbf{S}(N,\mathbf{I})^{\mathrm{ord}}$ by $\sigma_d(\boldsymbol{f})_{\chi\in\widehat{\Delta}} := (\chi(d)\boldsymbol{f})_{\chi\in\widehat{\Delta}}$. Then the ordinary $\mathbf{I}$-adic cuspidal Hecke algebra $\mathbf{T}(N,\mathbf{I})$ is defined to be the $\mathbf{I}$-subalgebra of $\mathrm{End}_{\mathbf{I}}\mathbf{S}(N,\mathbf{I})^{\mathrm{ord}}$ generated over $\mathbf{I}$ by T_ℓ for $\ell \mid Np$, $\mathbf{U}_\ell$ for $\ell \mid Np$ and the diamond operators σ_d for $d \in \Delta$. Let $Q \in \mathfrak{X}_{\mathbf{I}}^{\mathrm{ari}}$ be an arithmetic point. Every $t \in \mathbf{T}(N,\mathbf{I})$ commutes with the specialization: $(t \cdot \boldsymbol{f})_Q = t \cdot \boldsymbol{f}_Q$. For $\chi \in \widehat{\Delta}_{Np}$, let $\wp_{Q,\chi}$ be the ideal of $\mathbf{T}(N,\mathbf{I})$ generated by $\wp_Q$ and $\{\sigma_d - \chi(d)\}_{d\in\Delta}$. A classical result [Hid88c, Theorem 3.4] in Hida theory asserts that

$$\mathbf{T}(N,\mathbf{I})/\wp_{Q,\chi} \simeq \mathbb{T}_{k_Q}(Np^e, \chi\boldsymbol{\omega}^{-k_Q}\epsilon_Q) \otimes_{\mathcal{O}} \mathcal{O}(Q).$$

Let $\boldsymbol{f} \in e\mathbf{S}(N,\chi,\mathbf{I})$ be a primitive Hida family of tame level N and character χ and let $\lambda_{\boldsymbol{f}} : \mathbf{T}(N,\mathbf{I}) \to \mathbf{I}$ be the corresponding homomorphism defined by $\lambda_{\boldsymbol{f}}(T_\ell) = \mathbf{a}(\ell,\boldsymbol{f})$ for $\ell \nmid Np$, $\lambda_{\boldsymbol{f}}(\mathbf{U}_\ell) = \mathbf{a}(\ell,\boldsymbol{f})$ for $\ell \mid Np$ and $\lambda_{\boldsymbol{f}}(\sigma_d) = \chi(d)$ for $d \in \Delta$. Let $\mathfrak{m}_{\boldsymbol{f}}$ be the maximal of $\mathbf{T}(N,\mathbf{I})$ containing $\mathrm{Ker}\,\lambda_{\boldsymbol{f}}$ and let $\mathbf{T}_{\mathfrak{m}_{\boldsymbol{f}}}$ be the localization of $\mathbf{T}(N,\mathbf{I})$ at $\mathfrak{m}_{\boldsymbol{f}}$. It is the local ring of $\mathbf{T}(N,\mathbf{I})$ through which $\lambda_{\boldsymbol{f}}$ factors. Recall that the congruence ideal $C(\boldsymbol{f})$ of the morphism $\lambda_{\boldsymbol{f}} : \mathbf{T}_{\mathfrak{m}_{\boldsymbol{f}}} \to \mathbf{I}$ is defined by

$$C(\boldsymbol{f}) := \lambda_{\boldsymbol{f}}(\mathrm{Ann}_{\mathbf{T}_{\mathfrak{m}_{\boldsymbol{f}}}}(\mathrm{Ker}\,\lambda_{\boldsymbol{f}})) \subset \mathbf{I}.$$

The Hecke algebra $\mathbf{T}_{\mathfrak{m}_{\boldsymbol{f}}}$ is a local finite flat Λ-alegba and there is an algebra direct sum decomposition

$$\lambda : \mathbf{T}_{\mathfrak{m}_{\boldsymbol{f}}} \otimes_{\mathbf{I}} \mathrm{Frac}\,\mathbf{I} \simeq \mathrm{Frac}\,\mathbf{I} \oplus \mathscr{B},\ t \mapsto \lambda(t) = (\lambda_{\boldsymbol{f}}(t), \lambda_{\mathscr{B}}(t)), \tag{3.1}$$

where $\mathscr{B}$ is a finite dimensional $(\mathrm{Frac}\,\mathbf{I})$-algebra ([Hid88c, Corollary 3.7]). By definition we have

$$C(\boldsymbol{f}) = \lambda_{\boldsymbol{f}}(\mathbf{T}_{\mathfrak{m}_{\boldsymbol{f}}} \cap \lambda^{-1}(\mathrm{Frac}\,\mathbf{I} \oplus \{0\})).$$

Now we impose the following

Hypothesis (CR)**.** The residual Galois representation $\overline{\rho}_{\boldsymbol{f}}$ of $\rho_{\boldsymbol{f}}$ is absolutely irreducible and p-distiniguished.

Under the above hypothesis, $\mathbf{T}_{\mathfrak{m}_{\boldsymbol{f}}}$ is Gorenstein by [Wil95, Corollay 2, page 482], and with this Gorenstein property of $\mathbf{T}_{\mathfrak{m}_{\boldsymbol{f}}}$, Hida in [Hid88b] proved that the congruence ideal $C(\boldsymbol{f})$ is generated by a nonzero element $\eta_{\boldsymbol{f}} \in \mathbf{I}$, called the congruence number for $\boldsymbol{f}$. Let $1^*_{\boldsymbol{f}}$ be the unique element in $\mathbf{T}_{\mathfrak{m}_{\boldsymbol{f}}} \cap \lambda^{-1}(\mathrm{Frac}\,\mathbf{I} \oplus \{0\})$ such that $\lambda_{\boldsymbol{f}}(1^*_{\boldsymbol{f}}) = \eta_{\boldsymbol{f}}$. Then $1_{\boldsymbol{f}} := \eta_{\boldsymbol{f}}^{-1} 1^*_{\boldsymbol{f}}$ is the idempotent in $\mathbf{T}_{\mathfrak{m}_{\boldsymbol{f}}} \otimes_{\mathbf{I}} \mathrm{Frac}\,\mathbf{I}$ corresponding to the direct summand $\mathrm{Frac}\,\mathbf{I}$ of (3.1) and $1_{\boldsymbol{f}}$ does not depend on the choice of a generator of $C(\boldsymbol{f})$. Moreover, for each arithmetic point Q, it is also shown by Hida that the specialization $\eta_{\boldsymbol{f}}(Q) \in \mathcal{O}(Q)$ is the congruence number for $\boldsymbol{f}_Q$ and

$$1_f := \eta_{\boldsymbol{f}}^{-1} 1_{\boldsymbol{f}}^* (\mathrm{mod}\ \wp_{\chi,Q}) \in \mathbb{T}^{\mathrm{ord}}_{k_Q}(Np^r, \chi\boldsymbol{\omega}^{-k_Q}\epsilon_Q) \otimes_{\mathcal{O}} \mathrm{Frac}\,\mathcal{O}(Q)$$

is the idempotent with $\lambda_f(1_f) = 1$.

§4. A three variable p-adic family of Eisenstein series

4.1. Eisenstein series

We recall the construction of Eisenstein series described in [Jac72, §19]. Let (μ_1, μ_2) be a pair of Dirichlet characters. We shall identify (μ_1, μ_2) with their adelizations as described in §2.2. Let $\mathcal{B}(\mu_1, \mu_2, s)$ denote the space consisting of smooth and $\mathrm{SO}(2, \mathbf{R})$-finite functions $f\colon \mathrm{GL}_2(\mathbf{A}) \to \mathbf{C}$ such that

$$f(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}) = \mu_1(a)\mu_2(d) \left|\frac{a}{d}\right|_{\mathbf{A}}^{s+\frac{1}{2}} f(g).$$

For each place v and a positive integer n, denote by $\mathcal{S}(\mathbf{Q}_v^n)$ the space of Bruhat-Schwartz functions on $\mathbf{Q}_v^n$. For every Bruhat-Schwartz function $\Phi = \otimes_v \Phi_v \in \mathcal{S}(\mathbf{A}^2) = \otimes'_v \mathcal{S}(\mathbf{Q}_v^2)$, define the Godement section $f_{\mu_1,\mu_2,\Phi,s} = \otimes_v f_{\mu_{1,v},\mu_{2,v},\Phi_v,s} : \mathrm{GL}_2(\mathbf{A}) \to \mathbf{C}$ given by

$$\begin{aligned} &f_{\mu_{1,v},\mu_{2,v},\Phi_v,s}(g_v) \\ &:= \mu_{1,v}|\cdot|^{s+\frac{1}{2}}(\det g_v) \int_{\mathbf{Q}_v^\times} \Phi_v((0,t_v)g_v)\mu_{1,v}\mu_{2,v}^{-1}|\cdot|_{\mathbf{Q}_v}^{2s+1}(t_v)\mathrm{d}^\times t_v. \end{aligned} \tag{4.1}$$

Then $f_{\mu_1,\mu_2,\Phi,s}$ belongs to $\mathcal{B}(\mu_1, \mu_2, s)$. The Eisenstein seres associated to the section $f_{\mu_1,\mu_2,\Phi,s}$ is defined by the formal series

$$E_{\mathbf{A}}(g, f_{\mu_1,\mu_2,\Phi,s}) := \sum_{\gamma \in B\backslash \mathrm{GL}_2(\mathbf{Q})} f_{\mu_1,\mu_2,\Phi,s}(\gamma g), \quad (g \in \mathrm{GL}_2(\mathbf{A}), s \in \mathbf{C}).$$

The above series converges absolutely for $\mathrm{Re}(s) \gg 0$ and has meromorphic continuation to $s \in \mathbf{C}$. Define the Whittaker function of $f_{\mu_1,\mu_2,\Phi,s}$ by

$$W(g, f_{\mu_1,\mu_2,\Phi,s}) = \prod_v W(g_v, f_{\mu_{1,v},\mu_{2,v},\Phi_v,s}), \quad (g = (g_v) \in \mathrm{GL}_2(\mathbf{A})),$$

where

$$W(g_v, f_{\mu_{1,v},\mu_{2,v},\Phi_v,s}) = \int_{\mathbf{Q}_v} f_{\mu_{1,v},\mu_{2,v},\Phi_v,s}(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & x_v \\ 0 & 1 \end{pmatrix} g_v)\psi_{\mathbf{Q}_v}(-x_v)\mathrm{d}x_v.$$

The Eisenstein series $E_{\mathbf{A}}(g, f_{\mu_1,\mu_2,\Phi,s})$ admits the Fourier expansion

$$\begin{aligned} &E_{\mathbf{A}}(g, f_{\mu_1,\mu_2,\Phi,s}) \\ &= f_{\mu_1,\mu_2,\Phi,s}(g) + f_{\mu_2,\mu_1,\widehat{\Phi},-s}(g) + \sum_{\alpha\in\mathbf{Q}^\times} W(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g, f_{\mu_1,\mu_2,\Phi,s}), \end{aligned} \tag{4.2}$$

where $\widehat{\Phi} := \otimes\widehat{\Phi}_v$ is the symplectic Fourier transform defined by

$$\widehat{\Phi}_v(x,y) = \int_{\mathbf{Q}_v}\int_{\mathbf{Q}_v} \Phi(s,t)\psi_{\mathbf{Q}_v}(sy - tx)\mathrm{d}s\mathrm{d}t.$$

4.2. The Eisenstein series $E_k^{\pm}(\mu_1, \mu_2, C)$

If v is place of $\mathbf{Q}$ and $\phi \in \mathcal{S}(\mathbf{Q}_v)$, the usual Fourier transform $\widehat{\phi} \in \mathcal{S}(\mathbf{Q}_v)$ is defined by

$$\widehat{\phi}(x) := \int_{\mathbf{Q}_v} \phi(y)\psi_{\mathbf{Q}_v}(yx)\mathrm{d}y.$$

Note that if $\Phi = \phi_1 \otimes \phi_2 \in \mathcal{S}(\mathbf{Q}_v^2)$, then $\widehat{\Phi}(x,y) = \widehat{\phi}_2(-x)\widehat{\phi}_1(y)$. If $v = \ell$ is a finite place and $\mu : \mathbf{Q}_\ell^\times \to \mathbf{C}^\times$ is a character, we define $\phi_\mu \in \mathcal{S}(\mathbf{Q}_\ell)$ by

$$\phi_\mu(x) := \mathbb{I}_{\mathbf{Z}_\ell^\times}(x)\mu(x).$$

It is easy to verify that

$$\widehat{\phi}_\mu(x) = \chi^{-1}(x)\mathbb{I}_{\varpi^{-c(\mu)}\mathbf{Z}_\ell^\times}(x) \cdot \varepsilon(1, \mu^{-1}).$$

Now we fix a pair (C_1, C_2) of two positive integers such that $\gcd(C_1, C_2) = 1$ and $p \mid C_1C_2$. Let $C = C_1C_2$ and let k be a positive integer such that $\mu_1\mu_2^{-1}(-1) = (-1)^k$. We recall a construction of certain classical Eisenstein series $E_k^{\pm}(\mu_1, \mu_2)$ of weight k, level $\Gamma_1(C_1C_2)$ and nebentypes $\mu_1^{-1}\mu_2^{-1}$ by using suitable Godement sections as above. In the remainder of this section, we assume the following conditions:

- μ_1 is unramified outside p,
- $\mathfrak{c}(\mu_2) \mid p^\infty C_2$.

Definition 4.1. To the positive integer k and the Eisenstein datum

$$\mathcal{D} := (\mu_1, \mu_2, C_1, C_2),$$

we associate the Bruhat-Schwartz function $\Phi^{[k]}_{\mathcal{D}} = \Phi^{[k]}_\infty \otimes_\ell \Phi_{\mathcal{D},\ell} \in \mathcal{S}(\mathbf{A}^2)$ defined as follows:

- $\Phi^{[k]}_\infty(x, y) = 2^{-k}(x + \sqrt{-1}y)^k e^{-\pi(x^2+y^2)}$,
- $\Phi_{\mathcal{D},p}(x, y) = \phi_{\mu_{1,p}^{-1}} \otimes \widehat{\phi}_{\mu_{2,p}^{-1}}$,
- $\Phi_{\mathcal{D},\ell}(x, y) = \mathbb{I}_{C_1\mathbf{Z}_\ell}(x)\mathbb{I}_{\mathbf{Z}_\ell}(y)$ if $\ell \nmid pC_2$,
- $\Phi_{\mathcal{D},\ell}(x, y) = \mathbb{I}_{C_2\mathbf{Z}_\ell}(x)\phi_{\mu_{2,\ell}}(y)$ if $\ell \mid C_2$.

Define the local Godement sections

$$f^{[k]}_{\mathcal{D},s,\infty} := f_{\mu_{1,\infty},\mu_{2,\infty},\Phi^{[k]}_\infty,s}; \quad f_{\mathcal{D},s,\ell} := f_{\mu_{1,\ell},\mu_{2,\ell},\Phi_{\mathcal{D},\ell},s}$$

and define the Godement section attached to k and $\mathcal{D} = (\mu_1, \mu_2, C_1, C_2)$ by

$$f^{[k]}_{\mathcal{D},s} = f^{[k]}_{\mathcal{D},s,\infty} \bigotimes_{\ell<\infty} f_{\mathcal{D},s,\ell}.$$

Remark 4.2. For each place v, denote by $\mathcal{B}_v(\mu_1, \mu_2, s)$ the space of smooth functions $f : \mathrm{GL}_2(\mathbf{Q}_v) \to \mathbf{C}$ such that

$$f(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g) = \mu_{1,v}(a)\mu_{2,v}(d) \left|\frac{a}{d}\right|_{\mathbf{Q}_v}^{s+\frac{1}{2}} f(g).$$

Then $f^{[k]}_{\mathcal{D},s,\infty}$ is the unique function in $\mathcal{B}_\infty(\mu_1, \mu_2, s)$ such that

$$f^{[k]}_{\mathcal{D},s,\infty}(\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}) = e^{\sqrt{-1}k\theta} \cdot 2^{-k}(\sqrt{-1})^k \pi^{-(s+\frac{k+1}{2})}\Gamma\left(s + \frac{k+1}{2}\right).$$

For a finite prime ℓ, $f_{\mathcal{D},s,\ell} \in \mathcal{B}_\ell(\mu_1, \mu_2, s)$ is invariant by $\mathcal{U}_1(C)$ under the right translation, where $\mathcal{U}_1(C)$ is the open-compact subgroupof $\mathrm{GL}_2(\mathbf{Z}_\ell)$ given by

$$\mathcal{U}_1(C) = \mathrm{GL}_2(\mathbf{Z}_\ell) \cap \begin{pmatrix} \mathbf{Z}_\ell & \mathbf{Z}_\ell \\ C\mathbf{Z}_\ell & 1 + C\mathbf{Z}_\ell \end{pmatrix}.$$

Definition 4.3. Define the classical Eisenstein series $E^\pm_k(\mu_1, \mu_2) : \mathfrak{H} \to \mathbf{C}$ by

$$E^+_k(\mu_1, \mu_2)(x + \sqrt{-1}y) := y^{-\frac{k}{2}} E_{\mathbf{A}}(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}, f^{[k]}_{\mathcal{D},s})|_{\frac{k-1}{2}},$$

$$E^-_k(\mu_1, \mu_2)(x + \sqrt{-1}y) := y^{-\frac{k}{2}} E_{\mathbf{A}}(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}, f^{[k]}_{\mathcal{D},s})|_{\frac{1-k}{2}}.$$

Remark 4.2 implies that $E_k^{\pm}(\mu_1,\mu_2) \in \mathcal{M}_k(C, \mu_1^{-1}\mu_2^{-1})$, and by definition

$$\Phi(E_k^{+}(\mu_1,\mu_2)) = E(g,\, f_{\mathcal{D},s}^{[k]})|_{\frac{k-1}{2}}; \quad \Phi(E_k^{-}(\mu_1,\mu_2)) = E(g,\, f_{\mathcal{D},s}^{[k]})|_{\frac{1-k}{2}},$$

where Φ is the adelic lift map in (2.4).

Proposition 4.4. *For every non-negative integer t, we have*

$$\Phi(\delta_k^t E_k^{\pm}(\mu_1,\mu_2)) = E_{\mathbf{A}}(g, f_{\mathcal{D},s}^{[k+2t]})|_{s=\pm\frac{k-1}{2}} \in \mathcal{A}_k(C, \mu_1\mu_2).$$

PROOF. For the differential operator V^+ in (2.1), we have the relation $V^+ f_{\mathcal{D},s,\infty}^{[k]} = f_{\mathcal{D},s,\infty}^{[k+2]}$ (see [JL70, Lemma 5.6 (iii)]), and hence the assertion follows from (2.5). Q.E.D.

4.3. Fourier coefficients of Eisenstein series

Lemma 4.5. *For $a \in \mathbf{R}^\times$, we have*

$$W(f_{\mathcal{D},s,\infty}^{[k]}, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix})|_{s=\frac{k-1}{2}} = W(f_{\mathcal{D},s,\infty}^{[k]}, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix})|_{s=\frac{1-k}{2}} = a^{\frac{k}{2}} e^{-2\pi a} \cdot \mathbb{I}_{\mathbf{R}_+}(a).$$

PROOF. By definition, $W(f_{\mathcal{D},s,\infty}^{[k]}, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix})$ equals

$$\begin{aligned} & 2^{-k}\mu_1|\cdot|^{s+\frac{1}{2}}(a) \\ & \times \int_{\mathbf{R}}\int_{\mathbf{R}^\times} t^k(a+\sqrt{-1}x)^k e^{-\pi t^2(x^2+a^2)}\,\mathrm{sgn}(t)^k\,|t|^{2s+1}\,\psi_\infty(-x)\mathrm{d}^\times t\mathrm{d}x \\ =& \mu_1|\cdot|^{s+\frac{1}{2}}(a)\cdot(-2\sqrt{-1})^{-k}\cdot\Gamma(s+\frac{k+1}{2})\pi^{-(s+\frac{k+1}{2})} \\ & \times \int_{\mathbf{R}} (x+\sqrt{-1}a)^{-(s+\frac{k+1}{2})}(x-\sqrt{-1}a)^{-(s-\frac{k-1}{2})}\psi_\infty(-x)\mathrm{d}x. \end{aligned}$$

By an elementary calculation, we find that

$$\begin{aligned} & W(f_{\mathcal{D},s,\infty}^{[k]}, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix})|_{s=\frac{k-1}{2}} \\ =& \mu_1|\cdot|^{\frac{k}{2}}(a)\cdot(-2\pi\sqrt{-1})^{-k}\cdot\Gamma(k)\int_{\mathbf{R}}\frac{e^{-2\pi\sqrt{-1}x}}{(x+\sqrt{-1}a)^k}\mathrm{d}x \\ =& \mu_1(a)\cdot a^{\frac{k}{2}} e^{-2\pi a}\cdot\mathbb{I}_{\mathbf{R}_+}(a), \end{aligned}$$

and that

$$\begin{aligned}&W(f^{[k]}_{\mathcal{D},s,\infty},\begin{pmatrix}a&0\\0&1\end{pmatrix})|_{s=\frac{1-k}{2}}\\=&\mu_1|\cdot|^{1-\frac{k}{2}}(a)(-2\sqrt{-1})^k\pi^{-1}\int_{\mathbf{R}}\frac{(x-\sqrt{-1}a)^{k-1}e^{-2\pi\sqrt{-1}x}}{x+\sqrt{-1}a}\mathrm{d}x\\=&\mu_1(a)\cdot a^{\frac{k}{2}}e^{-2\pi a}\cdot\mathbb{I}_{\mathbf{R}_+}(a).\end{aligned}$$

Since $\mu_1(a)=\mathrm{sgn}(a)^{\pm}$, the lemma follows. Q.E.D.

Lemma 4.6. *Let ℓ be a finite prime. Let $\chi=\mu_{1,\ell}^{-1}\mu_{2,\ell}$. Let $a\in\mathbf{Q}_\ell^\times$ and $m=\mathrm{ord}_\ell(a)$. Then we have the following:*

- *If $\ell\nmid pC_2$, then*

$$W(f_{\mathcal{D},s,\ell},\begin{pmatrix}a&0\\0&1\end{pmatrix})=\mu_1|\cdot|^{s+\frac{1}{2}}(a)\sum_{j=0}^{m-\mathrm{ord}_\ell(C_1)}\chi|\cdot|^{-2s}(\ell^j).$$

- *If $c(\mu_{2,\ell})=0$ and $q\mid C_2$, then*

$$\begin{aligned}&W(f_{\mathcal{D},s,\ell},\begin{pmatrix}a&0\\0&1\end{pmatrix})\\=&\mu_1|\cdot|^{s+\frac{1}{2}}(a)\cdot\mathbb{I}_{C_2\ell^{-1}\mathbf{Z}_\ell}(a)\left(-\chi|\cdot|^{2s+1}(\ell)+(1-|\ell|)\sum_{j=0}^{m-\mathrm{ord}_\ell(C_2)}\chi|\cdot|^{-2s}(\ell^j)\right).\end{aligned}$$

- *If $c(\mu_{2,\ell})>0$, then $\ell\mid C_2$ and*

$$W(f_{\mathcal{D},s,\ell},\begin{pmatrix}a&0\\0&1\end{pmatrix})=\mu_1|\cdot|^{s+\frac{1}{2}}(a)\cdot\chi(-1)\varepsilon(2s+1,\chi)\cdot\mathbb{I}_{C\ell^{-c(\chi)}\mathbf{Z}_\ell}(a).$$

- *If $\ell=p$, then*

$$W(f_{\mathcal{D},s,p},\begin{pmatrix}a&0\\0&1\end{pmatrix})=\mathbb{I}_{\mathbf{Z}_p^\times}(a).$$

Proof. Write $\mu_1=\mu_{1,\ell}$ and $\mu_2=\mu_{2,\ell}$ for simplicity. Note that if $\Phi=\Phi_1\otimes\Phi_2\in\mathcal{S}(\mathbf{Q}_\ell)\otimes\mathcal{S}(\mathbf{Q}_\ell)$, then

$$\begin{aligned}&W(f_{\mu_1,\mu_2,\Phi_\ell,s},\begin{pmatrix}a&0\\0&1\end{pmatrix})\\&\qquad=\mu_1|\cdot|^{s+\frac{1}{2}}(a)\int_{\mathbf{Q}_\ell^\times}\Phi_1(ta)\widehat{\Phi}_2(-t^{-1})\mu_1\mu_2^{-1}|\cdot|^{2s}(t)\mathrm{d}^\times t.\end{aligned}$$

If $\ell \nmid pC_2$, $\Phi_{\mathcal{D},\ell} = \mathbb{I}_{C_1\mathbf{Z}_\ell} \otimes \mathbb{I}_{\mathbf{Z}_\ell}$, and hence

$$\begin{aligned}
&W(f_{\mathcal{D},s,\ell}, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}) \\
&\qquad = \mu_1|\cdot|^{s+\frac{1}{2}}(a) \int_{\mathbf{Q}_\ell^\times} \mathbb{I}_{C_1\mathbf{Z}_\ell}(t^{-1}a)\mathbb{I}_{\mathbf{Z}_\ell}(-t)\mu_1^{-1}\mu_2|\cdot|^{-2s}(t)\mathrm{d}^\times t \\
&\qquad = \mu_1|\cdot|^{s+\frac{1}{2}}(a) \sum_{j=0}^{m-\mathrm{ord}_p(C_1)} \mu_1^{-1}\mu_2|\cdot|^{-2s}(\ell^j).
\end{aligned}$$

Consider the case $\ell \mid C_2$. Recall that μ_1 is unramified at $\ell \neq p$ by our assumption. Let $c = c(\chi) = c(\mu_2)$. Recall that

$$\widehat{\phi}_{\mu_2}(x) = \begin{cases} \widehat{\phi}_{\mu_2}(x) = \mathbb{I}_{\mathbf{Z}_\ell} - |\ell|\,\mathbb{I}_{\ell^{-1}\mathbf{Z}_\ell} & \text{if } c = 0, \\ \varepsilon(1,\mu_2^{-1})\mu_2(x^{-1})\mathbb{I}_{\ell^{-c}\mathbf{Z}_\ell^\times}(x) & \text{if } c > 0. \end{cases}$$

If $c = 0$, then $W(f_{\mathcal{D},s,\ell}, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix})$ equals

$$\begin{aligned}
&= \mu_1|\cdot|^{s+\frac{1}{2}}(a) \left(\sum_{j=0}^{m-\mathrm{ord}_\ell(C_2)} \mu_1^{-1}\mu_2|\cdot|^{-2s}(\ell^j) - |\ell| \sum_{j=-1}^{m-\mathrm{ord}_\ell(C_2)} \mu_1^{-1}\mu_2|\cdot|^{-2s}(\ell^j) \right) \\
&= \mu_1|\cdot|^{s+\frac{1}{2}}(a) \cdot \mathbb{I}_{C_2\ell^{-1}\mathbf{Z}_\ell}(a) \\
&\quad \times \left(-\mu_1\mu_2^{-1}|\cdot|^{2s+1}(\ell) + (1-|\ell|) \sum_{j=0}^{m-\mathrm{ord}_\ell(C_2)} \mu_1^{-1}\mu_2|\cdot|^{-2s}(\ell^j) \right).
\end{aligned}$$

If $c > 0$, then $W(f_{\mathcal{D},s,\ell}, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix})$ equals

$$\begin{aligned}
&\mu_1|\cdot|^{s+\frac{1}{2}}(a) \int_{\mathbf{Q}_\ell^\times} \mathbb{I}_{C\mathbf{Z}_\ell}(at)\widehat{\phi}_{\mu_2}(-t^{-1})\mu_1\mu_2^{-1}|\cdot|^{2s}(t)\mathrm{d}^\times t \\
&= \mu_1|\cdot|^{s+\frac{1}{2}}(a)\varepsilon(1,\mu_2^{-1})\mu_2(-1)\mu_1|\cdot|^{2s}(\ell^c)\mathbb{I}_{C\ell^{-c}\mathbf{Z}_\ell}(a).
\end{aligned}$$

Finally, at the p-adic place, a similar calculation shows that

$$\begin{aligned}
W(f_{\mathcal{D},s,p}, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}) &= \mu_1|\cdot|^{s+\frac{1}{2}}(a) \int_{\mathbf{Q}_p^\times} \phi_{\mu_1^{-1}}(at)\phi_{\mu_2^{-1}}(t^{-1})\mu_1\mu_2^{-1}|\cdot|^{2s}(t)\mathrm{d}^\times t \\
&= \mathbb{I}_{\mathbf{Z}_p^\times}(a).
\end{aligned}$$

The assertion follows immediately from the above expressions of $W(f_{\mathcal{D},s,\ell}, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix})$. Q.E.D.

For each positive integer n, we define the polynomials $\mathcal{P}_{n,\chi,\ell} \in \mathbf{Z}_{(p)}[X, X^{-1}]$ by

$$\begin{aligned}
\mathcal{P}_{n,\ell}(X) =& \sum_{j=0}^{\mathrm{ord}_\ell(nC_1^{-1})} \ell^{-j} X^j \text{ if } \ell \nmid pC_2, \\
\mathcal{P}_{n,\ell}(X) =& \sum_{j=0}^{\mathrm{ord}_\ell(nC_2^{-1})} \ell^{-j} X^j - \sum_{j=-1}^{\mathrm{ord}_\ell(nC_2^{-1})} \ell^{-j-1} X^j \text{ if } \ell \mid C_2, \\
\mathcal{P}_{n,p}(X) =& 1.
\end{aligned}$$

For a Dirichlet character χ, we set

$$\mathcal{G}_{\chi,\ell}(X) = \varepsilon(0, \chi_\ell)\chi_\ell(-1) \cdot X^{-c(\chi_\ell)}.$$

In the above equation, we have identified χ with its adelization as in §2.2 and χ_ℓ is the ℓ-component of χ.

Corollary 4.7. *We have the following Fourier expansion*

$$E_k^{\pm}(\mu_1, \mu_2)(q) = \sum_{n>0,\, p\nmid n} a_n^{\pm}(\mu_1, \mu_2, k) \cdot q^n \quad (q = e^{2\pi i \tau}),$$

where

$$\begin{aligned}
a_n^+(\mu_1, \mu_2, k) =& \mu_1^{-1}(n) \prod_{\ell \nmid \mathfrak{c}(\mu_2)} \mathcal{P}_{n,\ell}(\mu_1\mu_2^{-1}(\ell) \cdot \ell^k) \prod_{\ell \mid \mathfrak{c}(\mu_2), \ell \mid n} \mathcal{G}_{\mu_1\mu_2^{-1},\ell}(\ell^k); \\
a_n^-(\mu_1, \mu_2, k) =& n^{k-1} \cdot \mu_1^{-1}(n) \prod_{\ell \nmid \mathfrak{c}(\mu_2)} \mathcal{P}_{n,\mu_1^{-1}\mu_2,\ell}(\mu_1\mu_2^{-1}(\ell)\ell^{2-k})) \\
& \times \prod_{\ell \mid \mathfrak{c}(\mu_2),\, \ell \mid n} \mathcal{G}_{\mu_1\mu_2^{-1},\ell}(\ell^{2-k}).
\end{aligned}$$

Proof. Note that at the distinguished prime p, $\Phi_{\mathcal{D},p}(0, y) = 0$ and $\widehat{\Phi}_{\mathcal{D},p}(0, y) = \phi_{\mu_2^{-1}}(0)\widehat{\phi}_{\mu_1^{-1}}(y) = 0$, so we see that $f_{\mu_{1,p},\mu_{2,p},\Phi_{\mathcal{D},p},s}(1) = f_{\mu_{2,p},\mu_{1,p},\widehat{\Phi}_{\mathcal{D},p},-s}(1) = 0$. This in particular implies that

$$f_{\mu_1,\mu_2,\Phi_{\mathcal{D}},s}(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}) = f_{\mu_{2,p},\mu_{1,p},\widehat{\Phi}_{\mathcal{D},p},-s}(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}) = 0.$$

In view of (4.2) and Lemma 4.5, we find that

$$a_n^{\pm}(\mu_1,\mu_2,k) = n^{\frac{k}{2}} \prod_{\ell<\infty} W(f_{\mathcal{D},s,\ell}, \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix})|_{s=\pm\frac{k-1}{2}}.$$

The assertion follows from Lemma 4.6 by noting that $\mu_{1,\ell}^{-1}\mu_{2,\ell}(\ell) = \mu_1\mu_2^{-1}(\ell)$. Q.E.D.

4.4. A three variable p-adic family of Eisenstein series

Let (χ_1,χ_2) be a pair of Dirichlet characters modulo p and pC_2 respectively. Define the $\mathcal{O}[\![T_1,T_2,T_3]\!]$-adic q-expansion by

$$\boldsymbol{E}(\chi_1,\chi_2)(q) := \sum_{n>0,p\nmid n} \mathcal{A}_n(\chi_1,\chi_2)q^n \in \mathcal{O}[\![T_1,T_2,T]\!][\![q]\!],$$

where $\mathcal{A}_n(\chi_1,\chi_2) \in \mathcal{O}[\![T_1,T_2,T_3]\!]$ is given by

$$\mathcal{A}_n(\chi_1,\chi_2) = \langle n\rangle_{T_1}\,\langle n\rangle_{T_3}^{-1}\,\chi_1(n) \prod_{\ell\nmid\mathfrak{c}(\chi_2)} \mathcal{P}_{n,\ell}(\chi_1^{-1}\chi_2^{-1}(\ell)\,\langle\ell\rangle_{T_1}^{-1}\,\langle\ell\rangle_{T_2}^{-1}\,\langle\ell\rangle_{T_3}^{2})$$
$$\times \prod_{\ell|\mathfrak{c}(\chi_2),\ell|n} \mathcal{G}_{\chi_1^{-1}\chi_2^{-1},\ell}(\langle\ell\rangle_{T_1}^{-1}\,\langle\ell\rangle_{T_2}^{-1}\,\langle\ell\rangle_{T_3}^{2}).$$

Proposition 4.8. *For every $(Q_1,Q_2,P) \in \mathfrak{X}_\Lambda\times\mathfrak{X}_\Lambda\times\mathfrak{X}_\Lambda$ with*

$$0 < k_{Q_2} < k_P \le k_{Q_1},$$

we have the interpolation

$$\boldsymbol{E}(\chi_1,\chi_2)(Q_1,Q_2,P) = \begin{cases} \theta^{k_{Q_1}-k_P} E^{+}_{2k_P-k_{Q_1}-k_{Q_2}}(\chi_1^{-1}\epsilon_{Q_1}^{-1}\epsilon_P\boldsymbol{\omega}^{k_{Q_1}-k_P}, \chi_2\epsilon_{Q_2}\epsilon_P^{-1}\boldsymbol{\omega}^{k_P-k_{Q_2}}) \\ \quad \text{if } 2k_P > k_{Q_1}+k_{Q_2}, \\ \theta^{k_P-k_{Q_2}-1} E^{-}_{k_{Q_1}+k_{Q_2}-2k_P+2}(\chi_1^{-1}\epsilon_{Q_1}^{-1}\epsilon_P\boldsymbol{\omega}^{k_{Q_1}-k_P}, \chi_2\epsilon_{Q_2}\epsilon_P^{-1}\boldsymbol{\omega}^{k_P-k_{Q_2}}) \\ \quad \text{if } 2k_P \le k_{Q_1}+k_{Q_2}. \end{cases}$$

Here θ is the theta operator $\theta(\sum_n a_nq^n) = \sum_n na_nq^n$.

PROOF. Let $\mu_1 = \chi_1^{-1}\epsilon_{Q_1}^{-1}\epsilon_P\boldsymbol{\omega}^{k_{Q_1}-k_P}$ and $\mu_2 = \chi_2\epsilon_{Q_2}\epsilon_P^{-1}\boldsymbol{\omega}^{k_P-k_{Q_2}}$. Put $\mathbf{k} = 2k_P - k_{Q_1} - k_{Q_2}$. For an integer n prime to p, we have

$$\mathcal{A}_n(\chi_1,\chi_2)(Q_1,Q_2,P) = n^{k_{Q_1}-k_P}\mu_1^{-1}(n) \prod_{\ell\nmid\mathfrak{c}(\chi_2)} \mathcal{P}_{n,\ell}(\mu_1\mu_2^{-1}(\ell)\ell^{\mathbf{k}})$$
$$\times \prod_{\ell|\mathfrak{c}(\chi_2),\ell|n} \mathcal{G}_{n,\chi_1^{-1}\chi_2^{-1},\ell}(\chi_1\chi_2\mu_1\mu_2^{-1}(\ell)\ell^{\mathbf{k}}).$$

Since $\chi_1\chi_2\mu_1\mu_2^{-1}$ is a Dirichlet character modulo a power of p, one verifies that

$$\mathcal{G}_{\chi_1^{-1}\chi_2^{-1},\ell}(\chi_1\chi_2\mu_1\mu_2^{-1}(\ell)X)=\mathcal{G}_{\mu_1\mu_2^{-1},\ell}(X).$$

By Corollary 4.7, we find that

$$\mathcal{A}_n(\chi_1,\chi_2)(Q_1,Q_2,P)=\begin{cases} n^{k_{Q_1}-k_P}\cdot a_n^+(\mu_1,\mu_2,\mathbf{k}) & \text{if } \mathbf{k}>0,\\ n^{\mathbf{k}-1+k_{Q_1}-k_P}\cdot a_n^-(\mu_1,\mu_2,2-\mathbf{k}) & \text{if } \mathbf{k}<0.\end{cases}$$

The proposition follows immediately. Q.E.D.

§5. The construction of p-adic Rankin-Selberg L-functions

5.1. The construction of the p-adic L-function

Let $\mathcal{O}=\mathcal{O}_F$ for some finite extension F of $\mathbf{Q}_p$. For $i=1,2$, let $\mathbf{I}_i$ be a normal domain finite flat over Λ and let $\psi_i:(\mathbf{Z}/pN_i\mathbf{Z})^\times\to\mathcal{O}^\times$ be Dirichlet characters with $\psi_i(-1)=1$. We let

$$\boldsymbol{F}:=(\boldsymbol{f},\boldsymbol{g})\in e\mathbf{S}(N_1,\psi_1,\mathbf{I}_1)\times e\mathbf{S}(N_2,\psi_2,\mathbf{I}_2)$$

be a pair of primitive Hida families of tame conductors (N_1,N_2) and branch characters (ψ_1,ψ_2). In this section, we recall Hida's construction of the Rankin-Selberg p-adic L-function for $\boldsymbol{F}$. Fixing a topological generator γ_0 of $1+p\mathbf{Z}_p$ once and for all, we put $T=\gamma_0-1\in\Lambda$ and let

$$\mathcal{R}=\mathbf{I}_1\widehat{\otimes}_{\mathcal{O}}\mathbf{I}_2[\![T_3]\!]$$

be a finite extension over the three variable Iwasawa algebra

$$\Lambda\widehat{\otimes}_{\mathcal{O}}\Lambda\widehat{\otimes}_{\mathcal{O}}\Lambda=\mathcal{O}[\![T_1,T_2,T_3]\!],$$
$$(T_1=T\otimes 1\otimes 1,\ T_2=1\otimes T\otimes 1,\ T=1\otimes 1\otimes T).$$

Let $N=\mathrm{lcm}(N_1,N_2)$. Decompose the finite set $\mathrm{supp}(N)=\Sigma_{(\mathrm{i})}\sqcup\Sigma_{(\mathrm{ii})}\sqcup\Sigma_{(\mathrm{iii})}$, where

(5.1)

$\Sigma_{(\mathrm{i})}:=$ the set of primes $\ell\,|\,N$ such that $\pi_{\boldsymbol{f}_{Q_1},\ell}$ and $\pi_{\boldsymbol{g}_{Q_2},\ell}$ are principal series, $\mathrm{ord}_\ell(N_1)=\mathrm{ord}_\ell(N_2)>0$, $c_\ell(\psi_1\psi_2)=0$,

$\Sigma_{(\mathrm{ii})}=\left\{\ell\text{ prime}\mid\pi_{\boldsymbol{f}_{Q_1},\ell},\pi_{\boldsymbol{g}_{Q_2},\ell}\text{: discrete series and } L(s,\pi_{\boldsymbol{f}_{Q_1},\ell}\times\pi_{\boldsymbol{g}_{Q_2},\ell})\neq 1\right\}$,

$\Sigma_{(\mathrm{iii})}=\left\{\ell\text{ prime factor of } N\mid \ell\notin\Sigma_{(\mathrm{i})}\sqcup\Sigma_{(\mathrm{ii})}\right\}$.

Define the auxiliary integers C_1 and C_2 in the definition of an Eisenstein datum by

$$(5.2)\qquad \begin{aligned} C_1 &:= \prod_{\ell\in\Sigma_{(\mathrm{i})}} \ell^{\max\{\mathrm{ord}_\ell(N_1),\mathrm{ord}_\ell(N_2)\}} \prod_{\ell\in\Sigma_{(\mathrm{ii})}} \ell^{\lceil \frac{\mathrm{ord}_\ell(N_1)}{2}\rceil};\\ C_2 &:= \prod_{\ell\in\Sigma_{(\mathrm{iii})}} \ell^{\max\{\mathrm{ord}_\ell(N_1),\mathrm{ord}_\ell(N_2)\}}. \end{aligned}$$

If $\ell \in \Sigma_{(\mathrm{i})} \sqcup \Sigma_{(\mathrm{ii})}$, then $c_\ell(\psi_1\psi_2) = 0$ in view of [GJ78, Propositions (1.2) and (1.4)]). We have

- $C_1C_2 \mid N$,
- ψ_2 is a Dirichlet character modulo pC_2.

For any integer $a \in \mathbf{Z}/(p-1)\mathbf{Z}$, we define the power series $\boldsymbol{H}_a$

$$\boldsymbol{H}_a := \boldsymbol{g}\cdot \boldsymbol{E}(\psi_{1,(p)}\boldsymbol{\omega}^{-a}, \psi_2^{-1}\psi_1^{-1}\psi_{1,(p)}\boldsymbol{\omega}^{a}),$$

where $\psi_{1,(p)}$ is the p-primary component of ψ_1 in §2.2. By the arguments in [Hid93, page 226-227] and [Hid93, Lemma 1 in page 328], we can deduce that the power series $\boldsymbol{H}_a$ indeed belongs to $\mathcal{S}(N,\psi_1^{-1}\psi_{1,(p)}^2,\mathbf{I}_1)\widehat{\otimes}_{\mathbf{I}_1}\mathcal{R}$ (*cf.* [Hsi17, Lemma 3.4]). Therefore, one can apply the ordinary projector e to $\boldsymbol{H}_a$ and obtain $e\boldsymbol{H}_a$ an ordinary Λ-adic modular form with coefficients in $\mathcal{R}$. Let $\breve{\boldsymbol{f}} \in e\mathcal{S}(N_1,\psi_1^{-1}\psi_{1,(p)}^2,\mathbf{I}_1)$ be the primitive Hida family corresponding to the twist $\boldsymbol{f}|[\psi_1^{-1}\psi_{1,(p)}] := \sum_{n>0}\mathbf{a}(n,\boldsymbol{f})\psi_1^{-1}\psi_{1,(p)}(n)q^n$.

Definition 5.1. Fixing a generator $\eta_{\boldsymbol{f}}$ of the congruence ideal of $\boldsymbol{f}$, the p-adic Rankin-Selberg L-function $\mathscr{L}^{\boldsymbol{f}}_{\boldsymbol{F},a}$ is defined by

$$\mathscr{L}^{\boldsymbol{f}}_{\boldsymbol{F},a} := \text{ the first Fourier coefficient of } \eta_{\boldsymbol{f}}\cdot 1_{\breve{\boldsymbol{f}}}\mathrm{Tr}_{N/N_1}(e\boldsymbol{H}_a)) \in \mathcal{R},$$

where $\mathrm{Tr}_{N/N_1} : e\mathcal{S}(N,\psi_1^{-1}\psi_{1,(p)}^2,\mathbf{I}_1) \to e\mathcal{S}(N_1,\psi_1^{-1}\psi_{1,(p)}^2,\mathbf{I}_1)$ is the trace map (cf. [Hid88a, page 14]). Note that $\eta_{\boldsymbol{f}}\cdot 1_{\breve{\boldsymbol{f}}}$ is an integral Hecke operator since $\boldsymbol{f}$ and $\breve{\boldsymbol{f}}$ share the same congruence ideal (*cf.* [Hsi17, (3.2)]).

5.2. The interpolation formula and Rankin-Selberg integral

Define the weight space for the pair $(\boldsymbol{f},\boldsymbol{g})$ in the $\boldsymbol{f}$-dominated range by

$$(5.3)\qquad \mathfrak{X}^{\boldsymbol{f}}_{\mathcal{R}} := \left\{(Q_1,Q_2,P)\in \mathfrak{X}^+_{\mathbf{I}_1}\times\mathfrak{X}^{\mathrm{cls}}_{\mathbf{I}_2}\times\mathfrak{X}_\Lambda \mid k_{Q_2} < k_P \le k_{Q_1}\right\}.$$

Let $(Q_1, Q_2, P) \in \mathfrak{X}_{\mathcal{R}}^{\boldsymbol{f}}$. We relate the value $\mathscr{L}_{\boldsymbol{F}}^{\boldsymbol{f}}(Q_1, Q_2, P)$ to a global Rankin-Selberg integral. Let

$$(k_1, k_2, k_3) = (k_{Q_1}, k_{Q_2}, k_P)$$

and let r be an integer greater than $\max\{1, c_p(\epsilon_{Q_1}), c_p(\epsilon_{Q_2})\}$. Recall that the specializations

$$(f, g) := (\boldsymbol{f}_{Q_1}, \boldsymbol{g}_{Q_2}) \in \mathcal{S}_{k_1}(N_1 p^r, \chi_f) \times \mathcal{S}_{k_2}(N_2 p^r, \chi_g)$$

are p-stabilized cuspidal newforms with characters (χ_f, χ_g) modulo Np^r given by

$$\chi_f = \psi_1 \epsilon_{Q_1} \boldsymbol{\omega}^{-k_1},\ \chi_g = \psi_2 \epsilon_{Q_2} \boldsymbol{\omega}^{-k_2}.$$

Let $\varphi_f = \Phi(f)$ and $\varphi_g = \Phi(g)$ be the associated automorphic cusp forms as in (2.3). Then

$$(\varphi_1, \varphi_2) \in \mathcal{A}_{k_1}^0(N_1 p^r, \omega_1) \times \mathcal{A}_{k_2}^0(N_2 p^r, \omega_2)$$

and the central characters ω_1, ω_2 are the adelizations

$$\omega_1 = (\chi_f^{-1})_{\mathbf{A}},\ \omega_2 = (\chi_g^{-1})_{\mathbf{A}}$$

of χ_f^{-1} and χ_g^{-1}. Put

$$\omega = \omega_1 \omega_2.$$

Let $\omega_{1,(p)}$ be the p-primary component of ω_1 (so $\omega_{1,(p)}$ is the adelization of $\chi_{f,(p)}^{-1}$). Define the matrices $\mathcal{J}_\infty$ and $t_n \in \mathrm{GL}_2(\mathbf{A})$ for each integer n by

(5.4)

$$\mathcal{J}_\infty = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbf{R}),\ t_n = \begin{pmatrix} 0 & p^{-n} \\ -p^n & 0 \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Q}_p) \hookrightarrow \mathrm{GL}_2(\mathbf{A}).$$

Proposition 5.2. *Let $\mathcal{D}$ be the Eisenstein datum*

$$\mathcal{D} = (\epsilon_P \boldsymbol{\omega}^{a-k_P} \omega_{1,(p)}, \epsilon_P^{-1} \boldsymbol{\omega}^{-a+k_P} \omega^{-1} \omega_{1,(p)}, C_1, C_2). \tag{5.5}$$

Then we have

$$\begin{aligned} &\mathscr{L}_{\boldsymbol{F},a}^{\boldsymbol{f}}(Q_1, Q_2, P) \\ =& \left\langle \rho(\mathcal{J}_\infty t_n) \varphi_f \,,\, \varphi_g \cdot E_{\mathbf{A}}(-, f_{\mathcal{D}, s-\frac{1}{2}}^{[k_1-k_2]}) \otimes \omega_{f,(p)}^{-1} \right\rangle |_{s = \frac{2k_3-k_2-k_1}{2}} \\ &\times \frac{\zeta_{\mathbf{Q}}(2)[\mathrm{SL}_2(\mathbf{Z}) : \Gamma_0(N)](-2\sqrt{-1})^{k_1+1}}{\Omega_f} \cdot \frac{\zeta_p(1)}{\omega_{1,p}^{-1} \alpha_f^2 |\cdot|_{\mathbf{Q}_p}(p^n) \zeta_p(2)}. \end{aligned}$$

for any sufficiently large positive n, where $\langle\,,\,\rangle : \mathcal{A}^0_{-k_1}(N,\omega_1\omega_{1,(p)}^{-2}) \times \mathcal{A}_{k_1}(N,\omega_1^{-1}\omega_{1,(p)}^2) \to \mathbf{C}$ is the pairing defined in (2.2) *and α_f is the unramified character of $\mathbf{Q}_p^\times$ in* (2.10).

PROOF. Let $\boldsymbol{E} := \boldsymbol{E}(\boldsymbol{\omega}^{-a}\psi_{1,(p)}, \psi_1^{-1}\psi_2^{-1}\boldsymbol{\omega}^a\psi_{1,(p)})$. Since $\breve{\boldsymbol{f}}_{Q_1}$ is a p-stabilized newform, by the multiplicity one for new and ordinary vectors, we have

$$(5.6)\qquad \eta_f \cdot 1_{\breve{\boldsymbol{f}}_{Q_1}} \mathrm{Tr}_{N/N_1}(e(g\cdot \boldsymbol{E}(Q_1,Q_2,P))) = \mathscr{L}^{\boldsymbol{f}}_{\boldsymbol{F},a}(Q_1,Q_2,P)\cdot \breve{\boldsymbol{f}}_{Q_1}.$$

Put $\chi = \epsilon_P \boldsymbol{\omega}^{a-k_3}$. By Proposition 4.8, we have

$$\boldsymbol{E}(Q_1,Q_2,P) = \begin{cases} \theta^{k_1-k_3} E^+_{2k_3-k_1-k_2}(\chi\omega_{1,(p)}, \chi^{-1}\omega^{-1}\omega_{1,(p)}) \\ \text{ if } 2k_3 > k_1+k_2, \\ \theta^{k_3-k_2-1} E^-_{k_1+k_2-2k_3+2}(\chi\omega_{1,(p)}, \chi^{-1}\omega^{-1}\omega_{1,(p)}) \\ \text{ if } 2k_3 \le k_1+k_2. \end{cases}$$

We put

$$E^\dagger := \begin{cases} \delta^{k_1-k_3}_{2k_3-k_1-k_2} E^+_{2k_3-k_1-k_2}(\chi\omega_{1,(p)}, \chi^{-1}\omega^{-1}\omega_{1,(p)}) & \text{if } 2k_3 > k_1+k_2, \\ \delta^{k_3-k_2-1}_{k_1+k_2-2k_3+2} E^-_{k_1+k_2-2k_3+2}(\chi\omega_{1,(p)}, \chi^{-1}\omega^{-1}\omega_{1,(p)}) & \text{if } 2k_3 \le k_1+k_2, \end{cases}$$

where δ^m_k is the Maass-Shimura differential operator. The argument in [Hid93, equation (2), page 330] shows that

$$(5.7)\qquad e(g\cdot \boldsymbol{E}(Q_1,Q_2,P)) = e\mathrm{Hol}(g\cdot E^\dagger),$$

where Hol is the holomorphic projection as in [Hid93, (8a), page 314]. Put

$$\breve{\varphi}_f = \Phi(\breve{\boldsymbol{f}}_{Q_1}) \in \mathcal{A}^0_{k_1}(N_1p^r, \omega_1^{-1}\omega_{1,(p)}^2).$$

Pairing with the form $\rho(\mathcal{J}_\infty t_n)\varphi_f \otimes \omega_{1,(p)}^{-1}$ on the adelic lifts on both sides of (5.6), we obtain that

$$(5.8)\qquad \begin{aligned} &\langle \rho(\mathcal{J}_\infty t_n)\varphi_f \otimes \omega_{1,(p)}^{-1}, \breve{\varphi}_f\rangle \cdot \mathscr{L}^{\boldsymbol{f}}_{\boldsymbol{F},a}(Q_1,Q_2,P) \\ &= \langle \rho(\mathcal{J}_\infty t_n)\varphi_f \otimes \omega_{1,(p)}^{-1}, 1^*_{\breve{\boldsymbol{f}}_{Q_1}} \mathrm{Tr}_{N/N_1} e\Phi(\mathrm{Hol}(g\cdot E^\dagger))\rangle. \end{aligned}$$

Let $H = g\cdot E^\dagger$. Note that H is a nearly holomorphic modular form of weight k_1 and its adelic $\Phi(H) \in \mathcal{A}_{k_1}(Np^r, \omega_1^{-1}\omega_{1,(p)}^2)$ has a decomposition

$$\Phi(H) = \mathrm{Hol}(\Phi(H)) + V_+\varphi'_1 + V_+^2\varphi'_2 + \cdots + V_+^n\varphi'_n,$$

where $\mathrm{Hol}(\varPhi(H))$ and $\{\varphi_j\}_{j=1,\cdots,n}$ are holomorphic automorphic forms. It follows that $\mathrm{Hol}(\varPhi(H)) = \varPhi(\mathrm{Hol}(H))$. Let $1_f^* \in \mathbb{T}^{\mathrm{ord}}(N_1p^r, \mu_f)$ be the specializations of $1_{\boldsymbol{f}}^*$ at Q_1. As a consequence of strong multiplicity one theorem for modular forms, the idempotent $1_f = \eta_f^{-1}1_f^* \in \mathbb{T}^{\mathrm{ord}}(N_1p^r, \mu_f)\otimes_{\mathcal{O}}\mathrm{Frac}\mathcal{O}(Q_1)$ is generated by the Hecke operators T_ℓ for $\ell \nmid Np$, and this implies that hat 1_f is the left adjoint operator of $1_{\breve{\boldsymbol{f}}_{Q_1}}$ for the pairing $\langle - \otimes \omega_{1,(p)}^{-1}, - \rangle$. Hence, the right hand side of (5.8) equals

$$\begin{aligned}\eta_f\langle \mathrm{Tr}_{N/N_1}&\left(1_f \cdot \rho(\mathcal{J}_\infty t_n)\varphi_f \otimes \omega_{1,(p)}^{-1}\right), \mathrm{Hol}(\varPhi(H))\rangle \\ =&\eta_f[U_0(N_1) : U_0(N)] \cdot \langle \rho(\mathcal{J}_\infty t_n)\varphi_f \otimes \omega_{1,(p)}^{-1}, e\mathrm{Hol}(\varPhi(H))\rangle.\end{aligned} \tag{5.9}$$

On the other hand, it is straightforward to verify that for all sufficiently large n

$$\begin{aligned}\langle \rho(t_n)\mathbf{U}_p\varphi, \varphi' \rangle =& \langle \varphi, \mathbf{U}_p\varphi' \rangle, \\ \langle \rho(\mathcal{J}_\infty)\varphi, V_+\varphi' \rangle =& - \langle \rho(\mathcal{J}_\infty)V_-\varphi, \varphi' \rangle\end{aligned}$$

(cf. [Hid85, (5.4)]). It follows that the pairing on the right hand side of (5.9) equals

$$\begin{aligned}&\langle \rho(\mathcal{J}_\infty t_n)\varphi_f \otimes \omega_{1,(p)}^{-1}, e\mathrm{Hol}(\varPhi(H))\rangle \\ &= \langle \rho(\mathcal{J}_\infty t_n)\varphi_f \otimes \omega_{1,(p)}^{-1}, \varPhi(H)\rangle = \langle \rho(\mathcal{J}_\infty t_n)\varphi_f \otimes \omega_{1,(p)}^{-1}, \varPhi(H)\rangle.\end{aligned}$$

On the other hand, by Proposition 4.4,

$$\varPhi(H) = \varphi_g \cdot \varPhi(E^\dagger) = \varphi_g \cdot E_{\mathbf{A}}(g, f_{\mathcal{D},s-\frac{1}{2}}^{[k_1-k_2]})|_{s=\frac{2k_3-k_1-k_2}{2}}.$$

We obtain

$$\begin{aligned}&\langle \rho(\mathcal{J}_\infty t_n)\varphi_f \otimes \omega_{1,(p)}^{-1}, \breve{\varphi}_f \rangle \cdot \mathscr{L}_{\boldsymbol{F},a}^{\boldsymbol{f}}(Q_1, Q_2, P) \\ =&\eta_f[\Gamma_0(N_1) : \Gamma_0(N)] \cdot \langle \rho(\mathcal{J}_\infty t_n)\varphi_f, \varphi_g \cdot E_{\mathbf{A}}(g, f_{\mathcal{D},s-\frac{1}{2}}^{[k_1-k_2]}) \otimes \omega_{1,(p)}^{-1}\rangle|_{s=\frac{2k_3-k_1-k_2}{2}}.\end{aligned}$$

By the formula for $\langle \rho(\mathcal{J}_\infty t_n)\varphi_f \otimes \omega_{1,(p)}^{-1}, \breve{\varphi}_f \rangle$ in [Hsi17, Lemma 3.6] and the definition of $\Omega_f = \Omega_{\boldsymbol{f}_{Q_1}}$ in (1.5), we have

$$\begin{aligned}&\langle \rho(\mathcal{J}_\infty t_n)\varphi_f \otimes \omega_{1,(p)}^{-1}, \breve{\varphi}_f \rangle \\ &= \frac{\zeta_{\mathbf{Q}}(2)^{-1}}{[\mathrm{SL}_2(\mathbf{Z}) : \Gamma_0(N_1)]} \cdot \eta_f \cdot (-2\sqrt{-1})^{-k_1-1}\Omega_f \cdot \frac{\omega_{1,p}^{-1}\alpha_f^2|\cdot|_{\mathbf{Q}_p}(p^n)\zeta_p(2)}{\zeta_p(1)}.\end{aligned}$$

Putting the above equations together, we get the proposition. Q.E.D.

5.3. Rankin-Selberg L-functions for $\mathrm{GL}_2 \times \mathrm{GL}_2$

In this subsection, we review briefly Jacquet's approach to Rankin-Selberg L-functions. Let $(\pi_1, \mathcal{A}(\pi_1))$ and $(\pi_2, \mathcal{A}(\pi_2))$ be irreducible cuspidal automorphic representations of $\mathrm{GL}_2(\mathbf{A})$ and let χ be a Hecke character of $\mathbf{Q}^\times \backslash \mathbf{A}^\times$. Put $\omega = \omega_1\omega_2$. For $\varphi_1 \in \mathcal{A}(\pi_1)$ and $\varphi_2 \in \mathcal{A}(\pi_2)$, let W_{φ_1} and W_{φ_2} be the Whittaker functions of φ_1 and φ_2 defined in (2.9) respectively. Assume that W_{φ_1} and W_{φ_2} are decomposable, i.e.

$$W_{\varphi_1} = \prod_v W_{\varphi_1,v}, \quad W_{\varphi_2} = \prod_v W_{\varphi_2,v}$$

with $W_{\varphi_i,v} \in \mathcal{W}(\pi_{i,v})$ for $i = 1, 2$. Let $\Phi = \otimes\Phi_v \in \mathcal{S}(\mathbf{A}^2)$. For each place v of $\mathbf{Q}$, define the local zeta integral $\Psi(W_{\varphi_1,v}, W_{\varphi_2,v}, f_{\chi_v,\chi_v\omega_v^{-1},\Phi_v,s})$ (cf. [Jac72, (14.5)]) by

(5.10)

$$\Psi(W_{\varphi_1,v}, W_{\varphi_2,v}, f_{\chi_{1,v},\chi^{-1}\omega^{-1},\Phi_v,s}) := \int_{N(\mathbf{Q}_v)\backslash \mathrm{PGL}_2(\mathbf{Q}_v)} W_{\varphi_1,v}(g_v) W_{\varphi_2,v}(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} g_v) f_{\chi_v,\chi^{-1}\omega_{2,v}^{-1},\Phi_v,s-\frac{1}{2}}(g_v) \mathrm{d}g_v,$$

where $\mathrm{d}g_v$ is the quotient measure of the Haar measure of $\mathrm{PGL}_2(\mathbf{Q}_v)$ by the additive Haar measure of $\mathbf{Q}_v$ defined in §2.6.1. It is well-known that the local zeta integrals converge absolutely for $\mathrm{Re}(s) \gg 0$ and has meromorphic continuation to $s \in \mathbf{C}$. A standard unfolding argument shows that

(5.11)

$$\left\langle \varphi_1, \varphi_2 \cdot E_{\mathbf{A}}(-, f_{\chi,\chi^{-1}\omega^{-1},\Phi,s-\frac{1}{2}}) \right\rangle = \frac{1}{\zeta_{\mathbf{Q}}(2)} \prod_v \Psi(W_{\varphi_1,v}, W_{\varphi_2,v}, f_{\chi_v,\chi_v^{-1}\omega_v^{-1},\Phi_v,s})$$

as meromorphic functions in $s \in \mathbf{C}$.

For each place v of $\mathbf{Q}$, let $L(s, \pi_{1,v} \times \pi_{2,v} \otimes \chi_v)$ be the local L-factor of $\pi_{1,v} \times \pi_{2,v}$ twisted by χ_v. The Rankin-Selberg L-function of $\pi_1 \times \pi_2$ twisted by χ is defined by

$$L(s, \pi_1 \times \pi_2 \otimes \chi) := \prod_v L(s, \pi_{1,v} \times \pi_{2,v} \otimes \chi_v).$$

Note that if π_1, π_2, and χ are unitary, then $L(s, \pi_1 \otimes \pi_2 \otimes \chi)$ is an entire function if and only if π_1 and $\pi_2^\vee \otimes \chi^{-2}$ are not isomorphic up to unramified twist (cf. [JS81, Proposition 3.3]). Let S be a finite set of places of $\mathbf{Q}$ containing the archimedean place such that for all $v \notin S$,

- $\pi_{1,v}$ and $\pi_{2,v}$ are spherical, and χ_v is unramified,
- $W_{\varphi_1,v} = W_{\pi_1,v}$, $W_{\varphi_2,v} = W_{\pi_2,v}$ are the normalized local Whittaker newforms, and $\Phi_v = \mathbb{I}_{\mathbf{Z}_v \oplus \mathbf{Z}_v}$.

By [Jac72, Proposition 15.9], for all $v \notin S$ we have

$$\Psi(W_{\varphi_1,v}, W_{\varphi_2,v}, f_{\chi_v,\chi_v^{-1}\omega_v^{-1},\Phi_v,s}) = L(s, \pi_{1,v} \times \pi_{2,v} \otimes \chi_v).$$

It follows that (5.11) can be rephrased in the following form

$$\begin{aligned} &\left\langle \varphi_1, \varphi_2 \cdot E_{\mathbf{A}}(-, f_{\chi,\chi^{-1}\omega^{-1},\Phi,s-\frac{1}{2}}) \right\rangle \\ &= \frac{L(s, \pi_1 \times \pi_2 \otimes \chi)}{\zeta_{\mathbf{Q}}(2)} \prod_{v \in S} \frac{\Psi(W_{\varphi_1,v}, W_{\varphi_2,v}, f_{\chi_v,\chi_v^{-1}\omega_v^{-1},\Phi_v,s})}{L(s, \pi_{1,v} \times \pi_{2,v} \otimes \chi_v)}. \end{aligned} \tag{5.12}$$

5.4. The interpolation formula and Rankin-Selberg L-values

Now we return to the setting in §5.2 and keep the notation there. Let π_1 and π_2 be the cuspidal automorphic representation generated by the automorphic forms φ_f and φ_g associated with p-stabilized newforms $f = \boldsymbol{f}_{Q_1}$ and $g = \boldsymbol{g}_{Q_2}$ respectively. From the discussion in §2.7, we know the Whittaker functions of φ_f and φ_g can be factorized into a product of local Whittaker newforms and p-ordinary Whittaker functions

$$W_{\varphi_f} = W^{\mathrm{ord}}_{\pi_1,p} \prod_{v \neq p} W_{\pi_1,v},\ W_{\varphi_g} = W^{\mathrm{ord}}_{\pi_2,p} \prod_{v \neq p} W_{\pi_2,v}.$$

Let $\chi = \epsilon_P \boldsymbol{\omega}^{a-k_3}$ and $\mathcal{D}$ be the Eisenstein datum $(\chi\omega_{1,(p)}, \chi^{-1}\omega^{-1}\omega_{1,(p)}, C_1, C_2)$.Define the local Godement section

$$f^\star_{\mathcal{D},s,\infty} := f_{\chi_\infty,\chi_\infty^{-1}\omega_\infty^{-1},\Phi_\infty^{[k_1-k_2]},s}; \quad f^\star_{\mathcal{D},s,\ell} := f_{\chi,\chi_\ell^{-1}\omega_\ell^{-1},\Phi_{\mathcal{D},\ell},s} \tag{5.13}$$

with the Bruhat-Schwartz function $\Phi_\infty^{[k_1-k_2]}$ and $\Phi_{\mathcal{D},p}$ in Definition 4.1. By definition,

$$f^\star_{\mathcal{D},s} := \prod_v f^\star_{\mathcal{D},s,v} = f_{\chi,\chi^{-1}\omega^{-1},\Phi_{\mathcal{D}}^{[k_1-k_2]},s} = f^{[k_1-k_2]}_{\mathcal{D},s} \otimes \omega^{-1}_{1,(p)} \in \mathcal{B}(\chi, \chi^{-1}\omega^{-1}, s)$$

is the Godement section attached to $\Phi_{\mathcal{D}}^{[k_1-k_2]}$.

Proposition 5.3. *For every sufficiently large positive integer n, we have*

$$\begin{aligned} &\mathscr{L}^{\boldsymbol{f}}_{\boldsymbol{F},a}(Q_1, Q_2, P) \\ =&L(\frac{2k_3-k_1-k_2}{2}, \pi_1 \times \pi_2 \otimes \chi) \cdot (-1)^{a+1} \psi_{2,(p)}(-1) \cdot \frac{(\sqrt{-1})^{-2k_3+k_2+1}}{\Omega_{\boldsymbol{f}_{Q_1}}}, \\ &\times \Psi^{\mathrm{ord}}_p(s) \cdot \prod_{\ell | N} \Psi^*_\ell(s) \Big|_{s=\frac{2k_3-k_1-k_2}{2}}, \end{aligned}$$

where $\Psi_p^{\mathrm{ord}}(s)$ *and* $\Psi_\ell^*(s)$ *are normalized local zeta integrals given by*

$$\begin{aligned}\Psi_p^{\mathrm{ord}}(s) =&\frac{\Psi(\rho(t_n)W^{\mathrm{ord}}_{\pi_{1,p}}, W^{\mathrm{ord}}_{\pi_{2,p}}, f^\star_{\mathcal{D},s,p})}{L(s,\pi_{1,p}\otimes\pi_{2,p}\otimes\chi_p)}\cdot\frac{\omega_{2,p}(-1)\cdot\zeta_p(1)}{\omega_{1,p}^{-1}\alpha_f^2|\cdot|_{\mathbf{Q}_p}(p^n)\zeta_p(2)},\\ \Psi_\ell^*(s) =&\frac{\zeta_{\mathbf{Q}_\ell}(1)}{\zeta_{\mathbf{Q}_\ell}(2)\,|N|_{\mathbf{Q}_\ell}}\cdot\frac{\Psi(W_{\pi_{1,\ell}}, W_{\pi_{2,\ell}}, f^\star_{\mathcal{D},s,\ell})}{L(s,\pi_{1,\ell}\otimes\pi_{2,\ell}\otimes\chi_\ell)}.\end{aligned}$$

PROOF. By (5.11), we find that

$$\begin{aligned}&\langle\rho(\mathcal{J}_\infty t_n)\varphi_f,\varphi_g\cdot E_{\mathbf{A}}(-,f^{[k_1-k_2]}_{\mathcal{D},s-\frac{1}{2}})\otimes\omega^{-1}_{f,(p)}\rangle\\ =&\langle\rho(\mathcal{J}_\infty t_n)\varphi_f,\varphi_g\cdot E_{\mathbf{A}}(-,f^\star_{\mathcal{D},s-\frac{1}{2}})\rangle\\ =&\zeta_{\mathbf{Q}}(2)^{-1}\Psi(\rho(\mathcal{J}_\infty)W_{\pi_1,\infty},W_{\pi_2,\infty},f^\star_{\mathcal{D},s,\infty})\cdot\Psi(\rho(t_n)W^{\mathrm{ord}}_{\pi_1,p},W^{\mathrm{ord}}_{\pi_2,p},f^\star_{\mathcal{D},s,p})\\ &\times\prod_{\ell\neq p}\Psi(W_{\pi_1,\ell},W_{\pi_2,\ell},f^\star_{\mathcal{D},s,\ell}).\end{aligned}$$

We must calculate the archimedean local zeta integral. Note that $\pi_{1,\infty}$ and $\pi_{2,\infty}$ are discrete series of weight k_1 and k_2, and the corresponding Whittaker newforms $W_{\pi_{1,\infty}}$ and $W_{\pi_{2,\infty}}$ are given in (2.8). Let $k' = k_1 - k_2$. In addition, $f^\star_{\mathcal{D},s,\infty}$ is the Godement section attached to $\Phi_\infty^{[k']}$. Therefore, we can compute the integral

$$\begin{aligned}&\Psi(\rho(\mathcal{J}_\infty)W_{\pi_1,\infty},W_{\pi_2,\infty},f^\star_{\mathcal{D},s,\infty})\\ =&\int_{\mathbf{R}^\times}\int_{\mathrm{O}_2(\mathbf{R})}W_{\pi_{1,\infty}}(\begin{pmatrix}-y&0\\0&1\end{pmatrix}u)W_{\pi_{2,\infty}}(\begin{pmatrix}-y&0\\0&1\end{pmatrix}u)\\ &\times\chi_\infty|\cdot|_{\mathbf{R}}^{s-1}(y)f^\star_{\mathcal{D},s-\frac{1}{2},\infty}(u)\mathrm{d}u\,\mathrm{d}^\times y\\ =&\chi_\infty(-1)2^{-k'}(\sqrt{-1})^{k'}\pi^{-s-k'/2}\Gamma\left(s+\frac{k'}{2}\right)\\ &\times\int_{\mathbf{R}^\times}W_{\pi_{1,\infty}}(\begin{pmatrix}y&0\\0&1\end{pmatrix})W_{\pi_{2,\infty}}(\begin{pmatrix}y&0\\0&1\end{pmatrix})\,|y|_{\mathbf{R}}^{s-1}\,\mathrm{d}^\times y\\ =&(-1)^{a+k_3}2^{-t}(\sqrt{-1})^{k'}\pi^{-s-k'/2}\Gamma\left(s+\frac{k'}{2}\right)\int_{\mathbf{R}_+}y^{s+\frac{k_1+k_2}{2}-1}e^{-4\pi y}\mathrm{d}^\times y\\ =&(-1)^a(\sqrt{-1})^{2k_3+k_1-k_2}2^{-k_1-1}\Gamma_{\mathbf{C}}\left(s+\frac{k_1-k_2}{2}\right)\Gamma_{\mathbf{C}}\left(s+\frac{k_1+k_2}{2}-1\right)\\ =&(-1)^a(\sqrt{-1})^{2k_3+k_1-k_2}2^{-k_1-1}\cdot L(s,\pi_{1,\infty}\times\pi_{2,\infty}\otimes\chi_\infty).\end{aligned}$$

Combining Proposition 5.2, (5.12) and the fact that $\omega_{2,p}(-1) = \psi_{2,(p)}(-1)(-1)^{k_2}$ and $[\mathrm{SL}_2(N):\Gamma_0(N)] = \prod_{\ell|N}\frac{\zeta_{\mathbf{Q}_\ell}(1)}{\zeta_{\mathbf{Q}_\ell}(2)|N|_{\mathbf{Q}_\ell}}$, we get the proposition. Q.E.D.

§6. The calculation of local zeta integrals

In this section, we calculate the normalized local zeta integrals $\Psi_p^{\mathrm{ord}}(s)$ and $\Psi_\ell^*(s)$ in Proposition 5.3. The notation is as in §5.2. We will continue to use the local Haar measures normalized in §2.6.1.

6.1. The p-adic place

Lemma 6.1. *For all sufficiently large n, we have*

$$\Psi(\rho(t_n)W^{\mathrm{ord}}_{\pi_{1,p}}, W^{\mathrm{ord}}_{\pi_{2,p}}, f^\star_{\mathcal{D},s,p}) = \frac{\zeta_p(2)}{\zeta_p(1)} \cdot \frac{\alpha_f^2\omega_{1,p}^{-1}|\cdot|_{\mathbf{Q}_p}(p^n)\omega_{2,p}(-1)}{\gamma(s,\pi_{2,p}\otimes\alpha_f\chi_p)}.$$

In particular, $\Psi_p^{\mathrm{ord}}(s) = \mathcal{E}_p^1(s,f,g\otimes\chi)$, where $\mathcal{E}_p^1(s,f,g\otimes\chi)$ is the modified p-Euler factor given by

(6.1)

$$\begin{aligned}&\mathcal{E}_p^1(s,f,g\otimes\chi)\\ =&\frac{L(s,\pi_{2,p}\otimes\alpha_f\chi_p)}{\varepsilon(s,\pi_{2,p}\otimes\alpha_f\chi_p)L(1-s,\pi_{2,p}^\vee\otimes\alpha_f^{-1}\chi_p^{-1})}\cdot\frac{1}{L(s,\pi_{1,p}\times\pi_{2,p}\otimes\chi_p)}.\end{aligned}$$

PROOF. To simply the notation, we omit the subscript p in the proof. For example, we write $\pi_1, \omega_1, f^\star_{\mathcal{D},s}, |\cdot|$ for $\pi_{1,p}, \omega_{1,p}, f^\star_{\mathcal{D},s,p}, |\cdot|_{\mathbf{Q}_p}$. According to Definition 4.1, $f^\star_{\mathcal{D},s} = f_{\chi,\chi^{-1}\omega^{-1},\Phi_{\mathcal{D}},s}$ be the Godement section associated with

$$\Phi_{\mathcal{D},p} = \phi_{\chi^{-1}\omega_{1,(p)}^{-1}} \otimes \widehat{\phi}_{\chi\omega\omega_{1,(p)}^{-1}}.$$

A direct computation shows that

$$f^\star_{\mathcal{D},s}(\begin{pmatrix}0 & -1\\ 1 & x\end{pmatrix}) = \widehat{\phi}_{\chi\omega\omega_{1,(p)}^{-1}}(x).$$

Let $W_i^{\mathrm{ord}} = W_{\pi_i}^{\mathrm{ord}}$ for $i=1,2$. By (2.10), we have

$$W_1(\begin{pmatrix}y & 0\\ 0 & 1\end{pmatrix}) = \alpha_f|\cdot|^{\frac{1}{2}}(y)\mathbb{I}_{\mathbf{Z}_p}(y);\quad W_2(\begin{pmatrix}y & 0\\ 0 & 1\end{pmatrix}) = \alpha_g|\cdot|^{\frac{1}{2}}(y)\mathbb{I}_{\mathbf{Z}_p}(y).$$

Using the integration formula

$$\int_{N\backslash\mathrm{PGL}_2(\mathbf{Q}_p)} F(g)\mathrm{d}g = \frac{\zeta_p(2)}{\zeta_p(1)}\int_{\mathbf{Q}_p^\times}\int_{\mathbf{Q}_p} F(\begin{pmatrix}y & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & x\end{pmatrix})\mathrm{d}x\mathrm{d}^\times y$$

for $F \in L^1(N\backslash \mathrm{PGL}_2(\mathbf{Q}_p))$, we obtain

$$\begin{aligned}
&\Psi(\rho(t_n)W_1^{\mathrm{ord}}, W_2^{\mathrm{ord}}, f^\star_{\mathcal{D},s}) \\
=&\frac{\zeta_p(2)}{\zeta_p(1)}\int_{\mathbf{Q}_p^\times}\int_{\mathbf{Q}_p} W_1^{\mathrm{ord}}(\begin{pmatrix} y & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & x\end{pmatrix} t_n)W_2^{\mathrm{ord}}(\begin{pmatrix} -y & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & x\end{pmatrix}) \\
&\times \chi|\cdot|^s(y) f^\star_{\mathcal{D},s-\frac{1}{2}}(\begin{pmatrix} 0 & -1 \\ 1 & x\end{pmatrix})\,|y|^{-1}\,\mathrm{d}x\mathrm{d}^\times y \\
=&\frac{\zeta_p(2)}{\zeta_p(1)}\int_{\mathbf{Q}_p}\int_{\mathbf{Q}_p^\times} W_1^{\mathrm{ord}}(\begin{pmatrix} yp^n & 0 \\ 0 & p^{-n}\end{pmatrix}\begin{pmatrix} 1 & 0 \\ -p^{2n}x & 1\end{pmatrix})W_2^{\mathrm{ord}}(\begin{pmatrix} -y & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & x\end{pmatrix}) \\
&\times \chi|\cdot|^{s-1}(y)\widehat{\phi}_{\chi\omega\omega_{1,(p)}^{-1}}(x)\mathrm{d}^\times y\mathrm{d}x \\
=&\frac{\zeta_p(2)\alpha_f|\cdot|^{\frac{1}{2}}(p^{2n})\omega_1^{-1}(p^n)}{\zeta_p(1)} \\
&\times \int_{\mathbf{Q}_p}\int_{\mathbf{Q}_p^\times} W_2^{\mathrm{ord}}(\begin{pmatrix} -y & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & x\end{pmatrix})\chi\alpha_f|\cdot|^{s-\frac{1}{2}}(y)\widehat{\phi}_{\chi\omega\omega_{1,(p)}^{-1}}(x)\mathrm{d}^\times y\mathrm{d}x
\end{aligned}$$

for sufficiently large n. By the local functional equation for GL(2) [Bum98, Theorem 4.7.5], the above integral equals

$$\begin{aligned}
&\frac{\zeta_p(2)\alpha_f^2\omega_1^{-1}|\cdot|(p^n)}{\zeta_p(1)}\cdot\frac{\chi(-1)}{\gamma(s,\pi_2\otimes\alpha_f\chi)} \\
&\times\int_{\mathbf{Q}_p}\int_{\mathbf{Q}_p^\times} W_2^{\mathrm{ord}}(\begin{pmatrix} y & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix})\omega_2^{-1}\chi^{-1}\alpha_f^{-1}|\cdot|^{\frac{1}{2}-s}(y)\widehat{\phi}_{\chi\omega\omega_{1,(p)}^{-1}}(x)\mathrm{d}^\times y\mathrm{d}x \\
=&\frac{\zeta_p(2)\alpha_f^2\omega_1^{-1}|\cdot|(p^n)}{\zeta_p(1)}\cdot\frac{\chi(-1)}{\gamma(s,\pi_2\otimes\alpha_f\chi)} \\
&\times\int_{\mathbf{Q}_p}\int_{\mathbf{Q}_p^\times}\psi(yx)\mu_2\omega_2^{-1}\chi^{-1}\alpha_f^{-1}|\cdot|^{1-s}(y)\widehat{\phi}_{\chi\omega\omega_{1,(p)}^{-1}}(x)\mathrm{d}^\times y\mathrm{d}x \\
=&\frac{\zeta_p(2)\alpha_f^2\omega_1^{-1}|\cdot|(p^n)}{\zeta_p(1)}\cdot\frac{\chi(-1)}{\gamma(s,\pi_2\otimes\alpha_f\chi)} \\
&\times\int_{\mathbf{Q}_p^\times}\alpha_f^{-1}\alpha_g\omega_2^{-1}\chi^{-1}|\cdot|^{1-s}(y)\phi_{\chi\omega\omega_{1,(p)}^{-1}}(-y)\mathrm{d}^\times y.
\end{aligned}$$

Since α_f, α_g and $\omega_1\omega_{1,(p)}^{-1}$ are unramified, we find that

$$\Psi(\rho(t_n)W_1^{\mathrm{ord}}, W_2^{\mathrm{ord}}, f^\star_{\mathcal{D},s}) = \frac{\zeta_p(2)\alpha_f^2\omega_1^{-1}|\cdot|(p^n)}{\zeta_p(1)}\cdot\frac{\omega_2(-1)}{\gamma(s,\pi_2\otimes\alpha_f\chi)}.$$

This completes the proof. Q.E.D.

6.2. The ℓ-adic case with $\ell \mid N$

In this subsection, we compute the local zeta integral $\Psi_\ell^*(s)$ under certain minimal Hypothesis (M) below. Recall that an irreducible admissible representation π of $\mathrm{GL}_2(\mathbf{Q}_\ell)$ is called minimal if the exponent of the conductor $c(\pi)$ of π is minimal among the twists $\pi \otimes \chi$ for all characters χ of $\mathbf{Q}_\ell^\times$. In this subsection, we assume the following *minimal* hypothesis for (f, g)

Hypothesis (M)**.** For every $\ell \mid N$, there exists a rearrangement $\{\pi_1, \pi_2\} = \{\pi_{f,\ell}, \pi_{g,\ell}\}$ such that

- π_1 is minimal,
- Either π_1 is discrete series or π_1 and π_2 are both principal series.
- If π_1 and π_2 are both principal series with $L(s, \pi_1 \times \pi_2) \neq 1$, then π_2 is also minimal.

Remark 6.2. Note that if the above hypothesis holds for (f, g), then it holds for the specialization of $(\boldsymbol{f}, \boldsymbol{g})$ at any classical point by the rigidity of automorphic types for Hida families described in [FO12, Lemma 2.14] (See also [Hsi17, Remark 3.1]). Moreover, one can always find a Dirichlet character λ such that $(\pi_f \otimes \lambda, \pi_g \otimes \lambda^{-1})$ satisfies Hypothesis (M).

Let ℓ be a prime factor of N. So ℓ belongs to $\Sigma_{(\mathrm{i})} \sqcup \Sigma_{(\mathrm{ii})} \sqcup \Sigma_{(\mathrm{iii})}$ as described in (5.1). Note that in this case χ is unramified at ℓ, and $\Psi_\ell^*(s)$ is symmetric for $(\pi_{f,\ell}, \pi_{g,\ell})$. Let (π_1, π_2) as above. For $i = 1, 2$, let $c_i = c(\pi_i)$ be the exponent of the conductor of π_i. Set $c = \max\{c_1, c_2\} = \mathrm{ord}_\ell(N) > 0$. We write

$$a(t) = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix},\ u(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix},\ w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (t \in \mathbf{Q}_\ell^\times,\ x \in \mathbf{Q}_\ell).$$

For a non-negative integer n, let

$$\mathcal{U}_0(\ell^n) = \mathrm{GL}_2(\mathbf{Z}_\ell) \cap \begin{pmatrix} \mathbf{Z}_\ell & \mathbf{Z}_\ell \\ \ell^n \mathbf{Z}_\ell & \mathbf{Z}_\ell \end{pmatrix}.$$

In what follows, we assume Hypothesis (M) throughout this subsection. We often omit the subscript ℓ as before. We first treat the case $\ell \in \Sigma_{(\mathrm{i})}$.

Lemma 6.3. *If $\ell \in \Sigma_{(\mathrm{i})}$, then we have*

$$\Psi\left(W_{\pi_1}, W_{\pi_2}, f_{\mathcal{D},s}^\star\right) = \frac{\zeta_{\mathbf{Q}_\ell}(2)\, |N|_{\mathbf{Q}_\ell}}{\zeta_{\mathbf{Q}_\ell}(1)} \cdot L(s, \pi_1 \times \pi_2 \otimes \chi),$$

and hence $\Psi_\ell^*(s) = 1$.

PROOF. Write W_1 and W_2 for W_{π_1} and W_{π_2} respectively. Recall that $f^\star_{\mathcal{D},s} = f_{1,\omega^{-1},\Phi_{\mathcal{D}},s}$ is the Godement section attached to the Bruhat-Schwartz function $\Phi_{\mathcal{D}} = \mathbb{I}_{\ell^c\mathbf{Z}_\ell} \otimes \mathbb{I}_{\mathbf{Z}_\ell}$. Since χ is unramified, we may assume $\chi = 1$. For $k \in \mathrm{GL}_2(\mathbf{Q}_\ell)$, put

$$\mathcal{I}(u) = \int_{\mathbf{Q}_\ell^\times} W_1(a(t)u)W_2(a(-t)u)\,|t|^{s-1}\,\mathrm{d}^\times t.$$

Then $\mathcal{I}$ and $f^\star_{\mathcal{D},s}$ are indeed functions on $N(\mathbf{Q}_\ell)\backslash \mathrm{GL}_2(\mathbf{Q}_\ell)/\mathcal{U}_0(\ell^c)$, so we have

$$\text{(6.2)}\quad \Psi(W_1, W_2, f^\star_{\mathcal{D},s}) = \mathrm{vol}(\mathcal{U}_0(\ell^c), \mathrm{d}u) \sum_{u\in \mathrm{GL}_2(\mathbf{Z}_\ell)/\mathcal{U}(\ell^c)} \mathcal{I}(u) f^\star_{\mathcal{D},s-\frac{1}{2}}(u).$$

To evaluate the above sum, we give an explicit formula for the function $f^\star_{\mathcal{D},s-\frac{1}{2}}$ and $\mathcal{I}$. First of all, it is easy to verify that $f^\star_{\mathcal{D},s-\frac{1}{2}}(1) = L(2s,\omega)$,

$$\begin{aligned} f^\star_{\mathcal{D},s-\frac{1}{2}}(w) =& f^\star_{\mathcal{D},s-\frac{1}{2}}(w\begin{pmatrix}1&0\\x&1\end{pmatrix}) = L(2s,\omega)\omega|\cdot|^{2s}(\ell^c),\\ f^\star_{\mathcal{D},s-\frac{1}{2}}(\begin{pmatrix}1&0\\\ell^n&1\end{pmatrix}) =& L(2s,\omega)\cdot\omega|\cdot|^{2s}(\ell^{c-n}) \end{aligned}$$

and that

$$\begin{aligned} \mathcal{I}(w) = \mathcal{I}\left(\begin{pmatrix}1&0\\1&1\end{pmatrix}\right) =& \mathcal{I}\left(w\begin{pmatrix}1&0\\x&1\end{pmatrix}\right);\\ \mathcal{I}\left(\begin{pmatrix}1&0\\\ell^n&1\end{pmatrix}\right) =& \mathcal{I}\left(\begin{pmatrix}1&0\\\ell^n t&1\end{pmatrix}\right), \end{aligned}$$

for $x \in \mathbf{Q}_\ell$, $t \in \mathbf{Z}_\ell^\times$, and $n \in \mathbf{Z}$. In view of the definition of $\Sigma_{(\mathrm{i})}$ in (5.1) and Hypothesis (M), we must have that $\pi_1 = \mu_1 \boxplus \nu_1$ and $\pi_2 = \mu_2 \boxplus \nu_2$ are principal series such that

$$c = c(\nu_1) = c(\nu_2) > 0; \quad \mu_1, \mu_2 \text{ and } \omega_1\omega_2 \text{ are unramified.}$$

For $i = 1, 2$ we have

$$\begin{aligned} W_i(a(t)) &= \mu_i|\cdot|^{\frac{1}{2}}(t)\mathbb{I}_{\mathbf{Z}_\ell}(t),\\ W_i(a(t)w) &= \mu_i^{-1}|\cdot|^{\frac{1}{2}}(\ell^c)\cdot\varepsilon\left(\frac{1}{2},\pi_i\right)\nu_i|\cdot|^{1/2}(t)\mathbb{I}_{\ell^{-c}\mathbf{Z}_\ell}(t). \end{aligned}$$

Combined with the equation $\varepsilon(1/2,\pi_1)\varepsilon(1/2,\pi_2) = \nu_1\nu_2(\ell^c)\nu_2(-1)$, we obtain

$$\begin{aligned} \mathcal{I}(1) &= (1-\mu_1\mu_2(\ell)\,|\ell|^s)^{-1},\\ \mathcal{I}(w) &= \mu_1\mu_2(\ell)^{-c}\,|\ell|^{c-cs}\,(1-\nu_1\nu_2(\ell)\,|\ell|^s)^{-1}. \end{aligned}$$

Moreover, for $0 < n < c$, from the explicit formula for Whittaker newforms in Proposition 6.4 (ii) below, we deduce by a straightforward computation that

$$\mathcal{I}\left(\begin{pmatrix} 1 & 0 \\ \ell^n & 1 \end{pmatrix}\right) = \zeta_\ell(1)\mu_1\mu_2(\ell)^{n-c}|\ell^{n-c}|^{s-1}.$$

With the above formulae, using a complete coset representatives for $\mathrm{GL}_2(\mathbf{Z}_\ell)/\mathcal{U}_0(\ell^c)$ given by

$$\left\{ \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}, w\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \;\middle|\; x \in \mathbf{Z}_\ell/\ell^c\mathbf{Z}_\ell, y \in \ell\mathbf{Z}_\ell/\ell^c\mathbf{Z}_\ell \right\}.$$

and the relation $\omega = \mu_1\mu_2\nu_1\nu_2$, we find that the sum in (6.2) equals

$$\begin{aligned} & f^\star_{\mathcal{D},s-\frac{1}{2}}(1)\mathcal{I}(1) + \sum_{n=0}^{c-1}(\ell-1)\ell^{c-n-1}f^\star_{\mathcal{D},s-\frac{1}{2}}(\begin{pmatrix} 1 & 0 \\ \ell^n & 1 \end{pmatrix})\mathcal{I}\left(\begin{pmatrix} 1 & 0 \\ \ell^n & 1 \end{pmatrix}\right) \\ & + \ell^{c-1}f^\star_{\mathcal{D},s-\frac{1}{2}}(w)\mathcal{I}(w) \\ =& L(2s,\omega)\left(L(s,\mu_1\mu_2) + \sum_{n=1}^{c-1}\nu_1\nu_2(\ell^n)\,|\ell|^{sn} + \nu_1\nu_2(\ell^c)\,|\ell|^{sc}\,L(s,\nu_1\nu_2)\right) \\ =& L(s,\mu_1\mu_2)L(s,\nu_1\nu_2) = L(s,\pi_1\times\pi_2). \end{aligned}$$

Hence we obtain

$$\begin{aligned} \Psi(W_1, W_2, f^\star_{\mathcal{D},s}) &= \mathrm{vol}(\mathcal{U}_0(\ell^c), \mathrm{d}u)L(s,\pi_1\times\pi_2) \\ &= [\mathrm{GL}_2(\mathbf{Z}_\ell) : \mathcal{U}_0(\ell^n)]^{-1}\cdot L(s,\pi_1\times\pi_2). \end{aligned}$$

The lemma follows. Q.E.D.

Proposition 6.4. *Let $\widehat{\mathbf{Z}_\ell^\times}$ be the set of continuous characters of $\mathbf{Z}_\ell^\times$. For $\xi \in \widehat{\mathbf{Z}_\ell^\times}$, we extend ξ to a character of $\mathbf{Q}_\ell^\times$ so that $\xi(\ell) = 1$. Let $\pi = \mu \boxplus \nu$ with $c(\mu) = 0$ and $c(\nu) = c > 0$. Let $W_\pi \in \mathcal{W}(\pi,\psi)$ be the Whittaker newform. For $n \geq 0$, let $W_\pi^{(n)} = \rho\left(\begin{pmatrix} 1 & 0 \\ \ell^n & 1 \end{pmatrix}\right)W_\pi$.*

(i) *For $n = 0$, $W_\pi^{(0)}(a(t))$ is supported in $q^{-c}\mathbf{Z}_\ell$. For $t \in \mathbf{Z}_\ell^\times$ and $-c \leq m < -1$, we have*

$$\begin{aligned} W_\pi^{(0)}(a(\ell^m t)) = |\ell|^{c/2}\,\zeta_\ell(1)\mu(\ell)^{-c}\varepsilon\left(\frac{1}{2},\pi\right) \\ \times \sum_{\xi\in\widehat{\mathbf{Z}_\ell^\times}, c(\nu\xi^{-1})=-m} \varepsilon\left(\frac{1}{2},\nu\xi^{-1}\right)^{-1}\xi(-t), \end{aligned}$$

and

$$W_\pi^{(0)}(a(\ell^{-1}t)) = |\ell|^{c/2}\zeta_\ell(1)\mu(\ell)^{-c}\varepsilon\left(\frac{1}{2},\pi\right)\sum_{\substack{\xi\in\widehat{\mathbf{Z}_\ell^\times},c(\xi)=c\\ c(\nu\xi^{-1})=1}}\varepsilon\left(\frac{1}{2},\nu\xi^{-1}\right)^{-1}\xi(-t)$$
$$-|\ell|^{c/2+1/2}\,\zeta_\ell(1)\mu(\ell)^{-c}\varepsilon\left(\frac{1}{2},\pi\right)\nu(-q^{-1}t).$$

For $t\neq 0\in\mathbf{Z}_\ell$, *we have*

$$W_\pi^{(0)}(a(t)) = |\ell|^{c/2}\,\mu(\ell)^{-c}\nu(-1)\varepsilon\left(\frac{1}{2},\pi\right)\nu|\cdot|^{1/2}(t).$$

(ii) *For* $0<n<c$, *we have*

$$W_\pi^{(n)}(a(t)) = \mathbb{I}_{\ell^{n-c}\mathbf{Z}_\ell^\times}(t)\cdot\zeta_\ell(1)\,|\ell|^{c/2-n/2}\,\mu(\ell)^{n-c}\varepsilon\left(\frac{1}{2},\pi\right)$$
$$\times\sum_{\xi\in\widehat{\mathbf{Z}_\ell^\times},0\leq c(\xi)\leq c-n}\varepsilon\left(\frac{1}{2},\nu\xi^{-1}\right)^{-1}\xi(-t).$$

(iii) *For* $n\geq c$, $W_\pi^{(n)}=W_\pi$.

Proof. Let $n\geq 0$ and $m\in\mathbf{Z}$. For $\xi\in\widehat{\mathbf{Z}_\ell^\times}$, put

$$A_m^{(n)}(\xi)=\int_{\mathbf{Z}_\ell^\times}W_\pi^{(n)}(a(\ell^m t))\xi^{-1}(t).$$

For $t\in\mathbf{Z}_\ell^\times$, we have

$$W_\pi^{(n)}(a(\ell^m t))=\sum_{\xi\in\widehat{\mathbf{Z}_\ell^\times}}A_m^{(n)}(\xi)\cdot\xi(t).$$

Recall the local functional equation for GL(2)

$$\int_{\mathbf{Q}_\ell^\times}W_\pi^{(n)}(a(t))\xi^{-1}(t)\,|t|^{s-1/2}\,\mathrm{d}^\times t=\frac{L(s,\pi\otimes\xi^{-1})}{L(1-s,\pi^\vee\otimes\xi)\varepsilon(s,\pi\otimes\xi^{-1})}$$
$$\times\int_{\mathbf{Q}_\ell^\times}W_\pi^{(n)}\,(a(t)w)\,\xi\omega_\pi^{-1}(t)\,|t|^{1/2-s}\,\mathrm{d}^\times t.$$

Let $\omega_\pi=\mu\nu$ be the central character of π. Note that (cf. [Sch02, §2.4])

$$W_{\pi^\vee}(a(t)) = \mu^{-1}|\cdot|^{1/2}(t)\mathbb{I}_{\mathbf{Z}_\ell}(t),$$
$$\rho\,(w)\,W_\pi(a(t)) = \varepsilon\left(\frac{1}{2},\pi\right)\omega_\pi(t)W_{\pi^\vee}(a(\ell^c t)).$$

Therefore, the local functional equation for GL(2) implies that

$$\begin{aligned}&\sum_{m\in\mathbf{Z}} A_m^{(n)}(\xi)\cdot|\ell|^{(s-1/2)m}\\ =&\frac{L(s,\pi\otimes\xi^{-1})}{L(1-s,\pi^\vee\otimes\xi)\varepsilon(s,\pi\otimes\xi^{-1})}\cdot\varepsilon\left(\frac{1}{2},\pi\right)\cdot\xi(-1)\cdot|\ell|^{(s-1/2)c}\\ &\times\int_{\mathbf{Q}_\ell^\times}\psi(\ell^{n-c}t)\xi\mu^{-1}(t)\,|t|^{1-s}\,\mathbb{I}_{\mathbf{Z}_\ell}(t)\mathrm{d}^\times t.\end{aligned}$$

A direct calculation shows that

$$\begin{aligned}&\int_{\mathbf{Q}_\ell^\times}\psi(\ell^{n-c}t)\xi\mu^{-1}(t)\,|t|^{1-s}\,\mathbb{I}_{\mathbf{Z}_\ell}(t)\mathrm{d}^\times t\\ =&\begin{cases}|\ell|^{c(\xi)/2}\,\zeta_\ell(1)\mu(\ell)^{n-c}\left|q^{c-n-c(\xi)}\right|^{1-s}\varepsilon\left(\frac{1}{2},\mu\xi^{-1}\right)\text{ if }0<c(\xi)\le c-n,\\ \zeta_\ell(1)\mu(\ell)^{n-c}\left|q^{c-n}\right|^{1-s}(1-\mu(\ell)q^{-s})(1-\mu^{-1}(\ell)q^{s-1})^{-1}\\ \quad\text{if }\xi=1\text{ and }c-n\ge 1,\\ (1-\mu^{-1}(\ell)q^{s-1})^{-1}\text{ if }\xi=1\text{ and }c=n,\\ 0\text{ if }c(\xi)>c,\end{cases}\end{aligned}$$

and for $0\le c(\xi)\le c$

$$\begin{aligned}&\frac{L(s,\pi\otimes\xi^{-1})}{L(1-s,\pi^\vee\otimes\xi)\varepsilon(s,\pi\otimes\xi^{-1})}\\ =&\begin{cases}|\ell|^{(c(\xi)+c(\nu\xi^{-1}))(1/2-s)}\,\varepsilon\left(\frac{1}{2},\mu\xi^{-1}\right)^{-1}\varepsilon\left(\frac{1}{2},\nu\xi^{-1}\right)^{-1}\\ \quad\text{if }0<c(\chi)\le c\text{ and }c(\nu\xi^{-1})\neq 0,\\ (1-\nu^{-1}(\ell)\ell^{s-1})(1-\nu(\ell)\ell^{-s})^{-1}\,|\ell^c|^{1/2-s}\,\varepsilon\left(\frac{1}{2},\mu\xi^{-1}\right)\\ \quad\text{if }c(\xi)=c\text{ and }c(\nu\xi^{-1})=0,\\ (1-\mu^{-1}(\ell)\ell^{s-1})(1-\mu(\ell)\ell^{-s})^{-1}\,|\ell^c|^{1/2-s}\,\varepsilon\left(\frac{1}{2},\nu\right)\text{ if }\xi=1.\end{cases}\end{aligned}$$

We conclude that for $c(\xi)>c$,

$$A_m^{(n)}(\xi)=0.$$

For $0<c(\xi)\le c-n$ and $c(\nu\xi^{-1})>0$, $A_m^{(n)}(\xi)$ equals

$$=\begin{cases}|\ell|^{c/2-n/2}\,\zeta_\ell(1)\mu(\ell)^{-c+n}\varepsilon\left(\frac{1}{2},\pi\right)\varepsilon\left(\frac{1}{2},\nu\xi^{-1}\right)^{-1}\xi(-1) & \text{if }m=n-c,\\ 0 & \text{if }m\neq n-c.\end{cases}$$

For $c(\xi)=c$ and $c(\nu\xi^{-1})=0$, $A_m^{(n)}(\xi)$ equals

$$=|\ell|^{c/2}\zeta_\ell(1)\mu(\ell)^{-c}\varepsilon\left(\frac{1}{2},\pi\right)\begin{cases}-|\ell|^{1/2}\,\nu^{-1}(\ell) & \text{if } m=-1 \text{ and } n=0,\\ \zeta_\ell(1)^{-1}|q|^{m/2}\nu(\ell)^m & \text{if } m\geq 0 \text{ and } n=0,\\ 0 & \text{if } m<-1 \text{ or } n\neq 0.\end{cases}$$

For $\chi=1$ and $c-n\geq 1$,

$$A_m^{(n)}(\xi)=\begin{cases}|\ell|^{c/2-n/2}\,\zeta_\ell(1)\mu(\ell)^{n-c} & \text{if } m=n-c,\\ 0 & \text{if } m\neq n-c.\end{cases}$$

For $\chi=1$ and $c=n$,

$$A_m^{(n)}(\xi)=\begin{cases}|\ell|^{m/2}\,\mu(\ell)^m & \text{if } m\geq 0,\\ 0 & \text{if } m<0.\end{cases}$$

This completes the proof. Q.E.D.

Now we suppose that $\ell\in\Sigma_{(\mathrm{ii})}$, so $\mathrm{ord}_\ell(C_1)=\lceil\frac{c}{2}\rceil$ by (5.2). In addition, π_1 is minimal and π_1,π_2 are discrete series with $L(s,\pi_1\times\pi_2)\neq 1$. This in particular implies that $\pi_2=\pi_1^\vee\otimes\xi$ for some unramified character ξ of $\mathbf{Q}_\ell^\times$ by [GJ78, Propositions (1.2) and (1.4)], and hence $c=c_1=c_2$. Let $\tau_{\mathbf{Q}_{q^2}}$ be the unramified quadratic character associated to the unramified quadratic extension of $\mathbf{Q}_\ell$. A discrete serires representation π is said to be of type **1** if $\pi\simeq\pi\otimes\tau_{\mathbf{Q}_{q^2}}$, and of type **2** if $\pi\not\simeq\pi\otimes\tau_{\mathbf{Q}_{q^2}}$. Note that a special representation is always of type **2**.

Lemma 6.5. *If $\ell\in\Sigma_{(\mathrm{ii})}$, then we have*

$$\Psi\left(W_{\pi_1},W_{\pi_2},f^\star_{\mathcal{D},s}\right)$$
$$=\ell^{c-\lceil\frac{c}{2}\rceil}\cdot\frac{\zeta_{\mathbf{Q}_\ell}(2)\,|N|_{\mathbf{Q}_\ell}}{\zeta_{\mathbf{Q}_\ell}(1)}L(s,\pi_1\times\pi_2\otimes\chi)\begin{cases}1+|\ell| & \textit{if } \pi_1 \textit{ is of type } \mathbf{1},\\ 1 & \textit{if } \pi_1 \textit{ is of type } \mathbf{2}.\end{cases}$$

Proof. We may assume $\chi=1$ as in the previous case. Let f^0 be the spherical section in $\mathcal{B}(1,\omega^{-1},s-\frac{1}{2})$ normalized so that $f^0(1)=1$. Let $r=\lceil\frac{c}{2}\rceil=\mathrm{ord}_p(C_1)$. Then $f^\star_{\mathcal{D},s}$ is the Godement section associated with $\mathbb{I}_{\ell^r\mathbf{Z}_\ell}\otimes\mathbb{I}_{\mathbf{Z}_\ell}$ according to Definition 4.1. It is easy to verify that

$$f^\star_{\mathcal{D},s-\frac{1}{2}}=|\ell^r|^s\,L(2s,\omega)\cdot\rho(a(\ell^{-r}))f^0.$$

It is computed in [Hsi17, Proposition 6.9] that $\Psi(W_{\pi_1},W_{\pi_2},\rho\left(a(\ell^{-r})\right)f^\circ)$ equals

$$\frac{|\ell|^{r(1-s)}}{1+|\ell|}\cdot\frac{L(s,\pi_1\times\pi_2)}{L(2s,\omega)}\begin{cases}1+|\ell| & \text{if } \pi_1 \text{ is of type } \mathbf{1},\\ 1 & \text{if } \pi_1 \text{ is of type } \mathbf{2}.\end{cases}$$

The lemma follows. Q.E.D.

Finally, we consider the last case $\ell \in \Sigma_{(\mathrm{iii})}$.

Lemma 6.6. *If $\ell \in \Sigma_{(\mathrm{iii})}$, then*

$$\Psi(W_{\pi_1}, W_{\pi_2}, f^{\star}_{\mathcal{D},s}) = \frac{\zeta_{\mathbf{Q}_\ell}(2)\,|N|_{\mathbf{Q}_\ell}}{\zeta_{\mathbf{Q}_\ell}(1)} \cdot L(s, \pi_1 \times \pi_2 \otimes \chi),$$

and hence $\Psi^*_\ell(s) = 1$.

PROOF. We may assume $\chi = 1$ and write $W_i = W_{\pi_i}$, $i = 1, 2$ as before. Since $\ell \in \Sigma_{(\mathrm{iii})}$, by definition $\ell \mid C_2$, and $f^{\star}_{\mathcal{D},s-\frac{1}{2}}$ is the Godement section associated with $\mathbb{I}_{\ell^c\mathbf{Z}_\ell} \otimes \phi_{\omega^{-1}}$ according to Definition 4.1. It is easy to see that

$$f^{\star}_{\mathcal{D},s}(k) = \omega(d)\mathbb{I}_{\mathcal{U}_0(\ell^c)}(k) \text{ for } k = \begin{pmatrix} y & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Z}_\ell).$$

Therefore, we obtain

$$\begin{aligned}\Psi(W_1, W_2, f^{\star}_{\mathcal{D},s}) &= \int_{\mathrm{GL}_2(\mathbf{Z}_\ell)} \int_{\mathbf{Q}_\ell^\times} f^{\star}_{\mathcal{D},s-\frac{1}{2}}(k) W_1(a(t)u) W_2(a(-t)u)\,|t|^{s-1}\,\mathrm{d}^\times t \mathrm{d}u \\ &= [GL_2(\mathbf{Z}_\ell) : \mathcal{U}_0(\ell^c)]^{-1} \int_{\mathbf{Q}_\ell^\times} W_1\left(a(t)\right) W_2\left(a(-t)\right) |t|^{s-1}\,\mathrm{d}^\times t.\end{aligned}$$

Hence the lemma follows immediately if we can prove the following equality

$$\int_{\mathbf{Q}_\ell^\times} W_1\left(a(t)\right) W_2\left(a(-t)\right) |t|^{s-1}\,\mathrm{d}^\times t = L(s, \pi_1 \times \pi_2). \tag{6.3}$$

To show (6.3), we first consider the case π_1 is spherical. Then π_2 is not spherical, and hence (6.3) can be verified easily. Now suppose that π_1 is special $\mu_1|\cdot|^{-\frac{1}{2}}\mathrm{St}$ or ramified principal series $\mu_1 \boxplus \nu_1$ with ν_1 ramified. By the minimality of π_1, μ_1 is unramified and $W_1(a(t)) = \mu_1|\cdot|^{\frac{1}{2}}(a)\mathbb{I}_{\mathbf{Z}_\ell}(a)$. We thus conclude that the left hand side of (6.3) equals $L(s, \pi_2 \otimes \mu_1)$. It remains to see

$$L(s, \pi_2 \otimes \mu_1) = L(s, \pi_1 \times \pi_2). \tag{6.4}$$

If π_2 is supercuspidal, (6.4) is clear. If $\pi_2 = \mu_2|\cdot|^{-\frac{1}{2}}\mathrm{St}$ is special, then (6.4) fails only when π_1 is special and $\mu_1\mu_2$ is unramified, which contradicts to the fact that $\ell \notin \Sigma_{(\mathrm{ii})}$. If $\pi_2 = \mu_2 \boxplus \nu_2$ is principal series, then the failure of (6.4) implies that $L(s, \pi_2 \otimes \nu_1) \neq 1$, and then π_2 is minimal by Hypothesis (M) and $\mu_1\mu_2\nu_1\nu_2$ is unramified. This implies $\ell \in \Sigma_{(\mathrm{i})}$, a contradiction. We thus shows (6.4), and hence (6.3) if π_1 is not supercuspidal. Finally, suppose that π_1 is supercuspidal. In this case, $W_1(t(a)) = \mathbb{I}_{\mathbf{Z}_\ell^\times}(a)$, and we must have $L(s, \pi_1 \times \pi_2) = 1$ as $\ell \in \Sigma_{(\mathrm{iii})}$. Thus the left hand side equals

$$\int_{\mathbf{Z}_\ell^\times} W_2(t(a))\mathrm{d}^\times t = 1 = L(s, \pi_1 \times \pi_2).$$

This completes the proof. Q.E.D.

§7. The interpolation formulae

We prove the main result in this paper with the setting and the notation in the introduction and §5.1. Recall that the finite set Σ_{exc} in the introduction is

$$\Sigma_{\mathrm{exc}} = \left\{\ell \in \Sigma_{(\mathrm{ii})} \mid \pi_{f,\ell} \simeq \pi_{f,\ell} \otimes \tau_{\mathbf{Q}_{\ell^2}}\right\}.$$

We continue to suppose that $\boldsymbol{f}$ satisfies the Hypothesis (CR) and fix a generator $\eta_{\boldsymbol{f}}$ of the congruence ideal of $\boldsymbol{f}$ used in the definition of Hida's canonical periods $\Omega_{\boldsymbol{f}_Q}$ for $Q \in \mathfrak{X}^+_{\mathbf{I}_1}$. We have the following

Theorem 7.1. *For each $a \in \mathbf{Z}/(p-1)\mathbf{Z}$, there exists a unique element $\mathcal{L}^{\boldsymbol{f}}_{\boldsymbol{F},a} \in \mathcal{R} = \mathbf{I}_1 \widehat{\otimes}_{\mathcal{O}} \mathbf{I}_2[\![T]\!]$ such that for every $\underline{Q} = (Q_1, Q_2, P) \in \mathfrak{X}^{\boldsymbol{f}}_{\mathcal{R}}$, we have*

$$\begin{aligned}
&\mathcal{L}^{\boldsymbol{f}}_{\boldsymbol{F},a}(Q_1, Q_2, P) \\
&= \Gamma_{\mathbf{V}_{\underline{Q}}}(0) \cdot \frac{L(\mathbf{V}_{\underline{Q}}, 0)}{(\sqrt{-1})^{2k_P - k_{Q_2} - 1}\Omega_{\boldsymbol{f}_{Q_1}}} \cdot \mathcal{E}^{\boldsymbol{f}}_p(\mathrm{Fil}^+ \mathbf{V}_{\underline{Q}}) \\
&\times (-1)^{a+1}\psi_{2,(p)}(-1) \prod_{\ell \in \Sigma_{\mathrm{exc}}} (1 + \ell^{-1}),
\end{aligned}$$

where $\mathcal{E}^{\boldsymbol{f}}_p(\mathrm{Fil}^+ \mathbf{V}_{\underline{Q}})$ is the modified p-Euler factor defined in (1.3).

Proof. For every Dirichlet character λ unramified at p, let $\boldsymbol{f} \otimes \lambda$ be the primitive Hida family associated with $\boldsymbol{f}|[\lambda]$. Then it is proved in [Hsi17, Proposition 7.5] that there exists a generator $\eta^0_{\boldsymbol{f}\otimes\lambda}$ of the congruence ideal of $\boldsymbol{f} \otimes \lambda$ such that for every arithmetic point Q, we have $\Omega_{(\boldsymbol{f}\otimes\lambda)_Q} = \Omega_{\boldsymbol{f}_Q}$. Therefore, we can deduce that up to a units in $\mathbf{I}$, the right hand side of the equation in the theorem is invariant under prime-to-p twists.

According to the discussion in Remark 6.2, we can choose a Dirichlet character λ of conductor $\mathfrak{c}(\lambda)$ such that $\mathfrak{c}(\lambda)^2 \mid N$ and for every arithmetic points (Q_1, Q_2), the pair $(\boldsymbol{f}_{Q_1} \otimes \lambda, \boldsymbol{g}_{Q_2} \otimes \lambda^{-1})$ satisfies the Hypothesis (M). Therefore, we may replace $\boldsymbol{F}$ by the twist $(\boldsymbol{f} \otimes \lambda, \boldsymbol{g} \otimes \lambda^{-1})$ with such λ and then define

$$\mathcal{L}^{\boldsymbol{f}}_{\boldsymbol{F},a} := \mathscr{L}^{\boldsymbol{f}}_{\boldsymbol{F},a} \cdot \prod_{\ell \in \Sigma_{(\mathrm{ii})}} \left| NC_1^{-1} \right|_{\mathbf{Q}_\ell} \in \mathcal{R}.$$

We put

$$(\pi_1, \pi_2, \chi) = (\pi_{\boldsymbol{f}_{Q_1}}, \pi_{\boldsymbol{g}_{Q_2}}, \epsilon_P \boldsymbol{\omega}^{a-k_P}); \quad s_0 = \frac{2k_P - k_{Q_1} - k_{Q_2}}{2}.$$

Then from Proposition 5.3 combined with the local calculation Lemma 6.1, Lemma 6.3, Lemma 6.5 and Lemma 6.6, we deduce that $\mathcal{L}^{\boldsymbol{f}}_{\boldsymbol{F},a}(\underline{Q})$ equals

$$\frac{L(s_0, \pi_1 \times \pi_2 \otimes \chi)}{\Omega_f} \cdot \mathcal{E}^1_p(s_0, \pi_1 \times \pi_2 \otimes \chi) \cdot (-1)^{a+1} (\sqrt{-1})^{-2k_3+k_2+1} \prod_{\ell \in \Sigma_{\mathrm{exc}}} (1 + \ell^{-1}).$$

Finally, a simple computation of local Langlands parameters associated with π_1 and π_2 shows that

$$L(s, \pi_1 \times \pi_2 \otimes \chi) = \Gamma_{\mathbf{V}_{\underline{Q}}}(s + \frac{k_{Q_1} + k_{Q_2}}{2}) \cdot L(\mathbf{V}_{\underline{Q}}, s + \frac{k_{Q_1} + k_{Q_2}}{2}),$$

where $\Gamma_{\mathbf{V}^\dagger_{\underline{Q}}}(s) = L(s + 1 - \frac{k_{Q_1}+k_{Q_2}}{2}, \pi_{1,\infty} \times \pi_{2,\infty} \otimes \chi_\infty)$ is the Γ-factor of $\mathbf{V}_{\underline{Q}}$ in (1.1) and that

$$\mathcal{E}^1_p(s_0, \pi_1 \times \pi_2 \otimes \chi) = \mathcal{E}^{\boldsymbol{f}}_p(\mathrm{Fil}^+ \mathbf{V}_{\underline{Q}})$$

in view of the definitions (1.3) and (6.1). Now the theorem follows. Q.E.D.

We proceed to establish the functional equation of the primitive p-adic L-functions. We first introduce the $\mathcal{R}$-adic root number for Rankin-Selberg convolution. To begin with, it follows from [Hsi17, Lemma 6.11] that there exists $\varepsilon^{(p\infty)}(\boldsymbol{f} \otimes \boldsymbol{g} \otimes \boldsymbol{\omega}^a) \in (\mathbf{I}_1 \widehat{\otimes}_{\mathcal{O}} \mathbf{I}_2)^\times$ such that

$$\varepsilon^{(p\infty)}(\boldsymbol{f} \otimes \boldsymbol{g} \otimes \boldsymbol{\omega}^a)(Q_1, Q_2) = \prod_{\ell \nmid p} \varepsilon(1 - \frac{k_{Q_1} + k_{Q_2}}{2}, \pi_{\boldsymbol{f}_{Q_1},\ell} \times \pi_{\boldsymbol{g}_{Q_2},\ell} \otimes \boldsymbol{\omega}^a).$$

Let $N_{\boldsymbol{fg}}$ be the tame conductor of $\pi_{\boldsymbol{f}_{Q_1}} \times \pi_{\boldsymbol{g}_{Q_2}}$ for any arithmetic specialization (Q_1, Q_2).[3] Then define the $\mathcal{R}$-adic root number $\varepsilon^{(p\infty)}(\mathbf{V}) \in (\mathbf{I}_1 \widehat{\otimes} \mathbf{I}_2[\![\Gamma]\!])^\times$ by

$$\varepsilon^{(p\infty)}(\mathbf{V}) := N_{\boldsymbol{fg}} \langle N_{\boldsymbol{fg}} \rangle_T^{-1} \cdot \varepsilon^{(p\infty)}(\boldsymbol{f} \otimes \boldsymbol{g} \otimes \boldsymbol{\omega}^a).$$

which satisfies the following interpolation property

(7.1)

$$\varepsilon^{(p\infty)}(\mathbf{V})(Q_1, Q_2, P) = \prod_{\ell \neq p\infty} \varepsilon(k_P - \frac{k_{Q_1} + k_{Q_2}}{2}, \pi_{\boldsymbol{f}_{Q_1},\ell} \times \pi_{\boldsymbol{g}_{Q_2},\ell} \otimes \epsilon_{P,\ell} \boldsymbol{\omega}_\ell^{a-k_P}).$$

Corollary 7.2. *Suppose that* $\psi_{1,(p)}\psi_{2,(p)} = \boldsymbol{\omega}^{a_0}$. *Let*

$$\breve{\boldsymbol{f}} = \boldsymbol{f} \otimes \overline{\psi}_1^{(p)}; \quad \breve{\boldsymbol{g}} = \boldsymbol{g} \otimes \overline{\psi}_2^{(p)}$$

and let $\breve{\boldsymbol{F}} = (\breve{\boldsymbol{f}}, \breve{\boldsymbol{g}})$. *Then we have*

$$\begin{aligned}&\mathcal{L}^{\breve{\boldsymbol{f}}}_{\breve{\boldsymbol{F}},1+a_0-a}(\boldsymbol{\varepsilon}_{\mathrm{cyc}}(\gamma_0)(1+T)^{-1}(1+T_1)(1+T_2))\\ =&\psi_{1,(p)}(-1)(-1) \cdot \varepsilon^{(p\infty)}(\mathbf{V}) \cdot \mathcal{L}^{\boldsymbol{f}}_{\boldsymbol{F},a}(T).\end{aligned}$$

Proof. It suffices to show the equation of both sides are equal after specialized at all arithmetic points $\underline{Q} = (Q_1, Q_2, P) \in \mathfrak{X}^{\boldsymbol{f}}_{\mathcal{R}}$. Let $\underline{\breve{Q}} = (Q_1, Q_2, \breve{P}) \in \mathfrak{X}^{\breve{\boldsymbol{f}}}_{\mathcal{R}}$ be the arithmetic point defined by

$$k_{\breve{P}} = -k_P + k_{Q_2} + k_P + 1, \quad \epsilon_{\breve{P}} = \epsilon_P^{-1}\epsilon_{Q_1}\epsilon_{Q_2}.$$

Let $\breve{\chi} = \boldsymbol{\omega}^{1+a_0-a}\epsilon_{\breve{P}}\boldsymbol{\omega}^{k_{\breve{P}}} = \chi^{-1}\omega_{1,(p)}^{-1}\omega_{2,(p)}^{-1}$. Then left hand side specialized at $\underline{Q}$ equals

$$\begin{aligned}&\mathcal{L}^{\breve{\boldsymbol{f}}}_{\breve{\boldsymbol{F}},1+a_0-a}(\underline{\breve{Q}})\\ =&\frac{L(1-s_0, \pi_1^\vee \times \pi_2^\vee \otimes \chi^{-1})}{(\sqrt{-1})^{2k_1+k_2-2k_3+1}\Omega_f} \cdot \mathcal{E}_p^1(1-s_0, \breve{f}, \breve{g} \otimes \breve{\chi}) \cdot (-1)^{2+a_0-a} \prod_{\ell \in \Sigma_{\mathrm{exc}}} (1+\ell^{-1}).\end{aligned}$$

We have the relation

$$\varepsilon(s_0, \pi_{2,p} \otimes \alpha_f \chi_p) \cdot \mathcal{E}_p(s_0, f, g \otimes \chi) = \varepsilon(1-s_0, \pi_{2,p}^\vee \otimes \alpha_f \omega_{1,p}^{-1}\chi_p^{-1}) \cdot \mathcal{E}_p(1-s, \breve{f}, \breve{g} \otimes \breve{\chi})$$

and

$$\varepsilon(s_0, \pi_{1,\infty} \times \pi_{2,\infty} \otimes \chi_\infty) = (\sqrt{-1})^{2k_{Q_1}-2} = (-1)^{k_1-1}.$$

[3]This is independent of any choice of arithmetic specializations by the rigidity of automorphic types in Hida families.

as $\pi_{1,\infty}$ and $\pi_{2,\infty}$ are discrete series of weight k_{Q_1} and k_{Q_2}. Therefore, by the functional equation

$$L(1-s,\pi_1^\vee\times\pi_2^\vee\otimes\chi^{-1})=\varepsilon(s,\pi_1\times\pi_2\otimes\chi)L(s,\pi_1\times\pi_2\otimes\chi)$$

and by (7.2), one verifies easily that

$$(-1)^{k_2+a_0-1}\omega_{2,p}(-1)\mathcal{L}^{\breve{\boldsymbol{f}}}_{\breve{\boldsymbol{F}},1+a_0-a}(\underline{\breve{Q}})=\prod_{v\neq p\infty}\varepsilon(s_0,\pi_{1,v}\times\pi_{2,v}\otimes\chi_v)\cdot\mathcal{L}^{\boldsymbol{f}}_{\boldsymbol{F},a}(\underline{Q}).$$

Keep in mind that $\omega_{2,p}(-1)=\psi_{2,(p)}(-1)(-1)^{k_{Q_2}}$, and we get the corollary. Q.E.D.

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