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Packing Directed Circuits through Prescribed Vertices Bounded-Fractionally

Naonori KAKIMURA*† and Ken-ichi KAWARABAYASHI‡§

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Abstract

A seminal result of Reed, Robertson, Seymour, and Thomas says that a directed graph has either k vertex-disjoint directed circuits or a set of at most $f(k)$ vertices meeting all directed circuits. This paper aims at generalizing their result to packing directed circuits through prescribed vertices. Even, Naor, Schieber, and Sudan showed a fractional version of packing such circuits. In this paper, we show that a fractionality can be bounded at most fifth: Given an integer k and a vertex subset S, whose size may not depend on k, we prove that either G has a $1/5$-integral packing of k disjoint circuits, each of which contains at least one vertex of S, or G has a vertex set X of order at most $f(k)$ (for some function f of k) such that $G - X$ has no such a circuit. We also give an FPT approximation algorithm for finding a $1/5$-integral packing of circuits through prescribed vertices. This algorithm finds a $1/5$-integral packing of size approximately k in polynomial time if it has a $1/5$-integral packing of size k for a given directed graph and an integer k.
Key Words: Disjoint Circuits, Feedback Vertex Sets, FPT Approximability

1 Introduction

This paper deals with packing vertex-disjoint circuits in a directed graph. We are given a directed graph (digraph), which is finite and may have loops and multiple edges. A *path* and a *circuit* of a digraph mean directed path and circuit, respectively. The *circuit packing problem* is the problem of finding k vertex-disjoint circuits for a given positive integer k.

A family $\mathcal{F}$ of subgraphs is said to have the *Erdős-Pósa property* if for every integer k there exists an integer $f(k, \mathcal{F})$ such that every (undirected or directed) graph G contains either k vertex disjoint subgraphs in $\mathcal{F}$ or a set X of at most $f(k, \mathcal{F})$ vertices such that $G \setminus X$ has no subgraph in $\mathcal{F}$. The term *Erdős-Pósa property* arose because in [2], Erdős and Pósa proved that the family of circuits in an undirected graph has this property. The Erdős-Pósa property is

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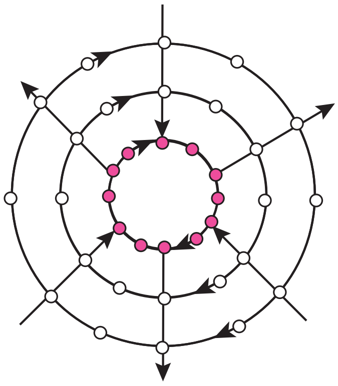

Figure 1: A Counterexample for the Erdős-Pósa property for S-circuits

concerned about both "packing", i.e., k vertex-disjoint subgraphs and "covering", i.e., at most $f(k)$ vertices that hit all the subgraphs in G. Starting with this result, there is a host of results in this direction. Packing appears almost everywhere in extremal graph theory, while covering leads to the well-known concept "feedback set" in theoretical computer science. Also, the circuit packing problem is a well-known problem too, e.g., [12].

For a directed graph, the Erdős-Pósa property for the family of directed circuits was conjectured by Younger [19] in 1973. Gallai [7] had previously conjectured the case of $k = 2$, which was shown by [13], and the planar case was resolved by Reed and Shepherd [16]. In 1996, Reed, Robertson, Seymour, and Thomas [15] settled Younger's conjecture:

Theorem 1.1 *For any positive integer k, there exists a constant t_k such that every digraph has either k vertex-disjoint circuits, or a vertex subset X with $|X| \leq t_k$ that meets all directed circuits.*

In this paper, we aim at extending Theorem 1.1 to packing vertex-disjoint circuits through prescribed vertices. Let $G = (V, E)$ be a digraph with vertex set V and arc set E, and S be a vertex subset. The *S-circuit packing problem* is the problem of finding k vertex-disjoint S-circuits for a given positive integer k, where an *S-circuit* is a circuit which has a vertex in S. We denote by $\nu_S(G)$ the maximum k such that G has k vertex-disjoint S-circuits. The minimum size of an *S-hitting set*, a vertex subset meeting all S-circuits, is denoted by $\tau_S(G)$. Then we consider the following natural generalization of Theorem 1.1: for a positive integer k, there is a constant t_k such that every digraph $G = (V, E)$ with $S \subseteq V$ satisfies $\nu_S(G) \geq k$ or $\tau_S(G) \leq t_k$. However, this statement does not hold. Indeed, as pointed out by Paul Wollan, a digraph G as in Figure 1 satisfies $\nu_S(G) = 1$ because any two S-circuits intersect by the planarity, while $\tau_S(G) = \Omega(\sqrt{|V|})$ holds.

We then relax an S-circuit packing to a *half-integral* packing as follows:

Conjecture 1.2 *Let k be a positive integer. There exists a constant t_k such that every digraph $G = (V, E)$ with prescribed vertices $S \subseteq V$ has either k S-circuits in which each vertex is used at most twice, or an S-hitting set X with $|X| \leq t_k$.*

In this paper, we discuss a fractional packing of S-circuits, and tackle to bound its fractionality. It should be noted that, for an undirected graph, it is shown in [11] that the family of S-circuits has the Erdős-Pósa property.

1.1 Fractional Packing of Circuits

A *fractional packing of (S-)circuits* in a digraph $G = (V, E)$ is a function q assigning a non-negative rational $q(C)$ to every (S-)circuit C, such that for every vertex v,

$$\sum\{q(C) \mid v \in V(C)\} \leq 1.$$

We define the *value* of q to be the summation of $q(C)$ over all directed (S-)circuits C. The maximum value of a fractional packing of S-circuits is denoted by $\nu_S^*(G)$ (ν_S^* if no ambiguity). Seymour [17] showed $\tau_S(G) = O(\nu_S^* \log \nu_S^* \log\log \nu_S^*)$ when $S = V$. This result was extended to the case of $S \subsetneq V$ by Even, Naor, Schieber, and Sudan [5], which means that a fractional version of the Erdős-Pósa property for S-circuits holds.

The main result of this paper is to show a bounded fractional version of the Erdős-Pósa property for S-circuits. Let q be a fractional packing of S-circuits. We say that q is a $1/p$-*integral packing*, where p is a positive integer, if $q(C)$ is restricted to $\{0, \frac{1}{p}, \frac{2}{p}, \ldots, \frac{p-1}{p}, 1\}$ for any S-circuit C. The maximum value of a $1/p$-integral packing is denoted by $\nu_S^p(G)$. We prove the following theorem in Section 3.

Theorem 1.3 *Let k be a positive integer. Then there exists a constant $f(k)$ such that, for any digraph $G = (V, E)$ with $S \subseteq V$, it holds that either $\nu_S^5(G) \geq k$ or $\tau_S(G) \leq f(k)$.*

We now review algorithmic aspects for the circuit packing problem. For an undirected graph, we can find k disjoint circuits in linear time for fixed k. Indeed, if a given graph has large tree width, then we can do this from the existence of a large grid minor, and otherwise we can use the dynamic programming to find disjoint circuits. In contrast, for a directed graph, the circuit packing problem is W[1]-hard, which follows from Slivkins [18]. Thus the circuit packing problem for a digraph is not fixed-parameter tractable (FPT) unless the class FPT equals the class W[1]. In the last section of [15], Reed, Robertson, Seymour, and Thomas mentioned an $O(n^{f(k)})$ time algorithm for this problem for some function f.

This paper discusses *parameterized approximability* for the S-circuit packing problem. An *FPT algorithm* is an algorithm which runs in $f(k)n^{O(1)}$, where f is a computable function, k is the parameter value, and n is the size of the input. An FPT algorithm for a parameterized maximization problem is an *FPT approximation algorithm with approximation ratio* ρ if given instance of the problem and a positive integer k the algorithm computes a solution of cost at least $k/\rho(k)$ if the instance has a solution of cost at least k, where ρ is a computable function such that $k/\rho(k)$ is nondecreasing and unbounded. Grohe and Grüber [10] devised an FPT approximation algorithm for the circuit packing problem based on the proof of Theorem 1.1. Our proof of Theorem 1.3 provides an FPT approximation algorithm for the problem of a 1/5-integral packing of S-circuits.

Theorem 1.4 *The problem of finding a* $1/5$*-integral S-circuit packing has an FPT approximation algorithm with polynomial running time.*

We will describe the proof of Theorem 1.4 in Section 4.

The linear programing dual of the fractional S-circuit packing problem is the fractional *subset feedback vertex set problem*, which is the problem of finding a minimum S-hitting set in a digraph. For this problem, Even, Naor, Rao, and Schieber [4] and Even, Naor, Schieber, and Sudan [5] gave $O(\log \nu_S^* \log\log \nu_S^*)$-approximation algorithms, which are used to obtain Theorem 1.4 in Section 4. For a planar digraph with $S = V$, Goemans and Williamson [8] provided a 9/4-approximation algorithm. Their algorithm does not work when $S \subsetneq V$, but their framework also leads to a 9/4-approximation algorithm for the subset feedback vertex set problem of a planar *undirected* graph. For non-planar undirected graphs, Even, Naor, and Zosin [6] presented an 8-approximation algorithm.

1.2 Preliminaries

The rest of this section is devoted to giving some notations and definitions. We assume that a path has at least one vertex, and no repeated vertices otherwise specified. A *linkage* L in a digraph G is a subdigraph consisting of vertex-disjoint paths. Let a linkage L consist of $P_1, \ldots, P_k$ and P_i be a path from a_i to b_i. Then we say that L *links* $(a_1, \ldots, a_k)$ to $(b_1, \ldots, b_k)$. If $A, B \subseteq V$ with $a_1, \ldots, a_k \in A$ and $b_1, \ldots, b_k \in B$, then L is called a linkage from A to B. The number k is the *size* of L.

A *separation* in G is an ordered pair (X, Y) of subsets of V with $X \cup Y = V$ so that no edge has the tail in $X \setminus Y$ and the head in $Y \setminus X$. Its *order* is $|X \cap Y|$. We shall frequently need the following version of Menger's theorem.

Theorem 1.5 *Let $G = (V, E)$ be a digraph with $A, B \subseteq V$, and $k \geq 0$ be an integer. Then exactly one of the following holds:*

- *there is a linkage from A to B of size k.*
- *there is a separation (X, Y) of G of order $< k$ with $A \subseteq X$ and $B \subseteq Y$.*

2 Two Linkages between Prescribed Vertices and a Hitting Set

Let $G = (V, E)$ be a digraph with $S \subseteq V$, and H be a minimum S-hitting set. For $J \subseteq S$, we denote by $\lambda_1(J)$ the maximum size of a linkage from J to H, and by $\lambda_2(J)$ to J from H. In this section, we show that there exists a sufficiently large subset $T \subseteq S$ with $\lambda_1(T) = \lambda_2(T) = |T|$.

We first observe the following lemma.

Lemma 2.1 *Let $G = (V, E)$ be a digraph with $S \subseteq V$, and H be a minimum S-hitting set with $|H| = \tau_S(G)$. If (X, Y) is a separation with $S \subseteq X$ and $H \subseteq Y$, then $|X \cap Y| \geq \tau_S(G)$ holds. Similarly, a separation (X, Y) with $H \subseteq X$ and $S \subseteq Y$ also has the order $\geq \tau_S(G)$.*

Proof: Since each S-circuit C meets both vertices in H and S, the circuit C has a vertex in $X \cap Y$, and hence $X \cap Y$ is an S-hitting set. Thus $\tau_S(G) \leq |X \cap Y|$. $\square$

Theorem 2.2 *Let $G = (V, E)$ be a digraph with $S \subseteq V$. Let H be a minimum S-hitting set with $|H| = \tau_S(G)$. Then there exists $T \subseteq S$ with $|T| \geq \tau_S(G)/2$ such that G has two linkages of order $|T|$ from T to H and from H to T.*

Proof: Take a maximum subset $T \subseteq S$ such that G has two linkages of order $|T|$ from T to H and from H to T, that is, $\lambda_1(T) = \lambda_2(T) = |T|$. Let $t = |T|$. We will show that $t \geq \tau_S(G)/2$.

Assume to the contrary that $t < \tau_S(G)/2$. Let

$$\begin{aligned} T_1 &= \{v \in S \mid \lambda_1(T \cup \{v\}) = \lambda_1(T)\}, \\ T_2 &= \{v \in S \mid \lambda_2(T \cup \{v\}) = \lambda_2(T)\}. \end{aligned}$$

Note that $T \subseteq T_i$ for $i = 1, 2$. We denote $S_i = S \setminus T_i$ for $i = 1, 2$. By the maximality of T, we have $S_1 \cap S_2 = \emptyset$.

We first show $\lambda_1(T_1) = \lambda_2(T_2) = t$. First assume that $\lambda_1(T_1) > t$, i.e., the maximum size of a linkage from T_1 to H is $> t$. Take a vertex $v \in T_1 \setminus T$, and let $T' = T \cup \{v\}$. Since G has no linkage of size $> t$ from T' to H, it follows from Theorem 1.5 that there exists a separation (A, B) with $T' \subseteq A$, $H \subseteq B$, and $|A \cap B| = t$. Among such separations, choose (A, B) such that A is maximal. The set $B \setminus A$ has a vertex u of T_1, otherwise (A, B) separates T_1 and H, which contradicts $\lambda_1(T_1) > t$. By Theorem 1.5 applied to $T \cup \{u\}$ and H, there exists a separation (A', B') with $T \cup \{u\} \subseteq A'$, $H \subseteq B'$, and $|A' \cap B'| = t$. Note that $(A \cup A', B \cap B')$ is a separation between T' and H, which implies that $|(A \cup A') \cap (B \cap B')| > |A \cap B|$ by $A \subset A \cup A'$ and the maximality of A. By $|A \cap B| + |A' \cap B'| = |(A \cup A') \cap (B \cap B')| + |(A \cap A') \cap (B \cup B')|$, we have $|(A \cap A') \cap (B \cup B')| < |A' \cap B'| = t$. Since $(A \cap A', B \cup B')$ is a separation between T and H, this contradicts $\lambda_1(T) = t$. Thus we obtain $\lambda_1(T_1) = t$. Similarly, $\lambda_2(T_2) = t$ holds.

By $\lambda_1(T_1) = t$, it follows from Theorem 1.5 that there exists a separation (X, Y) with $T_1 \subseteq X$, $H \subseteq Y$, and $|X \cap Y| = t$. We denote $W = X \cap Y$. Let $S_Y = S \setminus X$. Note that S_Y is nonempty, otherwise (X, Y) separates S and H, which contradicts Lemma 2.1 by $|X \cap Y| < \tau_S(G)$. Since $T_1 \subseteq X$, we have $S_Y \subseteq S_1$, and hence $T \cup S_Y \subseteq T_2$ because $S_1 \cap S_2 = \emptyset$. By $\lambda_2(T \cup S_Y) \leq \lambda_2(T_2) = t$, there exists a separation (X', Y') with $H \subseteq X'$, $T \cup S_Y \subseteq Y'$, and $|X' \cap Y'| \leq t$. Let $W' = X' \cap Y'$.

We claim that $W \cup W'$ is an S-hitting set. Let C be an S-circuit, and v be a vertex in $S \cap V(C)$. Since C has a vertex in H, the circuit C has a vertex in W if $v \in S \cap X$, and a vertex in W' if $v \in S_Y$. Thus each S-circuit intersects $W \cup W'$. Therefore, we obtain

$$\tau_S(G) \leq |W \cup W'| \leq |W| + |W'| \leq 2t.$$

By $2t < \tau_S(G)$, this is a contradiction. □

3 Fifth-Integral Packing of S-Circuits

In this section, we shall prove Theorem 1.3 by induction on k. If $k = 1$, then this statement is obvious. We suppose that, for $l \leq k - 1$, there exists $f(l)$ such that, for a digraph $G = (V, E)$ with $S \subseteq V$, it holds that $\nu_S^5(G) \geq l$ or $\tau_S(G) \leq f(l)$.

Lemma 3.1 *Assume that k is a positive integer such that $f(k-1)$ exists. Let $G=(V,E)$ be a digraph with $S \subseteq V$ such that $\nu_S^5(G) < k$, and H be a minimum S-hitting set with $|H| = \tau_S(G)$. Let $A, B \subseteq H$ be disjoint subsets with $|A| = |B| = r$, where $r \geq 2f(k-1)$. Then there exists a linkage from A to B of size r with no inner vertices in H.*

Proof: Suppose not. Let $Z = H \setminus (A \cup B)$. By applying Theorem 1.5 to $G \setminus Z$, the graph G has a separation (X, Y) with $A \subseteq X$, $B \subseteq Y$, $Z \subseteq X \cap Y$, and $|(X \cap Y) \setminus Z| < r$. Since $|A \cup (X \cap Y)| < |H| = \tau_S(G)$, there exists an S-circuit C_1 with $V(C_1) \cap (A \cup (X \cap Y)) = \emptyset$. By $V(C_1) \cap H \neq \emptyset$, we have $V(C_1) \cap B \neq \emptyset$, and hence $V(C_1) \cap Y \neq \emptyset$. Since (X, Y) is a separation and $X \cap Y \cap V(C_1) = \emptyset$, the set $V(C_1)$ does not meet X, so $V(C_1) \subseteq Y \setminus X$. Similarly G has an S-circuit C_2 such that $V(C_2) \subseteq X \setminus Y$.

Therefore, $\nu_S^5(G \setminus X) < k-1$ and $\nu_S^5(G \setminus Y) < k-1$ hold. Hence the induction hypothesis implies that $\tau_S(G \setminus X) \leq f(k-1)$ and $\tau_S(G \setminus Y) \leq f(k-1)$. Since every S-circuit that is not a circuit of $G \setminus X$ or $G \setminus Y$ meets $X \cap Y$, we have

$$\tau_S(G) \leq \tau_S(G \setminus X) + \tau_S(G \setminus Y) + |X \cap Y| < 2f(k-1) + |Z| + r = 2f(k-1) + |H| - 2r + r \leq |H|,$$

which contradicts $|H| = \tau_S(G)$. Thus the statement holds. $\square$

Let p be a positive integer. A *$1/p$-integral linkage* L in a digraph G is a subdigraph consisting of paths, which may not be simple, such that each end vertex of these paths appears exactly once in these paths, and each inner vertex is in at most p times of them. The *size* of a $1/p$-integral linkage is the number of paths. A linkage in which each path has a vertex in S is called a linkage *through* S. By Theorem 2.2, we show the following lemma.

Lemma 3.2 *Assume that k is a positive integer such that $f(k-1)$ exists. Let $G=(V,E)$ be a digraph with $S \subseteq V$ such that $\nu_S^5(G) < k$, and H be a minimum S-hitting set with $|H| = \tau_S(G)$. Let $A, B \subseteq H$ be disjoint subsets with $|A| = |B| = r$, where $r \geq 2f(k-1)$. Then, if $\tau_S(G) \geq 10r$, there exists a $1/4$-integral linkage of order r from A to B through S.*

Proof: By Theorem 2.2, there exists $T \subseteq S$ with $|T| = t$, where $t \geq \tau_S(G)/2 \geq 5r$, such that G has two linkages L_1 from H to T and L_2 from T to H, both of whose orders are t. We may assume that L_1 and L_2 have no inner vertices in H by taking minimal paths. Let $T = \{s_1, \ldots, s_t\}$. We denote by $H_1 = \{a_1, \ldots, a_t\} \subseteq H$ the set of the vertices such that L_1 links $(a_1, \ldots, a_t)$ to $(s_1, \ldots, s_t)$. Similarly, let $H_2 = \{b_1, \ldots, b_t\} \subseteq H$ denote the set of the vertices such that L_2 links $(s_1, \ldots, s_t)$ to $(b_1, \ldots, b_t)$. Define $H_1' = H_1 \setminus \{a_i, b_i \mid a_i \text{ or } b_i \in A \cup B\}$, and $H_2' = H_2 \setminus \{a_i, b_i \mid a_i \text{ or } b_i \in A \cup B\}$. Then $|H_1'| = |H_2'| \geq |T| - 2(|A| + |B|) \geq r$. Take $A' \subseteq H_1'$ with $|A'| = r$. It follows from Lemma 3.1 that there exists a linkage L_a of order r from A to A' with no inner vertex in H. We denote the end vertices of L_a in A' by $\{a_{i_1}, \ldots, a_{i_r}\}$. The linkage L_1 contains a linkage L_1' which links $(a_{i_1}, \ldots, a_{i_r})$ to $(s_{i_1}, \ldots, s_{i_r})$, and L_2 contains a linkage L_2' which links $(s_{i_1}, \ldots, s_{i_r})$ to $(b_{i_1}, \ldots, b_{i_r})$, where $\{b_{i_1}, \ldots, b_{i_r}\} \subseteq H_2'$. By Lemma 3.1, G has a linkage L_b of order r from $\{b_{i_1}, \ldots, b_{i_r}\}$ to B with no inner vertex in H. Therefore, the union $L_a \cup L_1' \cup L_2' \cup L_b$ consists of r non-simple paths from A to B such that each inner vertices are in at most four times of them. Thus G has a $1/4$-integral linkage of order r from A to B through S. $\square$

The following theorem is similar to the result of Reed, Robertson, Seymour, and Thomas [15].

Theorem 3.3 *Let $k \geq 1$ be an integer such that $f(k-1)$ exists, and $p \geq 1$ be an integer. Then there exists a constant $g(k,p)$ such that the following holds: For any digraph $G = (V, E)$ and $S \subseteq V$ with $\nu_S^5(G) < k$ and $\tau_S(G) \geq g(k,p)$, there exist disjoint vertices $a_1, \ldots, a_p$ and $b_1, \ldots, b_p$ such that*

- *There is a linkage L_1 from $(a_1, \ldots, a_p)$ to $(b_1, \ldots, b_p)$.*
- *There is a 1/4-integral linkage L_2 from $(b_1, \ldots, b_p)$ to one of $(a_1, \ldots, a_p)$ and $(a_p, \ldots, a_1)$ through S.*

In order to prove Theorem 3.3, we shall need Ramsey's theorem [14].

Proposition 3.4 *For all integers $q, l, r \geq 1$, there exists a (minimum) integer $R_l(r,q) \geq 0$ so that the following holds: For a set Z with $|Z| \geq R_l(r,q)$, a set Q with $|Q| = q$, and a function h from $X \subseteq Z$ with $|X| = l$ onto Q, there exist $T \subseteq Z$ with $|T| = r$ and $x \in Q$ so that $h(X) = x$ for all $X \subseteq T$ with $|X| = l$.*

We also need the following result by Erdős and Szekeres [3].

Proposition 3.5 *Let s, t be integers, and let $n = (s-1)(t-1)+1$, and let $a_1, \ldots, a_n$ be distinct integers. Then either*

- *there exist $1 \leq i_1 < \cdots < i_s \leq n$ so that $a_{i_1} < \cdots < a_{i_s}$,*
- *there exist $1 \leq i_1 < \cdots < i_t \leq n$ so that $a_{i_1} > \cdots > a_{i_t}$.*

Proof Theorem 3.3: Let

$$\begin{aligned} l &= (p-1)^2 + 1, \\ r &= \max\{2f(k-1), (p+1)l\}, \\ q &= (l!+1)^2, \end{aligned}$$

and define

$$g(k,p) = \max\{10r, R_l(r,q) + l\}.$$

We claim that $g(k,p)$ satisfies the theorem. Let G be a digraph such that $\nu_S^5(G) < k$ and $\tau_S(G) \geq g(k,p)$, and H be a minimum S-hitting set with $|H| = \tau_S(G)$. Choose $A \subseteq H$ with $|A| = l$ and $Z = H \setminus A$. Thus $|Z| \geq R_l(r,q)$.

Let $Z = \{z_i \mid 1 \leq i \leq |Z|\}$. For each $X = \{z_{i_1}, \ldots, z_{i_x}\} \subseteq Z$, where $i_1 < \cdots < i_x$, we denote by $\bar{X}$ the x-tuple $(z_{i_1}, \ldots, z_{i_x})$, and for $1 \leq h \leq x$, we denote z_{i_h} by $\bar{X}(h)$.

Let $X \subseteq Z$ with $|X| = l$. We define a function $p_1(X)$ as follows. If there exists a linkage $L_1(X)$ of order l from A to X with no vertex in $Z \setminus X$, then there exists a sequence $(a_1, \ldots, a_l)$ of A so that $L_1(X)$ links $(a_1, \ldots, a_l)$ to $\bar{X}$, and define $p_1(X) = (a_1, \ldots, a_l)$. If no such linkage from A to X, then define $p_1(X) = \emptyset$. We next define a function $p_2(X)$ as follows. If there exists a 1/4-integral linkage $L_2(X)$ of order l from X to A through S, then there exists a sequence $(b_1, \ldots, b_l)$ of A so that $L_2(X)$ links $\bar{X}$ to $(b_1, \ldots, b_l)$ 1/4-integrally, and define $p_2(X) = (b_1, \ldots, b_l)$. If no such 1/4-integral linkage from A to X, then $p_2(X) = \emptyset$.

We define $h(X)$ to be the pair $(p_1(X), p_2(X))$. Let Q be the set of all pairs (a, b) such that each of a, b is either the empty set or a sequence of l vertices in A. Then $|Q| = (l! + 1)^2$, and $h(X) \in Q$ for each $X \subseteq Z$ with $|X| = l$. By Proposition 3.4, there exists $T \subseteq Z$ with $|T| = r$ and $(a, b) \in Q$ such that $h(X) = (a, b)$ for any $X \subseteq T$ with $|X| = l$. Note that a and b are not the empty set. Indeed, suppose $a = \emptyset$. Then take a set A' with $A \subseteq A' \subseteq H \setminus T$ and $|A'| = r$. By Lemma 3.1, there exists a linkage of order r from A' to T with no vertex in $H \setminus (A' \cup T)$. This linkage includes a linkage of order l from A to a subset X of T with no vertex in $H \setminus (A \cup X)$, which contradicts $a = \emptyset$. Thus a is not the empty set. Similarly, Lemma 3.2 implies that there exists a 1/4-integral linkage of order r from T to A' through S. This 1/4-integral linkage includes a 1/4-integral linkage of order l from a subset X of T to A through S. Thus b is not the empty set.

Let $a = (a_1, \ldots, a_l)$ and $b = (b_1, \ldots, b_l)$. For $1 \leq i \leq l$, define j_i so that $b_{j_i} = a_i$. By Proposition 3.5, there exists $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ so that

$$j_{i_1} < j_{i_2} < \cdots < j_{i_k}$$

or

$$j_{i_1} > j_{i_2} > \cdots > j_{i_k}.$$

Define $(i'_1, i'_2, \ldots, i'_k) = (j_{i_1}, j_{i_2}, \ldots, j_{i_k})$ in the first case, $(i'_1, i'_2, \ldots, i'_k) = (j_{i_k}, j_{i_{k-1}}, \ldots, j_{i_1})$ in the second case.

Let $D = \{\bar{T}(l), \bar{T}(2l), \ldots, \bar{T}(kl)\}$. Choose $X \subseteq T$ with $|X| = l$ and $\bar{X}(i_h) = \bar{T}(hl)$ for $1 \leq h \leq k$. We can do this since there are $\geq l - 1$ items in $\bar{T}$ between two items in D. Then the linkage $L_1(X)$ links $(a_1, \ldots, a_l)$ to $\bar{X}$ which includes a linkage L_1 from $(a_{i_1}, \ldots, a_{i_k})$ to $\bar{D}$. Similarly, choose $Y \subseteq T$ with $\bar{Y}(i'_h) = \bar{T}(hl)$ for $1 \leq h \leq k$. Then the 1/4-integral linkage $L_2(Y)$ includes a 1/4-integral linkage from $\bar{D}$ to $(a_{i'_1}, \ldots, a_{i'_k})$. This completes the proof. □

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3: We prove this statement by induction on k. Assume that $k \geq 1$ and $f(k-1)$ exists. Let $f(k) = g(k, 10k)$, where g is as in Theorem 3.3. Suppose that $G = (V, E)$ and $S \subseteq V$ satisfies $\nu_S^5(G) < k$ and $\tau_S(G) > g(k, p)$. Then it follows from Theorem 3.3 that G has a linkage L_1 of order $10k$ and a 1/4-integral linkage L_2 of order $10k$ through S as in Theorem 3.3. Since the ith and $(10k + 1 - i)$th paths in both linkages L_1 and L_2 form an S-circuit which may not be simple, we obtain at least one simple S-circuit from these paths. Hence $L_1 \cup L_2$ contains at least $5k$ such S-circuits $C_1, \ldots, C_{5k}$ such that each vertex is contained in at most five of them. Therefore, by defining $q(C_i) = 1/5$ for $1 \leq i \leq 5k$ and $q(C) = 0$ for the other S-circuits, we know the maximum value of q is $\geq k$, which contradicts the assumption. Hence such G does not exists, and consequently the statement holds. □

4 FPT Approximation Algorithm for Fifth-integral Packing

In this section, we present an FPT approximation algorithm for a fifth-integral packing of S-circuits. Our algorithm is a recursive one derived from making the proof of Theorem 1.3 algorithmic.

The framework of this algorithm is designed by analogy with Grohe and Grüber [10]. We will first design an algorithm $\mathbb{A}$ that computes a 1/5-integral packing of S-circuits of size at least $\nu^*(G)/\rho(\nu^*(G))$ for a given digraph G and some computable function ρ such that the running time is bounded by $g(\nu^*(G)) \cdot |G|^{O(1)}$ for some computable function g and the input size $|G|$. It follows from Proposition 9 in [1] that the algorithm $\mathbb{A}$ implies an FPT approximation algorithm for the 1/5-integral S-circuit packing problem, which proves Theorem 1.4.

It remains to describe the algorithm $\mathbb{A}$. This algorithm is a recursive algorithm based on the following two lemmas. The first one below can be obtained from the proofs of Lemmas 3.1, 3.2, and Theorem 2.2. For the completeness, we give a sketch of the proof.

Lemma 4.1 *Let $G = (V, E)$ be a digraph with $S \subseteq V$, and H be an S-hitting set. Let $r \leq |H|/10$. Then at least one of the following holds:*

(i) *For all distinct $A, B \subseteq H$ with $|A| = |B| = r$, there exist a linkage from A to B of size r with no inner vertices in H and a 1/4-integral linkage from B to A through S of order r.*

(ii) *There is an S-hitting set H' with $|H'| < |H|$.*

(iii) *There are two vertex disjoint subgraphs G_i, S-circuits C_i of G_i, and S-hitting sets H_i of G_i for $i = 1, 2$ such that $|H_1| = |H_2| = r$ and, for any S-hitting sets H'_i of G_i with $i = 1, 2$ the set $H'_1 \cup H'_2 \cup (V \setminus (V(G_1) \cup V(G_2)))$ is an S-hitting set of size at most $|H'_1| + |H'_2| + |H| - (r + 1)$.*

Furthermore, we can decide in $3^{|H|} \cdot |G|^{O(1)}$ time if H satisfies (i), (ii), or (iii), where $|G|$ is the input size.

Sketch of Proof: By the proof of Theorem 2.2, if there is no set $T \subseteq S$ with $\lambda_1(T) = \lambda_2(T) \geq |H|/2$, then (ii) holds, which can be tested in polynomial time with the aid of network flow algorithms. Assume that such T exists. By the proof of Lemma 3.1, if there exists $A, B \subseteq H$ with $|A| = |B| = r$ such that G has no linkage from A to B of size $\geq r$ with no inner vertices in H, then we know that (ii) or (iii) holds. If there is no such A and B, then H satisfies (i) by Lemmas 3.1 and 3.2. Since we can find a linkage of size $< r$ between given two vertex sets in polynomial time, we can obtain such sets $A, B \subseteq H$ in $3^{|H|} \cdot |G|^{O(1)}$ time. $\square$

It is known in [9] that for integers $q, l, r \geq 1$ the value $R_l(r, q)$ in Proposition 3.4 is bounded by

$$R_l(r, q) \leq \exp^{(l)}(c_q r),$$

for some constants c_q, where $\exp^{(l)}(x)$ is the iterated exponential function defined inductively as $\exp^{(l)}(x) = x$ and $\exp^{(l)}(x) = 2^{\exp^{(l-1)}(x)}$ for $l \geq 2$. For a digraph G, we define $\kappa = \kappa(G)$ to be the maximum integer with

$$(\kappa - 1)^2 + 1 + \exp^{((\kappa-1)^2+1)}(c_{(((\kappa-1)^2+1)!+1)^2}(\kappa + 1)((\kappa - 1)^2 + 1)) \leq \nu_S^*(G).$$

We denote

$$\mu(G) = (\kappa + 1)((\kappa - 1)^2 + 1).$$

The following lemma follows from the same construction as the proof of Theorem 3.3.

Lemma 4.2 *There is a computable, nondecreasing, and unbounded function φ such that the following holds: Let $G = (V, E)$ be a digraph with $S \subseteq V$, $r \leq \mu(G)$, and H be an S-hitting set with $|H| \geq 10r$ such that (i) in Lemma 4.1 holds. Then there exists a $1/5$-integral packing of S-circuits of size at least $\varphi(r)$.*

Note that $\varphi(r)$ can be defined to be the maximum integer with $(10\varphi(r)+1)((10\varphi(r)-1)^2+1) \leq r$.

We now design the algorithm $\mathbb{A}$ by the following argument in Lemma 11 of [10]. The algorithm $\mathbb{A}$ starts with computing an S-hitting set H of G of size $O(\nu_S^* \log \nu_S^* \log\log \nu_S^*)$ using polynomial-time algorithms such as [4, 5]. Then we recursively do the following procedure for a pair (G, H) with a digraph G and an S-hitting set H: If H satisfies (i) of Lemma 4.1 then we obtain a 1/5-integral packing of S-circuits with a sufficiently large size by Lemma 4.2. If H satisfies (iii) then we split G into G_1 and G_2 with $|H_1| \geq |H_2|$, keep the S-circuit C_2, and repeat this procedure for (G_1, H_1). We call this step a *splitting step*. The remaining case is that H satisfies (ii). In this case, if G is the input of $\mathbb{A}$ then do the procedure for (G, H'). Otherwise, G is a subdigraph obtained by some splitting steps. In the last splitting step, let (G_0, H_0) produce (G, H) and the other subdigraph (G_1, H_1). Then construct an $(S \cap V(G_0))$-hitting set H_0' of G_0 using H' and H_1 as (iii) of Lemma 4.1. If $|H_0'| < |H_0|$ then do the procedure for (G_0, H_0'). Else if $|H'| < |H_1|$ apply the procedure for (G_1, H_1), and otherwise for (G, H'). We repeat this procedure until H satisfies (i) or G is a subdigraph obtained by $O(\log \mu(G))$ splitting steps. Note that this procedure has a step going back to a larger digraph, but such step reduces the size of an S-hitting set. Hence the algorithm $\mathbb{A}$ terminates in $g(\nu^*(G)) \cdot |G|^{O(1)}$ times for some computable function g. The output is a 1/5-integral packing of S-circuits with a sufficiently large size. Thus we obtain the desired algorithm $\mathbb{A}$.

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