MATHEMATICAL ENGINEERING
TECHNICAL REPORTS

Packing Cycles through Prescribed Vertices

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METR 2010–16
June 2010

WWW page: http://www.keisu.t.u-tokyo.ac.jp/research/techrep/index.html
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Abstract

The well-known theorem of Erdős and Pósa says that $G$ has either $k$ disjoint cycles or a vertex set $X$ of order at most $f(k)$ such that $G \setminus X$ is a forest. Starting with this result, there are many results concerning packing and covering cycles in graph theory and combinatorial optimization.

In this paper, we generalize Erdős-Pósa’s result. Given an integer $k$ and a vertex subset $S$ (possibly unbounded number of vertices) in a given graph $G$, we prove that either $G$ has $k$ disjoint cycles, each of which contains at least one vertex of $S$, or $G$ has a vertex set $X$ of order at most $f(k)$ such that $G \setminus X$ has no such a cycle. Our proof implies the function $f$ is bounded by a polynomial function, that is, $f(k) = O(k^d)$.

1 Introduction

Packing and covering vertex-disjoint cycles are one of the central areas in both graph theory and theoretical computer science. The starting point of this research area goes back to the following well-known theorem due to Erdős and Pósa [3] in early 1960’s.

Theorem 1.1 (Erdős and Pósa [3]) For any integer $k$ and any graph $G$, either $G$ contains $k$ vertex-disjoint cycles or a vertex set $X$ of order at most $f(k)$ (for some function $f$ of $k$) such that $G \setminus X$ is a forest.

In fact, Theorem 1.1 gives rise to the well-known Erdős-Pósa property. A family $\mathcal{F}$ of graphs is said to have the Erdős-Pósa property, if for every integer $k$ there is an integer $f(k, \mathcal{F})$ such that every graph $G$ contains either $k$ vertex-disjoint subgraphs each isomorphic to a graph in $\mathcal{F}$ or a set $C$ of at most $f(k, \mathcal{F})$ vertices such that $G \setminus C$ has no subgraph isomorphic to a graph in $\mathcal{F}$. The term Erdős-Pósa property arose because of Theorem 1.1 which proves that the family of cycles has this property.

Theorem 1.1 is about both “packing”, i.e., $k$ vertex-disjoint cycles and “covering”, i.e., at most $f(k)$ vertices that hit all the cycles in $G$. Starting with this result, there is a host of results in this
direction. Packing appears almost everywhere in extremal graph theory, while covering leads to the well-known concept “feedback set” in theoretical computer science. Also, the cycle packing problem, which asks whether or not there are $k$ vertex-disjoint cycles in an input graph $G$, is a well-known problem too, e.g., [5].

In addition to the feedback set problem, a natural generalization of the cycle packing problem has been studied extensively in theoretical computer science. The problem called “$S$-cycle packing” is that we are given a graph $G$ and a subset $S$ of its vertices, and the goal is to find among the cycles that intersect $S$ a maximum number of vertex-disjoint (or edge-disjoint) ones. See [5] for more details. As pointed out there, this problem is rather close to the well-known “the disjoint paths” problem [6], and approximation algorithms to find an $S$-cycle packing have been studied extensively. But on the other hand, it seems that the Erdős-Pósa type result has not been explored yet. This is our motivation of this paper. We prove that the Erdős-Pósa type result holds for the $S$-cycle packing problem. So this is a generalization of Theorem 1.1 to the “subset” version.

Let us formally define the $S$-cycle packing. Let $G = (V, E)$ be an undirected graph with vertex set $V$ and edge set $E$. For $S \subseteq V$, an $S$-cycle is a cycle which has a vertex in $S$. We denote by $\nu_S(G)$ the maximum $k$ such that $G$ has $k$ $S$-cycles that are pairwise disjoint. A vertex subset that meets all $S$-cycles is called an $S$-hitting set. The minimum size of an $S$-hitting set is denoted by $\tau_S(G)$.

In this paper, we show the following theorem. If $S = V$ then this coincides with Theorem 1.1.

**Theorem 1.2** Let $k$ be a positive integer. Then there exists a constant $f(k)$ such that any graph $G = (V, E)$ with $S \subseteq V$ satisfies $\nu_S(G) \geq k$ or $\tau_S(G) \leq f(k)$.

It should be noted that our proof yields a polynomial bound $f(k) = O(k^4 \log^2 k)$. Lower bound for the function $f(k)$ is $\Omega(k \log k)$ in the case of $S = V$ [3].

In the next section, we give some lemmas needed for the proof of Theorem 1.2. Our main proof follows in Section 3.

## 2 Preliminaries

### 2.1 Packing Paths through Prescribed Vertices

Let $G = (V, E)$ be a graph with $A, B \subseteq V$. A linkage $\mathcal{L}$ from $A$ to $B$ in $G$ is a subgraph consisting of vertex-disjoint paths each of which starts with $A$ and ends with $B$. The size of a linkage is the number of the disjoint paths. We assume that a path has at least one vertex and no repeated vertices. A separation in $G$ is an ordered pair $(X, Y)$ of subsets of $V$ with $X \cup Y = V$ such that $G$ has no edges between $X \setminus Y$ and $Y \setminus X$. Its order is $|X \cap Y|$. It is well known as Menger’s theorem that a graph $G = (V, E)$ with $A, B \subseteq V$ has either a linkage from $A$ to $B$ of size $k$, or a separation $(X, Y)$ of $G$ of order $< k$ with $A \subseteq X$ and $B \subseteq Y$.

For $S, T \subseteq V$ with $S \cap T = \emptyset$, an $S$-path with respect to $T$ is a path with end vertices in $T$ going through $S$. The end vertices of an $S$-path are called the terminals. We obtain the following theorem, which follows from the odd path theorem by Geelen, Gerards, Reed, Seymour, and Vetta [4].

**Theorem 2.1** Let $G = (V, E)$ be a graph, and $S, T \subseteq V$ with $S \cap T = \emptyset$. Then, if $G$ has no $k$ disjoint $S$-paths with respect to $T$, then there exists $Z \subseteq V$ with $|Z| \leq 2k - 2$ that intersects every $S$-path with respect to $T$. 


Theorem 2.2 (Geelen, Gerards, Reed, Seymour, and Vetta [4]) Let \( G = (V, E) \) be a graph with \( T \subseteq V \). Then, if \( G \) has no \( k \) disjoint paths each of which has an odd number of edges and its end points in \( T \), then there exists \( Z \subseteq V \) with \( |Z| \leq 2k - 2 \) that intersects every such path.

Proof of Theorem 2.1: We construct a graph \( G' \) from \( G \) as follows. We first subdivide every edge with a new vertex, and, for every vertex in \( S \), add an edge between it and all its original neighbors. Then if a path connecting two vertices of \( T \) in \( G' \) is odd, then the corresponding path in \( G \) contains a vertex of \( S \), i.e., an \( S \)-path. Moreover, an \( S \)-path with respect to \( T \) in \( G \) gives rise to an odd path connecting two vertices of \( T \) in \( G' \). Therefore, \( G' \) has \( k \) disjoint odd paths with end vertices in \( T \) if and only if \( G \) has \( k \) disjoint \( S \)-paths with respect to \( T \). Thus, by Theorem 2.2, we obtain Theorem 2.1. \( \square \)

2.2 Brambles and Well-attached Ladders

In this section, we first review brambles, established in the graph minor theory. A **bramble** in a graph \( G \) is a set \( B \) of connected subgraphs every two of which touch, that is, either intersect or are joined by an edge. A **transversal** of a bramble \( B \) is a set of vertices which meets each element of \( B \). The **order** of \( B \) is defined to be the minimum size of a transversal.

Given a bramble \( B \) of order \( r \) and a vertex subset \( X \) with \(|X| < r\), there is a subgraph in \( B \) which is disjoint from \( X \), and hence there is a component of \( G \setminus X \) containing a subgraph in \( B \). Since every pair of elements in \( B \) touch, this component is unique. We call such a component the **big component** of \( G \setminus X \). For an integer \( p \leq r \), we say that a subgraph is \( p \)-attached to \( B \) if this subgraph intersects the big component of \( G \setminus X \) for any \( X \) with \(|X| < p\).

A **ladder** of length \( h \) is defined to be a graph which is isomorphic to a subdivision of the graph \( L_h \) with vertex set \( V(L_h) = \{(i, j) \mid 1 \leq i \leq h, 1 \leq j \leq 2\} \) in which two vertices \((i, j)\) and \((i', j')\) are adjacent if and only if \(|i - i'| + |j - j'| = 1\) holds. A ladder of length \( h \) forms a \( 2 \times h \) wall. A **subladder** of a ladder is a subgraph which is a ladder. A **perimeter** of a ladder is the boundary cycle of the ladder.

We show that if \( G \) has a bramble of large order, then \( G \) has a ladder which is well-attached. More precisely, we show the following theorem.

**Theorem 2.3** Let \( h, p \) be a positive integer with \( h \geq 3p - 2 \). Define

\[
r = 4(h - 1)^2 + 4.
\]

Then, if \( G \) has a bramble \( B \) of order \( r \), then \( G \) has a ladder of length \( h \) such that the perimeter of any subladder of length \( \geq 3p - 2 \) is \( p \)-attached to \( B \).

To prove this theorem, we make use of the results by Birmelé, Bondy, and Reed [1]. For \( X \subseteq V \), an **\( X \)-sun** \((C, P_1, \ldots, P_q)\) consists of a cycle \( C \) together with \( q \) disjoint paths from \( V(C) \) to \( X \), all internally disjoint from \( C \). Note that the paths \( P_i \) could be trivial. The paths \( P_i \) are called the **rays** of the sun, and the end vertices of \( P_i \) in \( C \) are the **roots**. The value \( q \) is the **order** of the sun. The following lemmas are shown in [1].

**Lemma 2.4** Let \( B \) be a bramble of order \( r \geq 3 \), and \( F \) be its minimum transversal. Then there exists an \( F \)-sun of order \( r \).
Lemma 2.5 Let $F$ be a minimum transversal of $B$, and $F_1$ and $F_2$ be disjoint subsets of $F$ with $|F_1| = |F_2| = r$. Then there are $r$ disjoint paths linking $F_1$ and $F_2$.

We need the following result by Erdős and Szekeres [2].

Proposition 2.6 Let $s,t$ be integers, and let $n = (s - 1)(t - 1) + 1$, and let $a_1, \ldots, a_n$ be distinct integers. Then either

- there exist $1 < i_1 < \cdots < i_k \leq n$ so that $a_{i_1} < \cdots < a_{i_k}$,
- there exist $1 < i_1 < \cdots < i_k \leq n$ so that $a_{i_1} > \cdots > a_{i_k}$.

Proof of Theorem 2.3: We denote $r = 4r'$, that is, $r' = (h - 1)^2 + 1$. Let $F$ be a minimum transversal. It follows from Lemma 2.4 that $G$ has an $F$-sun of order $r$, denoted by $(C, P_1, \ldots, P_t)$. Let $C_1$ and $C_2$ be a partition of $C$, each containing the roots of at least $2r'$ rays of the sun. We denote by $F_i$ the set of vertices in $F$ reached by the rays rooted in $C_i$ for $i = 1, 2$. Lemma 2.5 implies that there exist $2r'$ disjoint paths, denoted by $Q_1, \ldots, Q_{2r'}$, from $F_1$ to $F_2$. The path $Q_i$ connects to two rays with end vertices in $F_1$ and $F_2$, respectively. These two rays, together with $Q_i$, yield a walk $W_i$ from $V(C_1)$ to $V(C_2)$. Since each vertex of $G$ is used in at most two of the $2r'$ walks from $V(C_1)$ to $V(C_2)$, there exists no separation $(X,Y)$ with $V(C_1) \subseteq X, V(C_2) \subseteq Y$, and $|X \cap Y| < r'$, and hence there exist $r'$ disjoint paths in the walks. By taking minimal paths from $C_1$ to $C_2$ in these paths, we may assume that these paths are internally disjoint from $C$. Such disjoint paths are denoted by $R_1, \ldots, R_{r'}$. Let $Z$ be the set of the end vertices of $R_1, \ldots, R_{r'}$ in $V(C_1)$.

By applying Proposition 2.6 to $Z$, there are $h$ disjoint paths $R_{m_1}, \ldots, R_{m_h}$ such that either two of them reach $C_2$ in the same order, or in the opposite order. We denote the end vertices of $R_{m_1}$ by $a_1 \in V(C_1)$ and $a_2 \in V(C_2)$, and the end vertices of $R_{m_h}$ by $b_1 \in V(C_1)$ and $b_2 \in V(C_2)$. Let $C'_1$ and $C'_2$ be the subpaths of $C_1$ and $C_2$ between $a_1$ and $b_1$ and between $a_2$ and $b_2$, respectively. In the both orderings of $R_{m_1}, \ldots, R_{m_h}$, the union of $R_{m_1}, \ldots, R_{m_h}$, $C'_1$ and $C'_2$ consists of a ladder of length $h$.

We next show that any subgraph $D$ containing at least $3p - 2$ vertices of $Z$ is $p$-attached to $B_H$, which completes the proof of this statement. Assume that $D$ is not $p$-attached. Then there is a vertex set $T$ with $|T| \leq p - 1$ such that the big component $T^*$ of $G \setminus T$ is disjoint from $D$. Since every element in $B$ intersects $T \cup T^*$, the set $(F \cap T^*) \cup T$ is a transversal. By the minimality of $F$, we obtain $|F \setminus T^*| \leq |T| \leq p - 1$, and hence one of $F_1$ and $F_2$ satisfies $|F_i \setminus T^*| \leq \lfloor (p - 1)/2 \rfloor$. We may assume that $|F_2 \setminus T^*| \leq \lfloor (p - 1)/2 \rfloor$.

Every vertex in $Z \cap V(D)$ is connected to $F_2$ by a path in the union of the walks $W_i$. Since each vertex in $G$ is used in at most two such paths, at most $2|F_2 \setminus T^*| \leq p - 1$ of these paths link $Z \cap V(D)$ to $F_2 \setminus T^*$, and at most $2|T| \leq 2p - 2$ paths link $Z \cap V(D)$ to $F_2 \cap T^*$. Hence $G$ has at most $3p - 3$ paths between $Z \cap V(D)$ and $F_2$, which contradicts that there are at least $3p - 2$ such paths.

Therefore, $G$ has a ladder of length $h$ such that the perimeter of each subladder of length $\geq 3p - 2$ is $p$-attached. □

3 Erdős-Pósa Property for Cycles through Prescribed Vertices

In this section, we shall prove Theorem 1.2 by induction on $k$. Throughout this section, $f(k)$ is defined as in Theorem 1.2. If $k = 1$ then this statement holds by $f(1) = 0$. We henceforth suppose that, for $\ell < k$, we have $f(\ell)$ such that, if $\nu_S(G) < \ell$, then $\tau_S(G) \leq f(\ell)$. Note that we may assume that each vertex in $S$ is contained in some $S$-cycle, otherwise we can delete it from $S$. 4
3.1 Defining a Bramble of Large Order

In this subsection, we construct a bramble of a large order if \( \tau_S(G) \) is large. For an integer \( k \geq 3 \), define

\[
\tilde{f}(k) = \max_{i=2,...,k-1} \{ f(i) + f(k-i+1) \},
\]

and define \( \tilde{f}(2) = 0 \). Note that, if \( f(\ell) \) is polynomial for \( \ell < k \), then so is \( \tilde{f}(k) \). We first show the following lemma.

Lemma 3.1 Assume that \( k \) is a positive integer such that \( f(\ell) \) exists for \( \ell < k \). Let \( G = (V, E) \) be a graph with \( S \subseteq V \) such that \( \nu_S(G) < k \), and \( H \) be an \( S \)-hitting set with \( |H| = \tau_S(G) \). Let \( H_1, H_2 \subseteq H \) be disjoint subsets with \( |H_1| = |H_2| = r \), where \( r \geq \tilde{f}(k) \). Then there exists a linkage from \( H_1 \) to \( H_2 \) of size \( r \) with no inner vertices in \( H \).

Proof: Suppose not. Let \( Z = H \setminus (H_1 \cup H_2) \). By Menger’s theorem applied to \( G \setminus Z \), the graph \( G \) has a separation \((X, Y)\) with \( H_1 \subseteq X \), \( H_2 \subseteq Y \), \( Z \subseteq X \cap Y \), and \( |X \cap Y | \setminus Z| < r \). Since \( |H_1 \cup (X \cap Y)| < |H| = \tau_S(G) \), there exists an \( S \)-cycle \( C_1 \) with \( V(C_1) \cap (H_1 \cup (X \cap Y)) = \emptyset \). By \( V(C_1) \cap H \neq \emptyset \), we have \( V(C_1) \cap H_2 \neq \emptyset \), and hence \( V(C_1) \cap Y \neq \emptyset \). Since \((X, Y)\) is a separation and \( X \cap Y \cap V(C_1) = \emptyset \), the set \( V(C_1) \) does not meet \( X \), so \( V(C_1) \subseteq Y \setminus X \). Similarly \( G \) has an \( S \)-cycle \( C_2 \) such that \( V(C_2) \subseteq X \setminus Y \). Thus \( k \geq 3 \).

These two \( S \)-cycles \( C_1 \) and \( C_2 \) imply \( \nu_S(G \setminus X) < k - 1 \) and \( \nu_S(G \setminus Y) < k - 1 \). More precisely, we have \( \nu_S(G \setminus X) < i \) and \( \nu_S(G \setminus Y) < k - i + 1 \) for some \( i \in \{2, \ldots, k-1\} \). Hence the induction hypothesis implies that \( \tau_S(G \setminus X) \leq f(i) \) and \( \tau_S(G \setminus Y) \leq f(k-i+1) \). Since every \( S \)-cycle that is not a cycle of \( G \setminus X \) or \( G \setminus Y \) meets \( X \cap Y \), we have

\[
\tau_S(G) \leq \tau_S(G \setminus X) + \tau_S(G \setminus Y) + |X \cap Y | \leq \tilde{f}(k) + |Z| + r = \tilde{f}(k) + |H| - 2r + r \leq |H|,
\]

which is a contradiction. Thus the statement holds. \( \square \)

Let \( r \) be a positive integer. Define \( H \) to be a vertex set of order \( \geq 3r \) such that there exists a linkage from \( H_1 \) to \( H_2 \) of size \( r \) with no inner vertices in \( H \) for any disjoint subsets \( H_1, H_2 \subseteq H \) with \( |H_1| = |H_2| = r \). For \( X \subseteq V \) with \( |X| < r \), the subgraph \( G \setminus X \) has a unique connected component \( G_X \) with \( |V(G_X) \cap H| \geq r \). We define \( B_H \) to be the set of such components for any \( X \subseteq V \) with \( |X| < r \). Then \( B_H \) forms a bramble of order \( \geq r \), because if we take any two components \( B_1, B_2 \) in \( B_H \) then these touch by the definition of \( H \). Thus we have the following lemma by Lemma 3.1.

Lemma 3.2 Assume that \( k \) is a positive integer such that \( f(\ell) \) exists for \( \ell < k \). Let \( G = (V, E) \) be a graph with \( S \subseteq V \) such that \( \tau_S(G) \geq 3r \), where \( r \geq \tilde{f}(k) \), and \( H \) be an \( S \)-hitting set with \( |H| = \tau_S(G) \). Then the set \( B_H \) is a bramble of order \( \geq r \).

The following lemma asserts that a long cycle with no vertices of \( S \) is well-attached to \( B_H \).

Lemma 3.3 Let \( k \) be a positive integer such that \( f(\ell) \) exists for \( \ell < k \), and \( h \) be a positive integer. Then there exists a positive integer \( r \) such that the following holds: Let \( G = (V, E) \) be a graph with \( S \subseteq V \) such that \( \nu_S(G) < k \) and \( \tau_S(G) \geq 3r \), and \( H \) be an \( S \)-hitting set with \( |H| = \tau_S(G) \). Then \( G \) has a cycle \( C \) of length \( \geq 3h - 2 \) with no vertices of \( S \) such that \( C \) is \( h \)-attached to a bramble \( B_H \).
Proof: Define \( r = \max\{4(k(3h - 2) - 1)^2 + 4, \hat{f}(k)\} \).

By \( r \geq \hat{f}(k) \), Lemma 3.2 implies that \( B_H \) is a bramble of order \( \geq r \). Therefore, it follows from Theorem 2.3 that \( G \) has a ladder of length \( k(3h - 2) \) such that the perimeter of each subladder of length \( 3h - 2 \) is \( h \)-attached to \( B_H \). By \( \nu_S(G) < k \), there exists at least one subladder of length \( 3h - 2 \) whose perimeter has no vertices of \( S \). Thus the statement holds. \( \square \)

### 3.2 Using a Well-attached Cycle of Long Length

In this section, we describe that having a well-attached long cycle without vertices of \( S \) implies \( \nu_S(G) \geq k \) or \( \tau_S(G) \leq g(k) \) for some function \( g \). This, together with Lemma 3.3, implies the proof of Theorem 1.2.

We first show the following lemma.

**Lemma 3.4** Let \( k \) be a positive integer, and define

\[
K = 4k \log_2(k + 10).
\]

Assume that \( G \) has a cycle \( C \) of length \( \geq 2K \) with no vertices of \( S \). If \( G \) has \( K \) disjoint \( S \)-paths with respect to \( V(C) \), then there exist \( k \) disjoint \( S \)-cycles.

**Proof:** Consider the subgraph \( G' \) of \( G \) formed by \( C \) and the \( K \) disjoint paths. Note that \( C \) is the only cycle in \( G' \) that is not an \( S \)-cycle and \( C \) intersects every other cycle in \( G' \), thus it is sufficient to show that \( G' \) has \( k \) disjoint cycles. Clearly, \( G' \) has \( 2K \) vertices of degree 3 and every other vertex is of degree 2. Therefore, by a result of Simonovits [7], \( G' \) has at least \( \lfloor \frac{1}{4}(2K)/\log_2(2K) \rfloor \) vertex-disjoint cycles. It can be checked that \( 2K \leq (k + 10)^2 \) for every \( k \geq 1 \), thus \( \lfloor \frac{1}{4}(2K)/\log_2(2K) \rfloor \geq [K/(2\log(k + 10)^2)] \geq k \), that is, there are \( k \) vertex-disjoint cycles in \( G' \).

Therefore, we may assume that \( G \) has no \( K \) disjoint \( S \)-paths with respect to vertices of a long cycle having no vertices of \( S \). For \( I \subseteq V \), we denote by \( G[I] \) the subgraph induced by \( I \).

**Lemma 3.5** Let \( k \) be a positive integer such that \( f(\ell) \) exists for \( \ell < k \), and \( K \) be a positive integer. Let \( G = (V, E) \) be a graph with \( S \subseteq V \) such that \( \nu_S(G) < k \), and \( H \) be a minimum \( S \)-hitting set such that \( B_H \) is a bramble of order \( \geq 4K \). Assume that \( G \) has a cycle \( C \) of length \( \geq 12K - 2 \) with no vertices of \( S \) such that \( C \) is \( 4K \)-attached to \( B_H \). If \( G \) has no \( K \) disjoint \( S \)-paths with respect to \( V(C) \), then \( \tau_S(G) \leq g(k) \) holds, where \( g(k) \) is defined to be

\[
g(k) = \max\{6K, \hat{f}(k) + 2K\}.
\]

**Proof:** We denote \( T = V(C) \). By Theorem 2.1, there is a vertex subset \( Z \subseteq V \) of size \( \leq 2K - 2 \) such that \( G \setminus Z \) has no \( S \)-path with respect to \( T \setminus Z \). We denote \( S' = S \setminus Z \) and \( T' = T \setminus Z \). Note that \( T' \) is nonempty by \( |T| > |Z| \).

For \( s \in S' \), the graph \( G \setminus Z \) has a separation \( (X_s, Y_s) \) of order at most one with \( s \in X_s \setminus Y_s \) and \( T' \subseteq Y_s \). Among such separations \( (X_s, Y_s) \) with minimum order, choose \( (X_s, Y_s) \) such that \( X_s \) is minimal. We denote by \( u_s \) the vertex in \( X_s \cap Y_s \) if \( X_s \cap Y_s \neq \emptyset \). Since \( (X_s, Y_s) \) is a minimum separation, there is a path from \( u_s \) to \( T' \) in \( G[Y_s] \). Define \( X = \bigcup_{s \in S'} X_s \) and \( Y = \bigcap_{s \in S'} Y_s \). Then we know \( S' \subseteq X \setminus Y \) and \( T' \subseteq Y \). Moreover, the pair \( (X, Y) \) is a separation of \( G \setminus Z \) with
\(X \cap Y = \{u_s \mid s \in S', X_s \cap Y_s \neq \emptyset\} \cap Y\). Indeed, each \(v \in X \setminus Y\) is not contained in \(Y_s\) for some \(s \in S'\), and hence \(v\) is in \(X_s \subseteq X\) and adjacent to no vertex in \(Y \setminus X \subseteq Y_s \setminus X_s\). Note that each vertex in \(X \cap Y\) is a cut vertex of \(G \setminus Z\) between a vertex in \(S'\) and the set \(T'\). We denote \(U = X \cap Y\).

**Claim 1** We may assume that \(|U| > 2K + 1\).

**Proof:** Assume to the contrary that \(|U| \leq 2K + 1\). Then \(|Z \cup U| \leq |Z| + |U| \leq 2K - 2 + 2K + 1 \leq 4K\) holds. The pair \((X', Y')\), where \(X' = X \cup Z\) and \(Y' = Y \cup Z\), is a separation of \(G\) with \(S \subseteq X' \setminus Y\) and \(T \subseteq Y'\), and its order is \(< 4K\). Since \(S \subseteq X'\) and a vertex in \(U\) is a cut vertex in \(G \setminus Z\), each \(S\)-cycle of \(G\) is contained in \(X'\), or has a vertex in \(Z\). Hence \((H \cap X') \cup Z\) is an \(S\)-hitting set. By the minimality of \(H\), we have \(|H| \leq |(H \cap X') \cup Z| \leq |H \cap X'| + |Z|\), and hence \(|H \setminus X'| \leq |Z| \leq 2K - 2\) holds. In addition, since \(C\) is \(4K\)-attached to \(B_H\) and \(V(C) \subseteq Y'\), the set \(Y' \setminus X'\) includes the big component of \(G \setminus (U \cup Z)\). This implies that \(|H \setminus X'| \geq |H|/3\), and hence we obtain

\[|H| \leq 3|H \setminus X'| \leq 6(K - 1) \leq g(k).\]

Thus Lemma 3.5 holds. \(\Box\)

By Claim 1, we know that \(|S'| \geq |U| > 2K + 1\).

**Claim 2** We may assume that there is \(s_0 \in S'\) such that \(G[X_{s_0}]\) contains an \(S\)-cycle.

**Proof:** Assume that \(G[X_s]\) contains no \(S\)-cycle for any \(s \in S'\). Then each \(S\)-cycle meets a vertex in \(Z\), which implies that \(\tau_S(G) \leq |Z| \leq 2K - 2 \leq g(k)\). Thus Lemma 3.5 holds. \(\Box\)

Let \(X'_{s_0} = X_{s_0} \setminus \{u_{s_0}\}\) and \(G' = G \setminus X'_{s_0}\). Note that \(X'_{s_0} \subseteq X \setminus Y\) and \(X'_{s_0} \cap U = \emptyset\). The pair \((X \setminus X'_{s_0}, Y)\) is a separation of \(G' \setminus Z\), and each vertex in \(U\) is a cut vertex of \(G' \setminus X'_{s_0}\) between a vertex of \(S'\) and the vertex set \(T'\).

**Claim 3** The subgraph \(G'\) has an \(S\)-cycle.

**Proof:** Let \(U = \{u_1, \ldots, u_m\}\), where \(m = |U|\). For \(1 \leq j \leq m\), let \((X_j, Y_j)\) be a separation of \(G' \setminus Z\) with \(X_j \cap Y_j = \{u_j\}\) such that \(T \subseteq Y_j\) and \(X_j \setminus Y_j\) contains some vertex of \(S'\). Choose \((X_j, Y_j)\) such that \(X_j\) is minimal. Then \(Y \subseteq Y_j\) holds and \(X_1, \ldots, X_m\) are disjoint. We may assume that \(G'[X_j]\) has no \(S\)-cycle for any \(j\), otherwise we are done. Since \(\{u_j\} = X_s \cap Y_s\) for some \(s \in S'\) and \(X_s\) is chosen minimal, this implies that \(X_j\) has a vertex \(s_j \in S'\) that connects to \(u_j\). Moreover, since \(s_j\) is contained in some \(S\)-cycle in \(G\), this assumption implies that \(G'\) has an edge connecting \(X_j\) and \(Z\), and hence \(Z\) is nonempty.

Let \(C_j\) be an \(S\)-cycle containing \(s_j\) in \(G\). The subgraph \(G'[X_j \cup Z]\) has a path \(P_j\) through \(s_j\) from \(u_j\) to a vertex in \(Z\) by using the edge \((s_j, u_j)\) and \(C_j\). We may suppose that \(P_j\) has no inner vertices in \(Z\) by taking a minimal path. By \(|U| > 2K + 1 \geq |Z| \geq 1\), there exist a vertex \(z\) in \(Z\) and two indices \(j_1, j_2\) such that both \(P_{j_1}\) and \(P_{j_2}\) end with \(z\). The path \(P_{j_i}\) is contained in \(G'[X_{j_i} \cup \{z\}]\) for \(i = 1, 2\), respectively.

The subgraph \(G'[Y_{j_i}]\) has a path \(P^i_{j_i}\) from \(u_{j_i}\) to a vertex \(w_{j_i}\) of \(T'\). Since each vertex in \(U\) is a cut vertex in \(G \setminus Z\), the path \(P^i_{j_i}\) has no vertex of \(Y_{j_i} \setminus Y\), and thus is contained in \(G'[Y]\). We may assume that \(P^i_{j_i}\) has no inner vertex in \(T'\). By \(T \subseteq Y \cup Z\), the subgraph \(G'[Y \cup Z]\) has two internally disjoint paths between \(w_{j_1}\) and \(w_{j_2}\) along \(C\), and hence one of these two paths, denoted by \(P\), does not have \(z\). Then the union of \(P^i_{j_1}, P\), and \(P^i_{j_2}\) includes a path in \(G[Y \cup Z \setminus \{z\}]\) from \(u_{j_1}\) to \(u_{j_2}\). This path, together with \(P_{j_1}\) and \(P_{j_2}\), yields an \(S\)-cycle in \(G'\).}\(\Box\)
Therefore, both $G[X_{s_0}]$ and $G'$, i.e., $G[Y_{s_0}]$ have $S$-cycles, and thus $k \geq 3$. These two $S$-cycles imply by $\nu_S(G) < k$ that $\nu_S(G \setminus X) < i$ and $\nu_S(G \setminus Y) < k - i + 1$ for some $i \in \{2, \ldots, k - 1\}$. By the induction hypothesis, it holds that $\tau_S(G \setminus X) \leq f(i)$ and $\tau_S(G \setminus Y) \leq f(k - i + 1)$. Since every $S$-cycle that is not a cycle of $G \setminus Y_{s_0}$ or $G \setminus X_{s_0}$ meets $X_{s_0} \cap Y_{s_0} = Z \cup \{s_0\}$, we have

$$\tau_S(G) \leq \tau_S(G \setminus X) + \tau_S(G \setminus Y) + |Z| + 1 \leq \bar{f}(k) + 2K - 1 \leq g(k).$$

Thus the statement holds.

\textbf{Proof of Theorem 1.2:} Define

$$K = 4k \log_2 (k + 10),$$
$$r_k = \max\{4(k(12K - 2) - 1)^2 + 4, \bar{f}(k)\},$$
$$f(k) = \max\{3r_k, \bar{f}(k) + 2K\}.$$ 

Note that $f(k) = \max\{3r_k, g(k)\}$, where $g(k)$ is defined as in Lemma 3.5. We will show that $f(k)$ satisfies Theorem 1.2. Assume to the contrary that there is a graph $G = (V, E)$ with $S \subseteq V$ satisfying $\nu_S(G) < k$ and $\tau_S(G) > f(k)$. By $\tau_S(G) \geq \bar{f}(k)$, the set $B_H$ forms a bramble of order $\geq r_k$. Moreover, by $\tau_S(G) \geq r_k$, it follows from Lemma 3.3 that $G$ has a cycle $C$ of length $\geq 12K - 2$ with no vertices of $S$ such that $C$ is $4K$-attached to $B_H$. If $G$ has $K$ disjoint $S$-paths with respect to $V(C)$, then $\nu_S(G) \geq k$ holds by Lemma 3.4. Otherwise, by Lemma 3.5, we have $\nu_S(G) \geq k$ or $\tau_S(G) \leq g(k) \leq f(k)$. Hence both cases have a contradiction. Thus the statement holds.

\textbf{References}


