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Packing Cycles through Prescribed Vertices

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Abstract

The well-known theorem of Erdős and Pósa says that G has either k disjoint cycles or a vertex set X of order at most $f(k)$ such that $G \setminus X$ is a forest. Starting with this result, there are many results concerning packing and covering cycles in graph theory and combinatorial optimization.

In this paper, we generalize Erdős-Pósa's result. Given an integer k and a vertex subset S (possibly unbounded number of vertices) in a given graph G, we prove that either G has k disjoint cycles, each of which contains at least one vertex of S, or G has a vertex set X of order at most $f(k)$ such that $G \setminus X$ has no such a cycle. Our proof implies the function f is bounded by a polynomial function, that is, $f(k) = \tilde{O}(k^4)$.

1 Introduction

Packing and covering vertex-disjoint cycles are one of the central areas in both graph theory and theoretical computer science. The starting point of this research area goes back to the following well-known theorem due to Erdős and Pósa [3] in early 1960's.

Theorem 1.1 (Erdős and Pósa [3]) *For any integer k and any graph G, either G contains k vertex-disjoint cycles or a vertex set X of order at most $f(k)$ (for some function f of k) such that $G \setminus X$ is a forest.*

In fact, Theorem 1.1 gives rise to the well-known Erdős-Pósa property. A family $\mathcal{F}$ of graphs is said to have the *Erdős-Pósa property*, if for every integer k there is an integer $f(k, \mathcal{F})$ such that every graph G contains either k vertex-disjoint subgraphs each isomorphic to a graph in $\mathcal{F}$ or a set C of at most $f(k, \mathcal{F})$ vertices such that $G \setminus C$ has no subgraph isomorphic to a graph in $\mathcal{F}$. The term *Erdős-Pósa property* arose because of Theorem 1.1 which proves that the family of cycles has this property.

Theorem 1.1 is about both "packing", i.e., k vertex-disjoint cycles and "covering", i.e., at most $f(k)$ vertices that hit all the cycles in G. Starting with this result, there is a host of results in this

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direction. Packing appears almost everywhere in extremal graph theory, while covering leads to the well-known concept "feedback set" in theoretical computer science. Also, the cycle packing problem, which asks whether or not there are k vertex-disjoint cycles in an input graph G, is a well-known problem too, e.g., [5].

In addition to the feedback set problem, a natural generalization of the cycle packing problem has been studied extensively in theoretical computer science. The problem called "S-cycle packing" is that we are given a graph G and a subset S of its vertices, and the goal is to find among the cycles that intersect S a maximum number of vertex-disjoint (or edge-disjoint) ones. See [5] for more details. As pointed out there, this problem is rather close to the well-known "the disjoint paths" problem [6], and approximation algorithms to find an S-cycle packing have been studied extensively. But on the other hand, it seems that the Erdős-Pósa type result has not been explored yet. This is our motivation of this paper. We prove that the Erdős-Pósa type result holds for the S-cycle packing problem. So this is a generalization of Theorem 1.1 to the "subset" version.

Let us formally define the S-cycle packing. Let $G = (V, E)$ be an undirected graph with vertex set V and edge set E. For $S \subseteq V$, an *S-cycle* is a cycle which has a vertex in S. We denote by $\nu_S(G)$ the maximum k such that G has k S-cycles that are pairwise disjoint. A vertex subset that meets all S-cycles is called an *S-hitting set*. The minimum size of an S-hitting set is denoted by $\tau_S(G)$.

In this paper, we show the following theorem. If $S = V$ then this coincides with Theorem 1.1.

Theorem 1.2 *Let k be a positive integer. Then there exists a constant $f(k)$ such that any graph $G = (V, E)$ with $S \subseteq V$ satisfies $\nu_S(G) \geq k$ or $\tau_S(G) \leq f(k)$.*

It should be noted that our proof yields a polynomial bound $f(k) = \mathrm{O}(k^4 \log^2 k)$. Lower bound for the function $f(k)$ is $\Omega(k \log k)$ in the case of $S = V$ [3].

In the next section, we give some lemmas needed for the proof of Theorem 1.2. Our main proof follows in Section 3.

2 Preliminaries

2.1 Packing Paths through Prescribed Vertices

Let $G = (V, E)$ be a graph with $A, B \subseteq V$. A *linkage* $\mathcal{L}$ from A to B in G is a subgraph consisting of vertex-disjoint paths each of which starts with A and ends with B. The *size* of a linkage is the number of the disjoint paths. We assume that a path has at least one vertex and no repeated vertices. A *separation* in G is an ordered pair (X, Y) of subsets of V with $X \cup Y = V$ so that G has no edges between $X \setminus Y$ and $Y \setminus X$. Its *order* is $|X \cap Y|$. It is well known as Menger's theorem that a graph $G = (V, E)$ with $A, B \subseteq V$ has either a linkage from A to B of size k, or a separation (X, Y) of G of order $< k$ with $A \subseteq X$ and $B \subseteq Y$.

For $S, T \subseteq V$ with $S \cap T = \emptyset$, an *S-path with respect to T* is a path with end vertices in T going through S. The end vertices of an S-path are called the *terminals*. We obtain the following theorem, which follows from the odd path theorem by Geelen, Gerards, Reed, Seymour, and Vetta [4].

Theorem 2.1 *Let $G = (V, E)$ be a graph, and $S, T \subseteq V$ with $S \cap T = \emptyset$. Then, if G has no k disjoint S-paths with respect to T, then there exists $Z \subseteq V$ with $|Z| \leq 2k - 2$ that intersects every S-path with respect to T.*

Theorem 2.2 (Geelen, Gerards, Reed, Seymour, and Vetta [4]) *Let $G = (V, E)$ be a graph with $T \subseteq V$. Then, if G has no k disjoint paths each of which has an odd number of edges and its end points in T, then there exists $Z \subseteq V$ with $|Z| \leq 2k - 2$ that intersects every such path.*

Proof of Theorem 2.1: We construct a graph G' from G as follows. We first subdivide every edge with a new vertex, and, for every vertex in S, add an edge between it and all its original neighbors. Then if a path connecting two vertices of T in G' is odd, then the corresponding path in G contains a vertex of S, i.e., an S-path. Moreover, an S-path with respect to T in G gives rise to an odd path connecting two vertices of T in G'. Therefore, G' has k disjoint odd paths with end vertices in T if and only if G has k disjoint S-paths with respect to T. Thus, by Theorem 2.2, we obtain Theorem 2.1. $\square$

2.2 Brambles and Well-attached Ladders

In this section, we first review brambles, established in the graph minor theory. A *bramble* in a graph G is a set $\mathcal{B}$ of connected subgraphs every two of which *touch*, that is, either intersect or are joined by an edge. A *transversal* of a bramble $\mathcal{B}$ is a set of vertices which meets each element of $\mathcal{B}$. The *order* of $\mathcal{B}$ is defined to be the minimum size of a transversal.

Given a bramble $\mathcal{B}$ of order r and a vertex subset X with $|X| < r$, there is a subgraph in $\mathcal{B}$ which is disjoint from X, and hence there is a component of $G \setminus X$ containing a subgraph in $\mathcal{B}$. Since every pair of elements in $\mathcal{B}$ touch, this component is unique. We call such a component the *big component* of $G \setminus X$. For an integer $p \leq r$, we say that a subgraph is *p-attached* to $\mathcal{B}$ if this subgraph intersects the big component of $G \setminus X$ for any X with $|X| < p$.

A *ladder* of length h is defined to be a graph which is isomorphic to a subdivision of the graph L_h with vertex set $V(L_h) = \{(i, j) \mid 1 \leq i \leq h\ 1 \leq j \leq 2\}$ in which two vertices (i, j) and (i', j') are adjacent if and only if $|i - i'| + |j - j'| = 1$ holds. A ladder of length h forms a $2 \times h$ wall. A *subladder* of a ladder is a subgraph which is a ladder. A *perimeter* of a ladder is the boundary cycle of the ladder.

We show that if G has a bramble of large order, then G has a ladder which is well-attached. More precisely, we show the following theorem.

Theorem 2.3 *Let h, p be a positive integer with $h \geq 3p - 2$. Define*

$$r = 4(h-1)^2 + 4.$$

Then, if G has a bramble $\mathcal{B}$ of order r, then G has a ladder of length h such that the perimeter of any subladder of length $\geq 3p - 2$ is p-attached to $\mathcal{B}$.

To prove this theorem, we make use of the results by Birmelé, Bondy, and Reed [1]. For $X \subseteq V$, an *X-sun* $(C, P_1, \ldots, P_q)$ consists of a cycle C together with q disjoint paths from $V(C)$ to X, all internally disjoint from C. Note that the paths P_i could be trivial. The paths P_i are called the *rays* of the sun, and the end vertices of P_i in C are the *roots*. The value q is the *order* of the sun. The following lemmas are shown in [1].

Lemma 2.4 *Let $\mathcal{B}$ be a bramble of order $r \geq 3$, and F be its minimum transversal. Then there exists an F-sun of order r.*

Lemma 2.5 *Let F be a minimum transversal of $\mathcal{B}$, and F_1 and F_2 be disjoint subsets of F with $|F_1| = |F_2| = r$. Then there are r disjoint paths linking F_1 and F_2.*

We need the following result by Erdős and Szekeres [2].

Proposition 2.6 *Let s, t be integers, and let $n = (s-1)(t-1)+1$, and let $a_1, \dots, a_n$ be distinct integers. Then either*

- *there exist $1 < i_1 < \cdots < i_s \leq n$ so that $a_{i_1} < \cdots < a_{i_s}$,*
- *there exist $1 < i_1 < \cdots < i_t \leq n$ so that $a_{i_1} > \cdots > a_{i_t}$.*

Proof of Theorem 2.3: We denote $r = 4r'$, that is, $r' = (h-1)^2 + 1$. Let F be a minimum transversal. It follows from Lemma 2.4 that G has an F-sun of order r, denoted by $(C, P_1, \dots, P_r)$. Let C_1 and C_2 be a partition of C, each containing the roots of at least $2r'$ rays of the sun. We denote by F_i the set of vertices in F reached by the rays rooted in C_i for $i = 1, 2$. Lemma 2.5 implies that there exist $2r'$ disjoint paths, denoted by $Q_1, \dots, Q_{2r'}$, from F_1 to F_2. The path Q_i connects to two rays with end vertices in F_1 and F_2, respectively. These two rays, together with Q_i, yield a walk W_i from $V(C_1)$ to $V(C_2)$. Since each vertex of G is used in at most two of the $2r'$ walks from $V(C_1)$ to $V(C_2)$, there exists no separation (X, Y) with $V(C_1) \subseteq X$, $V(C_2) \subseteq Y$, and $|X \cap Y| < r'$, and hence there exist r' disjoint paths in the walks. By taking minimal paths from C_1 to C_2 in these paths, we may assume that these paths are internally disjoint from C. Such disjoint paths are denoted by $R_1, \dots, R_{r'}$. Let Z be the set of the end vertices of $R_1, \dots, R_{r'}$ in $V(C_1)$.

By applying Proposition 2.6 to Z, there are h disjoint paths $R_{m_1}, \dots, R_{m_h}$ such that either two of them reach C_2 in the same order, or in the opposite order. We denote the end vertices of R_{m_1} by $a_1 \in V(C_1)$ and $a_2 \in V(C_2)$, and the end vertices of R_{m_h} by $b_1 \in V(C_1)$ and $b_2 \in V(C_2)$. Let C_1' and C_2' be the subpaths of C_1 and C_2 between a_1 and b_1 and between a_2 and b_2, respectively. In the both orderings of $R_{m_1}, \dots, R_{m_h}$, the union of $R_{m_1}, \dots, R_{m_h}$, C_1' and C_2' consists of a ladder of length h.

We next show that any subgraph D containing $\geq 3p-2$ vertices of Z is p-attached to $\mathcal{B}_H$, which completes the proof of this statement. Assume that D is not p-attached. Then there is a vertex set T with $|T| \leq p-1$ such that the big component T^* of $G \setminus T$ is disjoint from D. Since every element in $\mathcal{B}$ intersects $T \cup T^*$, the set $(F \cap T^*) \cup T$ is a transversal. By the minimality of F, we obtain $|F \setminus T^*| \leq |T| \leq p-1$, and hence one of F_1 and F_2 satisfies $|F_i \setminus T^*| \leq \lfloor (p-1)/2 \rfloor$. We may assume that $|F_2 \setminus T^*| \leq \lfloor (p-1)/2 \rfloor$.

Every vertex in $Z \cap V(D)$ is connected to F_2 by a path in the union of the walks W_i's. Since each vertex in G is used in at most two such paths, at most $2|F_2 \setminus T^*| \leq p-1$ of these paths link $Z \cap V(D)$ to $F_2 \setminus T^*$, and at most $2|T| \leq 2p-2$ paths link $Z \cap V(D)$ to $F_2 \cap T^*$. Hence G has at most $3p-3$ paths between $Z \cap V(D)$ and F_2, which contradicts that there are $\geq 3p-2$ such paths.

Therefore, G has a ladder of length h such that the perimeter of each subladder of length $\geq 3p-2$ is p-attached. $\square$

3 Erdős-Pósa Property for Cycles through Prescribed Vertices

In this section, we shall prove Theorem 1.2 by induction on k. Throughout this section, $f(k)$ is defined as in Theorem 1.2. If $k = 1$ then this statement holds by $f(1) = 0$. We henceforth suppose that, for $\ell < k$, we have $f(\ell)$ such that, if $\nu_S(G) < \ell$, then $\tau_S(G) \leq f(\ell)$. Note that we may assume that each vertex in S is contained in some S-cycle, otherwise we can delete it from S.

3.1 Defining a Bramble of Large Order

In this subsection, we construct a bramble of a large order if $\tau_S(G)$ is large. For an integer $k \geq 3$, define

$$\tilde{f}(k) = \max_{i=2,\dots,k-1} \{f(i) + f(k-i+1)\},$$

and define $\tilde{f}(2) = 0$. Note that, if $f(\ell)$ is polynomial for $\ell < k$, then so is $\tilde{f}(k)$. We first show the following lemma.

Lemma 3.1 *Assume that k is a positive integer such that $f(\ell)$ exists for $\ell < k$. Let $G = (V, E)$ be a graph with $S \subseteq V$ such that $\nu_S(G) < k$, and H be an S-hitting set with $|H| = \tau_S(G)$. Let $H_1, H_2 \subseteq H$ be disjoint subsets with $|H_1| = |H_2| = r$, where $r \geq \tilde{f}(k)$. Then there exists a linkage from H_1 to H_2 of size r with no inner vertices in H.*

Proof: Suppose not. Let $Z = H \setminus (H_1 \cup H_2)$. By Menger's theorem applied to $G \setminus Z$, the graph G has a separation (X, Y) with $H_1 \subseteq X$, $H_2 \subseteq Y$, $Z \subseteq X \cap Y$, and $|(X \cap Y) \setminus Z| < r$. Since $|H_1 \cup (X \cap Y)| < |H| = \tau_S(G)$, there exists an S-cycle C_1 with $V(C_1) \cap (H_1 \cup (X \cap Y)) = \emptyset$. By $V(C_1) \cap H \neq \emptyset$, we have $V(C_1) \cap H_2 \neq \emptyset$, and hence $V(C_1) \cap Y \neq \emptyset$. Since (X, Y) is a separation and $X \cap Y \cap V(C_1) = \emptyset$, the set $V(C_1)$ does not meet X, so $V(C_1) \subseteq Y \setminus X$. Similarly G has an S-cycle C_2 such that $V(C_2) \subseteq X \setminus Y$. Thus $k \geq 3$.

These two S-cycles C_1 and C_2 imply $\nu_S(G \setminus X) < k-1$ and $\nu_S(G \setminus Y) < k-1$. More precisely, we have $\nu_S(G \setminus X) < i$ and $\nu_S(G \setminus Y) < k-i+1$ for some $i \in \{2, \dots, k-1\}$. Hence the induction hypothesis implies that $\tau_S(G \setminus X) \leq f(i)$ and $\tau_S(G \setminus Y) \leq f(k-i+1)$. Since every S-cycle that is not a cycle of $G \setminus X$ or $G \setminus Y$ meets $X \cap Y$, we have

$$\tau_S(G) \leq \tau_S(G \setminus X) + \tau_S(G \setminus Y) + |X \cap Y| < \tilde{f}(k) + |Z| + r = \tilde{f}(k) + |H| - 2r + r \leq |H|,$$

which is a contradiction. Thus the statement holds. $\square$

Let r be a positive integer. Define H to be a vertex set of order $\geq 3r$ such that there exists a linkage from H_1 to H_2 of size r with no inner vertices in H for any disjoint subsets $H_1, H_2 \subseteq H$ with $|H_1| = |H_2| = r$. For $X \subseteq V$ with $|X| < r$, the subgraph $G \setminus X$ has a unique connected component G_X with $|V(G_X) \cap H| \geq r$. We define $\mathcal{B}_H$ to be the set of such components for any $X \subseteq V$ with $|X| < r$. Then $\mathcal{B}_H$ forms a bramble of order $\geq r$, because if we take any two components B_1, B_2 in $\mathcal{B}_H$ then these touch by the definition of H. Thus we have the following lemma by Lemma 3.1.

Lemma 3.2 *Assume that k is a positive integer such that $f(\ell)$ exists for $\ell < k$. Let $G = (V, E)$ be a graph with $S \subseteq V$ such that $\tau_S(G) \geq 3r$, where $r \geq \tilde{f}(k)$, and H be an S-hitting set with $|H| = \tau_S(G)$. Then the set $\mathcal{B}_H$ is a bramble of order $\geq r$.*

The following lemma asserts that a long cycle with no vertices of S is well-attached to $\mathcal{B}_H$.

Lemma 3.3 *Let k be a positive integer such that $f(\ell)$ exists for $\ell < k$, and h be a positive integer. Then there exists a positive integer r such that the following holds: Let $G = (V, E)$ be a graph with $S \subseteq V$ such that $\nu_S(G) < k$ and $\tau_S(G) \geq 3r$, and H be an S-hitting set with $|H| = \tau_S(G)$. Then G has a cycle C of length $\geq 3h - 2$ with no vertices of S such that C is h-attached to a bramble $\mathcal{B}_H$.*

Proof: Define

$$r = \max\{4(k(3h-2)-1)^2 + 4, \tilde{f}(k)\}.$$

By $r \geq \tilde{f}(k)$, Lemma 3.2 implies that $\mathcal{B}_H$ is a bramble of order $\geq r$. Therefore, it follows from Theorem 2.3 that G has a ladder of length $k(3h-2)$ such that the perimeter of each subladder of length $3h-2$ is h-attached to $\mathcal{B}_H$. By $\nu_S(G) < k$, there exists at least one subladder of length $3h-2$ whose perimeter has no vertices of S. Thus the statement holds. □

3.2 Using a Well-attached Cycle of Long Length

In this section, we describe that having a well-attached long cycle without vertices of S implies $\nu_S(G) \geq k$ or $\tau_S(G) \leq g(k)$ for some function g. This, together with Lemma 3.3, implies the proof of Theorem 1.2.

We first show the following lemma.

Lemma 3.4 *Let k be a positive integer, and define*

$$K = 4k \log_2(k+10).$$

Assume that G has a cycle C of length $> 2K$ with no vertices of S. If G has K disjoint S-paths with respect to $V(C)$, then there exist k disjoint S-cycles.

Proof: Consider the subgraph G' of G formed by C and by the K disjoint paths. Note that C is the only cycle in G' that is not an S-cycle and C intersects every other cycle in G', thus it is sufficient to show that G' has k disjoint cycles. Clearly, G' has $2K$ vertices of degree 3 and every other vertex is of degree 2. Therefore, by a result of Simonovits [7], G' has at least $\lfloor \frac{1}{4}(2K)/\log_2(2K) \rfloor$ vertex-disjoint cycles. It can be checked that $2K \leq (k+10)^2$ for every $k \geq 1$, thus $\lfloor \frac{1}{4}(2K)/\log_2(2K) \rfloor \geq \lfloor K/(2\log(k+10)^2) \rfloor \geq k$, that is, there are k vertex-disjoint cycles in G'. □

Therefore, we may assume that G has no K disjoint S-paths with respect to vertices of a long cycle having no vertices of S. For $I \subseteq V$, we denote by $G[I]$ the subgraph induced by I.

Lemma 3.5 *Let k be a positive integer such that $f(\ell)$ exists for $\ell < k$, and K be a positive integer. Let $G = (V, E)$ be a graph with $S \subseteq V$ such that $\nu_S(G) < k$, and H be a minimum S-hitting set such that $\mathcal{B}_H$ is a bramble of order $\geq 4K$. Assume that G has a cycle C of length $\geq 12K - 2$ with no vertices of S such that C is $4K$-attached to $\mathcal{B}_H$. If G has no K disjoint S-paths with respect to $V(C)$, then $\tau_S(G) \leq g(k)$ holds, where $g(k)$ is defined to be*

$$g(k) = \max\{6K, \tilde{f}(k) + 2K\}.$$

Proof: We denote $T = V(C)$. By Theorem 2.1, there is a vertex subset $Z \subseteq V$ of size $\leq 2K-2$ such that $G \setminus Z$ has no S-path with respect to $T \setminus Z$. We denote $S' = S \setminus Z$ and $T' = T \setminus Z$. Note that T' is nonempty by $|T| > |Z|$.

For $s \in S'$, the graph $G \setminus Z$ has a separation (X_s, Y_s) of order at most one with $s \in X_s \setminus Y_s$ and $T' \subseteq Y_s$. Among such separations (X_s, Y_s) with minimum order, choose (X_s, Y_s) such that X_s is minimal. We denote by u_s the vertex in $X_s \cap Y_s$ if $X_s \cap Y_s \neq \emptyset$. Since (X_s, Y_s) is a minimum separation, there is a path from u_s to T' in $G[Y_s]$. Define $X = \bigcup_{s \in S'} X_s$ and $Y = \bigcap_{s \in S'} Y_s$. Then we know $S' \subseteq X \setminus Y$ and $T' \subseteq Y$. Moreover, the pair (X, Y) is a separation of $G \setminus Z$ with

$X \cap Y = \{u_s \mid s \in S',\, X_s \cap Y_s \neq \emptyset\} \cap Y$. Indeed, each $v \in X \setminus Y$ is not contained in Y_s for some $s \in S'$, and hence v is in $X_s \subseteq X$ and adjacent to no vertex in $Y \setminus X \subseteq Y_s \setminus X_s$. Note that each vertex in $X \cap Y$ is a cut vertex of $G \setminus Z$ between a vertex in S' and the set T'. We denote $U = X \cap Y$.

Claim 1 *We may assume that $|U| > 2K + 1$.*

Proof: Assume to the contrary that $|U| \leq 2K+1$. Then $|Z \cup U| \leq |Z| + |U| \leq 2K - 2 + 2K + 1 < 4K$ holds. The pair (X', Y'), where $X' = X \cup Z$ and $Y' = Y \cup Z$, is a separation of G with $S \subseteq X' \setminus Y$ and $T \subseteq Y'$, and its order is $< 4K$. Since $S \subseteq X'$ and a vertex in U is a cut vertex in $G \setminus Z$, each S-cycle of G is contained in X', or has a vertex in Z. Hence $(H \cap X') \cup Z$ is an S-hitting set. By the minimality of H, we have $|H| \leq |(H \cap X') \cup Z| \leq |H \cap X'| + |Z|$, and hence $|H \setminus X'| \leq |Z| \leq 2K - 2$ holds. In addition, since C is $4K$-attached to $\mathcal{B}_H$ and $V(C) \subseteq Y'$, the set $Y' \setminus X'$ includes the big component of $G \setminus (U \cup Z)$. This implies that $|H \setminus X'| \geq |H|/3$, and hence we obtain

$$|H| \leq 3|H \setminus X'| \leq 6(K - 1) \leq g(k).$$

Thus Lemma 3.5 holds. $\square$

By Claim 1, we know that $|S'| \geq |U| > 2K + 1$.

Claim 2 *We may assume that there is $s_0 \in S'$ such that $G[X_{s_0}]$ contains an S-cycle.*

Proof: Assume that $G[X_s]$ contains no S-cycle for any $s \in S'$. Then each S-cycle meets a vertex in Z, which implies that $\tau_S(G) \leq |Z| \leq 2K - 2 \leq g(k)$. Thus Lemma 3.5 holds. $\square$

Let $X'_{s_0} = X_{s_0} \setminus \{u_{s_0}\}$ and $G' = G \setminus X'_{s_0}$. Note that $X'_{s_0} \subseteq X \setminus Y$ and $X'_{s_0} \cap U = \emptyset$. The pair $(X \setminus X'_{s_0}, Y)$ is a separation of $G' \setminus Z$, and each vertex in U is a cut vertex of $G' \setminus Z$ between a vertex of S' and the vertex set T'.

Claim 3 *The subgraph G' has an S-cycle.*

Proof: Let $U = \{u_1, \ldots, u_m\}$, where $m = |U|$. For $1 \leq j \leq m$, let (X_j, Y_j) be a separation of $G' \setminus Z$ with $X_j \cap Y_j = \{u_j\}$ such that $T \subseteq Y_j$ and $X_j \setminus Y_j$ contains some vertex of S'. Choose (X_j, Y_j) such that X_j is minimal. Then $Y \subseteq Y_j$ holds and $X_1, \ldots, X_m$ are disjoint. We may assume that $G'[X_j]$ has no S-cycle for any j, otherwise we are done. Since $\{u_j\} = X_s \cap Y_s$ for some $s \in S'$ and X_s is chosen minimal, this implies that X_j has a vertex s_j of S' that connects to u_j. Moreover, since s_j is contained in some S-cycle in G, this assumption implies that G' has an edge connecting X_j and Z, and hence Z is nonempty.

Let C_j be an S-cycle containing s_j in G. The subgraph $G'[X_j \cup Z]$ has a path P_j through s_j from u_j to a vertex in Z by using the edge (s_j, u_j) and C_j. We may suppose that P_j has no inner vertices in Z by taking a minimal path. By $|U| > 2K + 1 \geq |Z| \geq 1$, there exist a vertex z in Z and two indices j_1, j_2 such that both P_{j_1} and P_{j_2} end with z. The path P_{j_i} is contained in $G'[X_{j_i} \cup \{z\}]$ for $i = 1, 2$, respectively.

The subgraph $G'[Y_{j_i}]$ has a path P'_{j_i} from u_{j_i} to a vertex w_{j_i} of T'. Since each vertex in U is a cut vertex in $G \setminus Z$, the path P'_{j_i} has no vertex of $Y_{j_i} \setminus Y$, and thus is contained in $G'[Y]$. We may assume that P'_{j_i} has no inner vertex in T'. By $T \subseteq Y \cup Z$, the subgraph $G'[Y \cup Z]$ has two internally disjoint paths between w_{j_1} and w_{j_2} along C, and hence one of these two paths, denoted by P, does not have z. Then the union of P'_{j_1}, P, and P'_{j_2} includes a path in $G'[Y \cup Z \setminus \{z\}]$ from u_{j_1} to u_{j_2}. This path, together with P_{j_1} and P_{j_2}, yields an S-cycle in G'. $\square$

Therefore, both $G[X_{s_0}]$ and G', i.e., $G[Y_{s_0}]$ have S-cycles, and thus $k \geq 3$. These two S-cycles imply by $\nu_S(G) < k$ that $\nu_S(G \setminus X) < i$ and $\nu_S(G \setminus Y) < k - i + 1$ for some $i \in \{2, \ldots, k-1\}$. By the induction hypothesis, it holds that $\tau_S(G \setminus X) \leq f(i)$ and $\tau_S(G \setminus Y) \leq f(k - i + 1)$. Since every S-cycle that is not a cycle of $G \setminus Y_{s_0}$ or $G \setminus X_{s_0}$ meets $X_{s_0} \cap Y_{s_0} = Z \cup \{s_0\}$, we have

$$\tau_S(G) \leq \tau_S(G \setminus X) + \tau_S(G \setminus Y) + |Z| + 1 \leq \tilde{f}(k) + 2K - 1 \leq g(k).$$

Thus the statement holds. □

Proof of Theorem 1.2: Define

$$\begin{aligned} K &= 4k \log_2(k + 10), \\ r_k &= \max\{4(k(12K - 2) - 1)^2 + 4, \tilde{f}(k)\}, \\ f(k) &= \max\{3r_k, \tilde{f}(k) + 2K\}. \end{aligned}$$

Note that $f(k) = \max\{3r_k, g(k)\}$, where $g(k)$ is defined as in Lemma 3.5. We will show that $f(k)$ satisfies Theorem 1.2. Assume to the contrary that there is a graph $G = (V, E)$ with $S \subseteq V$ satisfying $\nu_S(G) < k$ and $\tau_S(G) > f(k)$. By $\tau_S(G) \geq \tilde{f}(k)$, the set $\mathcal{B}_H$ forms a bramble of order $\geq r_k$. Moreover, by $\tau_S(G) \geq r_k$, it follows from Lemma 3.3 that G has a cycle C of length $\geq 12K - 2$ with no vertices of S such that C is $4K$-attached to $\mathcal{B}_H$. If G has K disjoint S-paths with respect to $V(C)$, then $\nu_S(G) \geq k$ holds by Lemma 3.4. Otherwise, by Lemma 3.5, we have $\nu_S(G) \geq k$ or $\tau_S(G) \leq g(k) \leq f(k)$. Hence both cases have a contradiction. Thus the statement holds. □

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