A Mathematical Justification of a Thin Film Approximation for the Flow down an Inclined Plane

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Abstract

We consider a two-dimensional motion of a thin film flowing down an inclined plane under the influence of the gravity and the surface tension. In order to investigate the stability of such flow, we often use a thin film approximation, which is an approximation obtained by the perturbation expansion with respect to the aspect ratio of the film. The famous example of the approximate equations are the Burgers equation, Kuramoto–Sivashinsky equation, KdV–Burgers equation, KdV–Kuramoto–Sivashinsky equation, and so on. In this paper, we give a mathematically rigorous justification of a thin film approximation by establishing an error estimate between the solution of the Navier–Stokes equations and those of approximate equations.

1 Introduction

In this paper, we consider a two-dimensional motion of liquid film of a viscous and incompressible fluid flowing down an inclined plane under the influence of the gravity and the surface tension on the interface. The motion can be mathematically formulated as a free boundary problem for the incompressible Navier–Stokes equations. We assume that the domain $\Omega(t)$ occupied by the liquid at time $t \geq 0$, the liquid surface $\Gamma(t)$, and the rigid plane Σ are of the forms

$$\begin{cases} \Omega(t) = \{(x,y) \in \mathbb{R}^2 \mid 0 < y < h_0 + \eta(x,t)\}, \\ \Gamma(t) = \{(x,y) \in \mathbb{R}^2 \mid y = h_0 + \eta(x,t)\}, \\ \Sigma = \{(x,y) \in \mathbb{R}^2 \mid y = 0\}, \end{cases}$$

where h_0 is the mean thickness of the liquid film and $\eta(x,t)$ is the amplitude of the liquid surface. Here we choose a coordinate system (x,y) so that x axis is pointed to the streamwise direction and y axis is normal to the plane. We consider fluctuations of the Nusselt flat film solution, which is the stationary laminar flow given by

$$\eta_1 = 0, \quad u_1 = (\rho g \sin\alpha / 2\mu)(2h_0 y - y^2), \quad v_1 = 0, \quad p_1 = p_0 - \rho g \cos\alpha (y - h_0), \tag{1.1}$$

where ρ is a constant density of the liquid, g is the acceleration of the gravity, α is the angle of inclination, μ is the shear viscosity coefficient, and p_0 is an atmospheric pressure. Throughout this paper, we assume that the flow is l_0-periodic in the streamwise direction x. Rescaling the independent and dependent variables by using h_0, l_0, the typical amplitude of the liquid surface a_0, $U_0 = \rho g h_0^2 \sin\alpha / 2\mu$, and $P_0 = \rho g h_0 \sin\alpha$, the equations are written in the non-dimensional form

$$\begin{cases} \delta \boldsymbol{u}_t + \big((\bar{\boldsymbol{u}} + \varepsilon \boldsymbol{u}) \cdot \nabla_\delta\big)\boldsymbol{u} + (\boldsymbol{u} \cdot \nabla_\delta)\bar{\boldsymbol{u}} + \dfrac{2}{\mathrm{R}}\nabla_\delta p - \dfrac{1}{\mathrm{R}}\Delta_\delta \boldsymbol{u} = \boldsymbol{0} & \text{in} \quad \Omega_\varepsilon(t),\ t > 0, \\ \nabla_\delta \cdot \boldsymbol{u} = 0 & \text{in} \quad \Omega_\varepsilon(t),\ t > 0, \end{cases} \tag{1.2}$$

$$\begin{cases} \big(\boldsymbol{D}_\delta(\varepsilon \boldsymbol{u} + \bar{\boldsymbol{u}}) - \varepsilon p \boldsymbol{I}\big)\boldsymbol{n} & \\ \quad = \Big(-\dfrac{1}{\tan\alpha}\varepsilon\eta + \dfrac{\delta^2 \mathrm{W}}{\sin\alpha}\dfrac{\varepsilon \eta_{xx}}{(1 + (\varepsilon\delta\eta_x)^2)^{\frac{3}{2}}}\Big)\boldsymbol{n} & \text{on} \quad \Gamma_\varepsilon(t),\ t > 0, \\ \eta_t + \big(1 - (\varepsilon\eta)^2 + \varepsilon u\big)\eta_x - v = 0 & \text{on} \quad \Gamma_\varepsilon(t),\ t > 0, \end{cases} \tag{1.3}$$

$$\boldsymbol{u} = \mathbf{0} \quad \text{on} \quad \Sigma,\ t > 0. \tag{1.4}$$

Here, $\delta, \varepsilon, \mathrm{R}$, and W are non-dimensional parameters defined by

$$\delta = \frac{h_0}{l_0}, \quad \varepsilon = \frac{a_0}{h_0}, \quad \mathrm{R} = \frac{\rho U_0 h_0}{\mu}, \quad \mathrm{W} = \frac{\sigma}{\rho g h_0^2},$$

where σ is the surface tension coefficient. Note that δ is the aspect ratio of the film, ε represents the magnitude of nonlinearity, R is the Reynolds number, and W is the Weber number. Moreover, we used notations $\boldsymbol{u} = (u, \delta v)^{\mathrm{T}}, \bar{\boldsymbol{u}} = (\bar{u}, 0)^{\mathrm{T}}, \bar{u} = 2y - y^2, \nabla_\delta = (\delta \partial_x, \partial_y)^{\mathrm{T}}, \Delta_\delta = \nabla_\delta \cdot \nabla_\delta, \boldsymbol{D}_\delta \boldsymbol{f} = \frac{1}{2}\{\nabla_\delta(\boldsymbol{f}^{\mathrm{T}}) + (\nabla_\delta(\boldsymbol{f}^{\mathrm{T}}))^{\mathrm{T}}\}$, and $\boldsymbol{n} = (-\varepsilon\delta\eta_x, 1)^{\mathrm{T}}$. In this scaling, the liquid domain $\Omega_\varepsilon(t)$ and the liquid surface $\Gamma_\varepsilon(t)$ are of the forms

$$\begin{cases} \Omega_\varepsilon(t) = \{(x, y) \in \mathbb{R}^2 \mid 0 < y < 1 + \varepsilon\eta(x, t)\}, \\ \Gamma_\varepsilon(t) = \{(x, y) \in \mathbb{R}^2 \mid y = 1 + \varepsilon\eta(x, t)\}. \end{cases}$$

Concerning a mathematical analysis of the problem in the case of $\delta = \varepsilon = 1$, Teramoto [14] showed that the initial value problem to the Navier–Stokes equations (1.2)–(1.4) has a unique solution globally in time under the assumptions that the Reynolds number and the initial data are sufficiently small. Nishida, Teramoto, and Win [10] showed the exponential stability of the Nusselt flat film solution under the assumptions that the angle of inclination is sufficiently small and $x \in \mathbb{T}$ in addition to the assumptions in [14]. Furthermore, Uecker [15] studied the asymptotic behavior for $t \to \infty$ of the solution in the case of $x \in \mathbb{R}$ and showed that the perturbations of the Nusselt flat film solution decay like the self-similar solution of the Burgers equation under the assumptions that the initial data are sufficiently small and $\mathrm{R} < \mathrm{R}_c$. Here, $\mathrm{R}_c = \frac{4}{5}\frac{1}{\tan\alpha}$ is the critical Reynolds number given by Benjamin [2]. On the other hand, Ueno, Shiraishi, and Iguchi [16] derived a uniform estimate for the solution of (1.2)–(1.4) with respect to δ when the Reynolds number, the angle of inclination, and the initial data are sufficiently small.

Benney [3] derived the following single nonlinear evolution equation

$$\begin{aligned} &\eta_t + 2(1 + \varepsilon\eta)^2\eta_x - \frac{8}{15}(\mathrm{R}_c - \mathrm{R})\delta\eta_{xx} + C_1\delta^2\eta_{xxx} \\ &\quad + C_2\varepsilon\delta(\eta\eta_{xx} + \eta_x^2) + \frac{2}{3}\frac{\mathrm{W}}{\sin\alpha}\delta^3\eta_{xxxx} = O(\delta^3 + \varepsilon^2\delta + \varepsilon\delta^2) \end{aligned} \tag{1.5}$$

by using the method of perturbation expansion of the solution (u, v, p) with respect to δ under the thin film regime $\delta \ll 1$. Here, $C_1 = C_1(\mathrm{R}, \alpha)$ and $C_2 = C_2(\mathrm{R}, \alpha)$ are constants independent of δ, ε, and W. Explicit forms of C_1 and C_2 will be given in Section 3. Many approximate equations are obtained from (1.5) by assuming that parameters ε, W, and R have appropriate orders in δ. In the following, we assume $\varepsilon = \delta$ and $\mathrm{R} < \mathrm{R}_c$ and set

$$\eta(x, t) = \zeta(x - 2t, \varepsilon t). \tag{1.6}$$

I. Burgers equation

Assuming $\mathrm{W}_1 \le \mathrm{W} \le \delta^{-1}\mathrm{W}_2$ in (1.5), we have

$$\eta_t + 2\eta_x + 4\varepsilon\eta\eta_x - \frac{8}{15}(\mathrm{R}_c - \mathrm{R})\delta\eta_{xx} = O(\delta^2).$$

Plugging (1.6) in the above equation and passing to the limit $\varepsilon = \delta \to 0$, we obtain

$$\zeta_\tau + 4\zeta\zeta_x - \frac{8}{15}(\mathrm{R}_c - \mathrm{R})\zeta_{xx} = 0. \tag{1.7}$$

II. Burgers equation with a fourth order dissipation term

Assuming $\mathrm{W} = \delta^{-2}\mathrm{W}_2$ in (1.5), we have

$$\eta_t + 2\eta_x + 4\varepsilon\eta\eta_x - \frac{8}{15}(\mathrm{R}_c - \mathrm{R})\delta\eta_{xx} + \frac{2}{3}\frac{\mathrm{W}_2}{\sin\alpha}\delta\eta_{xxxx} = O(\delta^2).$$

Plugging (1.6) in the above equation and passing to the limit $\varepsilon = \delta \to 0$, we obtain

$$\zeta_\tau + 4\zeta\zeta_x - \frac{8}{15}(\mathrm{R}_c - \mathrm{R})\zeta_{xx} + \frac{2}{3}\frac{\mathrm{W}_2}{\sin\alpha}\zeta_{xxxx} = 0. \tag{1.8}$$

III. Burgers equation with dispersion and nonlinear terms

Assuming $\mathrm{W}_1 \le \mathrm{W} \le \mathrm{W}_2$ in (1.5), we have

$$\eta_t + 2\eta_x + 4\varepsilon\eta\eta_x - \frac{8}{15}(\mathrm{R}_c - \mathrm{R})\delta\eta_{xx} + C_1\delta^2\eta_{xxx} + C_2\varepsilon\delta(\eta\eta_{xx} + \eta_x^2) + 2\varepsilon^2\eta^2\eta_x = O(\delta^3).$$

Plugging (1.6) in the above equation and neglecting the terms of $O(\delta^3)$, we obtain

$$\zeta_\tau + 4\zeta\zeta_x - \frac{8}{15}(\mathrm{R}_c - \mathrm{R})\zeta_{xx} + \delta\bigl\{C_1\zeta_{xxx} + C_2\bigl(\zeta\zeta_{xx} + \zeta_x^2\bigr) + 2\zeta^2\zeta_x\bigr\} = 0. \tag{1.9}$$

IV. Burgers equation with fourth order dissipation, dispersion, and nonlinear terms

Assuming $\mathrm{W} = \delta^{-1}\mathrm{W}_2$ in (1.5), we have

$$\begin{aligned}&\eta_t + 2\eta_x + 4\varepsilon\eta\eta_x - \frac{8}{15}(\mathrm{R}_c - \mathrm{R})\delta\eta_{xx} \\ &\quad + C_1\delta^2\eta_{xxx} + C_2\varepsilon\delta(\eta\eta_{xx} + \eta_x^2) + 2\varepsilon^2\eta^2\eta_x + \frac{2}{3}\frac{\mathrm{W}_2}{\sin\alpha}\delta^2\eta_{xxxx} = O(\delta^3).\end{aligned}$$

Plugging (1.6) in the above equation and neglecting the terms of $O(\delta^3)$, we obtain

$$\begin{aligned}&\zeta_\tau + 4\zeta\zeta_x - \frac{8}{15}(\mathrm{R}_c - \mathrm{R})\zeta_{xx} \\ &\quad + \delta\left\{C_1\zeta_{xxx} + C_2\bigl(\zeta\zeta_{xx} + \zeta_x^2\bigr) + 2\zeta^2\zeta_x + \frac{2}{3}\frac{\mathrm{W}_2}{\sin\alpha}\zeta_{xxxx}\right\} = 0.\end{aligned} \tag{1.10}$$

We remark that (1.9) and (1.10) are higher order approximate equations to Burgers equation (1.7). In this paper, we assume $\mathrm{R} \ll \mathrm{R}_c$ in order to use a uniform estimate in δ for the solution of the Navier–Stokes equations, which is a severe restriction. Uniform estimates in δ for the solution play a most important role in the justification for these approximation. Here if we could assume $\mathrm{R} > \mathrm{R}_c$, then (1.8) would be the Kuramoto–Sivashinsky equation (see [9], [12], and [13]). If we could assume $\mathrm{R}_c - \mathrm{R} = \delta\tilde{\mathrm{R}} > 0$, then we would obtain the δ-independent KdV–Burgers equation

$$\zeta_\tau + 4\zeta\zeta_x - \frac{8\tilde{\mathrm{R}}}{15}\zeta_{xx} + C_1\zeta_{xxx} = 0 \tag{1.11}$$

by plugging (1.6) in (1.5) and passing to the limit $\varepsilon = \delta^2 \to 0$ under the assumption $\mathrm{W}_1 \le \mathrm{W} \le \mathrm{W}_2$. Moreover if we could assume $\mathrm{R}_c - \mathrm{R} = -\delta\tilde{\mathrm{R}} < 0$, we would obtain the δ-independent KdV–Kuramoto–Sivashinsky equation

$$\zeta_\tau + 4\zeta\zeta_x + \frac{8\tilde{\mathrm{R}}}{15}\zeta_{xx} + C_1\zeta_{xxx} + \frac{2}{3}\frac{\mathrm{W}_2}{\sin\alpha}\zeta_{xxxx} = 0 \tag{1.12}$$

by plugging (1.6) in (1.5) and passing to the limit $\varepsilon = \delta^2 \to 0$ under the assumption $\mathrm{W} = \delta^{-1}\mathrm{W}_2$. More details or a list of useful references about the thin film approximation can be found in [6, 7, 8, 11, 16].

In this paper, we will give a mathematically rigorous justification of these thin film approximations by establishing an error estimate between the solution of the Navier–Stokes equations (1.2)–(1.4) and those of the approximate equations (1.7)–(1.10). We note that we cannot just yet justify the Kuramoto–Sivashinsky equation, the δ-independent KdV–Burgers equation (1.11), and the KdV–Kuramoto–Sivashinsky equation (1.12) because without the assumption $\mathrm{R} \ll \mathrm{R}_{\mathrm{c}}$ we have not yet obtain a uniform estimate in δ for the solution. We also remark that Bresch and Noble [5] justified the shallow water model by proving that remainder terms converges to 0 as $\delta \to 0$ (see also [4]).

The plan of this paper is as follows. In Section 2, we give our main theorem after we transform the problem in a time dependent domain to a problem in a time independent domain. In Section 3, we derive approximate solutions by using Benney's method. In Section 4, we recall the energy estimate for the solution of the Navier–Stokes equations obtained in [16]. Finally, we give an error estimates in Section 5.

Notation. We put $\Omega = \mathbb{T} \times (0, 1)$ and $\Gamma = \mathbb{T} \times \{y = 1\}$, where $\mathbb{T}$ is the flat torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. For a Banach space X, we denote by $\|\cdot\|_X$ the norm in X. For $1 \le p \le \infty$, we put $\|u\|_{L^p} = \|u\|_{L^p(\Omega)}$, $\|u\| = \|u\|_{L^2}$, $|u|_{L^p} = \|u(\cdot, 1)\|_{L^p(\mathbb{T})}$, and $|u|_0 = |u|_{L^2}$. We denote by $(\cdot,\cdot)_\Omega$ and $(\cdot,\cdot)_\Gamma$ the inner products of $L^2(\Omega)$ and $L^2(\Gamma)$, respectively. For $s \ge 0$, we denote by $H^s(\Omega)$ and $H^s(\Gamma)$ the L^2 Sobolev spaces of order s on Ω and Γ, respectively. The norms of these spaces are denoted by $\|\cdot\|_s$ and $|\cdot|_s$. For a function $u = u(x, y)$ on Ω, a Fourier multiplier $P(D_x)$ ($D_x = -\mathrm{i}\partial_x$) is defined by $(P(D_x)u)(x, y) = \sum_{n\in\mathbb{Z}} P(n)\hat{u}_n(y)\mathrm{e}^{2\pi i n x}$, where $\hat{u}_n(y) = \int_0^1 u(x, y)\mathrm{e}^{-2\pi i n x}\,\mathrm{d}x$ is the Fourier coefficient in x. We put $\partial_y^{-1} f(x, y) = -\int_y^1 f(x, z)\mathrm{d}z$ and $D_\delta^k f = \{(\delta\partial_x)^i \partial_y^j f \mid i + j = k\}$. $f \lesssim g$ means that there exists a non-essential positive constant C such that $f \le Cg$ holds.

2 Main results

We rewrite the system (1.2)–(1.4) according to [1, 16]. Transforming the problem in the moving domain $\Omega(t)$ to a problem in the fixed domain Ω by using an appropriate diffeomorphism, and introducing new unknown function (u', v', p') to keep the solenoidal condition, we obtain

$$
(2.1)\qquad \begin{cases} \delta(u_t + \bar{u}u_x + \bar{u}_y v) + \dfrac{2}{\mathrm{R}}\delta p_x - \dfrac{1}{\mathrm{R}}(\delta^2 u_{xx} + u_{yy}) = \delta^2 f_1 & \text{in} \quad \Omega,\ t > 0, \\ \delta^2(v_t + \bar{u}v_x) + \dfrac{2}{\mathrm{R}}p_y - \dfrac{1}{\mathrm{R}}\delta(\delta^2 v_{xx} + v_{yy}) = \delta^2 f_2 & \text{in} \quad \Omega,\ t > 0, \\ u_x + v_y = 0 & \text{in} \quad \Omega,\ t > 0, \end{cases}
$$

$$
(2.2)\qquad \begin{cases} \delta^2 v_x + u_y - 2(1 + \varepsilon\eta)^2\eta = \delta^3 h_1 & \text{on} \quad \Gamma,\ t > 0, \\ p - \delta v_y - \dfrac{1}{\tan\alpha}\eta + \dfrac{\delta^2 \mathrm{W}}{\sin\alpha}\eta_{xx} = \delta^2 h_2 & \text{on} \quad \Gamma,\ t > 0, \\ \eta_t + \eta_x - v = \delta^2\eta^2\eta_x =: \delta^2 h_3 & \text{on} \quad \Gamma,\ t > 0, \end{cases}
$$

$$
(2.3)\qquad u = v = 0 \quad \text{on} \quad \Sigma,\ t > 0,
$$

where we dropped the prime sign in the notation and f_1, f_2, h_1, and h_2 are collections of nonlinear terms. See [16] for more details on the explicit form of these nonlinear terms. In the following, we will consider the initial value problem to (2.1)–(2.3) under the initial conditions

$$\eta|_{t=0} = \eta_0 \quad \text{on} \quad \Gamma, \qquad (u, v)^{\mathrm{T}}|_{t=0} = (u_0, v_0)^{\mathrm{T}} \quad \text{in} \quad \Omega. \tag{2.4}$$

Here, we assume $\int_0^1 \eta_0(x)\mathrm{d}x = 0$ and denote h_1 determined from initial data by $h_1^{(0)}$.

We impose the following assumption on the non-dimensional parameters and initial data.

Assumption 2.1. *Let* $\mathrm{R}_0, \mathrm{R}_1, \alpha_0, \mathrm{W}_1, c_0$*, and* M *be positive constants and* $m \geq 2$ *be an integer.*

(1) Conditions for parameters

Parameters $\mathrm{R}, \alpha, \mathrm{W}, \delta$*, and* ε *satisfy*

$$\mathrm{R}_1 \leq \mathrm{R} \leq \mathrm{R}_0, \quad 0 < \alpha \leq \alpha_0, \quad \mathrm{W}_1 \leq \mathrm{W}, \quad 0 < \varepsilon = \delta \leq 1.$$

(2) Smallness of initial data

Initial data (η_0, u_0, v_0) *and parameters* W *and* δ *satisfy*

$$\begin{aligned} &|(1+\delta|D_x|)^2\eta_0|_2 + \|(1+|D_x|)^2(u_0, \delta v_0)^{\mathrm{T}}\| + \|(1+|D_x|)^2 D_\delta (u_0, \delta v_0)^{\mathrm{T}}\| \\ &\quad + \|(1+|D_x|)^2 D_\delta^2 (u_0, \delta v_0)^{\mathrm{T}}\| + \delta^2 \mathrm{W} |(1+\delta|D_x|)\eta_{0x}|_3 + \sqrt{\delta^2\mathrm{W}}\|(1+|D_x|)^2 \delta v_{0xy}\| \leq c_0. \end{aligned}$$

(3) Regularity of initial data

Initial data (η_0, u_0, v_0) *satisfies*

$$\|(1+|D_x|)^{m+1}(u_0, v_0)^{\mathrm{T}}\|_{H^2(\Omega)} + |\eta_0|_{m+4} \leq M.$$

(4) Compatibility conditions

Initial data (η_0, u_0, v_0) *and parameters* δ *and* ε *satisfy*

$$\begin{cases} u_{0x} + v_{0y} = 0 & \text{in} \quad \Omega, \\ u_{0y} + \delta^2 v_{0x} - 2(1+\varepsilon\eta_0)^2\eta_0 = \delta^3 h_1^{(0)} & \text{on} \quad \Gamma, \\ u_0 = v_0 = 0 & \text{on} \quad \Sigma. \end{cases}$$

Remark 2.1. Under the assumption that there exist small positive constants R_0, α_0, and c_0 such that Assumption 2.1 is fulfilled, Ueno, Shiraishi, and Iguchi [16] proved the global in time uniform estimate with respect to δ for the solution of the Navier–Stokes equations (2.1)–(2.4). See also Proposition 4.2 in this paper.

For later use, we define the norm of a difference between the solution $(\eta^\delta, u^\delta, v^\delta, p^\delta)$ of the Navier–Stokes equations (2.1)–(2.4) and the solution ζ of the approximate equations as

$$\begin{aligned} \mathcal{D}(t; \zeta, u, v, p) :=& |\eta^\delta(t) - \zeta(\cdot - 2t, \varepsilon t)|_0^2 + \|(1+|D_x|)^m (u^\delta - u)(t)\|^2 \\ &+ \|(1+|D_x|)^{m-1}(v^\delta - v)(t)\|^2 + \|(1+|D_x|)^{m-1}(p^\delta - p)(t)\|^2, \end{aligned} \tag{2.5}$$

where (u, v, p) is an approximate solution constructed from ζ. Let $\zeta^I, \zeta^{II}, \zeta^{III}$, and ζ^{IV} be the solution of (1.7)–(1.10) under the initial condition $\zeta|_{\tau=0} = \eta_0$, respectively.

Now we are ready to state our main results in this paper. Note that the definitions of $u^I, v^I, p^I, u^{II}, \ldots$ appeared in the following statement will be given in Section 5.

Theorem 2.2. *There exist small positive constants* R_0 *and* α_0 *such that the following statement holds: Let* m *be an integer satisfying* $m \geq 2$, $0 < \mathrm{R}_1 \leq \mathrm{R}_0$, $0 < \mathrm{W}_1 \leq \mathrm{W}_2$, *and* $0 < \alpha \leq \alpha_0$. *There exists small positive constant* c_0 *such that if the initial data* (η_0, u_0, v_0) *and the parameters* δ, ε, R, *and* W *satisfy Assumption 2.1, then we have the following estimates.*

I. Burgers equation

If the parameters δ *and* W *and the initial data* η_0 *and* u_0 *satisfy*

$$\mathrm{W}_1 \leq \mathrm{W} \leq \delta^{-1}\mathrm{W}_2, \quad |\eta_0|_{m+7} + \delta^{-1}\|(1+|D|_x)^{m+1}u_{0yy}\| \leq M < \infty, \tag{2.6}$$

then the following error estimate holds.

$$\mathcal{D}(t; \zeta^I, u^I, v^I, p^I) \leq C\delta^2 \mathrm{e}^{-c\varepsilon t}. \tag{2.7}$$

II. Burgers equation with a fourth order dissipation term

If the parameters δ *and* W *and the initial data* η_0 *and* u_0 *satisfy*

$$\mathrm{W} = \delta^{-2}\mathrm{W}_2, \quad |\eta_0|_{m+12} + \delta^{-1}\|(1+|D|_x)^{m+1}u_{0yy}\| \leq M < \infty, \tag{2.8}$$

then the following error estimate holds.

$$\mathcal{D}(t; \zeta^{II}, u^{II}, v^{II}, p^{II}) \leq C\delta^2 \mathrm{e}^{-c\varepsilon t}. \tag{2.9}$$

III. Burgers equation with dispersion and nonlinear terms

If the parameters δ *and* W *and the initial data* η_0 *and* u_0 *satisfy*

$$\mathrm{W}_1 \leq \mathrm{W} \leq \mathrm{W}_2, \quad |\eta_0|_{m+13} + \delta^{-2}\|(1+|D|_x)^{m+1}(u_{0yy} - u^{III}_{yy}|_{t=0})\| \leq M < \infty, \tag{2.10}$$

then the following error estimate holds.

$$\mathcal{D}(t; \zeta^{III}, u^{III}, v^{III}, p^{III}) \leq C\delta^4 \mathrm{e}^{-c\varepsilon t}. \tag{2.11}$$

IV. Burgers equation with a fourth order dissipation, dispersion, and nonlinear terms

If the parameters δ *and* W *and the initial data* η_0 *and* u_0 *satisfy*

$$\mathrm{W} = \delta^{-1}\mathrm{W}_2, \quad |\eta_0|_{m+17} + \delta^{-2}\|(1+|D|_x)^{m+1}(u_{0yy} - u^{IV}_{yy}|_{t=0})\| \leq M < \infty, \tag{2.12}$$

then the following error estimate holds.

$$\mathcal{D}(t; \zeta^{IV}, u^{IV}, v^{IV}, p^{IV}) \leq C\delta^4 \mathrm{e}^{-c\varepsilon t}. \tag{2.13}$$

Here, positive constants C *and* c *depend on* $\mathrm{R}_1, \mathrm{W}_1, \mathrm{W}_2, \alpha$, *and* M *but are independent of* δ, ε, R, *and* W.

Remark 2.2. The assumptions for u_{0yy} in (2.6) and (2.8) represent the restriction on the initial profile of the velocity. Moreover, the assumptions for u_{0yy} in (2.10) and (2.12) mean that the initial profile of the velocity have to be equal to that of the approximate solution up to $O(\delta^2)$.

Remark 2.3. We see formally that the order of error terms in (1.7) is of $O(\delta)$, which implies that the error estimates (2.7) and (2.9) are natural. In a similar way, we see that the error estimates (2.11) and (2.13) are natural.

Remark 2.4. By introducing the slow time scale $\tau = \varepsilon t$, the norm decays exponentially and uniformly in τ.

3 Approximate solutions

In this section, following Benney's perturbation method [3] we will give approximate equations by constructing approximate solutions. Hereafter, we assume $\varepsilon = \delta$. By straightforward calculation, we can rewrite (2.1)–(2.3) as follows.

(3.1)

$$\begin{cases} \delta(u_t + \bar{u}u_x + \bar{u}_y v) + \dfrac{2}{\mathrm{R}}\delta p_x - \dfrac{1}{\mathrm{R}}(\delta^2 u_{xx} + u_{yy}) = -\delta\dfrac{2}{\mathrm{R}}\eta u_{yy} + \delta^2 f_1^{(2)} + \delta^3 f_1^{(3)} & \text{in} \quad \Omega,\ t>0, \\ \delta^2(v_t + \bar{u}v_x) + \dfrac{2}{\mathrm{R}}p_y - \dfrac{1}{\mathrm{R}}\delta(\delta^2 v_{xx} + v_{yy}) = \delta\dfrac{2}{\mathrm{R}}\eta p_y + \delta^2 f_2^{(2)} + \delta^3 f_2^{(3)} & \text{in} \quad \Omega,\ t>0, \\ u_x + v_y = 0 & \text{in} \quad \Omega,\ t>0, \end{cases}$$

(3.2)
$$\begin{cases} \delta^2 v_x + u_y - 2(1+\delta\eta)^2\eta = \delta^3 h_1 & \text{on} \quad \Gamma,\ t>0, \\ p - \delta v_y - \dfrac{1}{\tan\alpha}\eta + \dfrac{\delta^2 \mathrm{W}}{\sin\alpha}\eta_{xx} = \delta^2 h_2^{(2)} + \delta^3 h_2^{(3)} & \text{on} \quad \Gamma,\ t>0, \end{cases}$$

(3.3)
$$u = v = 0 \quad \text{on} \quad \Sigma,\ t>0,$$

(3.4)
$$\eta_t + \eta_x - v = \delta^2 h_3 \quad \text{on} \quad \Gamma,\ t>0,$$

where

(3.5)
$$\begin{cases} f_1^{(2)} = \dfrac{1}{\mathrm{R}}\big(3\eta^2 u_{yy} - 2\eta p_x + 2y\eta_x p_y\big) + \eta_t u + y\eta_t u_y \\ \qquad\qquad + y^2\eta_x u + 2y(y-1)\eta u_x - y^2(y-2)\eta_x u_y - uu_x - vu_y + 2(2y-1)\eta v, \\ f_2^{(2)} = \dfrac{1}{\mathrm{R}}\big(-2\eta^2 p_y + 2\eta_x u_y + 2\eta u_{xy}\big), \\ h_2^{(2)} = 2\eta\eta_x + \eta_x u + \eta u_x. \end{cases}$$

We proceed to derive the approximate equations following Benney [3]. Let $\eta = \eta(x,t)$ be a given function. For any $\delta \in (0,1]$, let (u,v,p) be the solution of (3.1)–(3.3) and we expand (u,v,p) as

(3.6)
$$\begin{cases} u = u_0 + \delta u_1 + \delta^2 u_2 + \cdots, \\ v = v_0 + \delta v_1 + \delta^2 v_2 + \cdots, \\ p = p_0 + \delta p_1 + \delta^2 p_0 + \cdots \end{cases}$$

and substitute these into (3.1)–(3.3), we obtain a sequence of perturbation equations for each order of δ. Here, u_0 and v_0 are different from initial data defined in (2.4) and hereafter we use this notation whenever it does not lead to confusion. By assuming $\mathrm{W} = O(1)$, the $O(1)$, $O(\delta)$, and $O(\delta^2)$ problems are as follows.

(3.7)
$$\begin{cases} u_{0yy} = 0, \quad p_{0y} = 0, \quad u_{0x} + v_{0y} = 0 & \text{in} \quad \Omega, \\ u_{0y} = 2\eta, \quad p_0 = \dfrac{1}{\tan\alpha}\eta & \text{on} \quad \Gamma, \\ u_0 = v_0 = 0 & \text{on} \quad \Sigma, \end{cases}$$

$$
(3.8)\qquad \begin{cases} u_{1yy} = \mathrm{R}(u_{0t} + (2y-y^2)u_{0x} + 2(1-y)v_0) + 2p_{0x} + 2\eta u_{0yy} & \text{in} \quad \Omega, \\ 2p_{1y} = v_{0yy} + 2\eta p_{0y}, \quad u_{1x} + v_{1y} = 0 & \text{in} \quad \Omega, \\ u_{1y} = 4\eta^2, \quad p_1 = -u_{0x} & \text{on} \quad \Gamma, \\ u_1 = v_1 = 0 & \text{on} \quad \Sigma, \end{cases}
$$

(3.9)

$$
\begin{cases} u_{2yy} = \mathrm{R}(u_{1t} + (2y-y^2)u_{1x} + 2(1-y)v_1) + 2p_{1x} + 2\eta u_{1yy} - u_{0xx} - \mathrm{R}f_1^{(2)}(\eta, u_0, v_0, p_0) & \text{in} \quad \Omega, \\ 2p_{2y} = v_{1yy} + 2\eta p_{1y} - \mathrm{R}\big(v_{0t} + (2y-y^2)v_{0x}\big) + \mathrm{R}f_2^{(2)}(\eta, u_0, v_0, p_0), \quad u_{2x} + v_{2y} = 0 & \text{in} \quad \Omega, \\ u_{2y} = -v_{0x} + 2\eta^3, \quad p_2 = -u_{1x} + h_2^{(2)}(\eta, u_0) - \dfrac{\mathrm{W}}{\sin\alpha}\eta_{xx} & \text{on} \quad \Gamma, \\ u_2 = v_2 = 0 & \text{on} \quad \Sigma. \end{cases}
$$

Solving the above boundary value problem for the ordinary differential equations, we have

$$
(3.10)\qquad \begin{cases} u_0 = 2y\eta, \\ v_0 = -y^2\eta_x, \\ p_0 = \dfrac{1}{\tan\alpha}\eta, \end{cases}
$$

$$
(3.11)\qquad \begin{cases} u_1 = \left(\frac{1}{3}y^3 - y\right)\mathrm{R}\eta_t + \left\{(y^2-2y)\frac{1}{\tan\alpha} + \left(\frac{1}{6}y^4 - \frac{2}{3}y\right)\mathrm{R}\right\}\eta_x + 4y\eta^2, \\ v_1 = \left(-\frac{1}{12}y^4 + \frac{1}{2}y^2\right)\mathrm{R}\eta_{xt} + \left\{\left(-\frac{1}{3}y^3 + y^2\right)\frac{1}{\tan\alpha} + \left(-\frac{1}{30}y^5 + \frac{1}{3}y^2\right)\mathrm{R}\right\}\eta_{xx} - 4y^2\eta\eta_x, \\ p_1 = -(1+y)\eta_x, \end{cases}
$$

(3.12)

$$
\begin{cases} u_2 = \left(\frac{1}{60}y^5 - \frac{1}{6}y^3 + \frac{5}{12}y\right)\mathrm{R}^2\eta_{tt} \\ \qquad + \left\{\left(\frac{1}{12}y^4 - \frac{1}{3}y^3 + \frac{2}{3}y\right)\frac{\mathrm{R}}{\tan\alpha} + \left(-\frac{1}{252}y^7 + \frac{1}{45}y^6 - \frac{1}{12}y^4 - \frac{1}{9}y^3 + \frac{101}{180}y\right)\mathrm{R}^2\right\}\eta_{xt} \\ \qquad + \left\{\left(-\frac{2}{3}y^3 - y^2 + 5y\right) + \left(-\frac{1}{90}y^6 + \frac{1}{15}y^5 - \frac{1}{6}y^4 + \frac{2}{5}y\right)\frac{\mathrm{R}}{\tan\alpha}\right. \\ \qquad\quad \left. + \left(-\frac{1}{560}y^8 + \frac{2}{315}y^7 - \frac{1}{18}y^4 + \frac{121}{630}y\right)\mathrm{R}^2\right\}\eta_{xx} \\ \qquad + 2y\eta^3 + \mathrm{R}\left(\frac{4}{3}y^3 - 4y\right)\eta\eta_t + \left\{\mathrm{R}(y^4 - 4y) + (3y^2 - 6y)\frac{1}{\tan\alpha}\right\}\eta\eta_x, \\ v_2 = \left(-\frac{1}{360}y^6 + \frac{1}{24}y^4 - \frac{5}{24}y^2\right)\mathrm{R}^2\eta_{xtt} \\ \qquad + \left\{\left(-\frac{1}{60}y^5 + \frac{1}{12}y^4 - \frac{1}{3}y^2\right)\frac{\mathrm{R}}{\tan\alpha} + \left(\frac{1}{2016}y^8 - \frac{1}{315}y^7 + \frac{1}{60}y^5 + \frac{1}{36}y^4 - \frac{101}{360}y^2\right)\mathrm{R}^2\right\}\eta_{xxt} \\ \qquad + \left\{\left(\frac{1}{6}y^4 + \frac{1}{3}y^3 - \frac{5}{2}y^2\right) + \left(\frac{1}{630}y^7 - \frac{1}{90}y^6 + \frac{1}{30}y^5 - \frac{1}{5}y^2\right)\frac{\mathrm{R}}{\tan\alpha}\right. \\ \qquad\quad \left. + \left(\frac{1}{5040}y^9 - \frac{1}{1260}y^8 + \frac{1}{90}y^5 - \frac{121}{1260}y^2\right)\mathrm{R}^2\right\}\eta_{xxx} \\ \qquad - 3y^2\eta^2\eta_x + \mathrm{R}\left(-\frac{1}{3}y^4 + 2y^2\right)(\eta_x\eta_t + \eta\eta_{tx}) + \left\{\mathrm{R}\left(-\frac{1}{5}y^5 + 2y^2\right) + (-y^3 + 3y^2)\frac{1}{\tan\alpha}\right\}(\eta_x^2 + \eta\eta_{xx}), \\ p_2 = \left(\frac{1}{2}y + \frac{1}{6}\right)\mathrm{R}\eta_{xt} + \left\{-\frac{\mathrm{W}}{\sin\alpha} + \left(-\frac{1}{2}y^2 + y + \frac{1}{2}\right)\frac{1}{\tan\alpha} + \left(-\frac{1}{10}y^5 + \frac{1}{6}y^4 + \frac{1}{3}y + \frac{1}{10}\right)\mathrm{R}\right\}\eta_{xx} \\ \qquad + \left\{\mathrm{R}(4y-4) - 5y + 3\right\}\eta\eta_x. \end{cases}
$$

Using the above expressions, we put

$$
(3.13)\qquad \begin{cases} u_0^{III}(y;\eta) := u_0, \quad v_0^{III}(y;\eta) := v_0, \quad p_0^{III}(y;\eta) := p_0, \\ u_1^{III}(y;\eta) := u_1, \quad v_1^{III}(y;\eta) := v_1, \quad p_1^{III}(y;\eta) := p_1, \\ u_2^{III}(y;\eta) := u_2, \quad v_2^{III}(y;\eta) := v_2, \quad p_2^{III}(y;\eta) := p_2. \end{cases}
$$

In view of the perturbation expansion (3.6) substituting $v = v_0^{III} + \delta v_1^{III} + \delta^2 v_2^{III}$ into (3.4), we obtain the approximate equation

$$
\eta_t + 2\eta_x + 4\varepsilon\eta\eta_x - \frac{8}{15}(\mathrm{R}_c - \mathrm{R})\delta\eta_{xx} + C_1\delta^2\eta_{xxx} + C_2\varepsilon\delta(\eta\eta_{xx} + \eta_x^2) + 2\varepsilon^2\eta^2\eta_x = O(\delta^3),
$$

where $C_1 = 2 + \frac{32}{63}\mathrm{R}^2 - \frac{40}{63}\frac{\mathrm{R}}{\tan\alpha}$ and $C_2 = \frac{16}{5}\mathrm{R} - \frac{2}{\tan\alpha}$.

Thus far we have assumed $\mathrm{W} = O(1)$. Taking into account that W is contained only in the second equation in (3.2) and modifying $O(\delta)$ problem under the assumption $\mathrm{W} \le O(\delta^{-1})$, we see that (u_0^I, v_0^I, p_0^I) and (u_1^I, v_1^I, p_1^I), which are defined by

$$
\begin{cases} u_0^I(y;\eta) := u_0, \quad v_0^I(y;\eta) := v_0, \quad p_0^I(y;\eta) := p_0, \\ u_1^I(y;\eta) := u_1, \quad v_1^I(y;\eta) := v_1, \quad p_1^I(y;\eta) := p_1 - \frac{\delta \mathrm{W}}{\sin\alpha}\eta_{xx}, \end{cases} \tag{3.14}
$$

are the solutions of the problems. Putting $v = v_0^I + \delta v_1^I$ and substituting this into (3.4), we obtain the approximate equation

$$
\eta_t + 2\eta_x + 4\varepsilon\eta\eta_x - \frac{8}{15}(\mathrm{R}_c - \mathrm{R})\delta\eta_{xx} = O(\delta^2).
$$

Similarly, modifying $O(1)$ and $O(\delta)$ problems under the assumption $\mathrm{W} = O(\delta^{-2})$ and putting

$$
\begin{cases} u_0^{II} := u_0, \quad v_0^{II} := v_0, \quad p_0^{II} := p_0 - \frac{\delta^2\mathrm{W}}{\sin\alpha}\eta_{xx}, \\ u_1^{II} := u_1 - \frac{\delta^2\mathrm{W}}{\sin\alpha}(y^2 - 2y)\eta_{xxx}, \quad v_1^{II} := v_1 + \frac{\delta^2\mathrm{W}}{\sin\alpha}\big(\frac{1}{3}y^3 - y^2\big)\eta_{xxxx}, \quad p_1^{II} := p_1, \end{cases} \tag{3.15}
$$

we obtain the approximate equation

$$
\eta_t + 2\eta_x + 4\varepsilon\eta\eta_x - \frac{8}{15}(\mathrm{R}_c - \mathrm{R})\delta\eta_{xx} + \frac{2}{3}\frac{\mathrm{W}_2}{\sin\alpha}\delta\eta_{xxxx} = O(\delta^2).
$$

Moreover, putting

$$
\begin{cases} u_0^{IV} := u_0, \quad v_0^{IV} := v_0, \quad p_0^{IV} := p_0, \\ u_1^{IV} := u_1, \quad v_1^{IV} := v_1, \quad p_1^{IV} := p_1 - \frac{\delta\mathrm{W}}{\sin\alpha}\eta_{xx}, \\ u_2^{IV} := u_2 - \frac{\delta\mathrm{W}}{\sin\alpha}(y^2 - 2y)\eta_{xxx}, \quad v_2^{IV} := v_2 + \frac{\delta\mathrm{W}}{\sin\alpha}\big(\frac{1}{3}y^3 - y^2\big)\eta_{xxxx}, \quad p_2^{IV} := p_2 + \frac{\mathrm{W}}{\sin\alpha}\eta_{xx} \end{cases} \tag{3.16}
$$

and $v = v_0^{IV} + \delta v_1^{IV} + \delta^2 v_2^{IV}$ and substituting this into (3.4), we obtain the approximate equation

$$
\begin{aligned} &\eta_t + 2\eta_x + 4\varepsilon\eta\eta_x - \frac{8}{15}(\mathrm{R}_c - \mathrm{R})\delta\eta_{xx} \\ &\quad + C_1\delta^2\eta_{xxx} + C_2\varepsilon\delta(\eta\eta_{xx} + \eta_x^2) + 2\varepsilon^2\eta^2\eta_x + \frac{2}{3}\frac{\mathrm{W}_2}{\sin\alpha}\delta^2\eta_{xxxx} = O(\delta^3) \end{aligned}
$$

under the assumption $\mathrm{W} = O(\delta^{-1})$.

4 Energy estimate

In this section, we will derive energy estimates. Let $\eta = \eta(x,t)$ be a fixed function and $(u, v, p) = (u(y;\eta), v(y;\eta), p(y;\eta))$ be an approximate solution constructed from η satisfying $u_x + v_y = 0$ and $u|_{y=0} = v|_{y=0} = 0$, which will be defined in the next section. Using the approximate solution, we define $\psi_1, \psi_2, \phi_1, \phi_2, \phi_3$ by the following equality.

$$
\begin{cases} \psi_1(y;\eta) := \dfrac{1}{\delta^3}\Big\{\delta(u_t + \bar{u}u_x + \bar{u}_y v) + \dfrac{2}{\mathrm{R}}\delta p_x - \dfrac{1}{\mathrm{R}}(\delta^2 u_{xx} + u_{yy}) - \delta f_1^{(1)}(\eta, u, v, p)\Big\}, \\ \psi_2(y;\eta) := \dfrac{1}{\delta^3}\Big\{\delta^2(v_t + \bar{u}v_x) + \dfrac{2}{\mathrm{R}}p_y - \dfrac{1}{\mathrm{R}}\delta(\delta^2 v_{xx} + v_{yy}) - \delta f_2^{(1)}(\eta, u, p)\Big\}, \\ \phi_1(\eta) := \dfrac{1}{\delta^3}\{\delta^2 v_x + u_y - 2(1+\delta\eta)^2\eta\}|_{y=1}, \\ \phi_2(\eta) := \dfrac{1}{\delta^3}\Big\{p - \delta v_y - \dfrac{1}{\tan\alpha}\eta + \dfrac{\delta^2\mathrm{W}}{\sin\alpha}\eta_{xx} - \delta^2 h_2^{(2)}(\eta, u)\Big\}\Big|_{y=1}, \\ \phi_3(\eta) := \dfrac{1}{\delta^3}\{\eta_t + \eta_x - v - \delta^2 h_3(\eta)\}|_{y=1}, \end{cases} \tag{4.1}
$$

where

$$f_1^{(1)} = -\frac{2}{\mathrm{R}}\eta u_{yy} + \delta f_1^{(2)}, \quad f_2^{(1)} = \frac{2}{\mathrm{R}}\eta p_y + \delta f_2^{(2)}. \tag{4.2}$$

Then the approximate solution satisfies the following equations.

$$\begin{cases} \delta(u_t + \bar{u}u_x + \bar{u}_y v) + \dfrac{2}{\mathrm{R}}\delta p_x - \dfrac{1}{\mathrm{R}}(\delta^2 u_{xx} + u_{yy}) = \delta f_1^{(1)}(\eta, u, v, p) + \delta^3\psi_1(y;\eta) & \text{in} \quad \Omega,\ t > 0, \\ \delta^2(v_t + \bar{u}v_x) + \dfrac{2}{\mathrm{R}}p_y - \dfrac{1}{\mathrm{R}}\delta(\delta^2 v_{xx} + v_{yy}) = \delta f_2^{(1)}(\eta, u, p) + \delta^3\psi_2(y;\eta) & \text{in} \quad \Omega,\ t > 0, \\ u_x + v_y = 0 & \text{in} \quad \Omega,\ t > 0, \end{cases} \tag{4.3}$$

$$\begin{cases} \delta^2 v_x + u_y - 2(1+\delta\eta)^2\eta = \delta^3\phi_1(\eta) & \text{on} \quad \Gamma,\ t > 0, \\ p - \delta v_y - \dfrac{1}{\tan\alpha}\eta + \dfrac{\delta^2 \mathrm{W}}{\sin\alpha}\eta_{xx} = \delta^2 h_2^{(2)}(\eta, u) + \delta^3\phi_2(\eta) & \text{on} \quad \Gamma,\ t > 0, \\ \eta_t + \eta_x - v = \delta^2 h_3(\eta) + \delta^3\phi_3(\eta) & \text{on} \quad \Gamma,\ t > 0, \end{cases} \tag{4.4}$$

$$u = v = 0 \quad \text{on} \quad \Sigma,\ t > 0. \tag{4.5}$$

Note that in the next section, we will give explicit forms of ψ_1, ψ_2, ϕ_1, ϕ_2, and ϕ_3. Let $(\eta^\delta, u^\delta, v^\delta, p^\delta)$ be the solution of (3.1)–(3.3) and we set

$$H := \eta^\delta - \eta, \quad U := u^\delta - u, \quad V := v^\delta - v, \quad P := p^\delta - p.$$

Taking the difference between (3.1)–(3.4) and (4.3)–(4.5), we have

$$\begin{cases} \delta(U_t + \bar{u}U_x + \bar{u}_y V) + \dfrac{2}{\mathrm{R}}\delta P_x - \dfrac{1}{\mathrm{R}}(\delta^2 U_{xx} + U_{yy}) \\ \qquad\qquad = F_1 + \delta^3 f_1^{(3)}(\eta^\delta, u^\delta, v^\delta, p^\delta) - \delta^3\psi_1(y;\eta) & \text{in} \quad \Omega,\ t > 0, \\ \delta^2(V_t + \bar{u}V_x) + \dfrac{2}{\mathrm{R}}P_y - \dfrac{1}{\mathrm{R}}\delta(\delta^2 V_{xx} + V_{yy}) \\ \qquad\qquad = F_2 + \delta^3 f_2^{(3)}(\eta^\delta, u^\delta, v^\delta, p^\delta) - \delta^3\psi_2(y;\eta) & \text{in} \quad \Omega,\ t > 0, \\ U_x + V_y = 0 & \text{in} \quad \Omega,\ t > 0, \end{cases} \tag{4.6}$$

$$\begin{cases} \delta^2 V_x + U_y - \big(2 + b(\eta^\delta, \eta)\big)H = \delta^3 h_1(\eta^\delta, u^\delta, v^\delta) - \delta^3\phi_1(\eta) & \text{on} \quad \Gamma,\ t > 0, \\ P - \delta V_y - \dfrac{1}{\tan\alpha}H + \dfrac{\delta^2 \mathrm{W}}{\sin\alpha}H_{xx} = G_2 + \delta^3 h_2^{(3)}(\eta^\delta, u^\delta, v^\delta) - \delta^3\phi_2(\eta) & \text{on} \quad \Gamma,\ t > 0, \\ H_t + H_x - V = G_3 - \delta^3\phi_3(\eta) & \text{on} \quad \Gamma,\ t > 0, \end{cases} \tag{4.7}$$

$$U = V = 0 \quad \text{on} \quad \Sigma,\ t > 0, \tag{4.8}$$

where

$$\begin{cases} F_1 = \delta\big(f_1^{(1)}(\eta^\delta, u^\delta, v^\delta, p^\delta) - f_1^{(1)}(\eta, u, v, p)\big), \quad F_2 = \delta\big(f_2^{(1)}(\eta^\delta, u^\delta, p^\delta) - f_2^{(1)}(\eta, u, p)\big), \\ b = 2\delta\big(\delta(\eta^\delta)^2 + (2+\delta\eta)\eta^\delta + \delta\eta^2 + 2\eta\big), \\ G_2 = \delta^2\big(h_2^{(2)}(\eta^\delta, u^\delta, v^\delta) - h_2^{(2)}(\eta, u, v)\big), \quad G_3 = \delta^2\big(h_3(\eta^\delta) - h_3(\eta)\big). \end{cases} \tag{4.9}$$

For convenience, we set

$$\boldsymbol{U} := (U, \delta V)^{\mathrm{T}}, \quad \boldsymbol{F} := (F_1, F_2)^{\mathrm{T}}, \quad \boldsymbol{f} := (f_1^{(3)}, f_2^{(3)})^{\mathrm{T}}, \quad \boldsymbol{\psi} := (\psi_1, \psi_2)^{\mathrm{T}}.$$

We proceed to derive an energy estimate to (4.6)–(4.8) following [16]. In view of the energies obtained in [16] (see (3.6)–(3.8) and (3.24) in [16]), we put

$$\begin{aligned}
\mathscr{E}_0(H, \boldsymbol{U}) := {}& \delta^2\|V\|^2 + \frac{2}{\mathrm{R}}\left(\frac{1}{\tan\alpha}|H|_0^2 + \frac{\delta^2\mathrm{W}}{\sin\alpha}|H_x|_0^2\right) \\
&+ \beta_1\left\{\delta^2\|\boldsymbol{U}_x\|^2 + \frac{2}{\mathrm{R}}\left(\frac{1}{\tan\alpha}\delta^2|H_x|_0^2 + \frac{\delta^2\mathrm{W}}{\sin\alpha}\delta^2|H_{xx}|_0^2\right)\right\} \\
&+ \beta_2\left\{\delta^4\|\boldsymbol{U}_{xx}\|^2 + \frac{2}{\mathrm{R}}\left(\frac{1}{\tan\alpha}\delta^4|H_{xx}|_0^2 + \frac{\delta^2\mathrm{W}}{\sin\alpha}\delta^4|H_{xxx}|_0^2\right)\right\} \\
&+ \beta_3\left\{\delta^2\|\boldsymbol{U}_t\|^2 + \frac{2}{\mathrm{R}}\left(\frac{1}{\tan\alpha}\delta^2|H_t|_0^2 + \frac{\delta^2\mathrm{W}}{\sin\alpha}\delta^2|H_{tx}|_0^2\right)\right\}, \\
\mathscr{F}_0(H, \boldsymbol{U}, P) := {}& \delta\|\boldsymbol{U}_x\|^2 + \delta\|\partial_y^{-1}P_x\|^2 + \delta|H_x|_0^2 + \delta^3\mathrm{W}|H_{xx}|_0^2 + \delta^5\mathrm{W}^2|H_{xxx}|_0^2 \\
&+ \delta\|\nabla_\delta\boldsymbol{U}_x\|^2 + \delta^3\|\nabla_\delta\boldsymbol{U}_{xx}\|^2 + \delta\|\nabla_\delta\boldsymbol{U}_t\|^2.
\end{aligned}$$

Here, β_1, β_2, and β_3 are appropriate positive constants (see (3.28) in [16]). Integrating by parts and using the third equation in (4.7) and Poincaré's inequality, we see that for any $\epsilon > 0$ there exists a positive constant C_ϵ such that

$$\begin{aligned}
&\delta^3|(\{\boldsymbol{F} + \delta^3\boldsymbol{f} - \delta^3\boldsymbol{\psi}\}_{xx}, \boldsymbol{U}_{xx})_\Omega| \le \epsilon\delta^5\|\boldsymbol{U}_{xxx}\|^2 + C_\epsilon\delta(\|\boldsymbol{F}_x\|^2 + \delta^6\|\boldsymbol{f}_x\|^2 + \delta^6\|\boldsymbol{\psi}_x\|^2), \\
&|(H, (bH)_x)_\Gamma| \le \epsilon\delta|H_x|_0^2 + C_\epsilon\delta^{-1}|(bH)_x|_0^2, \\
&\delta^2\mathrm{W}|(H_{xx}, (bH)_x)_\Gamma| \le \epsilon\delta^3\mathrm{W}|H_{xx}|_0^2 + C_\epsilon\delta\mathrm{W}|(bH)_x|_0^2, \\
&\delta^2\mathrm{W}|(H_{xx}, G_3 - \delta^3\phi_3)_\Gamma| \le \epsilon\delta^3\mathrm{W}|H_{xx}|_0^2 + C_\epsilon\delta\mathrm{W}(|G_3|_0^2 + \delta^6|\phi_3|_0^2), \\
&\delta^6\mathrm{W}|(H_{xxxx}, \delta^3\phi_{3xx})_\Gamma| \le \epsilon\delta^5\mathrm{W}^2|H_{xxx}|_0^2 + C_\epsilon\delta^{13}|\phi_{3xxx}|_0^2, \\
&\delta^4\mathrm{W}|(H_{xxt}, G_{3t} - \delta^3\phi_{3t})_\Gamma| \le \epsilon(\delta^5\mathrm{W}^2|H_{xxx}|_0^2 + \delta^5\|U_{xxx}\|_0^2) \\
&\qquad + C_\epsilon(1 + \mathrm{W}^2)\delta^3(|G_{3t}|_0^2 + \delta^6|\phi_{3t}|_0^2) + \delta^5(|G_{3xx}|_0^2 + \delta^6|\phi_{3xx}|_0^2).
\end{aligned}$$

Here, we used the inequality $|V(\cdot,1)|_0 = |V(\cdot,1) - V(\cdot,0)|_0 \le \|V_y\| = \|U_x\|$ thanks to the third equation in (4.6) and the second equation in (4.8). In the following, we use frequently this type of inequality without any comment. Taking into account the above inequality and (3.27) in [16], we need to estimate the following quantities.

$$\begin{aligned}
\text{(4.10)} \quad \mathscr{N}_0^1(Z_1) := {}& (\delta\mathrm{W} + \delta^{-1})|(bH)_x|_0^2 + \delta^3|(bH)_{xx}|_0^2 + \delta|(bH)_t|_0^2 \\
&+ \delta^{-1}|G_2|_0^2 + \delta|G_{2x}|_0^2 + \delta^2||D_x|^{\frac12}G_{2x}|_0^2 + \delta|(G_{2t}, \delta V_t)_\Gamma| \\
&+ \delta\mathrm{W}|G_3|_0^2 + \delta^3|G_{3x}|_0^2 + \delta^5|G_{3xx}|_0^2 + \delta^3\mathrm{W}^2|G_{3t}|_0^2 + \delta^6\mathrm{W}|(H_{xxxx}, G_{3xx})_\Gamma| \\
&+ \delta^{-1}\|\boldsymbol{F}\|^2 + \delta\|\boldsymbol{F}_x\|^2 + \delta|(\boldsymbol{F}_t, \boldsymbol{U}_t)_\Omega|,
\end{aligned}$$

$$\begin{aligned}
\text{(4.11)} \quad \mathscr{N}_0^2(Z_2) := {}& \delta^5|h_1|_0^2 + \delta^7|h_{1x}|_0^2 + \delta^8||D_x|^{\frac12}h_{1x}|_0^2 + \delta^4|(h_{1t}, U_t)_\Gamma| \\
&+ \delta^5|h_2^{(3)}|_0^2 + \delta^7|h_{2x}^{(3)}|_0^2 + \delta^8||D_x|^{\frac12}h_{2x}^{(3)}|_0^2 + \delta^4|(h_{2t}^{(3)}, \delta V_t)_\Gamma| \\
&+ \delta^5\|\boldsymbol{f}\|^2 + \delta^7\|\boldsymbol{f}_x\|^2 + \delta^4|(\boldsymbol{f}_t, \boldsymbol{U}_t)_\Omega|,
\end{aligned}$$

$$\begin{aligned}
\text{(4.12)} \quad \mathscr{N}_0^3(Z_3) := {}& \delta^5|\phi_1|_0^2 + \delta^7|\phi_{1x}|_0^2 + \delta^8||D_x|^{\frac12}\phi_{1x}|_0^2 + \delta^7|\phi_{1t}|_0^2 + \delta^5|\phi_2|_0^2 + \delta^7|\phi_{2x}|_0^2 \\
&+ \delta^8||D_x|^{\frac12}\phi_{2x}|_0^2 + \delta^7|\phi_{2t}|_0^2 + \delta^7\mathrm{W}|\phi_3|_0^2 + \delta^9|\phi_{3x}|_0^2 + \delta^{11}|\phi_{3xx}|_0^2 \\
&+ \delta^{13}|\phi_{3xxx}|_0^2 + \delta^9\mathrm{W}^2|\phi_{3t}|_0^2 + \delta^5\|\boldsymbol{\psi}\|^2 + \delta^7\|\boldsymbol{\psi}_x\|^2 + \delta^7\|\boldsymbol{\psi}_t\|^2,
\end{aligned}$$

where

$$Z_1 = (H, \boldsymbol{U}, bH, G_2, G_3, \boldsymbol{F}), \quad Z_2 = (\boldsymbol{U}, h_1, h_2^{(3)}, h_3, \boldsymbol{f}), \quad Z_3 = (\phi_1, \phi_2, \phi_3, \boldsymbol{\psi}).$$

For an integer $m \geq 2$, we set

$$\mathscr{E}_m(H, \boldsymbol{U}) := \sum_{k=0}^{m} \mathscr{E}_0(\partial_x^k H, \partial_x^k \boldsymbol{U}), \quad \mathscr{F}_m(H, \boldsymbol{U}, P) := \sum_{k=0}^{m} \mathscr{F}_0(\partial_x^k H, \partial_x^k \boldsymbol{U}, \partial_x^k P), \tag{4.13}$$

$$\mathscr{N}_m^1(H, \boldsymbol{U}, P; \eta) := \sum_{k=0}^{m} \{\mathscr{N}_0^1(\partial_x^k Z_1) + |(\partial_x^k H, \partial_x^k G_3)_\Gamma|\}, \tag{4.14}$$

$$\mathscr{N}_m^2(\boldsymbol{U}) := \sum_{k=0}^{m} \mathscr{N}_0^2(\partial_x^k Z_2), \tag{4.15}$$

$$\mathscr{N}_m^3(H; \eta) := \sum_{k=0}^{m} \{\mathscr{N}_0^3(\partial_x^k Z_3) + |(\partial_x^k H, \delta^3 \partial_x^k \phi_3)_\Gamma|\}. \tag{4.16}$$

Here, the terms $\sum_{k=0}^m |(\partial_x^k H, \partial_x^k G_3)_\Gamma|$ and $\sum_{k=0}^m |(\partial_x^k H, \delta^3 \partial_x^k \phi_3)_\Gamma|$ come from (3.30) in [16]. Applying ∂_x^k to (4.6)–(4.8), using [16, Proposition 3.2], and adding the resulting inequalities for $0 \leq k \leq m$, we obtain the following lemma.

Lemma 4.1. *There exist small positive constants* R_0 *and* α_0 *such that if* $0 < \mathrm{R}_1 \leq \mathrm{R} \leq \mathrm{R}_0$, $\mathrm{W}_1 \leq \mathrm{W}$, *and* $0 < \alpha \leq \alpha_0$, *then the solution* (H, U, V, P) *of* (4.6)–(4.8) *satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}_m + \mathscr{F}_m \leq C(\mathscr{N}_m^1 + \mathscr{N}_m^2 + \mathscr{N}_m^3), \tag{4.17}$$

where the constant C *is independent of* δ, R, *and* W.

For later use, we modify the energy and the dissipation functions $\mathscr{E}_m$ and $\mathscr{F}_m$ as

$$\tilde{\mathscr{E}}_m(H, \boldsymbol{U}) := \mathscr{E}_m(H, \boldsymbol{U}) + \|(1 + |D_x|)^m U\|^2 + \|(1 + |D_x|)^m U_y\|^2, \tag{4.18}$$

$$\begin{aligned} \tilde{\mathscr{F}}_m(H, \boldsymbol{U}, P) := {} & \mathscr{F}_m(H, \boldsymbol{U}, P) + \delta|(1 + \delta|D_x|)^{\frac{5}{2}} H_t|_m^2 + (\delta^2 \mathrm{W})^2 \delta^2 ||D_x|^{\frac{7}{2}} H|_m^2 \\ & + \delta^{-1}\|(1 + |D_x|)^m (1 + \delta|D_x|)(\nabla_\delta P, U_{yy})\|^2 + \delta\|(1 + |D_x|)^{m-1} \nabla_\delta P_t\|^2. \end{aligned} \tag{4.19}$$

We also introduce another energy function $\mathscr{D}_m$ by

$$\begin{aligned} \mathscr{D}_m(H, \boldsymbol{U}) := {} & |(1 + \delta|D_x|)^2 H|_m^2 + \delta^2\|(1 + |D_x|)^m V\|^2 + \delta^2\|(1 + |D_x|)^m \boldsymbol{U}_x\|^2 \\ & + \|(1 + |D_x|)^m D_\delta^2 \boldsymbol{U}\|^2 + (\delta^2 \mathrm{W})^2 |(1 + \delta|D_x|) H_x|_{m+1}^2 + \sqrt{\delta^2 \mathrm{W}}\|(1 + |D_x|)^m \delta V_{xy}\|^2, \end{aligned} \tag{4.20}$$

which does not include any time derivatives. Setting $\tilde{E}_m = \tilde{\mathscr{E}}_m(\eta^\delta, \boldsymbol{u}^\delta)$ and $\tilde{F}_m = \tilde{\mathscr{F}}_m(\eta^\delta, \boldsymbol{u}^\delta, p^\delta)$ and using [16, Theorem 2.2 and Proposition 6.1], the following uniform estimate holds.

Proposition 4.2. *There exist small positive constants* R_0 *and* α_0 *such that the following statement holds: Let* m *be an integer satisfying* $m \geq 2$, $0 < \mathrm{R}_1 \leq \mathrm{R}_0$, $0 < \mathrm{W}_1 \leq \mathrm{W}_2$, *and* $0 < \alpha \leq \alpha_0$. *There exists small positive constant* c_0 *such that if the initial data* (η_0, u_0, v_0) *and the parameters* δ, ε, R, *and* W *satisfy Assumption 2.1 and* $\mathrm{W} \leq \delta^{-2}\mathrm{W}_2$, *then the solution* $(\eta^\delta, u^\delta, v^\delta, p^\delta)$ *of* (2.1)–(2.4) *satisfies*

$$\tilde{E}_2(t) \leq c_0, \quad \sup_{t \geq 0} \tilde{E}_{m+1}(t) + \int_0^\infty \tilde{F}_{m+1}(t)\mathrm{d}t \leq C, \quad \tilde{E}_{m+1}(t) \leq C\mathrm{e}^{-c\delta t}.$$

Here, positive constants C *and* c *depend on* $\mathrm{R}_1, \mathrm{W}_1, \mathrm{W}_2, \alpha$, *and* M *but are independent of* δ, ε, R, *and* W.

Moreover, we easily obtain the following lemma.

Lemma 4.3. *Let $\alpha > 0$, $0 < \mathrm{R}_1 \le \mathrm{R} < \mathrm{R}_c$. There exists small positive constant c_1 such that if $s \ge 2$ and $|\eta_0|_2^2 \le c_1$, then the problems* (1.7)–(1.10) *under the initial condition $\zeta|_{\tau=0} = \eta_0$ have unique solutions ζ^I, ζ^{II}, ζ^{III}, and ζ^{IV}, respectively, which satisfy*

$$\sup_{\tau\ge 0}|\zeta^I(\tau)|_s^2 + \int_0^\infty |\zeta^I_x(\tau)|_s^2 \mathrm{d}\tau \le C|\eta_0|_s^2, \quad |\zeta^I(\tau)|_s^2 \le C|\eta_0|_s^2 \mathrm{e}^{-c\delta t},$$

$$\sup_{\tau\ge 0}|\zeta^{II}(\tau)|_s^2 + \int_0^\infty \big(|\zeta^{II}_x(\tau)|_s^2 + |\zeta^{II}_{xx}(\tau)|_s^2\big) \mathrm{d}\tau \le C|\eta_0|_s^2, \quad |\zeta^{II}(\tau)|_s^2 \le C|\eta_0|_s^2 \mathrm{e}^{-c\delta t},$$

$$\sup_{\tau\ge 0}|\zeta^{III}(\tau)|_s^2 + \int_0^\infty |\zeta^{III}_x(\tau)|_s^2 \mathrm{d}\tau \le C|\eta_0|_s^2, \quad |\zeta^{III}(\tau)|_s^2 \le C|\eta_0|_s^2 \mathrm{e}^{-c\delta t},$$

$$\sup_{\tau\ge 0}|\zeta^{IV}(\tau)|_s^2 + \int_0^\infty \big(|\zeta^{IV}_x(\tau)|_s^2 + \delta|\zeta^{IV}_{xx}(\tau)|_s^2\big) \mathrm{d}\tau \le C|\eta_0|_s^2, \quad |\zeta^{IV}(\tau)|_s^2 \le C|\eta_0|_s^2 \mathrm{e}^{-c\delta t}.$$

Here, $\mathrm{R}_c = \frac{5}{4}\frac{1}{\tan\alpha}$ is the critical Reynolds number and positive constants C and c are independent of δ and R.

5 Error estimate

We will show (2.11) under Assumption 2.1 and (2.10). We can show the other claims in Theorem 2.2 in the same way as the proof of (2.11) and we will comment about the difference at the end of this section. Let ζ^{III} be the solution of (1.9) under the initial condition $\zeta^{III}|_{\tau=0} = \eta_0$ and we put $\eta^{III}(x,t) := \zeta^{III}(x-2t, \varepsilon t)$ and

$$(5.1) \qquad \begin{cases} u^{III}(x,y,t) := u_0^{III}(y;\eta^{III}(x,t)) + \delta u_1^{III}(y;\eta^{III}(x,t)) + \delta^2 u_2^{III}(y;\eta^{III}(x,t)), \\ v^{III}(x,y,t) := u_0^{III}(y;\eta^{III}(x,t)) + \delta v_1^{III}(y;\eta^{III}(x,t)) + \delta^2 v_2^{III}(y;\eta^{III}(x,t)), \\ p^{III}(x,y,t) := p_0^{III}(y;\eta^{III}(x,t)) + \delta p_1^{III}(y;\eta^{III}(x,t)) + \delta^2 p_2^{III}(y;\eta^{III}(x,t)), \end{cases}$$

where $u_0^{III}, v_0^{III}, p_0^{III}, \dots$ were defined by (3.10)–(3.13). Then, we have

$$(5.2) \qquad \begin{aligned} \eta_t^{III} =& -2\eta_x^{III} + \frac{8}{15}(\mathrm{R}_c - \mathrm{R})\delta\eta_{xx}^{III} - C_1\delta^2\eta_{xxx}^{III} \\ & - 4\delta\eta^{III}\eta_x^{III} - \delta^2\big\{C_2\big(\eta^{III}\eta_{xx}^{III} + (\eta_x^{III})^2\big) + 2(\eta^{III})^2\eta_x^{III}\big\}. \end{aligned}$$

Using the approximate solutions (5.1), we define ψ_1, ψ_2, ϕ_1, ϕ_2, and ϕ_3 by (4.1). By using the equality (5.2) to eliminate the t derivatives of η^{III}, we can rewrite these terms as follows.

$$(5.3) \qquad \begin{cases} \psi_1(y;\eta^{III}) = \mathcal{C}_1(y)\partial_x^3\eta^{III} + \mathcal{C}_2(y)\delta\partial_x^4\eta^{III} + \cdots + \mathcal{C}_7(y)\delta^6\partial_x^9\eta^{III} + N_1^{III}, \\ \psi_2(y;\eta^{III}) = \mathcal{C}_8(y)\partial_x^3\eta^{III} + \mathcal{C}_9(y)\delta\partial_x^4\eta^{III} + \cdots + \mathcal{C}_{15}(y)\delta^7\partial_x^{10}\eta^{III} + N_2^{III}, \\ \phi_1(\eta^{III}) = \mathcal{C}_{16}\partial_x^3\eta^{III} + \mathcal{C}_{17}\delta\partial_x^4\eta^{III} + \cdots + \mathcal{C}_{21}\delta^5\partial_x^8\eta^{III} + N_3^{III}, \\ \phi_2(\eta^{III}) = \mathcal{C}_{22}\partial_x^3\eta^{III} + \mathcal{C}_{23}\delta\partial_x^4\eta^{III} + \cdots + \mathcal{C}_{26}\delta^4\partial_x^7\eta^{III} + N_4^{III}, \\ \phi_3(\eta^{III}) = \mathcal{C}_{27}\partial_x^4\eta^{III} + \mathcal{C}_{28}\delta\partial_x^5\eta^{III} + \cdots + \mathcal{C}_{30}\delta^3\partial_x^7\eta^{III} + N_5^{III}, \end{cases}$$

where $\mathcal{C}_1, \dots, \mathcal{C}_{15}$ are polynomials in y, $\mathcal{C}_{16}, \dots, \mathcal{C}_{30}$ are constants, and $N_1^{III}, \dots, N_5^{III}$ are collections of the nonlinear terms of the form

$$(5.4) \qquad \frac{1}{\delta^3}\Phi_0(\delta\eta^{III}, \delta^2\partial_x\eta^{III}, \dots, \delta^5\partial_x^4\eta^{III}; y)\Phi_0(\delta^2\partial_x\eta^{III}, \dots, \delta^{10}\partial_x^9\eta^{III}; y).$$

Here we generally denote polynomials of $\boldsymbol{f}$ by the same symbol $\Phi = \Phi(\boldsymbol{f})$ and Φ_0 is such a function satisfying $\Phi_0(\mathbf{0}) = 0$. We also use such a function Φ_0 depending also on $y \in [0,1]$ and denote it by $\Phi_0(\boldsymbol{f}; y)$, that is, $\Phi_0(\mathbf{0}; y) \equiv 0$. Let $(\eta^\delta, u^\delta, v^\delta, p^\delta)$ be the solution of (2.1)–(2.3) and we set $H^{III} := \eta^\delta - \eta^{III}$, $\boldsymbol{U}^{III} := (u^\delta - u^{III}, \delta(v^\delta - v^{III}))^{\mathrm{T}}$, $\tilde{\mathscr{E}}_m^{III} := \tilde{\mathscr{E}}_m(H^{III}, \boldsymbol{U}^{III})$, and so on. We prepare several lemmas to proceed the error estimate.

Lemma 5.1. *Under the same assumption as Proposition 4.2, for any $\epsilon > 0$ there exists a positive constant C_ϵ such that we have*

$$\mathscr{N}_m^2(\boldsymbol{U}^{III})(t) \le \epsilon \tilde{\mathscr{F}}_m(t) + C_\epsilon \delta^4 \tilde{E}_m(t) \tilde{F}_{m+1}(t), \tag{5.5}$$

where $\mathscr{N}_m^2$ is the collection of nonlinear terms defined by (4.15).

Proof. By the explicit form of $\boldsymbol{f}$, h_1, and $h_2^{(3)}$ (see (3.5) and Section 2), we can obtain the desired estimate in the same but more easier way as proving [16, Lemmas 5.11 and 5.12]. $\square$

Lemma 5.2. *Under the same assumption as Proposition 4.2, for any $\epsilon > 0$ there exists a positive constant C_ϵ such that we have*

$$\mathscr{N}_m^3(H; \eta^{III})(t) \le \epsilon \tilde{\mathscr{F}}_m(t) + C_\epsilon \delta^5 |\eta_x^{III}(t)|_{m+12}^2,$$

where $\mathscr{N}_m^3$ is the collection of nonlinear terms defined by (4.16).

Proof. By the well-known inequalities $\|\partial_x^k(fg)\| \lesssim \|f\|_{L^\infty}\|\partial_x^k g\| + \|g\|_{L^\infty}\|\partial_x^k f\|$ and $\|\partial_x^k \Phi_0(\boldsymbol{f}; y)\| \le C(\|\boldsymbol{f}\|_{L^\infty})\|\partial_x^k \boldsymbol{f}\|$, (5.2)–(5.4) lead to $\sum_{k=0}^m \mathscr{N}_0^3(\partial_x^k Z_3) \lesssim \big(1 + |\eta^{III}|_{m+12}^2\big)\delta^5 |\eta_x^{III}|_{m+12}^2$. Moreover, by Poincaré's inequality and (5.4), we see that $|(\partial_x^k H, \delta^3 \partial_x^k \phi_3)_\Gamma| \le \epsilon\delta|\partial_x^k H_x|_0^2 + C_\epsilon \delta^5 |\partial_x^k \phi_3|_0^2 \le \epsilon\tilde{\mathscr{F}}_m + C_\epsilon\big(1 + |\eta^{III}|_{m+12}^2\big)\delta^5|\eta_x^{III}|_{m+12}^2$. These together with Lemma 4.3 imply the desired inequality. $\square$

Lemma 5.3. *Under the same assumption as Proposition 4.2, for any $\epsilon > 0$ there exists a positive constant C_ϵ such that we have*

$$\begin{aligned} \mathscr{N}_m^1(H^{III}, \boldsymbol{U}^{III}, P^{III}; \eta^{III})(t) \le& (C_\epsilon \tilde{E}_2(t) + \epsilon)\tilde{\mathscr{F}}_m^{III}(t) + C_\epsilon\big\{\tilde{E}_m(t)\tilde{\mathscr{F}}_2^{III}(t) + \delta^4 \tilde{E}_m(t)\tilde{F}_{m+1}(t) \\ &+ \delta^5 |\eta_x^{III}(t)|_{m+12}^2 + (\tilde{F}_m(t) + \delta|\eta_x^{III}(t)|_{m+12}^2)\tilde{\mathscr{E}}_m^{III}(t)\big\}, \end{aligned} \tag{5.6}$$

where $\mathscr{N}_m^1$ is the collection of nonlinear terms defined by (4.14).

Proof. In this proof, we omit the symbol III appeared in a superscript of solutions for simplicity. By (3.5), (4.2), and (4.9), we see that $\boldsymbol{F}$ is consist of terms of the form

$$\begin{cases} \delta\Phi_0(\eta^\delta, \delta\eta_x^\delta; y)(\nabla_\delta U_y, \nabla_\delta P) + \delta^2(\eta^\delta)^2(U_{yy}, P_y), \\ \delta\Phi_0(\eta^\delta, \delta\eta_x^\delta, u^\delta; y)(\delta V, \delta U_x), \\ \delta\Phi_0(\delta\eta_x^\delta, \delta\eta_t^\delta, \delta v^\delta; y)(U, U_y), \\ \delta\Phi_0(\eta, \boldsymbol{u}, \nabla_\delta \boldsymbol{u}, \nabla_\delta u_y, \nabla_\delta p; y)(\delta H_x, \delta H_t, U, \delta V), \\ \delta^2\eta^\delta(u_{yy} + p_y)H \end{cases}$$

and that $G_2 = \delta^2\{\eta^\delta(2H_x + U_x) + \eta_x^\delta U + (2\eta_x + u_x)H + uH_x\}$, $G_3 = \delta^2\{(\eta^\delta)^2 H_x + (\eta^\delta + \eta)\eta_x H\}$, and $bH = 2\delta\big(\delta(\eta^\delta)^2 + (2 + \delta\eta)\eta^\delta + \delta\eta^2 + 2\eta\big)H$. Note that using (5.1) and (5.2), we can express the approximate solutions $\boldsymbol{u}, \nabla_\delta \boldsymbol{u}, u_{yy}$, and $\nabla_\delta p$ in terms of η and its x derivatives. In view of these, by putting

$$\begin{cases} \Phi^1 = \Phi(\eta^\delta, \delta\eta_x^\delta, \delta\eta_t^\delta, \delta^2\eta_{xx}^\delta, \delta^2\eta_{tx}^\delta, \boldsymbol{u}^\delta; y), \\ \Phi^2 = \Phi(\delta\eta_x^\delta, \delta\eta_t^\delta, \delta^2\eta_{xx}^\delta, \delta^2\eta_{tx}^\delta, \delta^2\eta_{tt}^\delta, \delta v^\delta, \delta\boldsymbol{u}_x^\delta, \delta\boldsymbol{u}_t^\delta; y), \\ \Phi^3 = \Phi(\eta^\delta, \delta\eta_x^\delta; y), \\ \Phi^4 = \Phi(\eta, \delta\eta_x, \dots, \delta^{10}\partial_x^{10}\eta; y), \end{cases}$$

$$\begin{cases} W := (\delta H_x, \delta H_t, \delta^2 H_{xx}, \delta^2 H_{tx}, \delta^3 H_{xxx}, \delta V, \delta\boldsymbol{U}_x, \delta\boldsymbol{U}_t, \delta\nabla_\delta U_x, \delta\nabla_\delta U_t, \nabla_\delta U_y, \nabla_\delta U_{xy}, \\ \qquad \nabla_\delta P, \nabla_\delta P_x, \delta U_x|_\Gamma, \delta U_t|_\Gamma, \delta^2 U_{xx}|_\Gamma, \delta^{5/2}|D_x|^{5/2}U|_\Gamma), \\ Q := (H, \delta H_x, \delta H_t, \delta^2 H_{xx}, \delta^2 H_{tx}, \delta^3 H_{xxx}, \boldsymbol{U}, \nabla_\delta \boldsymbol{U}, \delta\boldsymbol{U}_t, U|_\Gamma), \end{cases}$$

it suffices to estimate

$$\begin{cases} I_1 = \delta\|\partial_x^k(\Phi_0^1 W)\|^2, \\ I_2 = \delta\|\partial_x^k(\Phi_0^2 Q)\|^2, \\ I_3 = \delta^3|(\partial_x^k(\eta^\delta \boldsymbol{U}_{tx}), \partial_x^k V_t)_\Gamma|, \\ I_4 = \delta^2|(\partial_x^k(\Phi_0^3\nabla_\delta U_{ty}), \partial_x^k \boldsymbol{U}_t)_\Omega|, \\ I_5 = \delta^2|(\partial_x^k(\Phi_0^3\nabla_\delta P_t), \partial_x^k \boldsymbol{U}_t)_\Omega|, \\ I_6 = \delta\|\partial_x^k(\Phi^1\Phi_0^4 Q)\|^2, \\ I_7 = \delta^4|(\partial_x^k(\Phi_0^4 H_{tt}), \partial_x^k V_t)_\Gamma|, \\ I_8 = \delta^6 \mathrm{W}|(\partial_x^k H_{xxxx}, \partial_x^k G_{3xx})_\Gamma| \end{cases}$$

for $0 \le k \le m$.

By Proposition 4.2 and $\|(u,v)\|_{L^\infty} \lesssim \|(u_y, v_y)\| + \|(u_{xy}, v_{xy})\|$ thanks to the boundary condition $u|_{y=0} = v|_{y=0} = 0$, we obtain

$$\|\Phi_0^1\|_{L^\infty}^2 \lesssim \tilde{E}_2, \quad \|\partial_x^k \Phi_0^1\|^2 + \|\partial_x^k \Phi_{0y}^1\|^2 \lesssim \tilde{E}_m, \tag{5.7}$$

$$\|\Phi_0^2\|_{L^\infty}^2 \lesssim \tilde{F}_2, \quad \|\partial_x^k \Phi_0^2\|^2 + \|\partial_x^k \Phi_{0y}^2\|^2 \lesssim \tilde{F}_m. \tag{5.8}$$

In the same way as the proof of Lemma 5.2, we have

$$\delta\|\Phi_0^4\|_{L^\infty}^2 \lesssim \delta|\eta_x|_{m+12}^2, \quad \delta(\|\partial_x^k \Phi_0^4\|^2 + \|\partial_x^k \Phi_{0y}^4\|^2) \lesssim \delta|\eta_x|_{m+12}^2, \quad |\Phi_0^4|_{m-\frac{1}{2}}^2 \lesssim |\eta|_{m+12}^2. \tag{5.9}$$

On the other hand, it is easy to see that

$$\|W\|^2 + \|W_x\|^2 \lesssim \tilde{\mathscr{F}}_2, \quad \|\partial_x^k W\|^2 \lesssim \tilde{\mathscr{F}}_m, \tag{5.10}$$

$$\|Q\|^2 + \|Q_x\|^2 \lesssim \tilde{\mathscr{E}}_2, \quad \|\partial_x^k Q\|^2 \lesssim \tilde{\mathscr{E}}_m, \tag{5.11}$$

where we used the trace theorem $|f|_0^2 + \delta||D_x|^{\frac{1}{2}} f|_0^2 \lesssim \|f\|^2 + \delta^2\|f_x\|^2 + \|f_y\|^2$ to estimate the term $\delta^5||D_x|^{\frac{5}{2}} U|_0^2$. In the following, we often use the inequality

$$\|\partial_x^k(af)\| \lesssim \|a\|_{L^\infty}\|\partial_x^k f\| + (\|\partial_x^k a\| + \|\partial_x^k a_y\|)(\|f\| + \|f_x\|), \tag{5.12}$$

which have been shown in [16, Lemma 5.2].

As for I_1, by (5.7), (5.10), and (5.12), we have $I_1 \lesssim \tilde{E}_2\tilde{\mathscr{F}}_m + \tilde{E}_m\tilde{\mathscr{F}}_2$. As for I_2, by (5.8), (5.11), and (5.12), we have $I_2 \lesssim \tilde{F}_m\tilde{\mathscr{E}}_m$. As for I_3, by integration by parts, we have $I_3 \lesssim$

$C_\epsilon\delta^3|\eta^\delta U_{tx}|^2_{m-\frac12}+\epsilon\delta^3|V_t|^2_{m+\frac12}\le C_\epsilon(\tilde E_2\tilde{\mathscr F}_m+\tilde E_m\tilde{\mathscr F}_2)+\epsilon\tilde{\mathscr F}_m$. As for I_4, by integration by parts in y, we have

$$\begin{aligned}
I_4 &\le C_\epsilon\delta^2\big(\|\partial_x^k(\Phi_0^3\nabla_\delta U_t)\|^2+\|\partial_x^k(\Phi_{0y}^3\nabla_\delta U_t)\|^2\big)\\
&\quad+\delta^3|(\partial_x^k(\Phi_0^3U_{tx}),\partial_x^k\boldsymbol{U}_t)_\Gamma|+\delta^2|(\partial_x^k(\Phi_0^3U_{ty}),\partial_x^k\boldsymbol{U}_t)_\Gamma|+\epsilon\delta\|\partial_x^k\boldsymbol{U}_{ty}\|^2\\
&\le I_{4,1}+I_{4,2}+I_{4,3}+\epsilon\tilde{\mathscr F}_m,
\end{aligned}$$

where $I_{4,1}=C_\epsilon\delta^2\big(\|\partial_x^k(\Phi_0^3\nabla_\delta U_t)\|^2+\|\partial_x^k(\Phi_{0y}^3\nabla_\delta U_t)\|^2\big)$, $I_{4,2}=\delta^3|(\partial_x^k(\Phi_0^3U_{tx}),\partial_x^k\boldsymbol{U}_t)_\Gamma|$, and $I_{4,3}=\delta^2|(\partial_x^k(\Phi_0^3U_{ty}),\partial_x^k\boldsymbol{U}_t)_\Gamma|$. The estimates for $I_{4,1}$ and $I_{4,2}$ are reduced to the estimates for I_1 and I_3, respectively. Thus, taking into account that we can eliminate the term $U_y|_\Gamma$ in $I_{4,3}$ by the first equation in (4.7), this together with the estimates for I_2, I_3, δ^3h_1, and $\delta^3\phi_1$ yields $I_4\le\epsilon\tilde{\mathscr F}_m+C_\epsilon\big\{\tilde E_2\tilde{\mathscr F}_m+\tilde E_m(\tilde{\mathscr F}_2+\delta^4\tilde F_{m+1}+|\eta|^2_{m+12}\delta^5|\eta_x|^2_{m+12})\big\}$. As for I_5, it suffices to show the case of $k\ge1$ because we can treat easily the case of $k=0$. Integrating by parts in x, (5.7), and (5.12), we have $I_5\le\epsilon\delta^3\|\partial_x^k\boldsymbol{U}_{tx}\|^2+C_\epsilon\delta\|\partial_x^{k-1}(\Phi_0^3\nabla_\delta P_t)\|^2\le\epsilon\tilde{\mathscr F}_m+C_\epsilon\big(\tilde E_2\tilde{\mathscr F}_m+\tilde E_m\tilde{\mathscr F}_2\big)$. As for I_6, by (5.7), (5.9), (5.11), and (5.12), we have

$$\begin{aligned}
I_6 &\lesssim \delta\big\{\|\Phi_0^4\|^2_{L^\infty}(\|\partial_x^k\Phi^1\|^2+\|\partial_x^k\Phi^1_y\|^2)(\|Q\|^2+\|Q_x\|^2)\\
&\quad+\|\Phi^1\|^2_{L^\infty}(\|\partial_x^k\Phi_0^4\|^2+\|\partial_x^k\Phi_{0y}^4\|^2)(\|Q\|^2+\|Q_x\|^2)+\|\Phi^1\|^2_{L^\infty}\|\Phi_0^4\|^2_{L^\infty}\|\partial_x^kQ\|^2\big\}\\
&\lesssim (\tilde E_m+|\eta|^2_{m+12})\delta|\eta_x|^2_{m+12}\tilde{\mathscr E}_m.
\end{aligned}$$

As for I_7, it suffices to show the case of $k\ge1$ because we can treat easily the case of $k=0$. By the third equation in (4.7), integration by parts, and the trace theorem, we have

$$\begin{aligned}
I_7 &\le C_\epsilon\delta^4||D_x|^{\frac12}\partial_x^{k-1}(\Phi_0^4V_t)|_0^2+C_\epsilon\delta^5|\partial_x^k(\Phi_0^4H_{xt}+\Phi_0^4G_{3t})|_0^2+C_\epsilon\delta^5|\delta^3\partial_x^k\phi_{3t}|_0^2\\
&\quad+\epsilon\big(\delta^4||D_x|^{\frac12}\partial_x^kV_t|_0^2+\delta^3|\partial_x^kV_t|_0^2\big)\\
&\le I_{7,1}+I_{7,2}+I_{7,3}+\epsilon\tilde{\mathscr F}_m,
\end{aligned}$$

where $I_{7,1}=C_\epsilon\delta^4||D_x|^{\frac12}\partial_x^{k-1}(\Phi_0^4V_t)|_0^2$, $I_{7,2}=C_\epsilon\delta^5|\partial_x^k(\Phi_0^4H_{xt}+\Phi_0^4G_{3t})|_0^2$, and $I_{7,3}=C_\epsilon\delta^5|\delta^3\partial_x^k\phi_{3t}|_0^2$. By the trace theorem, the second equation in (4.6), and (5.9), we have

$$\begin{aligned}
I_{7,1} &\lesssim |\Phi_0^4|^2_{m-\frac12}\delta^3|V_t|^2_{L^\infty}+\delta|\Phi_0^4|^2_{L^\infty}\delta^3||D_x|^{\frac12}\partial_x^{k-1}V_t|_0^2\\
&\lesssim |\Phi_0^4|^2_{m-\frac12}\delta^3\|U_{txx}\|^2+\delta|\Phi_0^4|^2_{L^\infty}(\delta^2\|\partial_x^kU_t\|^2+\delta^4\|\partial_x^kV_t\|^2)\\
&\lesssim |\eta|^2_{m+12}\tilde{\mathscr F}_2+\delta|\eta_x|^2_{m+12}\tilde{\mathscr E}_m.
\end{aligned}$$

Recalling the explicit form of G_3, we see that the estimate of $I_{7,2}$ is reduced to I_6. Taking into account that we have already estimated $I_{7,3}$ in the proof of Lemma 5.2, we obtain $I_7\le C_\epsilon\big\{|\eta|^2_{m+12}\tilde{\mathscr F}_2+(\tilde E_m+|\eta|^2_{m+12})\delta|\eta_x|^2_{m+12}\tilde{\mathscr E}_m+|\eta|^2_{m+12}\delta^5|\eta_x|^2_{m+12}\big\}+\epsilon\tilde{\mathscr F}_m$. As for I_8, integration by parts, (5.7), and (5.9) lead to

$$\begin{aligned}
\delta^6\mathrm{W}|(\partial_x^kH_{xxxx},\partial_x^kG_{3xx})_\Gamma| &\le \epsilon(\delta^2\mathrm{W})^2\delta^2||D_x|^{\frac72}H|_m^2+C_\epsilon\delta^6||D_x|^{\frac52}G_3|_m^2\\
&\le \epsilon\tilde{\mathscr F}_m+C_\epsilon\big\{\delta^2(\tilde F_m+\delta|\eta_x|^2_{m+12})\tilde{\mathscr E}_2+\tilde E_2\tilde{\mathscr F}_m\big\}.
\end{aligned}$$

Therefore, by the boundedness of the terms $\tilde E_m$ and $|\eta|^2_{m+12}$ which comes from Proposition 4.2 and Lemma 4.3, the proof is complete. $\square$

Lemma 5.4. *Under the same assumption as Proposition 4.2, we have*

$$\tilde{\mathscr{E}}_m^{III}(t) \lesssim \mathscr{E}_m^{III}(t) + \delta^4(\tilde{E}_{m+1}(t) + |\eta^{III}(t)|_{m+12}^2), \tag{5.13}$$

$$\begin{aligned}\tilde{\mathscr{F}}_m^{III}(t) \lesssim{}& \mathscr{F}_m^{III}(t) + (\tilde{F}_m(t) + \delta|\eta_x^{III}(t)|_{m+12}^2)\tilde{\mathscr{E}}_m^{III}(t) \\ &+ \delta^4\tilde{E}_m(t)\tilde{F}_{m+1}(t) + \delta^5|\eta_x^{III}(t)|_{m+12}^2,\end{aligned} \tag{5.14}$$

$$\mathscr{E}_m^{III}(t) \lesssim \mathscr{D}_m^{III}(t) + \delta^4. \tag{5.15}$$

Proof. In view of the discrepancy of non-homogeneous terms in the equations, modifying the proof of (6.2) in [16, Lemma 6.2], we obtain (5.13). Taking into account that we can eliminate U_{yy} in $\mathscr{F}_m^{III}$ by using the first equation in (4.6), modifying the proof of (6.3) in [16, Lemma 6.2], it is not difficult to check that (5.14) holds. Moreover, modifying the proof of (6.10) in [16], we obtain (5.15). $\square$

Lemma 5.5. *Under the same assumption as Proposition 4.2, we have*

$$\mathscr{D}_m^{III}(0) \lesssim \delta^4.$$

Remark 5.1. This lemma together with (5.15) yields

$$\mathscr{E}_m^{III}(0) \lesssim \delta^4. \tag{5.16}$$

Proof. By the second and third equations in the compatibility conditions, we see that

$$\begin{aligned} u_0(x,y) &= yu_{0y}(x,1) - \int_0^y \int_z^1 u_{0yy}(x,w)\mathrm{d}w\mathrm{d}z \\ &= \big(2y\eta_0 + 4y\delta\eta_0^2 + 2y\delta^2\eta_0^3\big) + \delta y\big(-\delta v_{0x} + \delta^2 h_1^{(0)}\big) - \int_0^y \int_z^1 u_{0yy}(x,w)\mathrm{d}w\mathrm{d}z. \end{aligned} \tag{5.17}$$

It follows from (2.10) and $\|(1+|D_x|)^{m+1}u_{yy}^{III}|_{t=0}\| \lesssim \delta$ (see the explicit form of u^{III}, that is, (3.10)–(3.13) and (5.1)) that $\|(1+|D_x|)^{m+1}u_{0yy}\| \lesssim \delta$. Thus, by (5.17), the explicit form of u^{III}, (2.10), and the uniform estimate for $\delta^2|h_1^{(0)}|_{m+1}$ (see the proof of Lemma 5.1), we obtain $\|(1+|D_x|)^{m+1}U|_{t=0}\| \lesssim \delta$. Combining this and the first equation in the compatibility conditions leads to $\|(1+|D_x|)^m V|_{t=0}\| \lesssim \delta$. Therefore, in view of the definition of $\mathscr{D}_m$ (see (4.20)), using these and $H|_{t=0} = 0$, we obtain the desired estimate. $\square$

Proof of (2.11) *in Theorem 2.2.* By Proposition 4.2, Lemmas 4.1, 5.1–5.3, and (5.13) and (5.14) in Lemma 5.4, if c_0 and ϵ are sufficiently small, then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}_m^{III}(t) + \tilde{\mathscr{F}}_m^{III}(t) \le C_1\big(\varphi_1(t)\mathscr{E}_m^{III}(t) + \tilde{E}_m(t)\tilde{\mathscr{F}}_2^{III}(t) + \delta^4\varphi_2(t)\big), \tag{5.18}$$

where

$$\varphi_1(t) = \tilde{F}_m(t) + \delta|\eta_x^{III}(t)|_{m+12}^2, \quad \varphi_2(t) = \tilde{E}_m(t)\tilde{F}_{m+1}(t) + \delta|\eta_x^{III}(t)|_{m+12}^2. \tag{5.19}$$

By considering the case of $m=2$ in (5.18) and using Gronwall's inequality and Proposition 4.2, if c_0 is sufficiently small, then we have $\mathscr{E}_2^{III}(t) + \int_0^t \tilde{\mathscr{F}}_2^{III}(s)\mathrm{d}s \le \varphi_3(t)$, where

$$\varphi_3(t) = \mathscr{E}_2^{III}(0)\exp\left(C_1\int_0^t \varphi_1(s)\mathrm{d}s\right) + C_1\int_0^t \delta^4\varphi_2(s)\exp\left(C_1\int_s^t \varphi_1(\sigma)\mathrm{d}\sigma\right)\mathrm{d}s, \tag{5.20}$$

which leads to

$$\int_0^t \tilde{\mathscr{F}}_2^{III}(s)\mathrm{d}s \le \varphi_3(t). \tag{5.21}$$

Note that by Proposition 4.2 and Lemma 4.3, we have the exponential decay estimate for $\tilde{E}_{m+1}(t)$ and $|\eta^{III}(t)|^2_{m+13}$. This together with (5.18), Gronwall's inequality, and $\delta\mathscr{E}_m^{III} \lesssim \tilde{\mathscr{F}}_m^{III}$ which comes from $|H|_0 \lesssim |H_x|_0$ and $\|V\| \lesssim \|V_y\| = \|U_x\|$ (see (4.13) and (4.19)) yields

$$\mathscr{E}_m^{III}(t) \le \left\{\mathscr{E}_m^{III}(0)\exp\left(C_1\int_0^t \varphi_1(s)\mathrm{d}s\right) + \varphi_4(t)\right\}\mathrm{e}^{-c\delta t},$$

where

$$\varphi_4(t) = C_1\int_0^t \left(\tilde{\mathscr{F}}_2^{III}(s) + \delta^4\tilde{F}_{m+1}(s)\right)\exp\left(C_1\int_s^t \varphi_1(\sigma)\mathrm{d}\sigma\right)\mathrm{d}s. \tag{5.22}$$

Combining the above inequality and (5.13) and (5.15) in Lemma 5.4, we obtain

$$\tilde{\mathscr{E}}_m^{III}(t) \le C_2\left(\delta^4 + \mathscr{D}_m^{III}(0) + \varphi_4(t)\right)\mathrm{e}^{-c\delta t}. \tag{5.23}$$

Here, recalling the definition $\eta^{III}(x,t) = \zeta^{III}(x-2t,\varepsilon t)$ and the assumption $\varepsilon = \delta$ and using Lemma 4.3, we have $\int_0^\infty \delta|\eta_x^{III}(t)|_s^2\mathrm{d}t = \frac{1}{\varepsilon}\int_0^\infty \delta|\zeta_x^{III}(\tau)|_s^2\mathrm{d}\tau \lesssim |\eta_0|_s$. By this, the integrability of $\tilde{F}_{m+1}$ which comes from Proposition 4.2, and (5.16), we have $\varphi_3(t) \lesssim \delta^4$ (see (5.19) and (5.20)). This together with (5.21) leads to $\varphi_4(t) \lesssim \delta^4$ (see (5.22)). Combining this, (5.23), and Lemma 5.5, we have

$$\tilde{\mathscr{E}}_m^{III}(t) \le C_3\delta^4\mathrm{e}^{-c\varepsilon t}, \tag{5.24}$$

which implies $\mathcal{D}(t;\zeta^{III},u^{III},v^{III},p^{III}) \lesssim \delta^4\mathrm{e}^{-c\varepsilon t}$ (see (2.5) and (4.18)). Here, we used $\|V\| \lesssim \|V_y\| = \|U_x\|$. Moreover, by taking into account the equality $P(x,y,t) = P(x,1,t) - \int_y^1 P_y(x,z,t)\mathrm{d}z$ and using the second equation in (4.6), the second equation in (4.7), and the uniform estimate (5.24), we easily obtain $\|(1+|D_x|)^m(p^\delta - p^{III})(t)\|^2 \lesssim \delta^4\mathrm{e}^{-c\varepsilon t}$. Note that in the case of $O(\delta^{-1}) \le \mathrm{W} \le O(\delta^{-2})$ we can estimate the term $\frac{\delta^2\mathrm{W}}{\sin\alpha}\partial_x^m H_{xx}$ which comes from the second equation in (4.7) by $\tilde{\mathscr{E}}_{m+1}^{III}$. Therefore, the proof of (2.11) in Theorem 2.2 is complete. □

We proceed to prove (2.7), (2.9), and (2.13). Let ζ^I, ζ^{II}, and ζ^{IV} be the solution for (1.7), (1.8), and (1.10), respectively under the initial condition $\zeta^I|_{\tau=0} = \zeta^{II}|_{\tau=0} = \zeta^{IV}|_{\tau=0} = \eta_0$. We put $\eta^I(x,t) := \zeta^I(x-2t,\varepsilon t)$, $\eta^{II}(x,t) := \zeta^{II}(x-2t,\varepsilon t)$, $\eta^{IV}(x,t) := \zeta^{IV}(x-2t,\varepsilon t)$ and

$$\begin{cases} u^I(x,y,t) := u_0^I(y;\eta^I(x,t)) + \delta u_1^I(y;\eta^I(x,t)), \\ v^I(x,y,t) := u_0^I(y;\eta^I(x,t)) + \delta v_1^I(y;\eta^I(x,t)), \\ p^I(x,y,t) := p_0^I(y;\eta^I(x,t)) + \delta p_1^I(y;\eta^I(x,t)), \end{cases}$$

$$\begin{cases} u^{II}(x,y,t) := u_0^{II}(y;\eta^{II}(x,t)) + \delta u_1^{II}(y;\eta^{II}(x,t)), \\ v^{II}(x,y,t) := u_0^{II}(y;\eta^{II}(x,t)) + \delta v_1^{II}(y;\eta^{II}(x,t)), \\ p^{II}(x,y,t) := p_0^{II}(y;\eta^{II}(x,t)) + \delta p_1^{II}(y;\eta^{II}(x,t)), \end{cases}$$

$$\begin{cases} u^{IV}(x,y,t) := u_0^{IV}(y;\eta^{IV}(x,t)) + \delta u_1^{IV}(y;\eta^{IV}(x,t)) + \delta^2 u_2^{IV}(y;\eta^{IV}(x,t)), \\ v^{IV}(x,y,t) := u_0^{IV}(y;\eta^{IV}(x,t)) + \delta v_1^{IV}(y;\eta^{IV}(x,t)) + \delta^2 v_2^{IV}(y;\eta^{IV}(x,t)), \\ p^{IV}(x,y,t) := p_0^{IV}(y;\eta^{IV}(x,t)) + \delta p_1^{IV}(y;\eta^{IV}(x,t)) + \delta^2 p_2^{IV}(y;\eta^{IV}(x,t)), \end{cases}$$

where $u_0^I, v_0^I, p_0^I, \ldots$ were defined by (3.14)–(3.16). In view of this, by applying the same argument as showing (2.11), it is not difficult to check that (2.7), (2.9), and (2.13) holds. Therefore, the proof of Theorem 2.2 is complete. □

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