The Generalized t-Distribution on the Circle

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Abstract

An extended version of t-distribution on the unit circle is generated by conditioning a normal mixture distribution, which is broaden to include not only unimodality and symmetry, but also bimodality and asymmetry, depending on the values of parameters. After reparametrization, the distribution contains four circular distributions as special cases: symmetric Jones–Pewsey, generalized von Mises, generalized cardioid and generalized wrapped Cauchy distributions. As an illustrative example, the proposed model is fitted to the number of occurrences of the thunder in a day.
Key words: Cardioid distribution; Directional statistics; Generalized von Mises distribution; Normal mixture distribution; Wrapped Cauchy distribution

1 Introduction

- Von Mises distribution (Mardia and Jupp,1999; Jammalamadaka and SenGupta, 2001), the circular normal distribution, is unimodal and symmetric about the mean direction.

- Since the circular data are not always symmetric, the von Mises distribution is extended to a more flexible distribution, the generalized von Mises distribution (Maksimov, 1967; Rukhin, 1972; Yfantis and Borgman, 1982; Gatto and Jammalamadaka, 2006; Mardia and Sutton, 1975). It can be unimodal or bimodal, symmetric or asymmetric, depending on the values of parameters.

- The t-distribution is often being used in hypothesis testing and the theory of estimation, as well as in Bayesian analysis and applications due to its more realistic tails (Kotz and Nadarajah, 2004).

- A new distribution, namely the generalized t-distribution on the circle is derived, which covers symmetry and asymmetry, unimodality and bimodality. It also includes generalized von Mises, generalized cardioid and generalized wrapped Cauchy distributions as special cases.

2 Derivation

2.1 Ad-hoc Method

- Jones and Pewsey (2005) propose a family of symmetric distributions on the circle, whose probability density function (pdf) is given by

$$f(\theta) \propto \{1 + \tanh(\kappa\psi)\cos(\theta - \mu)\}^{1/\psi}, \quad 0 \le \theta < 2\pi, \tag{1}$$

  where $\psi \neq 0$, $\kappa \ge 0$ and $0 \le \mu < 2\pi$.

- It includes some well-known symmetric distributions, such as von Mises, cardioid and wrapped Cauchy distributions as special cases, for $\psi = 0, 1$ and -1, respectively.

- The generalized von Mises distribution (Maksimov, 1967; Rukhin, 1972) with pdf

$$f(\theta) \propto \exp\{\kappa_1 \cos(\theta - \mu_1) + \kappa_2 \cos 2(\theta - \mu_2)\}, \quad 0 \le \theta < 2\pi, \tag{2}$$

  where $\kappa_1 \ge 0,\ \kappa_2 \ge 0,\ 0 \le \mu_1 < 2\pi$ and $0 \le \mu_2 < \pi$.

- It is a flexible circular distribution, which can be unimodal symmetric, bimodal symmetric, unimodal asymmetric or bimodal asymmetric.

- Combining them, a new distribution can be obtained with pdf

$$f(\theta) \propto \{1 + \tanh(\kappa_1\psi)\cos(\theta - \mu) + \tanh(\kappa_2\psi)\cos 2(\theta - \nu)\}^{1/\psi}, \quad 0 \le \theta < 2\pi, \tag{3}$$

  under certain conditions on the parameters ψ, κ_1, κ_2, μ and ν.

2.2 Conditioning Method

- Let $\boldsymbol{W}|\,(U = u)$ be a random variable of a bivariate normal distribution $N_2(\boldsymbol{\eta}, \boldsymbol{\Sigma}/u)$, where $u > 0$ and

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}, \quad \sigma_1 > 0,\ \sigma_2 > 0,\ |\rho| < 1.$$

- The pdf (weighting function) of U is set to be

$$g(u) = \frac{\beta^\alpha}{\Gamma(\alpha)} u^{\alpha-1} e^{-\beta u}, \quad \alpha > 0,\ \beta > 0. \tag{4}$$

- Note that U follows a gamma distribution with shape α and scale $1/\beta$.

- Adopting (4), the bivariate Pearson type VII distribution is generated with pdf

$$\begin{aligned} f(\boldsymbol{w}) &= \int_0^\infty f(\boldsymbol{w}|u)\, g(u)\, du \\ &= \frac{\alpha}{2\pi\beta\,|\Sigma|^{1/2}} \left[1 + \frac{(\boldsymbol{w} - \boldsymbol{\eta})' \Sigma^{-1} (\boldsymbol{w} - \boldsymbol{\eta})}{2\beta}\right]^{-\alpha-1}. \end{aligned}$$

- To derive a distribution on the circle, we now make polar transformation from $\boldsymbol{W} = (W_1, W_2)' \in \mathbb{R}^2$ to a vector $(R, \Theta)'$:

$$\boldsymbol{W} = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = R \begin{pmatrix} \cos\Theta \\ \sin\Theta \end{pmatrix},$$

 where

 1. $R = \|\boldsymbol{W}\|$ is the length of $\boldsymbol{W}$,
 2. $(\cos\Theta, \sin\Theta)'$ is the direction cosine of $\boldsymbol{W}$,
 3. The domain of (R, Θ) is

$$\omega = \left\{(r, \theta)' : r \geq 0,\ 0 \leq \theta < 2\pi\right\}.$$

- Similarly, we transform $\boldsymbol{\eta} = \zeta\,(\cos\tau, \sin\tau)'$, for $\zeta \geq 0$ and $0 \leq \tau < 2\pi$.
- After some calculation, the conditional distribution of Θ given $R = r$ has pdf

$$\begin{aligned} f(\theta|r) &= \frac{f(r,\theta)}{f(r)} \\ &= C\left\{1 - \kappa_\beta \cos(\theta - \mu) - \lambda_\beta \cos 2(\theta - \nu)\right\}^{-\alpha-1}, \end{aligned} \tag{5}$$

 where $\alpha > 0$, $\kappa_\beta > 0$, and $\lambda_\beta > 0$.
- The normalizing constant C is given by

$$\begin{aligned} C^{-1} &= 2\pi\left[F_4\left(\frac{\alpha}{2} + \frac{1}{2}, \frac{\alpha}{2} + 1; 1, 1; \kappa_\beta^2, \lambda_\beta^2\right)\right. \\ &\quad + 2\sum_{r=1}^{\infty} \frac{(\kappa_\beta/2)^{2r}(\lambda_\beta/2)^r(\alpha+1)_{3r}}{r!(2r)!}\cos 2r(\mu - \nu) \\ &\quad \left. \times F_4\left(\frac{\alpha}{2} + \frac{3r+1}{2}, \frac{\alpha}{2} + \frac{3r}{2} + 1; r+1, 2r+1; \kappa_\beta^2, \lambda_\beta^2\right)\right], \end{aligned} \tag{6}$$

 where

$$F_4(a, b; c, d; x, y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(a)_{i+j}(b)_{i+j}}{(c)_i (d)_j}\frac{x^i y^j}{i!j!}, \quad \sqrt{x} + \sqrt{y} < 1,$$

 is Appell's double hypergeometric function.
- We regard the distribution with pdf (5) and normalizing constant (6) as the asymmetric Pearson type VII distribution with parameters κ_β, λ_β, α, μ and ν on the circle.
- The asymmetric t-distribution on the circle is a special case of the asymmetric Pearson type VII distribution when $\alpha = \beta = n/2$.
- α is limited to take only positive values. However, the density is still valid when α is extended to the whole real line.
- Reparametrize (5) as follows:

$$\begin{aligned} f(\theta) = C_\psi \left\{1 + \tanh(\kappa_1\psi)\cos(\theta - \mu) + \tanh(\kappa_2\psi)\cos 2(\theta - \nu)\right\}^{1/\psi}, \\ \psi \neq 0,\ \kappa_1 \geq 0,\ \kappa_2 \geq 0,\ 0 \leq \mu < 2\pi,\ 0 \leq \nu < \pi, \end{aligned} \tag{7}$$

where $|\tanh(\kappa_1\psi)| + |\tanh(\kappa_2\psi)| < 1$ and normalizing constant C_ψ is

$$C_\psi^{-1} = \int_0^{2\pi} (1 + \tanh(\kappa_1\psi)\cos(\theta-\mu) + \tanh(\kappa_2\psi)\cos 2(\theta-\nu))^{1/\psi} \, d\theta. \tag{8}$$

- We refer to the distribution with pdf (7) and normalizing constant (8) as the generalized t-distribution on the circle.
- Note that (8) can be expressed in terms of Appell's function as (6) for negative ψ by setting $\alpha = -(1/\psi + 1)$, $\kappa_\beta = -\tanh(\kappa_1\psi)$ and $\lambda_\beta = -\tanh(\kappa_2\psi)$.

3 Properties

3.1 Special Cases

- $\kappa_2 = 0$: Symmetric Jones–Pewsey distribution

$$f(\theta) = \frac{\{\cosh(\kappa_1\psi) + \sinh(\kappa_1\psi)\cos(\theta-\mu)\}^{1/\psi}}{2\pi P_{1/\psi}(\cosh(\kappa_1\psi))},$$

where $P_{1/\psi}(z)$ is the associated Legendre function of the first kind and degree $1/\psi$ and order 0.

- $\psi \to 0$: Generalized von Mises distribution For small $|\psi|$,

$$\begin{aligned}
&(1 + \tanh(\kappa_1\psi)\cos(\theta-\mu) + \tanh(\kappa_2\psi)\cos 2(\theta-\nu))^{1/\psi} \\
&\approx \exp\left\{\frac{1}{\psi}\log(1 + \kappa_1\psi\cos(\theta-\mu) + \kappa_2\psi\cos 2(\theta-\nu))\right\} \\
&\approx \exp\{\kappa_1\cos(\theta-\mu) + \kappa_2\cos 2(\theta-\nu)\},
\end{aligned}$$

The normalizing constant is

$$\left[2\pi\left\{I_0(\kappa_1)I_0(\kappa_2) + 2\sum_{r=1}^{\infty} I_{2r}(\kappa_1)I_r(\kappa_2)\cos 2r(\nu-\mu)\right\}\right]^{-1}.$$

- $\psi = 1$: Generalized cardioid distribution

$$f(\theta) = \frac{1}{2\pi}\{1 + \tanh(\kappa_1)\cos(\theta-\mu) + \tanh(\kappa_2)\cos 2(\theta-\nu)\}.$$

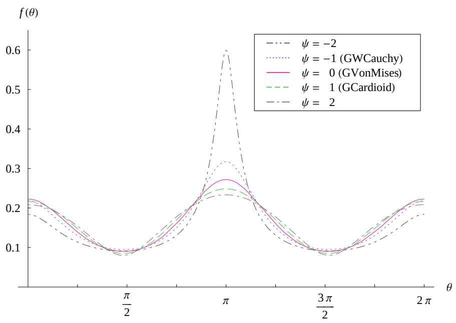
- $\psi = -1$: Generalized wrapped Cauchy distribution

$$f(\theta) = C_{-1}[1 - \tanh(\kappa_1)\cos(\theta-\mu) - \tanh(\kappa_2)\cos 2(\theta-\nu)]^{-1}.$$

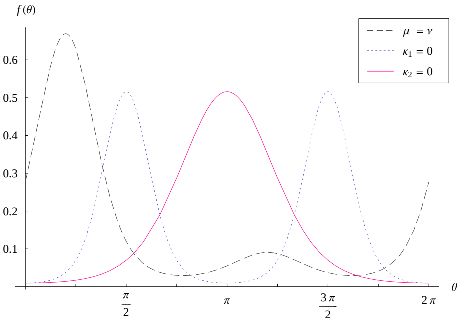
The normalizing constant C_{-1} can be expressed in terms of Appell's function, as given in (6), by setting $\alpha = 0$, $\kappa_\beta = \tanh(\kappa_1)$ and $\lambda_\beta = \tanh(\kappa_2)$.

3.2 Density plots

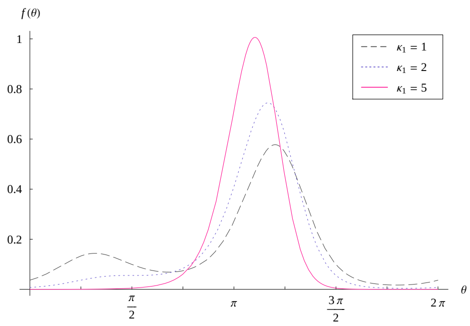
(1)



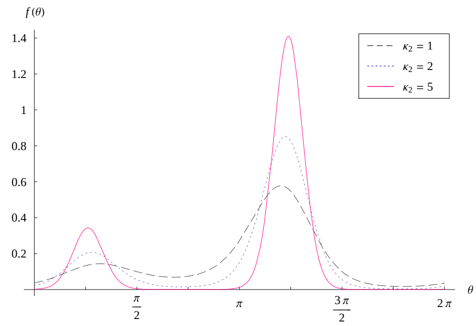
(2)



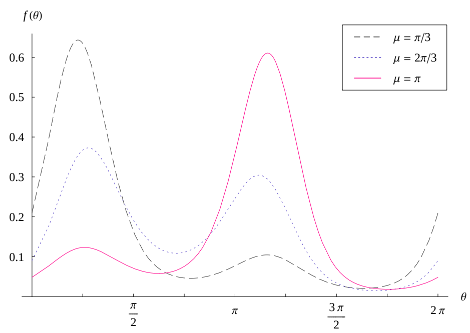
(3)



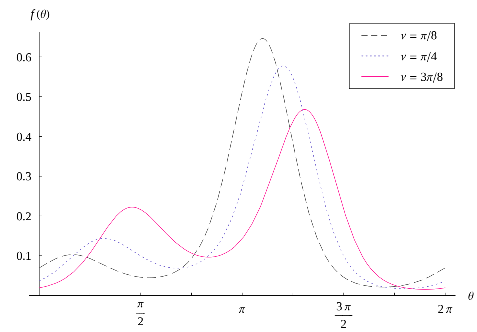
(4)



(5)



(6)



1. Density plots for $\psi = -2(1)2$, $\kappa_1 = 0.1$, $\kappa_2 = 0.5$ and $\mu = \nu = \pi$.
2. Some symmetrical density plots of the proposed distribution for $\psi = 0$. The dashed curve corresponding to the densities with $\kappa_1 = \kappa_2 = 1$, and $\mu = \nu = \pi/5$. The dotted curve for $\kappa_1 = 0$, $\kappa_2 = 2$, $\mu = \pi$ and $\nu = \pi/2$. The solid curve for $\kappa_1 = 2$, $\kappa_2 = 0$, $\mu = \pi$ and $\nu = \pi/2$.
3. Density plots for $\psi = 0$, $\kappa_1 = 1, 2, 5$, $\kappa_2 = 1$, $\mu = \pi$ and $\nu = \pi/4$.
4. Density plots for $\psi = 0$, $\kappa_1 = 1$, $\kappa_2 = 1, 2, 5$, $\mu = \pi$ and $\nu = \pi/4$.

5. Density plots for $\psi = 0$, $\kappa_1 = 1$, $\kappa_2 = 1$, $\mu = \pi/3, 2\pi/3, \pi$, and $\nu = \pi/4$.
6. Density plots for $\psi = 0$, $\kappa_1 = 1$, $\kappa_2 = 1$, $\mu = \pi$ and $\nu = \pi/8, \pi/4, 3\pi/8$.

3.3 Modality

- Let $a = \tanh(\kappa_1\psi)$ and $b = \tanh(\kappa_2\psi)$. Then, θ^*, the solution of
$$a\sin(\theta-\mu) + 2b\sin 2(\theta-\nu) = 0, \tag{9}$$
is a mode or antimode depending on the sign of $h(\theta^*)$, where
$$h(\theta) = \psi^{-1}[a\cos(\theta-\mu) + 4b\cos 2(\theta-\nu)].$$
- In general, (9) can be solved numerically for all combinations of μ and ν.

3.4 Trigonometric moments

- For negative ψ, the moments can be expressed in terms of Appell's function F_4:

$$\begin{aligned}
\alpha_k &= 2\pi C_\psi \left\{ \sum_{2i\geq k} \frac{\cos(2i(\nu-\mu)+k\mu)}{(2i-k)!i!} \left(-\frac{1}{\psi}\right)_{3i-k} \left(-\frac{\tanh(\kappa_1\psi)}{2}\right)^{2i-k} \left(-\frac{\tanh(\kappa_2\psi)}{2}\right)^{i} \right. \\
&\quad \times F_4\left(\frac{3i-k}{2}-\frac{1}{2\psi}, \frac{3i-k+1}{2}-\frac{1}{2\psi}; 2i-k+1, i+1; (\tanh(\kappa_1\psi))^2, (\tanh(\kappa_2\psi))^2\right) \\
&\quad + \sum_{i=0}^{\infty} \frac{\cos(2i(\nu-\mu)-k\mu)}{(2i+k)!i!} \left(-\frac{1}{\psi}\right)_{3i+k} \left(-\frac{\tanh(\kappa_1\psi)}{2}\right)^{2i+k} \left(-\frac{\tanh(\kappa_2\psi)}{2}\right)^{i} \\
&\quad \times F_4\left(\frac{3i+k}{2}-\frac{1}{2\psi}, \frac{3i+k+1}{2}-\frac{1}{2\psi}; 2i+k+1, i+1; (\tanh(\kappa_1\psi))^2, (\tanh(\kappa_2\psi))^2\right) \\
&\quad + \sum_{\substack{i+2j=k \\ i,j\geq 1}}\sum \frac{\cos(i\mu+2j\nu)}{i!j!} \left(-\frac{1}{\psi}\right)_{i+j} \left(-\frac{\tanh(\kappa_1\psi)}{2}\right)^{i} \left(-\frac{\tanh(\kappa_2\psi)}{2}\right)^{j} \\
&\quad \left. \times\, F_4\left(\frac{i+j}{2}-\frac{1}{2\psi}, \frac{i+j+1}{2}-\frac{1}{2\psi}; i+1, j+1; (\tanh(\kappa_1\psi))^2, (\tanh(\kappa_2\psi))^2\right) \right\},
\end{aligned}$$

$$\begin{aligned}
\beta_k &= 2\pi C_\psi \left\{ \sum_{2i\geq k} \frac{\sin(2i(\nu-\mu)+k\mu)}{(2i-k)!i!} \left(-\frac{1}{\psi}\right)_{3i-k} \left(-\frac{\tanh(\kappa_1\psi)}{2}\right)^{2i-k} \left(-\frac{\tanh(\kappa_2\psi)}{2}\right)^{i} \right. \\
&\quad \times F_4\left(\frac{3i-k}{2}-\frac{1}{2\psi}, \frac{3i-k+1}{2}-\frac{1}{2\psi}; 2i-k+1, i+1; (\tanh(\kappa_1\psi))^2, (\tanh(\kappa_2\psi))^2\right) \\
&\quad - \sum_{i=0}^{\infty} \frac{\sin(2i(\nu-\mu)-k\mu)}{(2i+k)!i!} \left(-\frac{1}{\psi}\right)_{3i+k} \left(-\frac{\tanh(\kappa_1\psi)}{2}\right)^{2i+k} \left(-\frac{\tanh(\kappa_2\psi)}{2}\right)^{i} \\
&\quad \times F_4\left(\frac{3i+k}{2}-\frac{1}{2\psi}, \frac{3i+k+1}{2}-\frac{1}{2\psi}; 2i+k+1, i+1; (\tanh(\kappa_1\psi))^2, (\tanh(\kappa_2\psi))^2\right) \\
&\quad + \sum_{\substack{i+2j=k \\ i,j\geq 1}}\sum \frac{\sin(i\mu+2j\nu)}{i!j!} \left(-\frac{1}{\psi}\right)_{i+j} \left(-\frac{\tanh(\kappa_1\psi)}{2}\right)^{i} \left(-\frac{\tanh(\kappa_2\psi)}{2}\right)^{j} \\
&\quad \left. \times\, F_4\left(\frac{i+j}{2}-\frac{1}{2\psi}, \frac{i+j+1}{2}-\frac{1}{2\psi}; i+1, j+1; (\tanh(\kappa_1\psi))^2, (\tanh(\kappa_2\psi))^2\right) \right\}.
\end{aligned}$$

4 Maximum likelihood estimation

- Let $t_1, t_2, \ldots, t_n$ be a random sample of size n from a circular distribution with pdf (7). Then the log-likelihood function is

$$\begin{aligned} L(\Psi) &= \frac{1}{\psi} \log \sum_{j=1}^{n} \left(1 + \tanh(\kappa_1 \psi) \cos(t_j - \mu)\right. \\ &\quad \left. + \tanh(\kappa_2 \psi) \cos 2(t_j - \nu)\right) - n \log C_\psi, \end{aligned}$$

  where $\Psi = (\psi, \kappa_1, \kappa_2, \mu, \nu)'$ is the parameter vector.

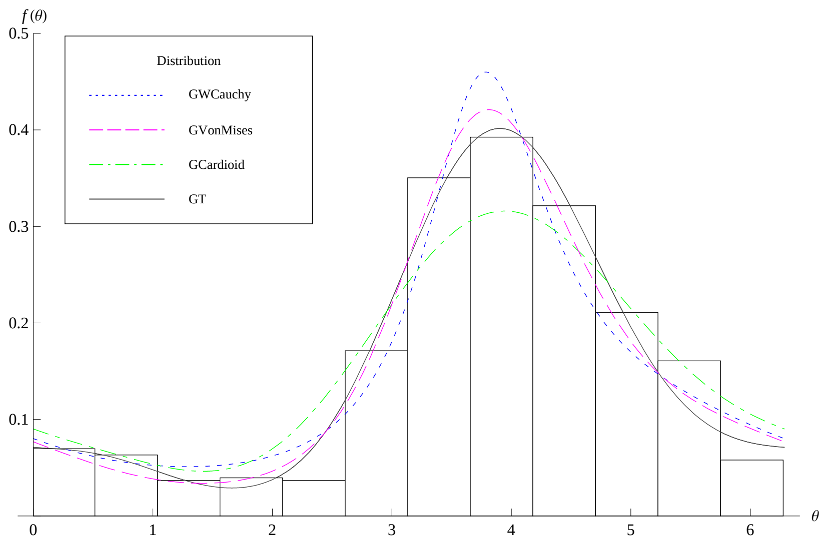
- The maximum likelihood estimates are obtainable by numerical methods.

5 Illustrative Example

- We consider a grouped dataset of size $n = 725$, in Table 1.3 of Mardia (1972), which is adapted from Bishop (1947), on the number of the occurrences of thunder in a day at Kew (England) in the summers of 1910–1935. The data were first recorded and grouped into two-hourly intervals of a day, then converted into angles.

- From the application of reflective symmetry test by Pewsey (2002), the test statistic has p-value of 2×10^{-5}, for which we reject firmly the null hypothesis that the underlying distribution of the data is symmetric.

- Maximum likelihood fits of the thunderstorms data for the: generalized wrapped Cauchy (GWCauchy), generalized von Mises (GVonMises), generalized cardioid (GCardioid) and generalized t-(GT) distributions.



Distribution	ψ	κ_1	κ_2	μ	ν	MLL	AIC
GWCauchy	-1	1.12	0.17	4.20	0	-1125.32	2258.63
GVonMises	0	1.15	0.29	4.14	0.31	-1116.90	2241.80
GCardioid	1	1.13	0.19	4.08	0.66	-1133.19	2274.37
GT	0.66	1.39	0.42	4.06	0.66	-1115.16	2240.31

- From the figure and table, we notice that the generalized t-distribution gave a better fit than its competitors. Hence, it is chosen to be the optimal model for the data.

6 Remarks

- In this study, we focus on generating the generalized t-distribution for circular data by conditioning a scale mixture of bivariate normal distribution with non-identity variance-covariance matrix.

- For some symmetric distributions, such as von Mises and Jones–Pewsey distributions, the extension from circular to the arbitrary dimensional spherical distributions is straightforward.

- To derive an asymmetric distribution on the p-dimensional sphere, $p\,(\geq 3)$, we can apply the similar techniques described in the earlier sections.

References

Bishop, B. V. (1947): The frequency of thunderstorms at Kew observatory. *Meteorological Magazine* **76**, 108–111.

Gatto, R. and Jammalamadaka, S. R. (2006): The generalized von Mises distribution. *Statistical Methodology* **4**, 341–353.

[**Jammalamadaka and SenGupta(2001)**] Jammalamadaka, S. R. and SenGupta, A. (2001): *Topics in Circular Statistics*. World Scientific: Singapore.

Jones, M. C. and Pewsey, A. (2005): A family of symmetric distributions on the circle. *Journal of the American Statistical Association* **100**, 1422–1428.

Kotz, S. and Nadarajah, S. (2004): *Multivariate t Distributions and Their Applications*. Cambridge University Press: Cambridge.

Maksimov, V. M. (1967): Necessary and sufficient statistics for a family of shifts of probability distributions on continuous bicompact groups. *Rossiĭskaya Akademiya Nauk. Teoriya Veroyatnosteĭ i ee Primeneniya* **12**, 307–321 (in Russian), English Translation: *Theory of Probability and Its Applications* **12**, 267–280.

Mardia, K. V. (1972): *Statistics of Directional Data*. Academic Press: London.

Mardia, K. V. and Jupp, P. E. (1999): *Directional Statistics*. Wiley: Chichester.

Mardia, K. V. and Sutton, T. W. (1975): On the modes of a mixture of two von Mises distributions. *Biometrika* **62**, 699–701.

Pewsey, A. (2002): Testing circular symmetry. *Canadian Journal of Statistics* **30**, 591–600.

Rukhin, A. L. (1972): Some statistical decisions about distributions on a circle for large samples. *Sanhkyā Series A*, **34**, 243–250.

Yfantis, E. A. and Borgman, L. E. (1982): An extension of the von Mises distribution. *Communications in Statistics–Theory and Methods* **11**, 1695–1706.