A structural credit risk valuation model with a multiple company debts structure

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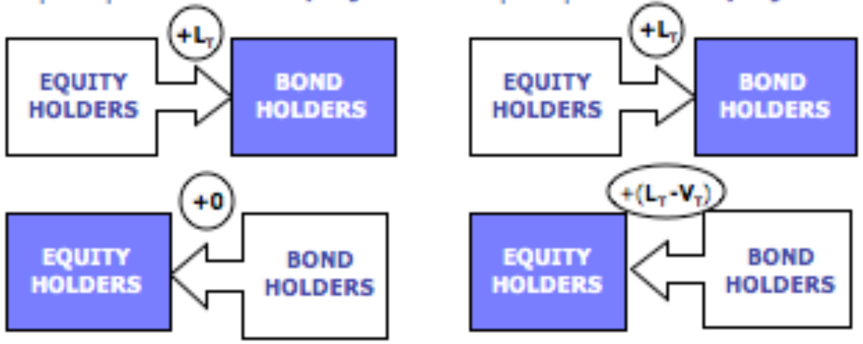
What is a **structural** credit risk model ?

Structural Models Concept

- They model the firm capital structure to assess default risk
- The first to use this kind of models was Merton in 1974. Suppose one company whose asset is $\{V_t\}$ has issued one zero-coupon bond paying some capital L_T at maturity T. For the bondholders, the credit risky zero-coupon bond can be simply decomposed as follow

$V_T \geq L_T \Leftrightarrow$ No bankruptcy at T $V_T < L_T \Leftrightarrow$ Bankruptcy at T



In both cases, for the bondholders

$$\min(L_T, V_T) = L_T - (L_T - V_T)_+$$
$$= ZC\ Bond - Put(V, L_T, T)$$

- If the company value is below L_T at T, it can only pay back the bondholders by liquidating its assets into cash. Then, the bondholders receive V_T. If the company value is above L_T, they receive normally the capital L_T. In other words, they receive $\min(L_T,V_T)$ at T
- They often assume the firm value $\{V_t\}$ follows a risk-neutral geometric Brownian motion

$$\frac{dV_t}{V_t} = r dt + \sigma dW_t \Leftrightarrow V_t = V_0 \exp\left[\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right]$$

Existing mathematical models 2/8

Merton's model

- Pay-off and price for a zero-coupon bond $\{X_t\}$ paying L_T at T and $X_t^* = L_T\, e^{-r(T-t)}$

$$\frac{X_T}{L_T} = 1_{\{V_T \geq L_T\}} + \frac{V_T}{L_T}\, 1_{\{V_T < L_T\}}$$

$$\frac{X_t}{X_t^*} = N\left(d_t - \sigma\sqrt{T-t}\right) + \frac{V_t}{L_T} e^{r(T-t)} N\left(-d_t\right) \qquad d_t = \frac{\log \frac{V_t}{L_T} + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

- Drawbacks: default prior to maturity impossible and only one single company debt

Black and Cox's model - An extension of Merton's model

- Pay-off and price for a zero-coupon bond $\{X_t\}$ paying L_T at T and $X_t^* = L_T\, e^{-r(T-t)}$

$$\frac{X_T}{L_T} = 1_{\{\tau \geq T,\, V_T \geq L_T\}} + \beta_1 \frac{V_T}{L_T}\, 1_{\{\tau \geq T,\, V_T < L_T\}} + \beta_2 \frac{K}{L_T} e^{-\gamma(T-t)} e^{r(T-\tau)} 1_{\{\tau < T\}}$$

- Presence of a random hitting time $\tau = \inf\left\{t > 0 \,/\, V_t < K e^{-\gamma(T-t)}\right\}$ $\quad \bar{v}_t = K e^{-\gamma(T-t)}$

$$X_t = X_t^* \left[N(a_{1_t}) - \left(\frac{V_t}{\bar{v}_t}\right)^{2\tilde{a}} N(a_{2_t})\right] + \beta_1 V_t\, \left[N(b_{1_t}) - N(b_{2_t})\right]$$

$$+\beta_1 V_t \left(\frac{V_t}{\bar{v}_t}\right)^{2(\tilde{a}+1)} \left[N(c_{1_t}) - N(c_{2_t})\right] + \beta_2 V_t \left[\left(\frac{V_t}{\bar{v}_t}\right)^{\theta+\phi} N(d_{1_t}) + \left(\frac{V_t}{\bar{v}_t}\right)^{\theta-\phi} N(d_{2_t})\right]$$

- Drawbacks : only one single company debt and meaning of the coefficients unclear

Our new model 3/8

1. Closer to accounting values and therefore easier calibration
   $\Rightarrow$ only (7+2n) parameters for n debts and only 2 to be estimated (β,σ)

2. Many company debts can be considered and this, without complexifying the prices resolution since one can show

   Pay-off [ZC bond n+1] = Pay-off [ZC bond n] - F (Pay-off [ZC bond n])

3. Both kinds of banruptcy can be envisaged

   $\Rightarrow$ Non-respect of the law, applicable at any time

   $\Rightarrow$ Shortage of cash to pay the bondholders, applicable at maturity

4. Semi-explicit formulas obtained for ZC bond prices, accurately estimated by classical numerical methods

Pay-offs and mathematical tools 4/8

One pay-off different for each debt

$$X^1_{T_1} L^{-1}_{T_1} = (1+\alpha)\beta\, 1_{\{\tau_1 < T_1\}} + \beta \frac{V_{T_1}}{D^1_{T_1}}\, 1_{\{\tau_1 \geq T_1,\, \beta V_{T_1} < L_{T_1}\}} + 1_{\{\tau_1 \geq T_1,\, \beta V_{T_1} \geq L_{T_1}\}}$$

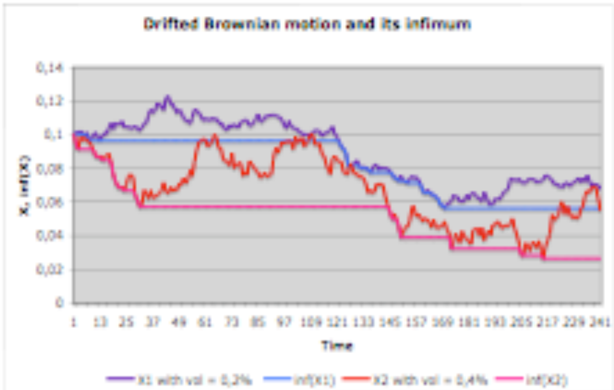
$$X^2_{T_2} L^{-1}_{T_2} = (1+\alpha)\beta\, 1_{\{\tau_1 < T_1\}} + \beta \frac{V_{T_1}}{D^1_{T_1}}\, 1_{\{\tau_1 \geq T_1,\, \beta V_{T_1} < L_{T_1}\}} + (1+\alpha)\beta\, 1_{\{\tau_1 \geq T_1,\, \beta V_{T_1} \geq L_{T_1},\, \tau_2 < T_2\}} + \beta \frac{V_{T_2}}{D^2_{T_2}}\, 1_{\{\tau_1 \geq T_1,\, \beta V_{T_1} \geq L_{T_1},\, \tau_2 \geq T_2,\, \beta V_{T_2} < L_{T_2}\}} + 1_{\{\tau_1 \geq T_1,\, \beta V_{T_1} \geq L_{T_1},\, \tau_2 \geq T_2,\, \beta V_{T_2} \geq L_{T_2}\}}$$

Use of the joint density of the drifted Brownian motion and its infimum

Let $X_t = \nu t + \sigma W_t \qquad M^X_t = \inf_{s \leq t} X_s$

Let $y < 0$ *and* $y \leq x$

$$f_{X_t, M^X_t}(x, y) = e^{\frac{\nu x}{\sigma^2} - \frac{\nu^2 t}{2\sigma^2}} \frac{2(x-2y)}{\sqrt{2\pi}(\sigma^2 t)^{\frac{3}{2}}} \exp\left[-\frac{(2y-x)^2}{2\sigma^2 t}\right]$$



Main assumptions of our new model 5/8

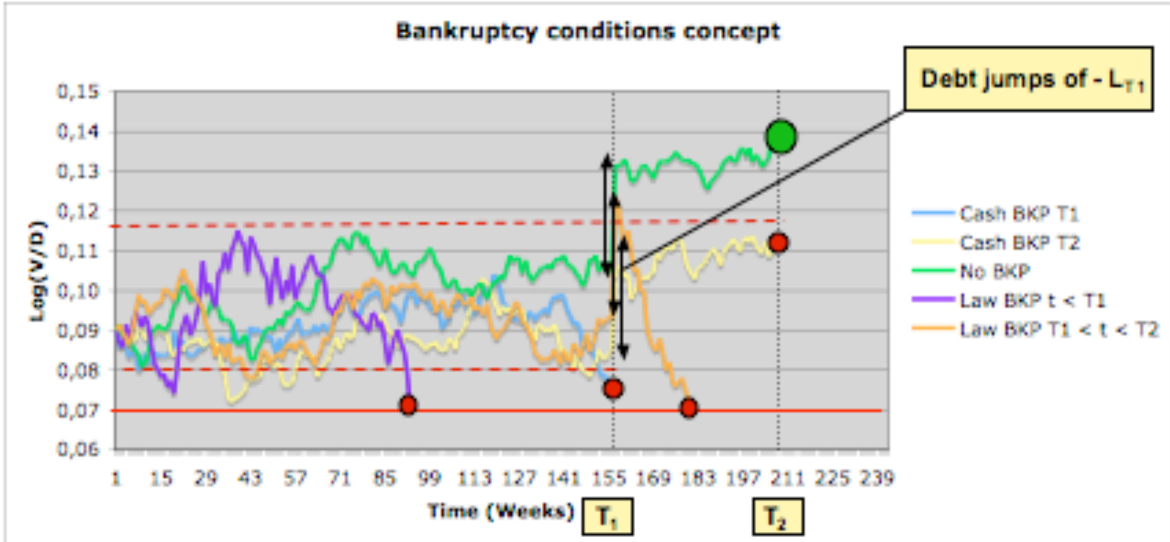
BALANCE SHEET	
ASSETS	LIABILITIES
FIXED ASSETS (FA_t)	EQUITY (EQ_t)
Machine Property	
CURRENT ASSETS (CA_t)	FINANCIAL DEBTS (D_t)
Stocks Debtors	
CASH (C_t)	CURRENT DEBTS (CD_t)
TOTAL ASSETS (A_t)	TOTAL LIABILITIES (L_t)

$$V_t = EQ_t + D_t = A_t - CD_t$$

$$\frac{dV_t}{V_t} = rdt + \sigma dW_t$$

$$\forall\, t < T_1 \qquad D_t = D_t^1 = L_{T_1} e^{-r(T_1 - t)} + L_{T_2} e^{-r(T_2 - t)}$$

$$\forall\, T_1 \leq t < T_2 \quad D_t = D_t^2 = L_{T_2} e^{-r(T_2 - t)}$$



Main assumptions of our new model 6/8

2 ways of going bankrupt

- A *legal condition* always applicable

  Company debts jumps problem solved by mean of several random hitting times and a parameter α that can be easily calibrated

  For a 2-debts problem

  $\tau_1 = \inf\{t > 0 \;/\; EQ_t < \alpha D_t^1\} \quad \tau_2 = \inf\{t > T_1 \;/\; EQ_t < \alpha D_t^2\}$

- A *counterparty condition* only applicable at maturity

  Even if the law is respected, mispricing of assets owing to a lack of liquidity can occur and lead to a default payment when paying back bondholders. Liquidity risk is estimated by a parameter β

  $\beta V_{T_i} < L_{T_i}$

1 way of recovering capital

- We suppose fairness of sharing out once the company has gone bankrupted. If τ is the time of bankruptcy, the recovered capital at τ is

  $\frac{\beta V_\tau}{D_\tau} \times L_{T_i} e^{-r(T_i - \tau)}$

Debts solutions 7/8

$$0 \le \alpha \qquad \bar{v}_t^1 = (1+\alpha) D_t^1 \qquad a_t = \frac{\log \frac{V_t}{\bar{v}_t^1} + \frac{1}{2}\sigma^2 (T_1 - t)}{\sigma\sqrt{T_1 - t}} \qquad \Gamma_t^1 = \frac{V_t}{\bar{v}_t^1} \quad J_{T_1} = \frac{D_{T_1}^1}{D_{T_1}^2}$$

$$0 < \beta \le 1$$

$$1 < (1+\alpha)\beta = \gamma \qquad \vartheta_t = \frac{1}{\sigma\sqrt{T_1 - t}} \log \frac{L_{T_1}}{\beta \bar{v}_{T_1}^1} \qquad b(x) = \frac{\log x + \frac{1}{2}\sigma^2 (T_2 - T_1)}{\sigma\sqrt{T_2 - T_1}} \qquad \theta = -\frac{1}{\sigma\sqrt{T_2 - T_1}} \log \gamma$$

First maturing debt price

$$X_t^1 = X_t^{1*} \left[1 - \Psi_\gamma \left(\Gamma_t^1,\, a_t,\, \sigma\sqrt{T_1 - t},\, 2\vartheta_t \right) \right]$$

$$\Psi_\gamma(t,u,v,w) = (1-\gamma)\left[1 - N(u-v) + t\,(1 - N(u))\right] + N(u-v) - N(u-v-w) - \left[N(u+w) - N(u-w)\right]$$

Second maturing debt price

$$X_t^2 = X_t^{2*} \left[1 - \Psi_\gamma \left(\Gamma_t^1,\, a_t,\, \sigma\sqrt{T_1 - t},\, 2\vartheta_t \right) - \int_A^{+\infty} \Psi_\gamma \left(x J_{T_1},\, b(x),\, \sigma\sqrt{T_2 - T_1},\, \theta \right) dQ_t^1(x) \right] \quad A = \frac{L_{T_1}}{\beta \bar{v}_{T_1}^1}$$

$$dQ_t^1(x) = d_x \left[N \left(\frac{\frac{1}{2}\sigma^2 (T_1 - t) + x - Y_t}{\sigma\sqrt{T_1 - t}} \right) \right] + Y_t^1\, d_x \left[N \left(\frac{-\frac{1}{2}\sigma^2 (T_1 - t) - x - Y_t}{\sigma\sqrt{T_1 - t}} \right) \right] \quad Y_t^1 = \log \Gamma_t^1$$

Approximation method and its error 8/8

Second maturing debt integral approximation

Owing to the presence of lognormal densities in the integral, and since the exponential of the lognormal tails are very thick, we approximate the infinite of the upperbound of the integral by a larbe number B, typically B ~ 20. Then we use a trapeze method to estimate the finite integral with a step n to be chosen to adjust accuracy

B ~ 20 and n ~ 150 gives | e | ~ 10^{-4} for the trapeze method approximation of a lognormal distribution function for instance

$$\varphi_t(x)\, dx = \Psi_\gamma\left(xJ_{T_1},\ b(x),\ \sigma\sqrt{T_2 - T_1},\ \theta\right) dQ_t^1(x)$$

$$\int_A^{+\infty} \varphi_t(x) dx = \frac{1}{2}\frac{B-A}{n}\sum_{k=1}^{n}\left[\varphi_t\left(A + k\frac{B-A}{n}\right) + \varphi_t\left(A + (k-1)\frac{B-A}{n}\right)\right]$$