Relative Error of the Generalized Pareto Approximation to Value-at-Risk

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Risk Management & Value-at-Risk

- Basel Accord (Basel Ⅱ)

  "The risk capital of a bank must be sufficient to cover losses on the bank's trading portfolio."

- Value-at-Risk

  Quantitative risk measure
  (Designated by Basel committee)

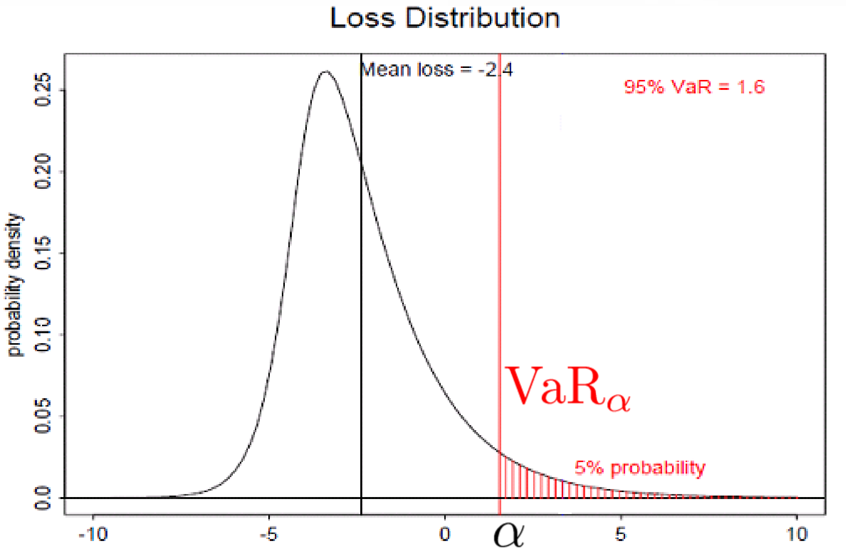
$$\mathrm{VaR}_\alpha = \inf\{x \in \mathbb{R} : F(x) \geq \alpha\}$$

F: Loss distribution

α: Confidence level (nearly 1)

$\Downarrow$

Extreme Risk



Generalized Pareto Approximation

Distribution function

$$F(x) = F(u) + F_u(x-u)(1-F(u))$$

$$\Uparrow$$

$$F_G(x) = F(u) + G_{\xi,\sigma(u)}(x-u)(1-F(u))$$

VaR (quantile)

$$\mathrm{VaR}_\alpha = x$$

$$\Uparrow$$

$$\widetilde{\mathrm{VaR}}_\alpha = F_G^{-1}(F(x))$$

- Genralized Pareto distribution

$$G_{\xi,\sigma}(x) = \begin{cases} 1-(1+\xi x/\sigma)^{-1/\xi}, & \xi > 0, \\ 1-\exp(-x/\sigma), & \xi = 0. \end{cases}$$

- Excess distribution

$$F_u(x) = (F(x+u) - F(u))/\bar{F}(u)$$

This approximation is frequently used
- MacNeil & Frey (2000) ･･･ Stock return
- Moscadelli (2004) ･･･ Operational risk
- Katz et al. (2002) ･･･ Hydrology etc

Generalized Pareto Approximation

F belongs to the maximum domain of attraction (MDA)

with tail index $\xi \in \mathbb{R}$

$\Updownarrow$

$$\lim_{u \to \infty} \sup_{u \le x < \infty} \left| F_u(x) - G_{\xi,\sigma(u)}(x) \right| = 0$$

(Pickands (1975), Balkema & de Haan (1974))

- $\sigma(u) = \bar{F}(u)/f(u)$ is possible when the von Mises' condition is satisfied.

(de Haan and Ferreira (2006))

- MDA covers almost all continuous distribution.
- **Fréchet class** ･･･ MDA with tail index $\xi > 0$.
- **Gumbell class** ･･･ MDA with tail index $\xi = 0$.

Evaluation of the Relative Error

- Relative approximation error

$$\varepsilon_{u,x} = 1 - \frac{\widetilde{\mathrm{VaR}}_\alpha}{\mathrm{VaR}\ _\alpha} = 1 - \frac{F_G^{-1}(F(x))}{x}$$

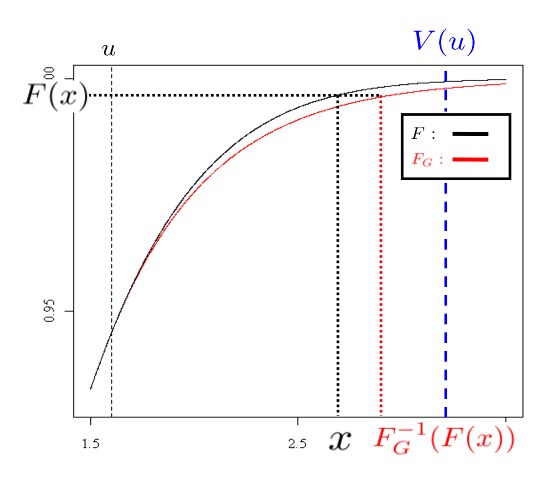
- Previous work
  - Beirlant et al. (2003) ・・・ Asymptotics in terms of a confidence level α (threshold: $u = u_\alpha$) , pointwise convergence
- To discuss the uniformity…

  Asymptotics in terms of a threshold u

  Set a upper bound V of x.

$$\boxed{\sup_{x \in (u, V(u))} \left|\varepsilon_{u,x}\right|}$$

$$V : \mathbb{R}^+ \to (u, \infty]$$



Main Results

■ Fréchet class with tail index $\xi > 0$

$$\bar{F}(x) = x^{-1/\xi} L(x)$$

$\left(\begin{array}{l} L\text{: slowly varying function} \\ \quad \Leftrightarrow \lim_{x\to\infty} \dfrac{L(tx)}{L(x)} = 1 \text{ for } t > 0 \end{array} \right)$

Sufficient conditions

- ✓ In terms of V

$$\limsup_{u\to\infty} \frac{V(u)}{u} < \infty \Longrightarrow \sup_{x\in(u,V(u))} |\varepsilon_{u,x}| \to 0, \text{ as } u \to \infty$$

- ✓ In terms of L

$$\lim_{x\to\infty} L(x) = c > 0 \Longrightarrow \sup_{x\in(u,\infty)} |\varepsilon_{u,x}| \to 0, \text{ as } u \to \infty$$

Main Results

Gumbel class

- **Weibull-type with index $\beta \geq 0$** **(Sub-class of the Gumbel class)**

$$\bar{F}(x) = \exp\left(-x^{\beta} L(x)\right),$$

- In case of $\beta = 1$

Sufficient conditions

✓ In terms of V

$$\limsup_{u \to \infty} \frac{V(u)}{u} < \infty \Longrightarrow \lim_{u \to \infty} \sup_{x \in (u, V(u))} |\varepsilon_{u,x}| = 0$$

✓In terms of L

$$\lim_{x \to \infty} L(x) = c > 0 \Longrightarrow \lim_{u \to \infty} \sup_{x \in (u, \infty)} |\varepsilon_{u,x}| = 0$$

Main Results

Gumbel class

- **Weibull-type with index** $\beta \geq 0$ **(Sub-class of the Gumbel class)**
  - In case of $\beta \neq 1$

Necessary and Sufficient condition

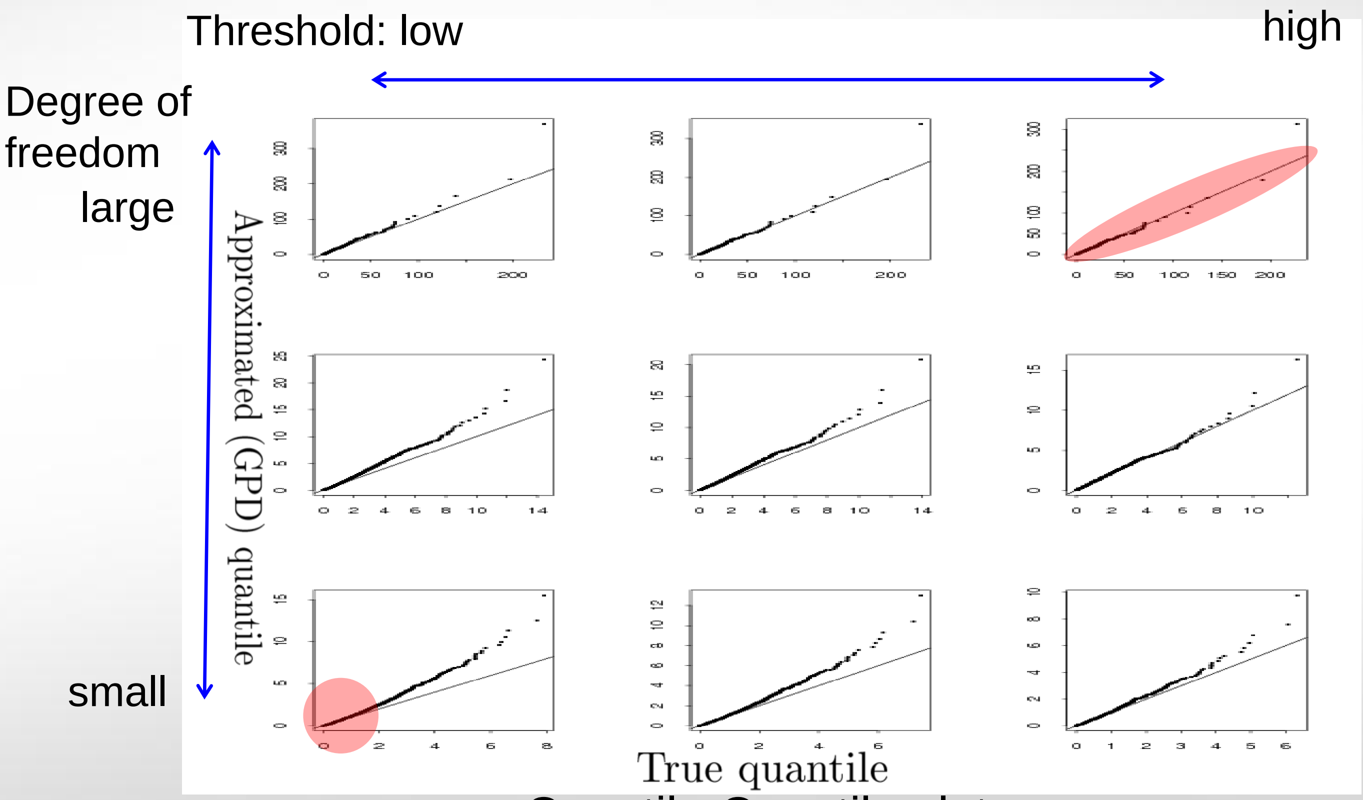
$$\lim_{u\to\infty} \frac{V(u)}{u} = 1 \Longleftrightarrow \lim_{u\to\infty} \sup_{x\in(u,V(u))} |\varepsilon_{u,x}| = 0$$

The uniform convergence holds true only in the neighborhood of threshold.

Numerical Result

- **t distribution (Fréchet class)**

$\lim_{x\to\infty} L(x) = \text{const.}$



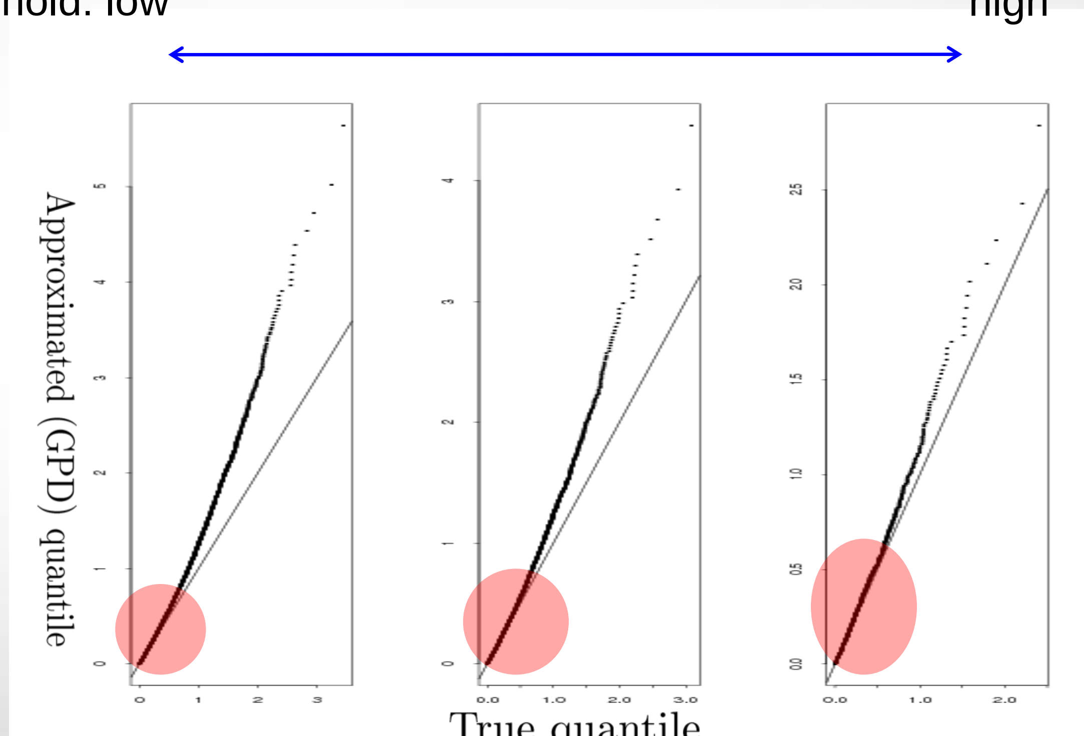
Quantile-Quantile plot

Numerical Result

- **Normal distribution (Gumbel class)**

  Weibull-type with $\beta = 2$

Threshold: low high



Quantile-Quantile plot

Conclusion

Our result suggests that

– generalized Pareto approximation does not always provide a good estimation of the VaR.

– it is safe to restrict our attention into $x \in (u, u + o(u))$.

References

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