A family of asymmetric distributions on the circle with links to, and applications arising from, Möbius transformation

Shogo Kato[1,*] and M. C. Jones[2]

[1] Keio University, Japan
[2] The Open University, UK

March 28, 2008

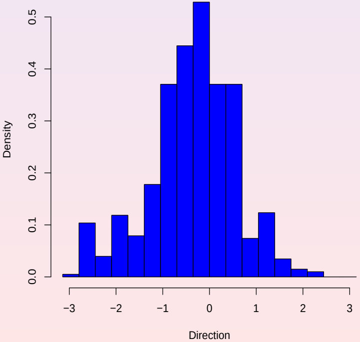
Outline

Asymmetric Circular Data

Circular data

Circular data are a set of observations expressed as angles $[-\pi, \pi)$.

Ex) wind directions, vanishing directions of migratory birds.



Asymmetric circular data

Circular data are often asymmetrically distributed.

Fig.1. azimuths of cross-beds in the rocks of the upper Kamthi river valley, India (SenGupta and Rao, 1966).

Asymmetric Distributions on the Circle

Existing model

Generalized von Mises distribution

- introduced as a statistical model by Maksimov (1967).
- discussed further, e.g., by Gatto & Jammalamadaka (2007).

Our goal

It would be ideal if an asymmetric distribution has the following properties:

- inclusion of some important symmetric models,
- wide range of the indices of skewness,
- mathematical tractability.

Definition of the Proposed Model

We propose a family of distributions on the circle by transforming the von Mises distribution via Möbius transformation.

Definition

Let $\tilde{\Theta} \sim$ von Mises distribution vM$(0, \kappa)$.

Then we define a family of distributions on the circle by

$$\Theta = \mu + \arg\left\{ \frac{e^{i\tilde{\Theta}} + re^{i\nu}}{re^{i(\tilde{\Theta}-\nu)} + 1} \right\},$$

where

$$-\pi \leq \mu, \nu < \pi, \quad 0 \leq r < 1.$$

What are the von Mises distribution and Möbius transformation?

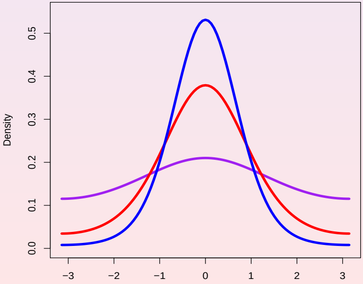
von Mises distribution

The von Mises distribution, vM(μ, κ), is given by the density

$$f(\theta) = \frac{1}{2\pi\mathcal{I}_0(\kappa)} \exp\{\kappa \cos(\theta - \mu)\}, \quad -\pi \le \theta < \pi,$$

where $\quad -\pi \le \mu < \pi, \quad \kappa \ge 0,$

$\mathcal{I}_j(\cdot)$: modified Bessel function of the first kind and order j.



Basic properties

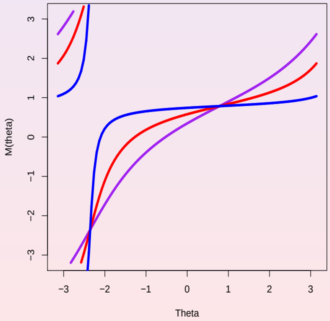
- unimodality
- symmetry about $\theta = \mu$
- mode (antimode) at $\theta = \mu$ $(\mu + \pi)$

Fig. 2. Density of von Mises with $\mu = 0$ and: $\kappa = 0.3$, $\kappa = 1.2$, $\kappa = 2.1$.

Möbius transformation

The Möbius transformation $\mathcal{M} : [-\pi, \pi) \to [-\pi, \pi)$ is defined by

$$\mathcal{M}(\theta; r, \nu) = \arg\left\{ \frac{e^{i\theta} + re^{i\nu}}{re^{i(\theta-\nu)} + 1} \right\}, \quad \theta \in [-\pi, \pi);\ r \in [0, 1),\ \nu \in [-\pi, \pi).$$



Interpretation of r and ν

The points on the circle are attracted towards ν with concentration r.

Fig. 3. Plot of $\mathcal{M}(\theta; r, \nu)$ for $\nu = \pi/4$ and: $r = 0.3$, $r = 0.6$, $r = 0.9$.

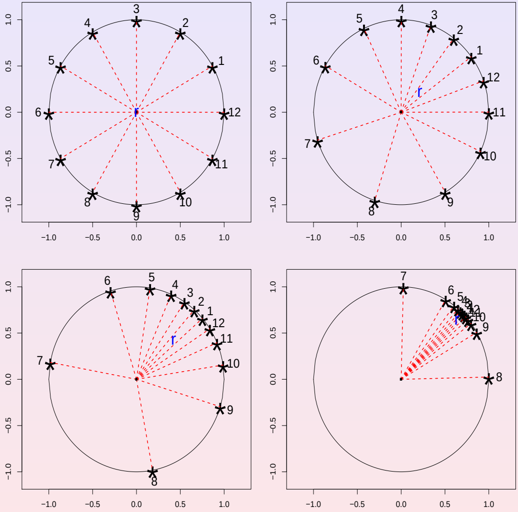


Fig. 4.

$$\tilde{\mathcal{M}}(\theta) = \frac{e^{i\theta} + re^{i\nu}}{re^{i(\theta-\nu)} + 1},$$

$$\theta = 2\pi k/12,$$

$$k = 1, \ldots, 12,$$

$$\nu = \pi/4,$$

1. $r = 0$ (above left),
2. $r = 0.3$ (above right),
3. $r = 0.6$ (below left),
4. $r = 0.9$ (below right).

Möbius transformation and circular uniform

McCullagh (1996)

$$\Theta \sim \text{circular uniform} \implies \arg\left\{\frac{e^{i\Theta} + re^{i\nu}}{re^{i(\Theta-\nu)} + 1}\right\} \sim \mathrm{WC}(\nu, r)$$

Wrapped Cauchy distribution

The wrapped Cauchy distribution, $\mathrm{WC}(\nu, r)$, is given by the density

$$f(\theta) = \frac{1}{2\pi}\frac{1-r^2}{1+r^2-2r\cos(\theta-\nu)}, \quad \theta \in [-\pi,\pi);\ r \in [0,1),\ \nu \in [\pi,-\pi).$$

This model is also a unimodal and symmetric distribution on the circle.

Definition of the Proposed Model (Revisited)

We propose a family of distributions on the circle by transforming the von Mises distribution via Möbius transformation.

Definition

Let $\tilde{\Theta} \sim$ von Mises distribution vM$(0, \kappa)$.

Then we define a family of distributions on the circle by

$$\Theta = \mu + \arg\left\{ \frac{e^{i\tilde{\Theta}} + re^{i\nu}}{re^{i(\tilde{\Theta}-\nu)} + 1} \right\},$$

where

$$-\pi \leq \mu, \nu < \pi, \quad 0 \leq r < 1.$$

We investigate some properties of the proposed model.

Probability Density Function

Probability density function

The density for Θ is given by

$$f(\theta;\mu,\kappa,r,\nu) = \frac{1-r^2}{2\pi\mathcal{I}_0(\kappa)}\exp\left[\frac{\kappa\{\xi\cos(\theta-\eta)-2r\cos\nu\}}{1+r^2-2r\cos(\theta-\gamma)}\right] \times\frac{1}{1+r^2-2r\cos(\theta-\gamma)},\quad -\pi\le\theta<\pi, \qquad (1)$$

where

$$\kappa\ge 0,\quad -\pi\le\mu,\nu<\pi,\quad 0\le r<1,\quad \xi=\sqrt{r^4+2r^2\cos(2\nu)+1},$$
$$\gamma=\mu+\nu,\quad \eta=\mu+\arg\{r^2\cos(2\nu)+1+ir^2\sin(2\nu)\}.$$

Probabilities

Probabilities

Let f: density for model (1),

g_{VM}: density for the von Mises vM$(0, \kappa)$.

Then probabilities of intervals under density f can be expressed as

$$\int_{t_1}^{t_2} f(\theta)\, d\theta = \int_{s_1}^{s_2} g_{VM}(\theta)\, d\theta,$$

where

$$s_j = \arg\left\{ \frac{re^{i\nu} - e^{i(t_j - \mu)}}{re^{i(t_j - \mu - \nu)} - 1} \right\}, \quad j = 1, 2.$$

Therefore we can evaluate the above probabilities by applying the numerical method for von Mises distribution developed by Hill (1977).

Special Cases

Proposed model

$$f(\theta) \propto \exp\left[\frac{\kappa\{\xi\cos(\theta-\eta)-2r\cos\nu\}}{1+r^2-2r\cos(\theta-\gamma)}\right]\frac{1}{1+r^2-2r\cos(\theta-\gamma)}, \quad \theta\in[-\pi,\pi);$$

$$\kappa\ge 0, \quad \mu,\nu\in[-\pi,\pi), \quad r\in[0,1), \quad \xi=\sqrt{r^4+2r^2\cos(2\nu)+1},$$

$$\gamma=\mu+\nu, \quad \eta=\mu+\arg\{r^2\cos(2\nu)+1+ir^2\sin(2\nu)\}.$$

Special cases

The proposed model includes the following as special cases:

- von Mises distribution $(r=0)$,
- wrapped Cauchy distribution $(\kappa=0)$,
- uniform distribution $(r=\kappa=0)$,
- point distribution at $\theta=\mu+\nu$ $(\kappa\to\infty$ or $r\to 1)$.

Graphs of Density

Proposed model

$$f(\theta) \propto \exp\left[\frac{\kappa\{\xi\cos(\theta-\eta)-2r\cos\nu\}}{1+r^2-2r\cos(\theta-\mu-\nu)}\right]\frac{1}{1+r^2-2r\cos(\theta-\mu-\nu)}.$$

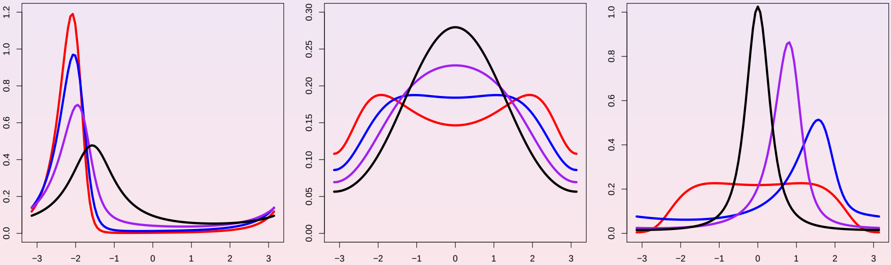


Fig. 5. (left) $\mu=\pi$, $r=0.5$, $\nu=\pi/2$ and: $\kappa=0$, $\kappa=1$, $\kappa=2$ and $\kappa=3$, (center) $\mu=0$, $\kappa=1$, $\nu=\pi$ and: $r=0.1$, $r=0.2$, $r=0.3$ and $r=0.4$, (right) $\mu=0$, $\kappa=1$, $r=0.5$ and: $\nu=0$, $\nu=\pi/3$, $\nu=2\pi/3$ and $\nu=\pi$.

Conditions for Symmetry/Unimodality

Conditions for symmetry

Density (1) is symmetric if and only if $r = 0, \ \nu = 0, \pi$ or $\kappa = 0$.

Conditions for unimodality

- Density (1) can be unimodal or bimodal.
- Modes of the density (1) can be calculated by solving a quartic equation.
- The condition for unimodality can be written out in terms of three parameters $r, \ \nu$ and κ.

Circular skewness

A measure of skewness, s, for circular r.v. Θ (Mardia, 1972).

$$s = \frac{E\left[\sin\left\{2(\Theta - \alpha)\right\}\right]}{(1-\rho)^{3/2}}; \quad \rho e^{i\alpha} = E(e^{i\Theta}), \quad \rho \geq 0, \quad \alpha \in [-\pi, \pi).$$

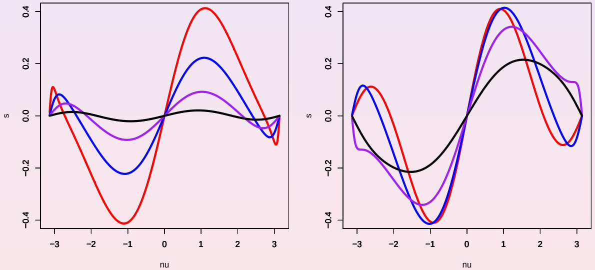


Fig. 6. Skewness, s, for the distribution (1) as a function of ν for $\mu = 0$ and: (left) $r = 0.5$ and: $\kappa = 0.3$, $\kappa = 0.6$, $\kappa = 0.9$ and $\kappa = 1.2$, (right) $\kappa = 1.16$ and: $r = 0.2$, $r = 0.4$, $r = 0.6$ and $r = 0.8$.

Random Variate Generation

First, put

$$a = 1 + (1 + 4\kappa^2)^{\frac{1}{2}}, \quad b = \frac{a - \sqrt{2a}}{2\kappa}, \quad \text{and} \quad \zeta = \frac{1 + b^2}{2b}.$$

Then the following steps generate the variables from model (1):

Algorithm

Step 1: $U_1, U_2 \sim i.i.d.\ U(0, 1)$.

Step 2: $Z = \cos(\pi U_1)$, $F = (\zeta z + 1)/(\zeta + z)$, $C = \kappa(\zeta - F)$.

Step 3: If $C(2 - C) - U_2 > 0$, then go to Step 5.

Step 4: If $\log(C/U_2) + 1 - C < 0$, return to Step 2.

Step 5: $U_3 \sim U(0, 1)$, $\Theta_T = \text{sign}(U_3 - 0.5) \cos^{-1}(F)$.

Step 6: $\Theta = \mu + \nu + 2 \arctan[\{(1 - r)/(1 + r)\} \tan\{\frac{1}{2}(\Theta_T - \nu)\}]$.

See Best & Fisher (1979) for the acceptance ratio for this algorithm.

Comparison with Generalized von Mises Distribution

Generalized von Mises (GvM) distribution

$$f_{GVM}(\theta) \propto \exp\left\{\kappa_1 \cos(\theta - \mu_1) + \kappa_2 \cos(2(\theta - \mu_2))\right\}, \quad \theta \in [-\pi, \pi);$$
$$\kappa_1, \kappa_2 \geq 0, \ \mu_1 \in [-\pi, \pi), \ \mu_2 \in [0, \pi).$$

Common properties of Model (1) and GvM distribution

- symmetry/asymmetry and unimodality/bimodality,
- inclusion of von Mises distribution,

Model (1) only

- inclusion of WC distribution,
- simple normalizing constant,

GvM distribution only

- a member of exponential family.

The Three-parameter Symmetric Special Case

We investigate detailed properties of the symmetric case of the distribution (1) corresponding to $\nu = 0$ or π.

Probability density function

In this case, the density (1) reduces to

$$f(\theta) = \frac{1-r^2}{2\pi\mathcal{I}_0(\kappa)} \exp\left[\frac{\kappa\{(1+r^2)\cos(\theta-\mu)-2r\}}{1+r^2-2r\cos(\theta-\mu)}\right] \frac{1}{1+r^2-2r\cos(\theta-\mu)}, \tag{2}$$

$$-\pi \le \theta < \pi,$$

where $\quad -\pi \le \mu < \pi, \quad \kappa > 0, \quad -1 < r < 1.$

Clearly, this subfamily also contains the von Mises and wrapped Cauchy distributions as special cases.

Limiting Distribution

Limiting distribution

Let $\Theta \sim$ Model (2). Then

$$\mu + \sqrt{\kappa}(\Theta - \mu) \xrightarrow{d} N\left(\mu, \left(\frac{1-r}{1+r}\right)^2\right) \quad \text{as} \quad \kappa \to \infty.$$

This is a generalized form of the usual result for the von Mises distribution ($r = 0$) given, e.g., by Mardia & Jupp (1999).

Conditions for Unimodality

Conditions for unimodality

The conditions for unimodality for the density (2) can be simply expressed as

$$\text{(i)} \quad \kappa > \frac{-2r}{(1+r)^2} \quad \text{or} \quad \text{(ii)} \quad \kappa < \frac{-2r}{(1-r)^2}.$$

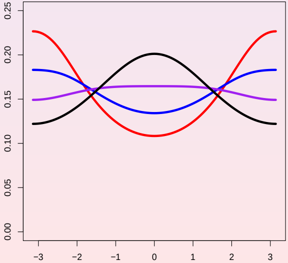


Fig. 7. Density (2) for $\mu = 0$, $\kappa = 1/4$, and: $r = 0$(i), $r = -0.1$(i), $r = -0.2$(ii) and $r = -0.3$(ii).

Fisher Information Matrix

Fisher information matrix

Let $\Theta_1, \ldots, \Theta_n \sim$ *i.i.d.* Model (2).

The elements of the expected Fisher information matrix of the parameters are given as follows:

$$\iota_{\mu,r} = \iota_{\mu,\kappa} = 0, \quad \iota_{\kappa,\kappa} = n\left\{1 - \frac{\mathcal{I}_1^2(\kappa)}{\mathcal{I}_0^2(\kappa)} - \frac{1}{\kappa}\frac{\mathcal{I}_1(\kappa)}{\mathcal{I}_0(\kappa)}\right\},$$

$$\iota_{r,r} = \frac{2n}{(1-r^2)^2}\left\{1 + 3\frac{\mathcal{I}_2(\kappa)}{\mathcal{I}_0(\kappa)}\right\}, \quad \iota_{r,\kappa} = \frac{n}{1-r^2}\left\{1 - \frac{\mathcal{I}_2(\kappa)}{\mathcal{I}_0(\kappa)}\right\},$$

where $\iota_{\alpha,\beta} = -E\left[\partial^2 logL/\partial\alpha\partial\beta\right]$.

Hence, μ and (r, κ) are orthogonal.

Asymptotic Correlation

The asymptotic correlation between $\hat{\kappa}$ and $\hat{r}$,

$$-\frac{\iota_{r,\kappa}}{\sqrt{\iota_{r,r}\iota_{\kappa,\kappa}}} = g(\kappa),$$

is a function of κ (not r).

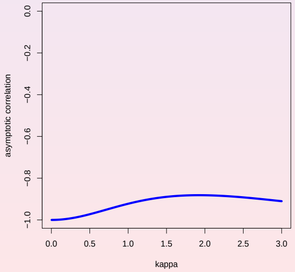


Fig. 8. Plot of asymptotic correlation between $\hat{\kappa}$ and $\hat{r}$ as a function of κ.

Problem

Is there a reparameterization which reduces the asymptotic correlation?

We can take advantage of the special structure of the Fisher information to attempt construction of an orthogonal reparameterization (Cox and Reid, 1987).

Reparameterization

Consider a reparameterization

$$(r, \kappa) \longrightarrow (s(r, \kappa), \kappa),$$

where

$$s(r, \kappa) = \sqrt{\frac{1+r}{1-r}} \exp\left\{ \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\kappa}{\sqrt{2}} \right) \right\}.$$

Then the asymptotic correlation between $\hat{s}$ and $\hat{\kappa}$ is approximately zero for sufficiently small κ.

Asymptotic correlation and asymptotic variance

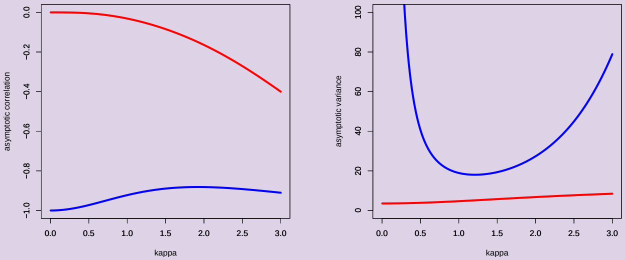


Fig. 9.
(left) asymptotic correlations between $\hat{\kappa}$ and $\hat{r}$ and between $\hat{\kappa}$ and $\hat{s}$,
(right) asymptotic variances of $\hat{\kappa}$ for the original parameterization and the new parameterization.

Therefore new parameterization reduces the asymptotic correlation and asymptotic variance.

Comparison with the Jones and Pewsey Distribution

Jones and Pewsey (2005) (J&P) distribution

$$f_{JP}(\theta) \propto \{1 + \tanh(\kappa\psi)\cos(\theta - \mu)\}^{1/\psi}, \ \theta, \mu \in [-\pi, \pi), \ \kappa \geq 0, \ \psi \in \mathbb{R}.$$

Common properties of Model (2) and J&P model

- symmetry and unimodality,
- inclusion of von Mises and wrapped Cauchy,

Model (2) only

- some properties vis-a-vis Möbius transformation,

J&P model only

- inclusion of the cardioid and power-of-cosine distributions.

A Three-parameter Asymmetric Special Case

We briefly discuss a three-parameter subfamily of the model (1) associated with $\nu = \pm\pi/2$.

Probability density function

In this case, the density (1) reduces to

$$f(\theta) = \frac{1-r^2}{2\pi\mathcal{I}_0(\kappa)} \exp\left\{ \frac{\kappa(1-r^2)\cos(\theta-\mu)}{1+r^2-2r\sin(\theta-\mu)} \right\} \frac{1}{1+r^2-2r\sin(\theta-\mu)}, \quad -\pi \le \theta < \pi,$$

where $\quad -\pi \le \mu < \pi, \quad \kappa > 0, \quad -1 < r < 1.$

Properties

- The density is, in general, asymmetrically distributed.
- The model includes the von Mises and wrapped Cauchy as special cases.

Illustrative Example

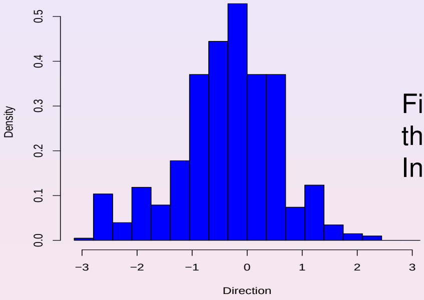


Fig.10. azimuths, $n = 580$, of cross-beds in the rocks of the upper Kamthi river valley, India (SenGupta and Rao, 1966).

As an illustrative example, we consider a dataset which Mardia & Jupp (1999) present as “an example of an asymmetrical distribution”.

Illustrative Example

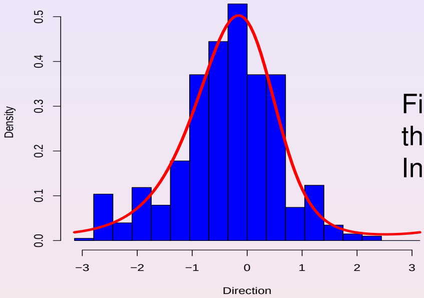


Fig.10. azimuths, $n = 580$, of cross-beds in the rocks of the upper Kamthi river valley, India (SenGupta and Rao, 1966).

Model	$\hat{\kappa}$	$\hat{r}$	$\hat{\nu}$	$\hat{\mu}$	log L	AIC
Full model (1)	1.93	0.130	1.87	5.74	-1380.59	2769.18
3-parameter symmetric (2)	1.66	0.0428	(0)	5.99	-1385.41	2776.82
3-parameter asymmetric (3)	1.78	0.122	$(\frac{\pi}{2})$	5.76	-1380.80	2767.60
von Mises	1.81	(0)	(0)	5.98	-1385.68	2775.36
Wrapped Cauchy	(0)	0.586	(0)	6.02	-1403.72	2811.44

Conclusions

A four-parameter asymmetric family

- closed form of the density, with simple normalizing constant,
- inclusion of von Mises and wrapped Cauchy distributions,
- clear conditions for symmetry/unimodality,
- wide range of indices of skewness,
- efficient algorithm for generating random variates.

The three-parameter symmetric special case

- simple form of the Fisher information matrix,
- reparameterization reducing the asymptotic correlation.

References I

[1] BEST, D.J. & FISHER, N.I. (1979). Efficient simulation of the von Mises distribution. *Appl. Statist.*, **28**, 152–157.

[2] COX, D.R. & REID, N.I. (1987). Parameter orthogonality and approximate conditional inference (with discussion). *J. Roy. Statist. Soc. Ser.* **B 49**, 1–39.

[3] GATTO, R. & JAMMALAMADAKA, S.R. (2007). The generalized von Mises distribution. *Statist. Meth.*, **4**, 341–353.

[4] HILL, G.W. (1977). Algorithm 518. Incomplete Bessel function I_0: the von Mises distribution. *ACM Trans. Math. Soft.*, **3**, 279–284.

[5] JONES, M.C. & PEWSEY, A. (2005). A family of symmetric distributions on the circle. *J. Amer. Statist. Assoc.* **100**, 1422–1428.

References II

[5] McCullagh, P. (1996). Möbius transformation and Cauchy parameter estimation. *Ann. Statist.* **24**, 787–808.

[6] Maksimov, V.M. (1967). Necessary and sufficient statistics for a family of shifts of probability distributions on continuous bicompact groups. *Teor. Veroyatn. Primen.*, **12**, 307–321 (in Russian). English translation: *Theory Probab. Appl.*, **12**, 267–280.

[7] Mardia, K.V. (1972). *Statistics of Directional Data*. Academic Press, New York.

[8] SenGupta, S. & Rao, J.S. (1996). Statistical analysis of crossbedding azimuths from the Kamthi formation around Bheemaran, Pranhati–Godavari valley *Sankhyā* **B 28**, 165–174.