A distribution for a pair of unit vectors generated by Brownian motion

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Introduction

Distribution for a pair of unit vectors

A distribution for a pair of d-dimensional unit vectors is a probability distribution which is defined on two unit spheres in $\mathbb{R}^d$, $S^{d-1} \times S^{d-1}$.

Data recorded as pairs of unit vectors

- Wind directions in Milwaukee at 6 a.m. and noon ($d = 2$).
- Directions of magnetic field in a rock sample before and after laboratory treatment ($d = 3$).

Purpose of the Study

Existing models

- Mardia (1975)
- Rivest (1988)
- Saw (1983)
- Shieh & Johnson (2005)

Purpose of the study

Our goal is to propose a distribution with the following features:

- new approach to generate a model
- easy interpretation of parameters
- mathematical tractability

Definition of the Proposed Model

We take a new approach to obtain a tractable model.

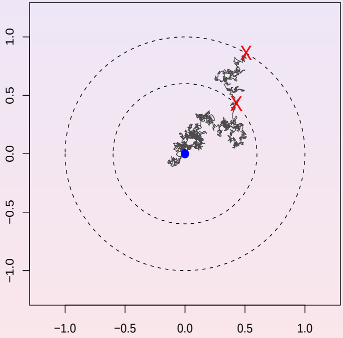


Fig. 1. Brownian motion ($d = 2$).

Definition

$\{B_t\,;\ t \geq 0\}$: $\mathbb{R}^d$-valued Brownian motion,

$B_0 = 0$,

$\tau_1 = \inf\{t \geq 0\,;\ \|B_t\| = \rho,\ \rho \in (0,1)\}$,

$\tau_2 = \inf\{t \geq 0\,;\ \|B_t\| = 1\}$.

The proposed model is defined by

$$\left(Q \frac{B_{\tau_1}}{\|B_{\tau_1}\|}, B_{\tau_2} \right),$$

where $Q \in O(d)$, $d \times d$ orthogonal matrices.

Probability Density Function

For brevity, write $(U, V) = \left(Q \dfrac{B_{\tau_1}}{\| B_{\tau_1} \|}, B_{\tau_2} \right)$.

Probability density function

The density for (U, V) is given by

$$c(u, v) = \frac{1}{A_{d-1}^2} \frac{1-\rho^2}{(1 - 2\rho u' Q v + \rho^2)^{d/2}}, \quad u, v \in S^{d-1}, \tag{1}$$

where $\rho \in [0, 1),\ Q \in O(d),\ S^{d-1} = \{x \in \mathbb{R}^d; \|x\| = 1\}$,
A_{d-1}: a surface area of S^{d-1}, i.e. $A_{d-1} = 2\pi^{d/2}/\Gamma(d/2)$,
u' : a transpose of u.

We write $(U, V) \sim BS_d(\rho Q)$ if r.v. (U, V) has density (1).

Interpretation of ρ

1. When $\rho = 0$, U and V are independent.
2. For any $\varepsilon > 0$, as ρ tends to 1, $P(\|U - QV\| < \varepsilon) \to 1$.

Parameter ρ influences the dependence between U and V.

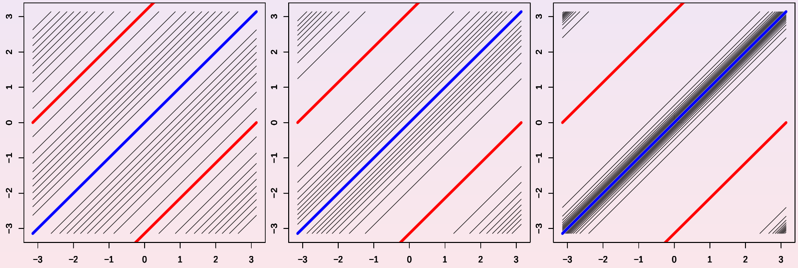


Fig. 2. A contour plot of density (1) (variables in radians) for $d = 2$, $Q = I$ and $\rho = 0.1$ (left), 0.4 (middle), 0.8 (right).

$$c(u,v) \propto \frac{1-\rho^2}{(1-2\rho u'Qv+\rho^2)^{d/2}}, \quad u,v \in S^{d-1};\ \rho \in [0,1),\ Q \in O(d).$$

Mode of density (1)

Density (1) takes maximum (minimum) values at $u = Qv$ $(u = -Qv)$.

Interpretation of Q

Orthogonal transformation Q consists of rotation and reflection.

For $d = 2$, the transformation can be expressed as

$$x \longmapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} x \quad \text{and} \quad x \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} x.$$

Marginals and Conditionals

Marginals of U and V

Suppose $(U, V) \sim BS_d(\rho Q)$. Then

$$U \sim \text{angular uniform}, \quad V \sim \text{angular uniform}.$$

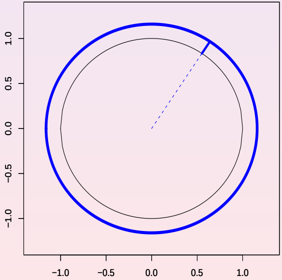


Fig. 3. Angular uniform ($d = 2$).

Angular uniform distribution

Angular uniform distribution is defined by density

$$f(x) = \frac{1}{A_{d-1}}, \quad x \in S^{d-1}.$$

Conditionals of $V|u$ and $U|v$

Suppose $(U, V) \sim BS_d(\rho Q)$. Then

$$U|v \sim \mathrm{Exit}_d(\rho Q v), \quad V|u \sim \mathrm{Exit}_d(\rho Q' u),$$

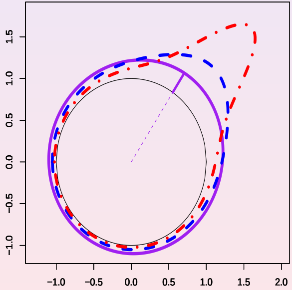
where

$\mathrm{Exit}_d(\cdot)$ denotes the exit distribution for d-dimensional sphere.

Exit distribution

Exit distribution for d-dimensional sphere, $\text{Exit}_d(\theta)$, is of the form

$$f(x) = \frac{1}{A_{d-1}} \frac{1 - \|\theta\|^2}{\|x - \theta\|^d}, \quad x \in S^{d-1}; \; \theta \in \{\eta \in \mathbb{R}^d ; \; \|\eta\| < 1\}.$$



Properties

- unimodality
- rotational symmetry about $x = \theta/\|\theta\|$
- mode at $x = \theta/\|\theta\|$;
  antimode at $x = -\theta/\|\theta\|$
- moment of X: $E(X) = \theta$

Fig. 4. Density of exit distribution for the circle ($d = 2$) with
$\theta = (0.2, 0.2)'$, $\theta = (0.4, 0.4)'$, $\theta = (0.55, 0.55)'$.

Conditionals of $V|u$ and $U|v$

Suppose $(U, V) \sim BS_d(\rho Q)$. Then

$$U|v \sim \mathrm{Exit}_d(\rho Q v), \quad V|u \sim \mathrm{Exit}_d(\rho Q' u),$$

where

$\mathrm{Exit}_d(\cdot)$ denotes the exit distribution for d-dimensional sphere.

Moments and Correlation Coefficient

Moments and correlation coefficient

Assume $(U, V) \sim BS_d(\rho Q)$. Then

$$E(U) = E(V) = 0, \quad E(UU') = E(VV') = d^{-1}I,$$

$$E(UV') = d^{-1}\rho Q.$$

Johnson & Wehrly (1977) coefficient of correlation, ρ_{JW}, is thus

$$\rho_{JW} \equiv \lambda^{1/2} = \rho,$$

where λ: the largest eigenvalue of $\Sigma_{UU}^{-1}\Sigma_{UV}\Sigma_{VV}^{-1}\Sigma_{UV}'$,
$\Sigma_{UU} = E(UU') - E(U)E(U')$, $\Sigma_{UV} = E(UV') - E(U)E(V')$,
$\Sigma_{VV} = E(VV') - E(V)E(V')$.

Parameter Estimation

Method of moments estimation

$$(U_j, V_j) \sim \; i.i.d. \; BS_d(\rho I), \quad j = 1, \ldots, n.$$

The method of moments estimator is obtained by equating

$$\begin{aligned} \text{theoretical moment} &= \text{sample moment}. \\ E(UV') &= \frac{1}{n} \sum_j U_j V_j'. \end{aligned}$$

Thus we get

$$\hat{\rho} = d \left| \det \left(\frac{1}{n} \sum_j U_j V_j' \right) \right|^{1/d}.$$

Maximum likelihood estimation

$(U_j, V_j) \sim$ *i.i.d.* $BS_d(\rho I), \quad j = 1, \ldots, n.$

The derivative of log-likelihood function with respect to ρ is

$$\frac{\partial}{\partial \rho} \log L = \frac{-2n\rho}{1-\rho^2} + d \sum_j \frac{x_j - \rho}{1 - 2\rho x_j + \rho^2},$$

where $x_j = u_j' v_j \in [-1, 1]$.

From this expression, we find that maximum likelihood estimation for $BS_d(\rho I)$ is essentially the same as that for Leipnik's (1947) distribution.

Pivotal Statistic

Pivotal statistic for (ρ, Q)

Suppose $(U, V) \sim BS_d(\rho Q)$. Define a random variable

$$T(\rho, Q) = \frac{U'QV - \rho}{1 - 2\rho U'QV + \rho^2},$$

Clearly, $0 < T(\rho, Q) < 1$ a.s. The r th moment of $T(\rho, Q)$ is given by

$$E\left\{T(\rho, Q)^r\right\} = \frac{B\{r + \frac{1}{2}(d-1), \frac{1}{2}\}}{B\{\frac{1}{2}(d-1), \frac{1}{2}\}},$$

where $B(\cdot, \cdot)$ is a beta function.

Since these moments are equal to those of a beta distribution $\mathrm{Beta}\{\frac{1}{2}(d-1), \frac{1}{2}\}$, it follows that T is a pivotal statistic for (ρ, Q) having $\mathrm{Beta}\{\frac{1}{2}(d-1), \frac{1}{2}\}$.

Bivariate Circular Case

This subsection focuses on the bivariate circular case ($d = 2$) of the proposed model.

Probability density function

Let $(U, V) \sim BS_2(\rho Q)$. The density for (U, V) is

$$c(u, v) = \frac{1}{4\pi^2} \frac{1 - \rho^2}{1 - 2\rho\, u' Q v + \rho^2}, \quad u, v \in S^1,$$

where

$\rho \in [0, 1),\ Q \in O(2),\ S^1 = \{x \in \mathbb{R}^2; \|x\| = 1\}.$

Transforming Random Vector and Parameters

To investigate further properties of model $BS_2(\rho Q)$, it is advantageous to transform the random vector and parameters as follows.

Transformation

Let $(U, V) \sim BS_2(\rho Q)$.

We transform random variables and parameters by putting

$$(Z_U, Z_V) = (U_1 + iU_2, V_1 + iV_2), \ \psi = \rho \exp\{i \arg(Q_{11} + iQ_{21})\},$$

where

$$U = (U_1, U_2)', \ V = (V_1, V_2)', \ Q_{ij} : (i, j) \text{ entry of } Q.$$

Then it is clear that $|\psi| < 1, \ Z_U, Z_V \in \Omega, \ \Omega = \{z \in \mathbb{C}; |z| = 1\}$.

Probability density function

Density of (Z_U, Z_V) is given by

$$c(z_u, z_v) = \frac{1}{4\pi^2} \frac{1 - |\psi|^2}{\left|1 - \psi z_v z_u^{-\det Q}\right|^2}, \quad z_u, z_v \in \Omega,$$

where

$$|\psi| < 1 \quad \text{and} \quad \Omega = \{z \in \mathbb{C}; |z| = 1\}.$$

For $\det Q = 1$, we write $(Z_U, Z_V) \sim BC_+(\psi)$.

For $\det Q = -1$, write $(Z_U, Z_V) \sim BC_-(\psi)$.

Properties of the Bivariate Circular Model

Multiplicative property

$$(Z_{U1}, Z_{V1}) \sim BC_+(\psi_1) \perp (Z_{U2}, Z_{V2}) \sim BC_+(\psi_2)$$

$$\Longrightarrow (Z_{U1}Z_{U2}, Z_{V1}Z_{V2}) \sim BC_+(\psi_1\psi_2).$$

Infinite divisibility

Model $BC_+(\psi)$ is infinitely divisible with respect to multiplication.

Proof: Assume $(Z_U, Z_V) \sim BC_+(\psi)$. Then for any positive integer n, the assumption $(Z_{Uj}, Z_{Vj}) \sim i.i.d.\ BC_+(\sqrt[n]{\psi}),\ (j = 1, \ldots, n)$ gives

$$\left(\prod_j Z_{Uj}, \prod_j Z_{Vj}\right) \stackrel{d}{=} (Z_U, Z_V).$$

Parameter Estimation

Trigonometric moment (t.m.)

$$(Z_U, Z_V) \sim BC_+(\psi) \implies E\left[{Z_U}^j {Z_V}^k\right] = \begin{cases} \psi^j, & j = -k, \\ 0, & otherwise. \end{cases}$$

Method of moments estimation

$(Z_{Uj}, Z_{Vj}) \sim i.i.d.\ BC_+(\psi) \quad (j = 1, \ldots, n).$

The method of moments estimator (MME) based on t.m. is obtained by equating

$$\text{theoretical t.m.} = \text{sample t.m.}$$

Thus we get

$$\hat{\psi} = \frac{1}{n} \sum_j Z_{Uj} \overline{Z_{Vj}}.$$

Maximum likelihood estimation

$(Z_{Uj}, Z_{Vj}) \sim$ *i.i.d.* $BC_+(\psi) \quad (j = 1, \ldots, n)$.

- For $n = 1$, MLE coincides with MME, i.e. $\hat{\psi} = Z_{U1}\overline{Z_{V1}}$.
- For $n \geq 2$, likelihood function can be expressed as

$$L \propto \prod_j \frac{1 - |\psi|^2}{\left| z_{uj}\overline{z_{vj}} - \psi \right|^2}.$$

Then maximum likelihood estimation for $BC_+(\psi)$ is essentially the same as that for wrapped Cauchy distribution.

Therefore we can get MLE by applying the algorithm by Kent & Tyler (1988).

A Related Distribution on $\mathbb{R}^2$

Bilinear fractional transformation of model $BC_{-}(\psi)$

Let $(Z_U, Z_V) \sim BC_{-}(\psi)$. Define a random vector (X, Y) as

$$X = i\frac{1 - Z_U}{1 + Z_U} \quad \text{and} \quad Y = i\frac{1 - Z_V}{1 + Z_V}.$$

Then $(X, Y) \in \mathbb{R}^2$. The density for (X, Y) is

$$f(x, y) = \frac{1}{\pi^2} \frac{\mathrm{Im}(\theta)}{|x + y + \theta(1 - xy)|^2}, \quad x, y \in \mathbb{R}, \tag{2}$$

where $\theta = i(1 - \psi)/(1 + \psi)$. Clearly, $\mathrm{Im}(\theta) > 0$.

Properties of model (2)

Model (2) has the following properties:

$$X \sim C(i), \quad Y \sim C(i),$$
$$X|y \sim C\left(\frac{\theta + y}{1 - \theta y}\right), \quad Y|x \sim C\left(\frac{\theta + x}{1 - \theta x}\right),$$

where $C(\phi)$ is a Cauchy distribution on $\mathbb{R}$ with median $\mathrm{Re}(\phi)$ and scale parameter $\mathrm{Im}(\phi)$.

Further properties of model (2) are obtainable by the inverse transformation $Z_U = (1 + iX)/(1 - iX)$ and $Z_V = (1 + iY)/(1 - iY)$.

Conclusion

Properties of the proposed model $BS_d(\rho Q)$

- Easy interpretation of parameters.
- Angular uniform marginals.
- Exit conditionals.
- Simple expression of moments and correlation coefficient.
- Pivotal statistic having a beta distribution.

Bivariate circular case ($d = 2$)

- Multiplicative property and infinite divisibility.
- Easy parameter estimation.

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