A distribution for a pair of unit vectors generated by Brownian motion

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March 15, 2007
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A distribution for a pair of unit vectors

A distribution for a pair of \(d\)-dimensional unit vectors is a probability distribution which is defined on two unit spheres in \(\mathbb{R}^d, \mathbb{S}^{d-1} \times \mathbb{S}^{d-1}\).

Data recorded as pairs of unit vectors

- Wind directions in Milwaukee at 6 a.m. and noon (\(d = 2\)).
- Directions of magnetic field in a rock sample before and after laboratory treatment (\(d = 3\)).
Purpose of the Study

Existing models

- Mardia (1975)
- Rivest (1988)
- Saw (1983)
- Shieh & Johnson (2005)

Purpose of the study

Our goal is to propose a distribution with the following features:

- new approach to generate a model
- easy interpretation of parameters
- mathematical tractability
Definition of the Proposed Model

We take a new approach to obtain a tractable model.

Definition

\[ \{ B_t \; ; \; t \geq 0 \} : \mathbb{R}^d \text{-valued Brownian motion,} \]
\[ B_0 = 0, \]
\[ \tau_1 = \inf \{ t \geq 0 \; ; \; \| B_t \| = \rho, \; \rho \in (0, 1) \}, \]
\[ \tau_2 = \inf \{ t \geq 0 \; ; \; \| B_t \| = 1 \}. \]

The proposed model is defined by

\[ \left( Q \frac{B_{\tau_1}}{\| B_{\tau_1} \|}, B_{\tau_2} \right), \]

where \( Q \in O(d), \; d \times d \) orthogonal matrices.
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Probability Density Function

For brevity, write $(U, V) = \left( Q \frac{B_{\tau_1}}{\|B_{\tau_1}\|}, B_{\tau_2} \right)$.

Probability density function

The density for $(U, V)$ is given by

$$c(u, v) = \frac{1}{A_{d-1}^2} \frac{1 - \rho^2}{(1 - 2\rho u'Qv + \rho^2)^{d/2}}, \quad u, v \in S^{d-1},$$

(1)

where $\rho \in [0, 1)$, $Q \in O(d)$, $S^{d-1} = \{x \in \mathbb{R}^d; \|x\| = 1\}$,

$A_{d-1}$: a surface area of $S^{d-1}$, i.e. $A_{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)}$,

$u'$: a transpose of $u$.

We write $(U, V) \sim BS_d(\rho Q)$ if r.v. $(U, V)$ has density (1).
**Interpretation of \( \rho \)**

1. When \( \rho = 0 \), \( U \) and \( V \) are independent.
2. For any \( \varepsilon > 0 \), as \( \rho \) tends to 1, \( P(\|U - QV\| < \varepsilon) \to 1 \).

Parameter \( \rho \) influences the dependence between \( U \) and \( V \).

**Fig. 2.** A contour plot of density (1) (variables in radians) for \( d = 2 \), \( Q = I \) and \( \rho = 0.1 \) (left), 0.4 (middle), 0.8 (right).
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$c(u, v) \propto \frac{1 - \rho^2}{(1 - 2\rho u'Qv + \rho^2)^{d/2}}, \quad u, v \in S^{d-1}; \quad \rho \in [0, 1), \quad Q \in O(d).$

Mode of density (1)

Density (1) takes maximum (minimum) values at $u = Qv$ ($u = -Qv$).

Interpretation of $Q$

Orthogonal transformation $Q$ consists of rotation and reflection.

For $d = 2$, the transformation can be expressed as

$$x \mapsto \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} x \quad \text{and} \quad x \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} x.$$
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Marginals and Conditionals

Marginals of $U$ and $V$

Suppose $(U, V) \sim BS_d(\rho Q)$. Then

$$U \sim \text{angular uniform}, \quad V \sim \text{angular uniform}.$$  

Angular uniform distribution

Angular uniform distribution is defined by density

$$f(x) = \frac{1}{A_{d-1}}, \quad x \in S^{d-1}.$$  

Fig. 3. Angular uniform ($d = 2$).
Conditionals of $V|u$ and $U|v$

Suppose $(U, V) \sim BS_d(\rho Q)$. Then

$$U|v \sim \text{Exit}_d(\rho Qv), \quad V|u \sim \text{Exit}_d(\rho Q'u),$$

where

$\text{Exit}_d(\cdot)$ denotes the exit distribution for $d$-dimensional sphere.
Exit distribution

Exit distribution for \( d \)-dimensional sphere, \( \text{Exit}_d(\theta) \), is of the form

\[
f(x) = \frac{1}{A_{d-1} \|x - \theta\|^{d}}, \quad x \in S^{d-1}; \quad \theta \in \{\eta \in \mathbb{R}^d; \|\eta\| < 1\}.
\]

Properties

- unimodality
- rotational symmetry about \( x = \theta/\|\theta\| \)
- mode at \( x = \theta/\|\theta\| \);
- antimode at \( x = -\theta/\|\theta\| \)
- moment of \( X: E(X) = \theta \)

Fig. 4. Density of exit distribution for the circle \( (d = 2) \) with
\[
\theta = (0.2, 0.2)', \quad \theta = (0.4, 0.4)', \quad \theta = (0.55, 0.55)'.
\]
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Conditionals of $V|u$ and $U|v$

Suppose $(U, V) \sim BS_d(\rho Q)$. Then

$$U|v \sim \text{Exit}_d(\rho Qv), \quad V|u \sim \text{Exit}_d(\rho Q'u),$$

where

$\text{Exit}_d(\cdot)$ denotes the exit distribution for $d$-dimensional sphere.
Moments and Correlation Coefficient

Assume \((U, V) \sim BS_d(\rho Q)\). Then

\[
E(U) = E(V) = 0, \quad E(UU') = E(VV') = d^{-1} I,
\]

\[
E(UV') = d^{-1} \rho Q.
\]

Johnson & Wehrly (1977) coefficient of correlation, \(\rho_{JW}\), is thus

\[
\rho_{JW} \equiv \lambda^{1/2} = \rho,
\]

where \(\lambda\): the largest eigenvalue of \(\Sigma_{UU}^{-1} \Sigma_{UV} \Sigma_{VV}^{-1} \Sigma'_{UV}\),

\[
\Sigma_{UU} = E(UU') - E(U)E(U'), \quad \Sigma_{UV} = E(UV') - E(U)E(V'), \\
\Sigma_{VV} = E(VV') - E(V)E(V').
\]
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Parameter Estimation

Method of moments estimation

$(U_j, V_j) \sim i.i.d. \; BS_d(\rho I), \; j = 1, \ldots, n.$

The method of moments estimator is obtained by equating

theoretical moment $= \; \text{sample moment}.$

$$E(\bar{U}V') = \frac{1}{n} \sum_j U_j V'_j.$$

Thus we get

$$\hat{\rho} = d \left| \det \left( \frac{1}{n} \sum_j U_j V'_j \right) \right|^{1/d}.$$
(U_j, V_j) \sim i.i.d. BS_d(\rho I), \quad j = 1, \ldots, n.

The derivative of log-likelihood function with respect to \( \rho \) is

\[
\frac{\partial}{\partial \rho} \log L = \frac{-2n\rho}{1-\rho^2} + d \sum_j \frac{x_j - \rho}{1 - 2\rho x_j + \rho^2},
\]

where \( x_j = u_j'v_j \in [-1, 1] \).

From this expression, we find that maximum likelihood estimation for \( BS_d(\rho I) \) is essentially the same as that for Leipnik's (1947) distribution.
Pivotal Statistic

Pivotal statistic for $(\rho, Q)$

Suppose $(U, V) \sim BS_d(\rho Q)$. Define a random variable

$$T(\rho, Q) = \frac{U'QV - \rho}{1 - 2\rho U'QV + \rho^2},$$

Clearly, $0 < T(\rho, Q) < 1$ a.s. The $r$ th moment of $T(\rho, Q)$ is given by

$$E \left\{ T(\rho, Q)^r \right\} = \frac{B\{r + \frac{1}{2}(d - 1), \frac{1}{2}\}}{B\{\frac{1}{2}(d - 1), \frac{1}{2}\}},$$

where $B(\cdot, \cdot)$ is a beta function.

Since these moments are equal to those of a beta distribution $Beta\{\frac{1}{2}(d - 1), \frac{1}{2}\}$, it follows that $T$ is a pivotal statistic for $(\rho, Q)$ having $Beta\{\frac{1}{2}(d - 1), \frac{1}{2}\}$. 
Bivariate Circular Case

This subsection focuses on the bivariate circular case \((d = 2)\) of the proposed model.

**Probability density function**

Let \((U, V) \sim BS_2(\rho Q)\). The density for \((U, V)\) is

\[
c(u, v) = \frac{1}{4\pi^2} \frac{1 - \rho^2}{1 - 2\rho \ u' Q v + \rho^2}, \quad u, v \in S^1,
\]

where

\[
\rho \in [0, 1), \ Q \in O(2), \ S^1 = \{x \in \mathbb{R}^2; \|x\| = 1\}.
\]
Transforming Random Vector and Parameters

To investigate further properties of model $BS_2(\rho Q)$, it is advantageous to transform the random vector and parameters as follows.

**Transformation**

Let $(U, V) \sim BS_2(\rho Q)$. We transform random variables and parameters by putting

$$(Z_U, Z_V) = (U_1 + iU_2, V_1 + iV_2), \quad \psi = \rho \exp\{i \arg(Q_{11} + iQ_{21})\},$$

where

$$U = (U_1, U_2)', \quad V = (V_1, V_2)', \quad Q_{ij} : (i, j) \text{ entry of } Q.$$ 

Then it is clear that $|\psi| < 1$, $Z_U, Z_V \in \Omega$, $\Omega = \{z \in \mathbb{C}; |z| = 1\}$. 

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Probability density function

Density of $(Z_U, Z_V)$ is given by

$$c(Z_u, Z_v) = \frac{1}{4\pi^2} \frac{1 - |\psi|^2}{|1 - \psi Z_v Z_u^{-1} \det Q|^2}, \quad Z_u, Z_v \in \Omega,$$

where

$|\psi| < 1 \quad \text{and} \quad \Omega = \{z \in \mathbb{C}; |z| = 1\}.$

For $\det Q = 1$, we write $(Z_U, Z_V) \sim BC_+(\psi)$.

For $\det Q = -1$, write $(Z_U, Z_V) \sim BC_-(\psi)$. 
Properties of the Bivariate Circular Model

### Multiplicative property

\[(Z_{U1}, Z_{V1}) \sim BC_+(\psi_1) \perp (Z_{U2}, Z_{V2}) \sim BC_+(\psi_2)\]

\[\implies (Z_{U1}Z_{U2}, Z_{V1}Z_{V2}) \sim BC_+(\psi_1\psi_2).\]

### Infinite divisibility

Model \(BC_+(\psi)\) is **infinitely divisible** with respect to multiplication.

**Proof**: Assume \((Z_U, Z_V) \sim BC_+(\psi)\). Then for any positive integer \(n\), the assumption \((Z_{Uj}, Z_{Vj}) \sim i.i.d. BC_+\left(n\sqrt{\psi}\right), (j = 1, \ldots, n)\) gives

\[
\left( \prod_j Z_{Uj}, \prod_j Z_{Vj} \right) \overset{d}{=} (Z_U, Z_V).
\]
Parameter Estimation

Trigonometric moment (t.m.)

\[(Z_U, Z_V) \sim BC_+(\psi) \implies E \left[ Z_U^j Z_V^k \right] = \begin{cases} \psi^j, & j = -k, \\ 0, & otherwise. \end{cases} \]

Method of moments estimation

\[(Z_{Uj}, Z_{Vj}) \sim i.i.d. \ BC_+(\psi) \quad (j = 1, \ldots, n). \]

The method of moments estimator (MME) based on t.m. is obtained by equating

\[ \text{theoretical t.m.} = \text{sample t.m.} \]

Thus we get

\[ \hat{\psi} = \frac{1}{n} \sum_{j} Z_{Uj} \overline{Z_{Vj}}. \]
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Maximum likelihood estimation

$(Z_{Uj}, Z_{Vj}) \sim i.i.d. \ BC_+(\psi) \ (j = 1, \ldots, n).$

- For $n = 1$, MLE coincides with MME, i.e. $\hat{\psi} = Z_{U1} \overline{Z_{V1}}.$
- For $n \geq 2$, likelihood function can be expressed as

$$L \propto \prod_j \frac{1 - |\psi|^2}{|Z_{Uj} \overline{Z_{Vj}} - \psi|^2}.$$ 

Then maximum likelihood estimation for $BC_+(\psi)$ is essentially the same as that for wrapped Cauchy distribution.

Therefore we can get MLE by applying the algorithm by Kent & Tyler (1988).
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Bilinear fractional transformation of model $BC_-(\psi)$

Let $(Z_U, Z_V) \sim BC_-(\psi)$. Define a random vector $(X, Y)$ as

$$X = i \frac{1 - Z_U}{1 + Z_U} \quad \text{and} \quad Y = i \frac{1 - Z_V}{1 + Z_V}.$$  

Then $(X, Y) \in \mathbb{R}^2$. The density for $(X, Y)$ is

$$f(x, y) = \frac{1}{\pi^2} \frac{\text{Im}(\theta)}{|x + y + \theta(1 - xy)|^2}, \quad x, y \in \mathbb{R}, \quad (2)$$

where $\theta = i(1 - \psi)/(1 + \psi)$. Clearly, $\text{Im}(\theta) > 0$. 

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Properties of model (2)

Model (2) has the following properties:

\[ X \sim C(i), \quad Y \sim C(i), \]
\[ X|y \sim C\left(\frac{\theta + y}{1 - \theta y}\right), \quad Y|x \sim C\left(\frac{\theta + x}{1 - \theta x}\right), \]

where \( C(\phi) \) is a Cauchy distribution on \( \mathbb{R} \) with median \( \text{Re}(\phi) \) and scale parameter \( \text{Im}(\phi) \).

Further properties of model (2) are obtainable by the inverse transformation \( Z_U = (1 + iX)/(1 - iX) \) and \( Z_V = (1 + iY)/(1 - iY) \).
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Properties of the proposed model $BS_d(\rho Q)$

- Easy interpretation of parameters.
- Angular uniform marginals.
- Exit conditionals.
- Simple expression of moments and correlation coefficient.
- Pivotal statistic having a beta distribution.

Bivariate circular case ($d = 2$)

- Multiplicative property and infinite divisibility.
- Easy parameter estimation.
References


