Some Reflections on "Symmetry" and its Role in the Analysis of Circular Data

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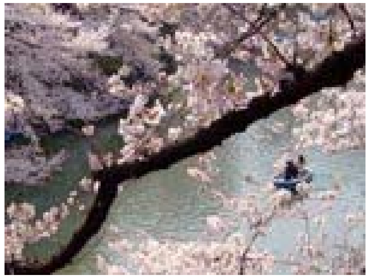
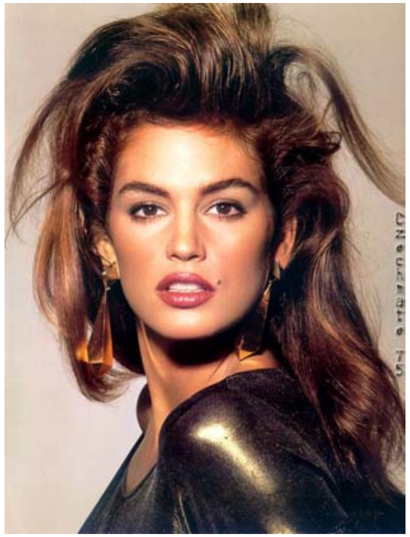
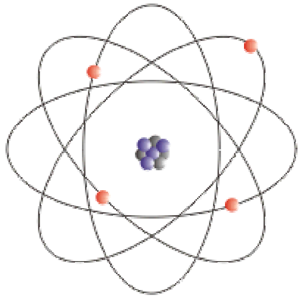
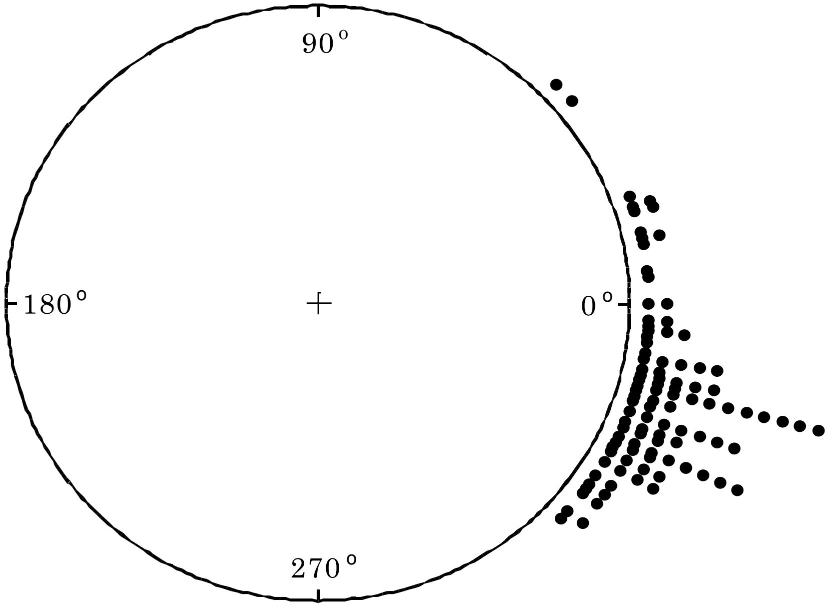
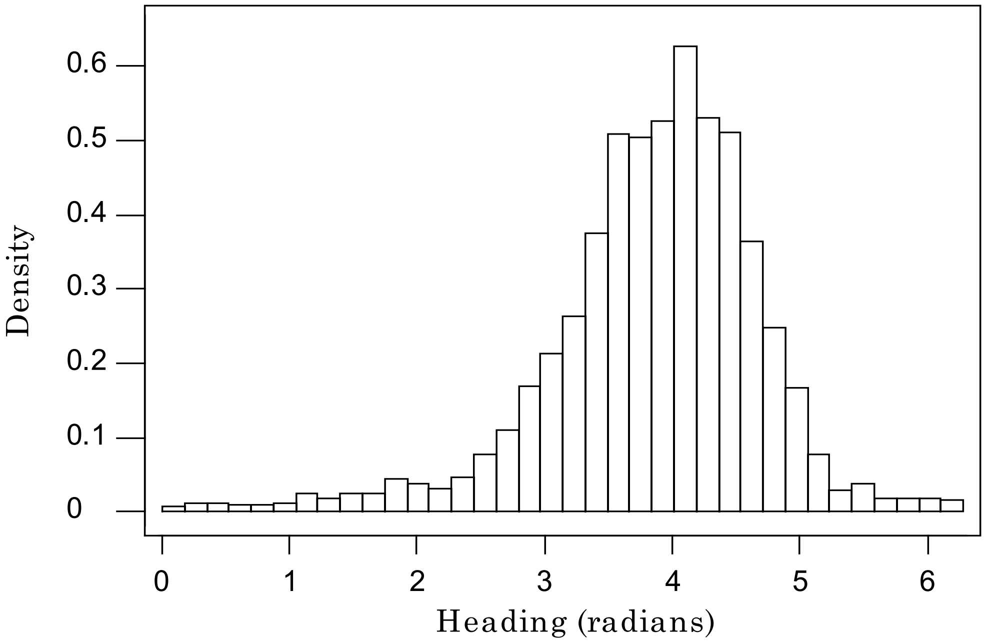


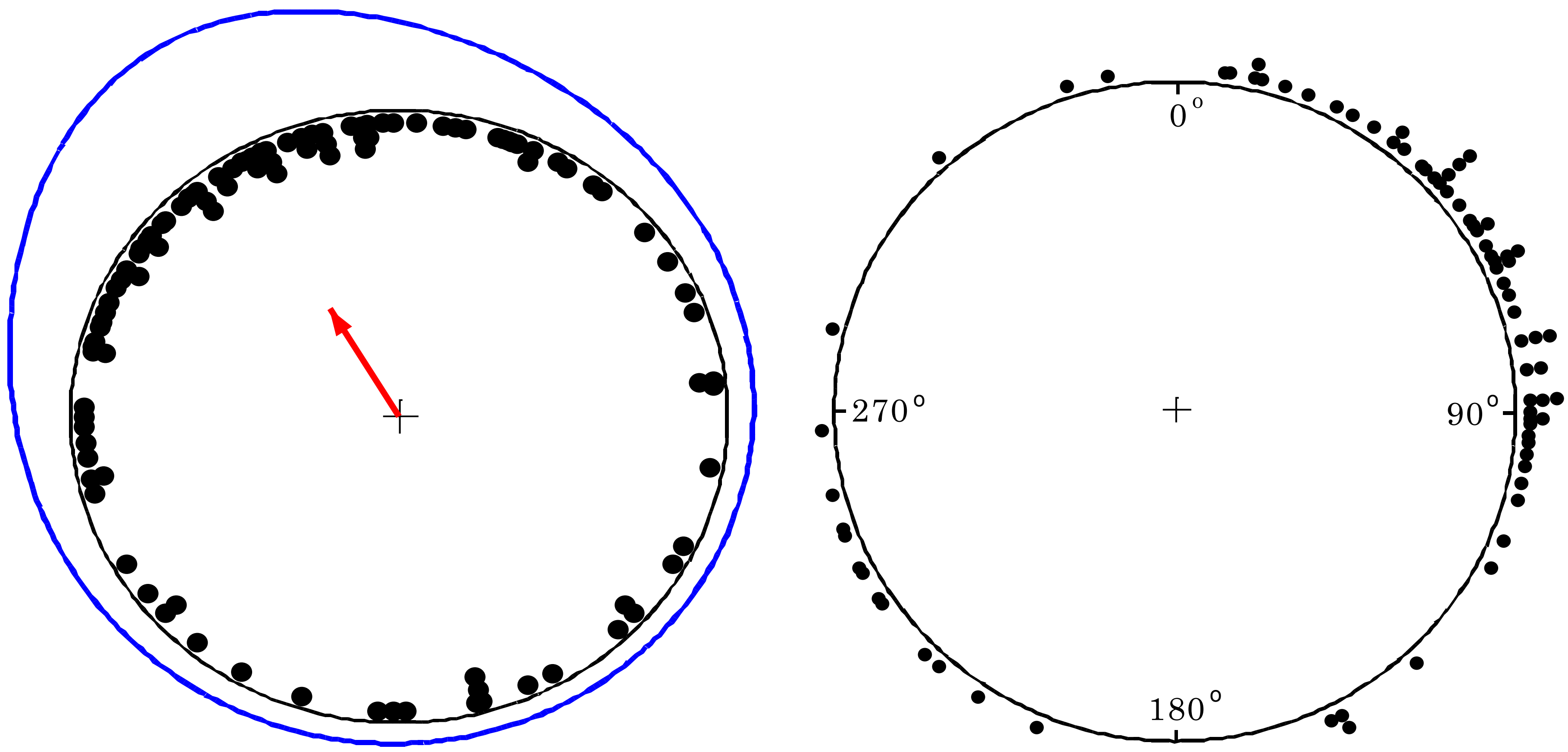

1. Circular Data

1.1 Asymmetric



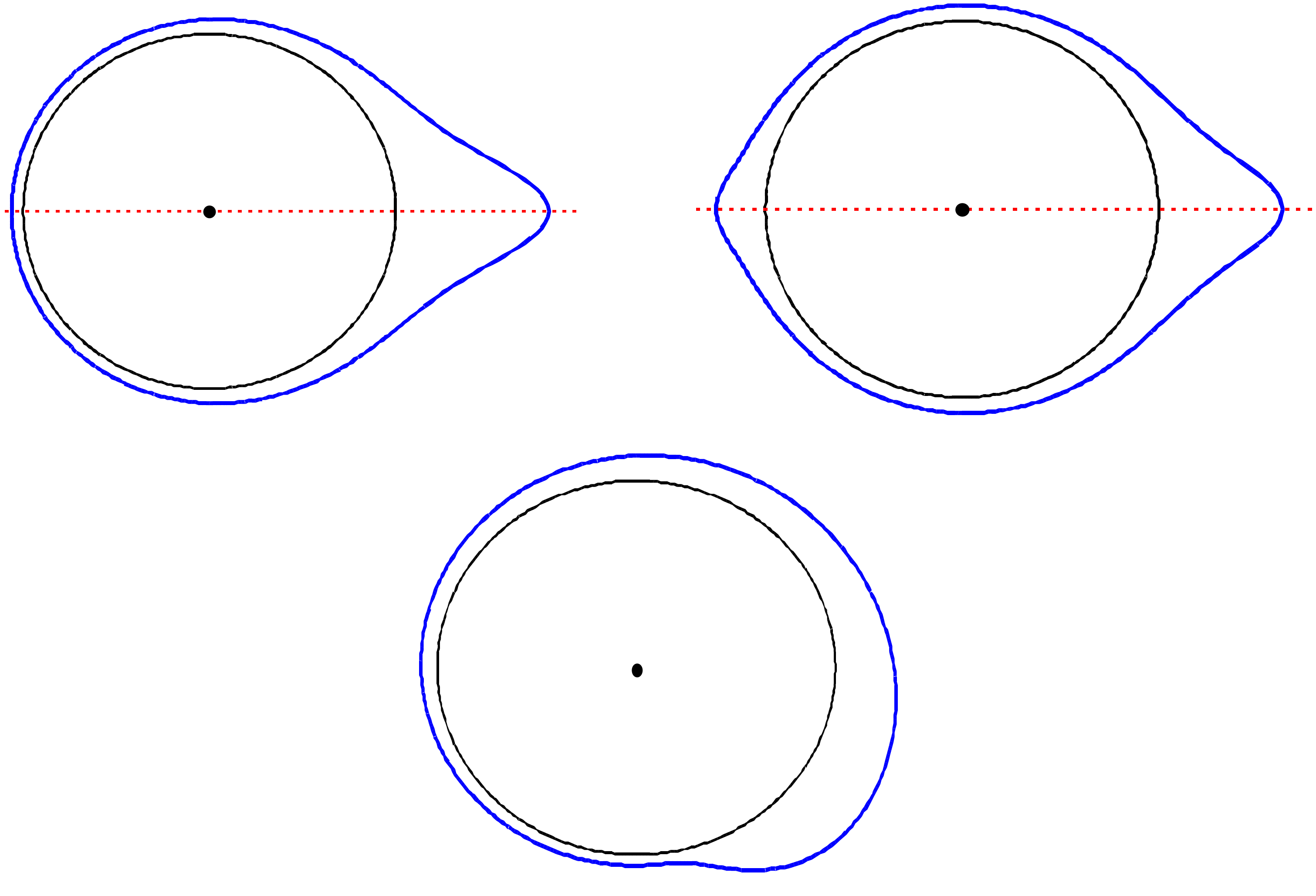


1.2 “Symmetric”

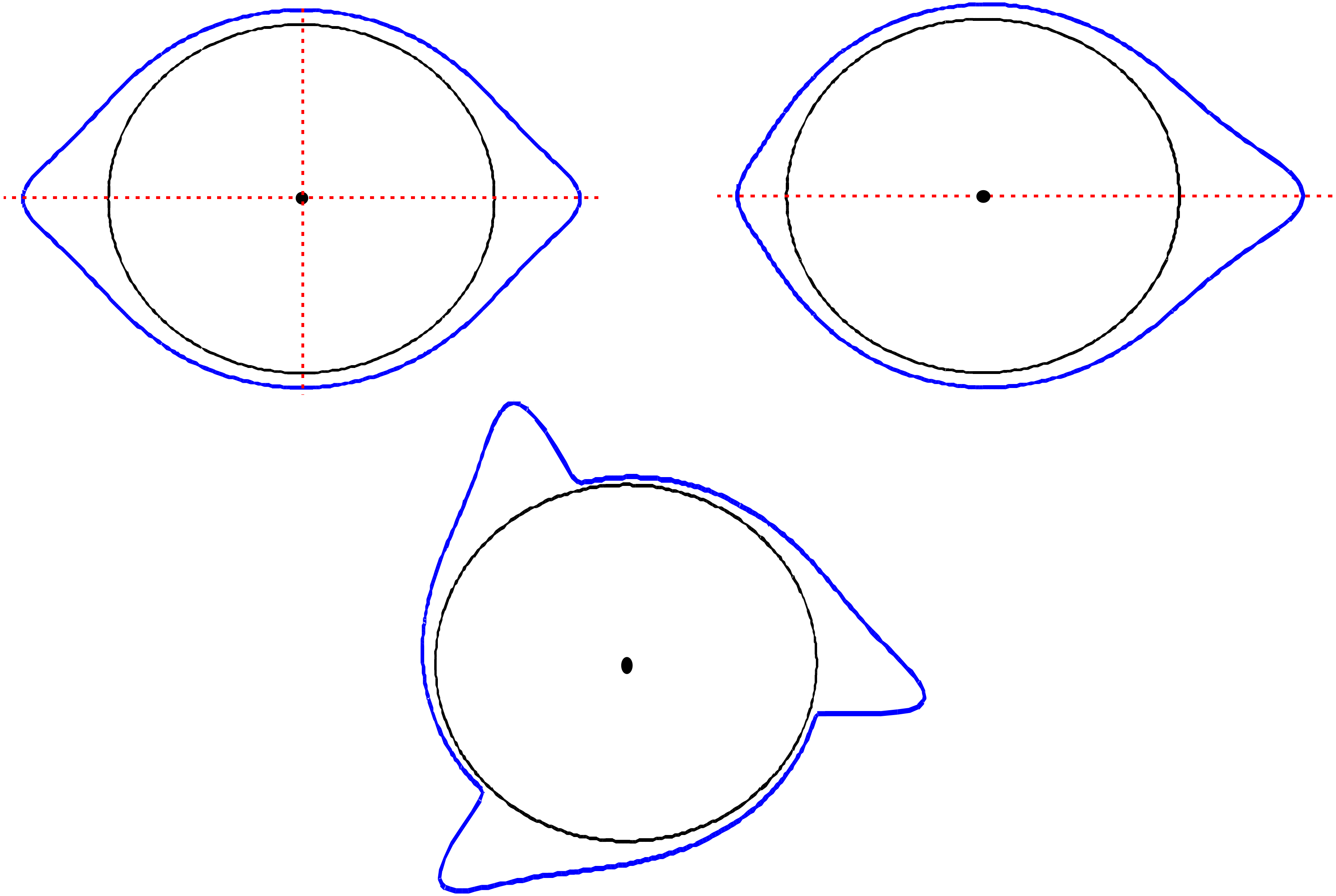


2. Two Forms of Symmetry

2.1 Reflective Symmetry $f(\theta) = f(2\phi - \theta)$



2.2 *l*-fold Symmetry $f(\theta) = f(\theta + 2\pi/l)$



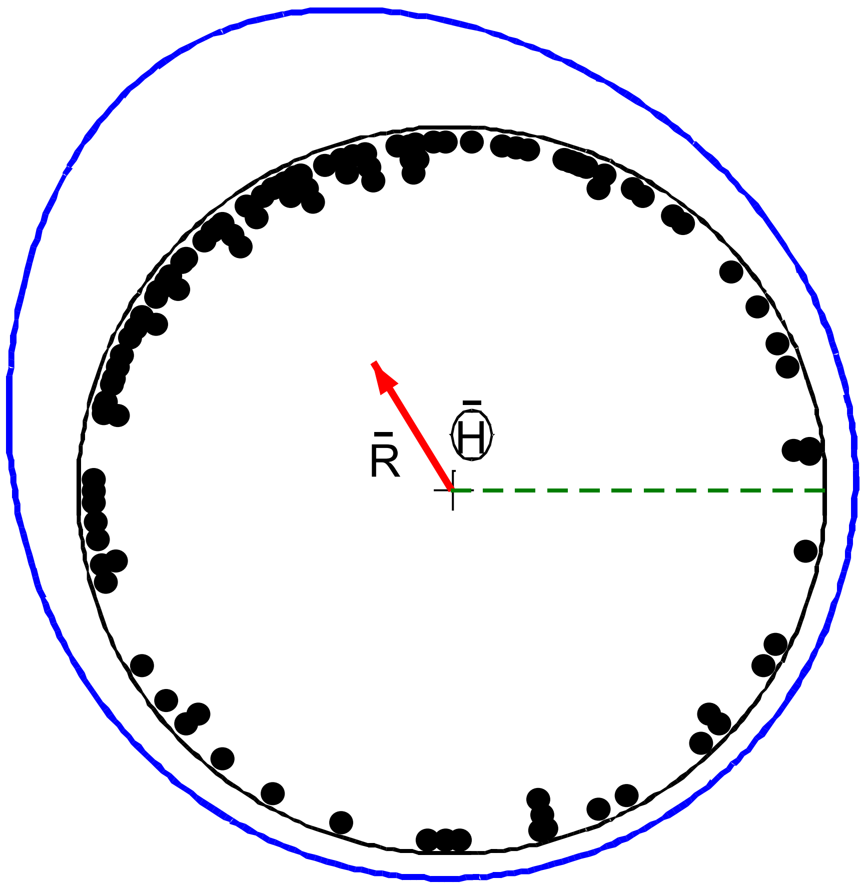
Structure of *l*-fold densities is, in general, highly restrictive (although 2-fold symmetry is appropriate for axial data).

Prefer to model multimodality using finite mixture distributions which are far more flexible.

l-fold symmetry can be tested for using any test of uniformity (or isotropy) once the data have been converted into “uniform scores”. Jupp & Spurr (1983)

3. Circular Statistics

$\theta_1,\ldots,\theta_n$ - random sample of n angular observations



"concentration"

Sample mean resultant length

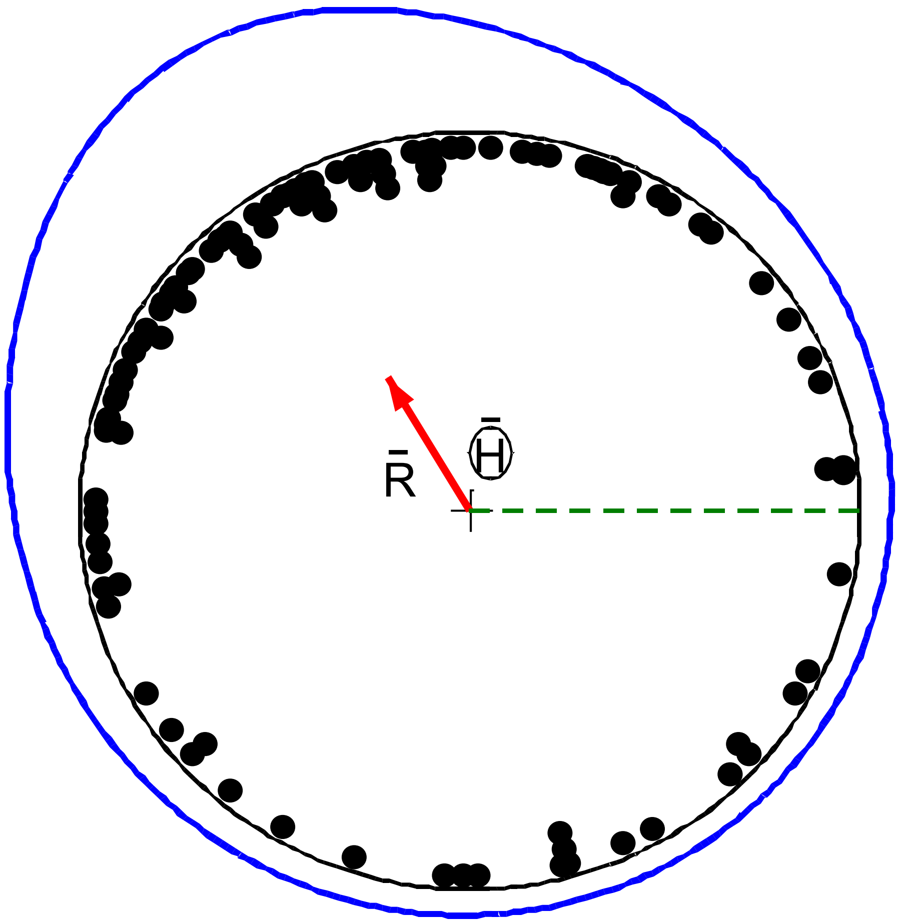
$$\bar{R} = \sqrt{a_1^2 + b_1^2} \in [0,1], \quad \text{where}$$

$$a_p = \frac{1}{n}\sum_{i=1}^{n}\cos\left(p\theta_i\right) \text{ and}$$

$$b_p = \frac{1}{n}\sum_{i=1}^{n}\sin\left(p\theta_i\right) \text{ (both } \in [-1,1]\text{)}$$

are the pth order (p = 1,2,...) trigonometric moments about the zero direction.

location



If $\overline{R} = 0$, sample mean direction, $\bar{\theta}$, is undefined.

If $\overline{R} > 0$,

$$\bar{\theta} = \begin{cases} \tan^{-1}\left(b_1 / a_1\right) & \text{if } a_1 \geq 0 \\ \pi + \tan^{-1}\left(b_1 / a_1\right) & \text{if } a_1 < 0' \end{cases}$$

where $\tan^{-1}(x) \in [-\pi/2, \pi/2]$.

The sample trigonometric moments about $\bar{\theta}$ are:

$$\bar{a}_p = \frac{1}{n}\sum_{i=1}^{n}\cos\{p(\theta_i - \bar{\theta})\}$$ and

$$\bar{b}_p = \frac{1}{n}\sum_{i=1}^{n}\sin\{p(\theta_i - \bar{\theta})\}$$

(all $\in [-1,1]$)

skewness

$$\bar{b}_2 = \frac{1}{n}\sum_{i=1}^{n}\sin\{2(\theta_i - \bar{\theta})\}$$

kurtosis

$$\bar{a}_2 = \frac{1}{n}\sum_{i=1}^{n}\cos\{2(\theta_i - \bar{\theta})\}$$

4. Population Measures

The statistics $\bar{R}$, $\bar{\theta}$, $\bar{a}_p$ and $\bar{b}_p$ are the sample analogues, and moment estimates, of the population measures:

Mean resultant length

$\rho = \sqrt{\alpha_1^2 + \beta_1^2} \in [0,1]$, where

$\alpha_p = E\{\cos(p\Theta)\}$, $\beta_p = E\{\sin(p\Theta)\}$

and Θ is a random angle.

Mean direction

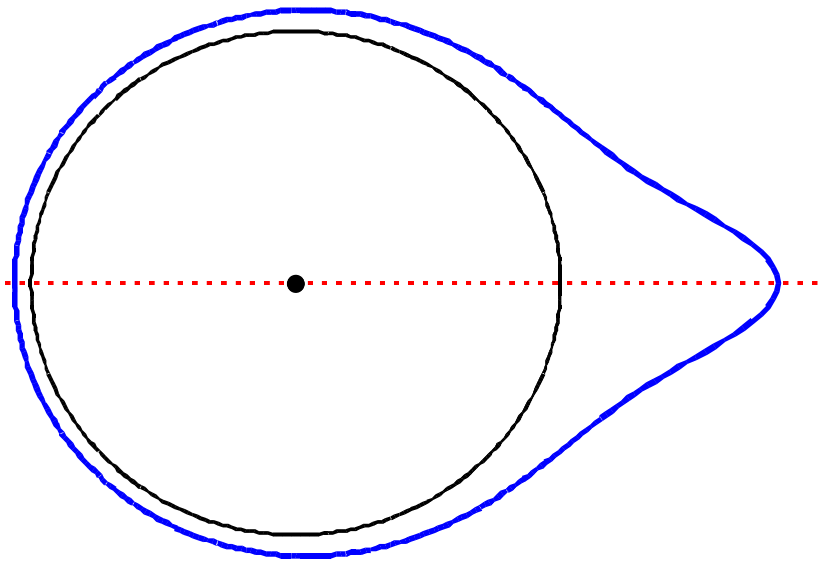
$$\mu_0 = \begin{cases} \tan^{-1}(\beta_1/\alpha_1) & \text{if } \alpha_1 \geq 0 \\ \pi + \tan^{-1}(\beta_1/\alpha_1) & \text{if } \alpha_1 < 0 \end{cases}.$$

Trigonometric moments about μ_0

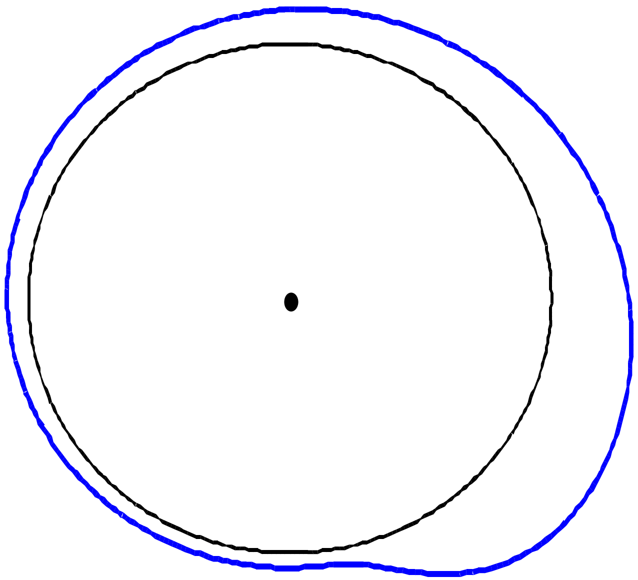
$\bar{\alpha}_p = E[\cos\{p(\Theta - \mu_0)\}]$ and

$\bar{\beta}_p = E[\sin\{p(\Theta - \mu_0)\}]$.

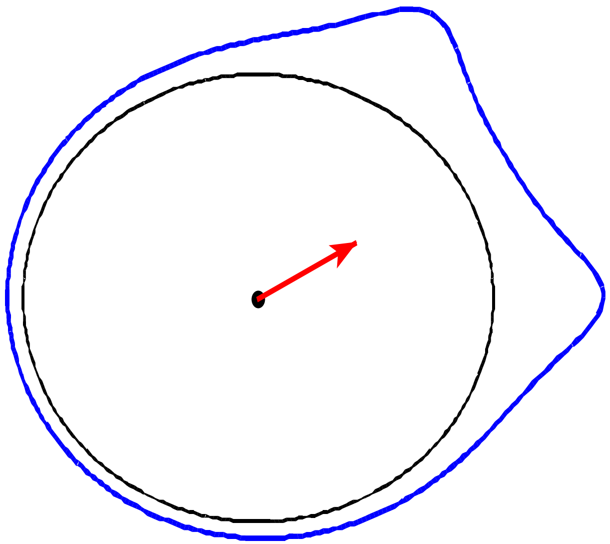
$\boxed{\bar{\beta}_p = 0}$, for all p, if distribution is reflectively symmetric.



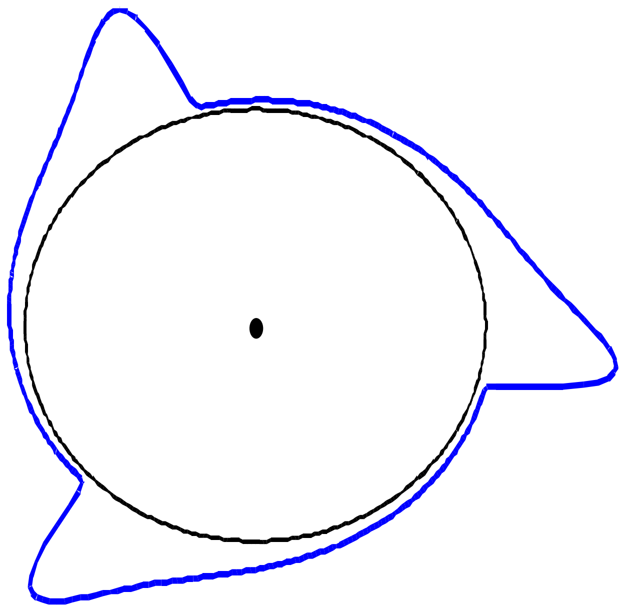
$\rho > 0,\ \mu_0 = 0,\ \overline{\beta}_2 = 0.$



$\rho > 0,\ \mu_0$ defined, $\overline{\beta}_2 \neq 0.$



$\rho > 0$, μ_0 defined, $\overline{\beta}_2 = 0$.



$\rho = 0$, μ_0 and $\overline{\beta}_2$ undefined.

But the distribution of $\boxed{3\Theta}$ is unimodal and asymmetric with:

$\rho > 0$, $\mu_0 \neq 0$, $\overline{\beta}_2 \neq 0$.

5. Omnibus Test for Reflective Symmetry

Results for the large-sample distribution of $(\bar{\theta}, \bar{R}, \bar{b}_2, \bar{a}_2)^{\mathrm{T}}$ are available in Pewsey (2004).

For $\rho \in (0,1)$, the marginal distribution of $\bar{b}_2$ is normal with, to order $O\left(n^{-3/2}\right)$, mean

$$\bar{\beta}_2 + \frac{1}{n\rho}\left(-\bar{\beta}_3 - \frac{\bar{\beta}_2}{\rho} + \frac{2\bar{\alpha}_2\bar{\beta}_2}{\rho^3}\right)$$

and variance

$$\frac{1}{n}\left[\frac{1-\bar{\alpha}_4}{2} - 2\bar{\alpha}_2 - \bar{\beta}_2^2 + \frac{2\bar{\alpha}_2}{\rho}\left\{\bar{\alpha}_3 + \frac{\bar{\alpha}_2\left(1-\bar{\alpha}_2\right)}{\rho}\right\}\right].$$

Substituting $\bar{R}$, $\bar{a}_p$ and $\bar{b}_p$ for ρ, α_p, β_p, a large-sample omnibus test of

H_0: Underlying distribution is reflectively symmetric

v

H_1: Underlying distribution is not reflectively symmetric

can be based on (Pewsey (2002)) a comparison of the test statistic

$$\bar{b}_2 \Bigg/ \left(\frac{1}{n} \left[\frac{1-\bar{a}_4}{2} - 2\bar{a}_2 + \frac{2\bar{a}_2}{\bar{R}} \left\{ \bar{a}_3 + \frac{\bar{a}_2 \left(1-\bar{a}_2\right)}{\bar{R}} \right\} \right] \right)^{1/2}$$

with the percentiles of the standard normal distribution. Large values of the test statistic lead to the rejection of H_0.

6. Symmetric Models

6.1 von Mises

$$f(\theta;\mu_0,\kappa)=\frac{1}{2\pi I_0(\kappa)}\exp\{\kappa\cos(\theta-\mu_0)\},$$

where $\kappa>0$ and $I_0(\kappa)$ is the modified Bessel function of the first kind and order *p*.

Maximum entropy distribution on the circle with given mean direction μ_0 and mean resultant length $\rho=A(\kappa)=I_1(\kappa)/I_0(\kappa)$.

Popular due to its mathematical tractability. Maximum likelihood (ML) estimates are $\hat{\mu}_0=\bar{\theta}$ and $\hat{\kappa}=A^{-1}(\bar{R})$.

6.2 Wrapped normal

$$f\left(\theta;\mu_0,\sigma^2\right)=\frac{1}{\sqrt{2\pi}\,\sigma}\sum_{k=-\infty}^{\infty}\exp\left\{\frac{-\left(\theta-\mu_0-2\pi k\right)^2}{2\sigma^2}\right\},\ \text{where } \mu_0=\mu(\bmod 2\pi).$$

Construction, properties and moment estimation

If $X \sim N\left(\mu,\sigma^2\right)$, $\phi_X(t)=\exp\left(i\mu t-t^2\sigma^2/2\right)$ and $\Theta=X(\bmod 2\pi)\sim WN\left(\mu_0,\sigma^2\right)$

has c.f. $\phi_p=\phi_X(p)=\exp\left(i\mu p-p^2\sigma^2/2\right)=\alpha_p+i\beta_p=\rho_p\exp\left(i\mu_p\right)$. Hence,

$$\alpha_p=E[\cos(p\Theta)]=\exp\left(-p^2\sigma^2/2\right)\cos(p\mu),\qquad \rho_p=\exp\left(-p^2\sigma^2/2\right),$$

$$\beta_p=E[\sin(p\Theta)]=\exp\left(-p^2\sigma^2/2\right)\sin(p\mu),\qquad \tan\left(\mu_p\right)=\beta_p/\alpha_p=\tan(p\mu).$$

So, $\rho=\rho_1=\exp\left(-\sigma^2/2\right)\Rightarrow\sigma^2=-2\log(\rho)$ and $\tilde{\sigma}^2=-2\log\left(\bar{R}\right)$. Also,

$\mu_0=\mu_1=\mu(\bmod 2\pi)$ and $\tilde{\mu}_0=\tilde{\mu}(\bmod 2\pi)=\bar{\theta}$.

6.3 Wrapped distributions in general

1. If X is a linear r.v. then its (circular) wrapped equivalent is $\Theta = X(\mathrm{mod}\, 2\pi)$.
2. If the c.f. of X is known then the trigonometric moments of Θ are easily identified.
3. Moment equations may be solvable with unique admissible solutions, solvable with unique but inadmissible solutions, solvable with more than one solution, or insolvable.
4. ML generally messy as density has to be represented as an (approximation to) an infinite sum. However, unless the tails of the distribution of X are very heavy, not many terms are required for a reasonable approximation.

6.4 Wrapped Cauchy

If $X \sim$ Cauchy with

$$f(x;\mu,a)=\frac{1}{\pi}\left(\frac{a}{a^2+(x-\mu)^2}\right), \text{ where } \mu\in\Re \text{ and } a>0,$$

then $\Theta = X(\bmod 2\pi)$ is wrapped Cauchy with

$$f(\theta;\mu_0,\rho)=\frac{1}{\pi}\sum_{k=-\infty}^{\infty}\left(\frac{a}{a^2+(\theta-\mu_0+2\pi k)^2}\right)=\frac{1}{2\pi}\left(\frac{1-\rho^2}{1+\rho^2-\rho\cos(\theta-\mu_0)}\right),$$

where $\mu_0=\mu(\bmod 2\pi)$ and $\rho=\exp(-a)$.

6.5 Wrapped symmetric stable

The wrapped normal and wrapped Cauchy distributions are special cases of the wrapped symmetric stable family (Mardia, 1972; Gatto & Jammalamadaka, 2003).

6.6 Jones-Pewsey

The von Mises and wrapped Cauchy distributions (amongst others) are special cases of the Jones-Pewsey family (Jones & Pewsey, 2005) with density

$$f(\theta;\mu_0,\kappa,\psi) \propto \{1+\tanh(\kappa\psi)\cos(\theta-\mu_0)\}^{1/\psi},$$

where $\mu_0 \in [0,2\pi)$, $\kappa \geq 0$ and $\psi \in \Re$. Scaling factor involves an associated Legendre function.

6.7 Wrapped t

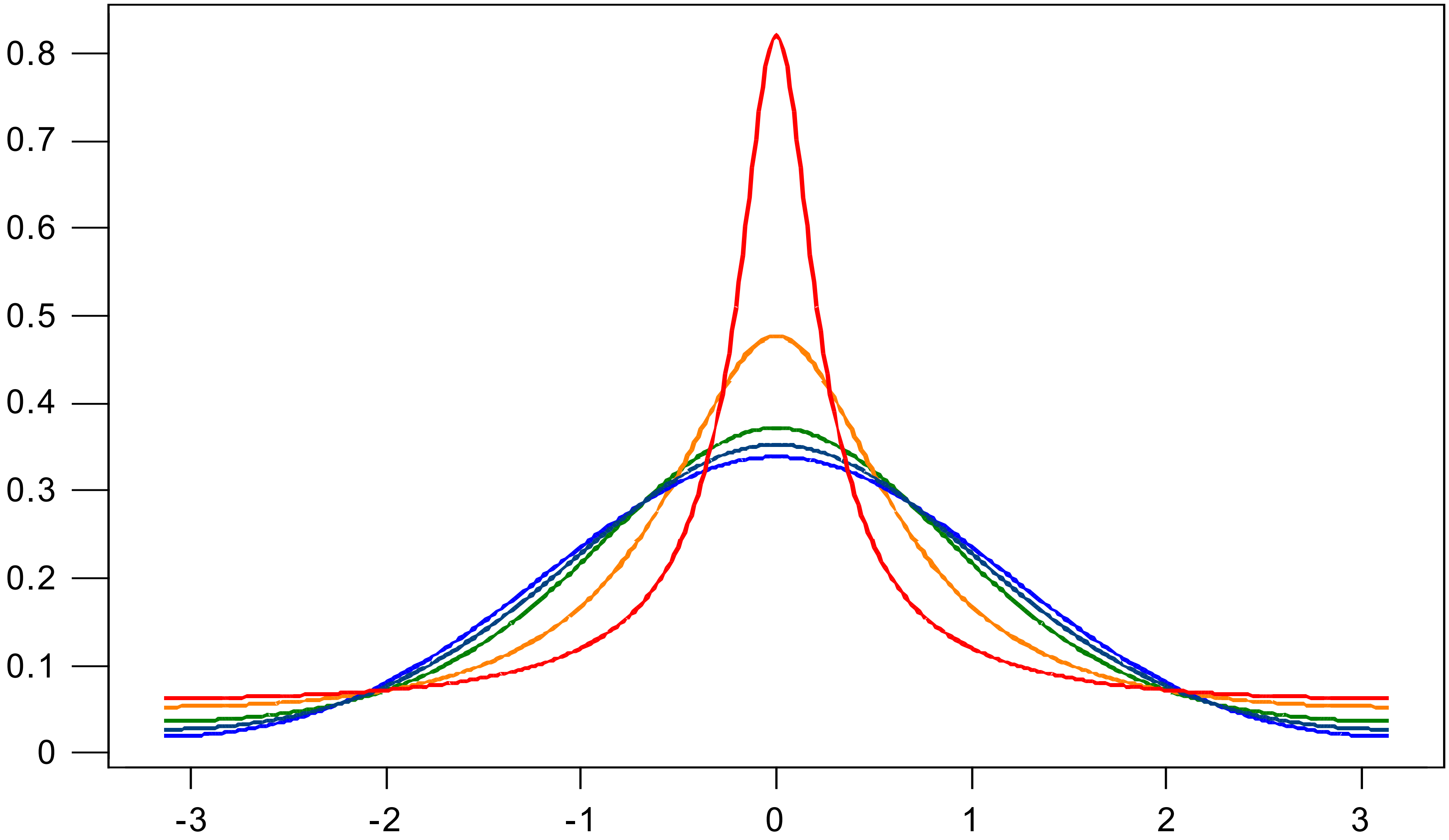
If $X \sim t_\nu$, where $\nu > 0$ (not necessarily integer), with density

$$f(x;\nu) = \frac{\Gamma((\nu+1)/2)}{\sqrt{\pi\nu}\,\Gamma(\nu/2)}\left(1 + \frac{x^2}{\nu}\right)^{-(\nu+1)/2},$$

and $Y = \mu + \lambda X$, then $\Theta = Y(\mathrm{mod}\, 2\pi)$ is wrapped t (Pewsey, Lewis & Jones, 2007) with density

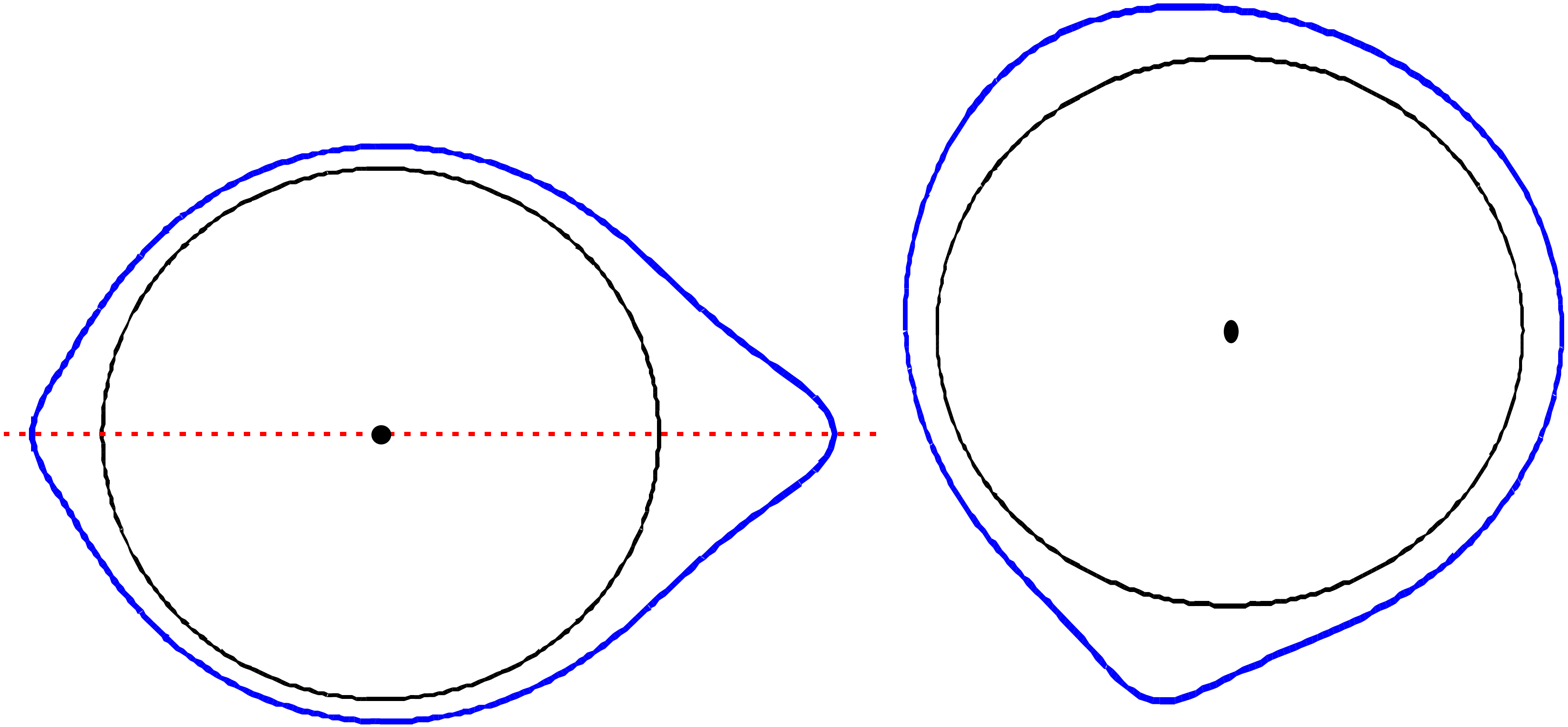
$$f(\theta;\mu_0,\lambda,\nu) = \frac{\Gamma((\nu+1)/2)}{\lambda\sqrt{\pi\nu}\;\Gamma(\nu/2)}\sum_{k=-\infty}^{\infty}\left(1 + \frac{(\theta + 2\pi k - \mu_0)^2}{\lambda^2\nu}\right)^{-(\nu+1)/2},$$

where $\mu_0 = \mu(\mathrm{mod}\, 2\pi)$.



wrapped normal ($\nu = \infty$), wrapped t ($\nu = 10$), von Mises ($\approx \nu = 3.94$), wrapped Cauchy ($\nu = 1$), wrapped t ($\nu = 0.4$). ($\rho = 0.5$)

6.8 Modelling multimodality using mixtures with symmetric components



1. Flexible.
2. Easy to interpret. Due to symmetry, the mean = mode = median is an easily interpreted measure of central location for each of the components. The interpretation of the weighting probabilities in any mixture is also simple.
3. Rather parameter heavy for modelling asymmetry (particularly heavy-tailed skew data). Need at least 5 parameters (generally) to model skew unimodal data whereas can often get away with using a 4 (or even 3) parameter asymmetric unimodal model.

7. Skew Models

7.1 Wrapped skew-normal

If $X \sim$ skew-normal with

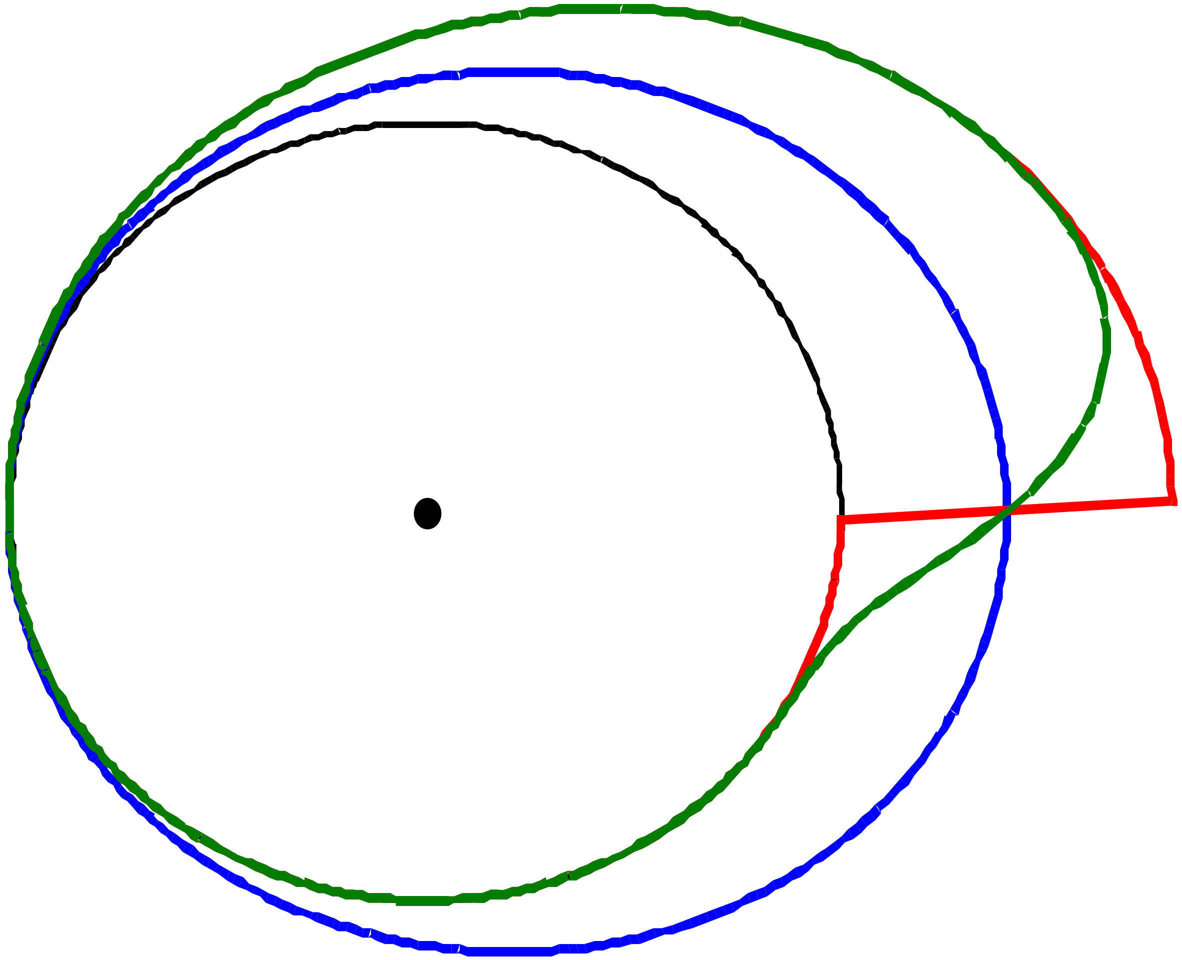
$$f(x;\xi,\eta,\lambda) = \frac{2}{\eta}\phi\left(\frac{x-\xi}{\eta}\right)\Phi\left\{\lambda\left(\frac{x-\xi}{\eta}\right)\right\},$$

where $\xi,\lambda \in \Re$, $\eta > 0$, then $\Theta = X(\mathrm{mod}\,2\pi)$ is wrapped skew-normal (Pewsey, 2000) with

$$f(\theta;\xi_0,\eta,\lambda) = \frac{2}{\eta}\sum_{k=-\infty}^{\infty}\phi\left(\frac{\theta+2\pi k-\xi_0}{\eta}\right)\Phi\left\{\lambda\left(\frac{\theta+2\pi k-\xi_0}{\eta}\right)\right\},$$

where $\xi_0 = \xi(\mathrm{mod}\,2\pi)$.

Wrapped normal $(\lambda = 0)$ and wrapped half-normal $(\lambda = \pm\infty)$ are limiting special cases.



wrapped normal $(\lambda = 0)$, wrapped skew-normal $(\lambda = 5)$, wrapped half-normal $(\lambda = \infty)$.

7.2 Wrapped exponential and Laplace

Wrapped Laplace distributions are appropriately scaled mixtures of two wrapped exponential distributions (Jammalamadaka & Kozubowski, 2004).

The behaviour around the mode of the densities of both distributions would appear to rule them out as useful models for real circular data.

7.3 Wrapped stable

$X \sim$ S(α, β, γ, δ) if its c.f. is given by

$$\phi_X(t) = \begin{cases} \exp\left(-\gamma^\alpha |t|^\alpha \left[1 + i\beta \,\mathrm{sign}(t) \tan\left(\frac{\pi\alpha}{2}\right) \left\{ (\gamma|t|)^{1-\alpha} - 1 \right\} \right] + i\delta t \right), & \alpha \neq 1, \\ \exp\left(-\gamma|t| \left[1 + i\beta \,\mathrm{sign}(t) \frac{2}{\pi} \log(\gamma|t|) \right] + i\delta t \right), & \alpha = 1. \end{cases}$$

$\alpha \in (0,2]$ (index of stability), $\beta \in [-1,1]$ (skewness),

$\gamma > 0$ (scale), $\delta \in \Re$ (location).

Includes normal (α=2, β=0) and Cauchy (α =1, β=0) distributions.

Distributions with β=0 are symmetric.

Density of the wrapped stable random variable $\Theta = X(\mathrm{mod}\, 2\pi)$ (Mardia, 1972; Pewsey, 2006) can be represented as

$$f(\theta) = \frac{1}{2\pi}\left\{1 + 2\sum_{p=1}^{\infty} \rho_p \cos\left(p\theta - \mu_p\right)\right\},$$

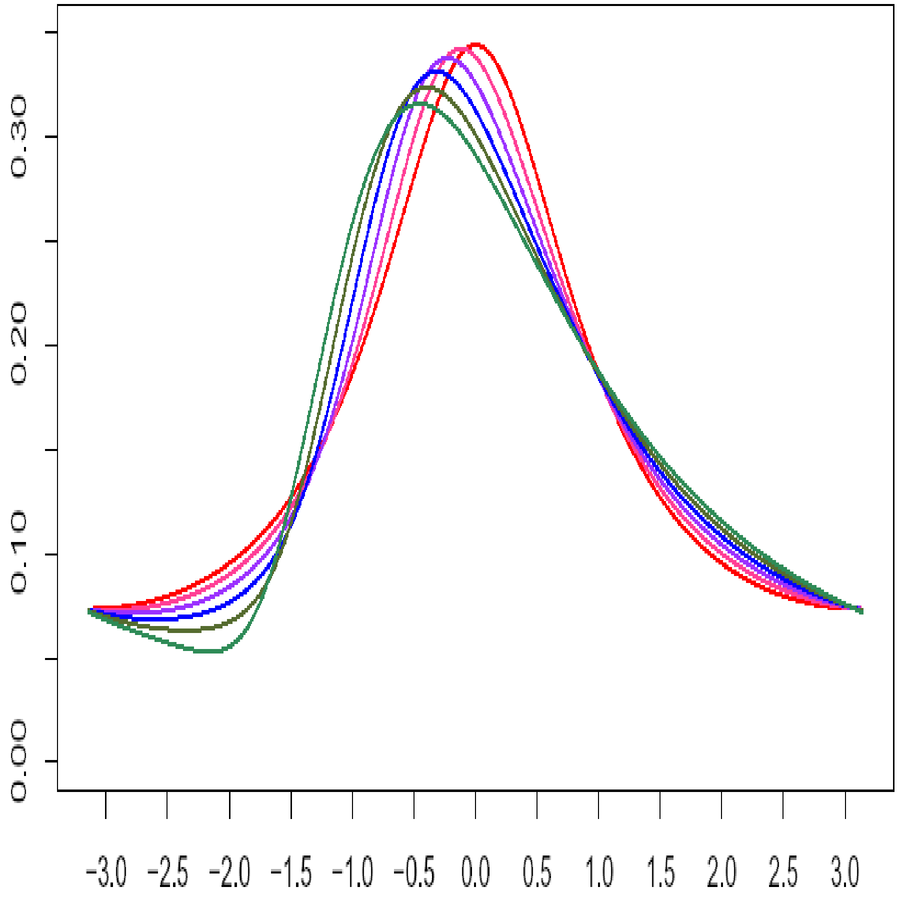
where

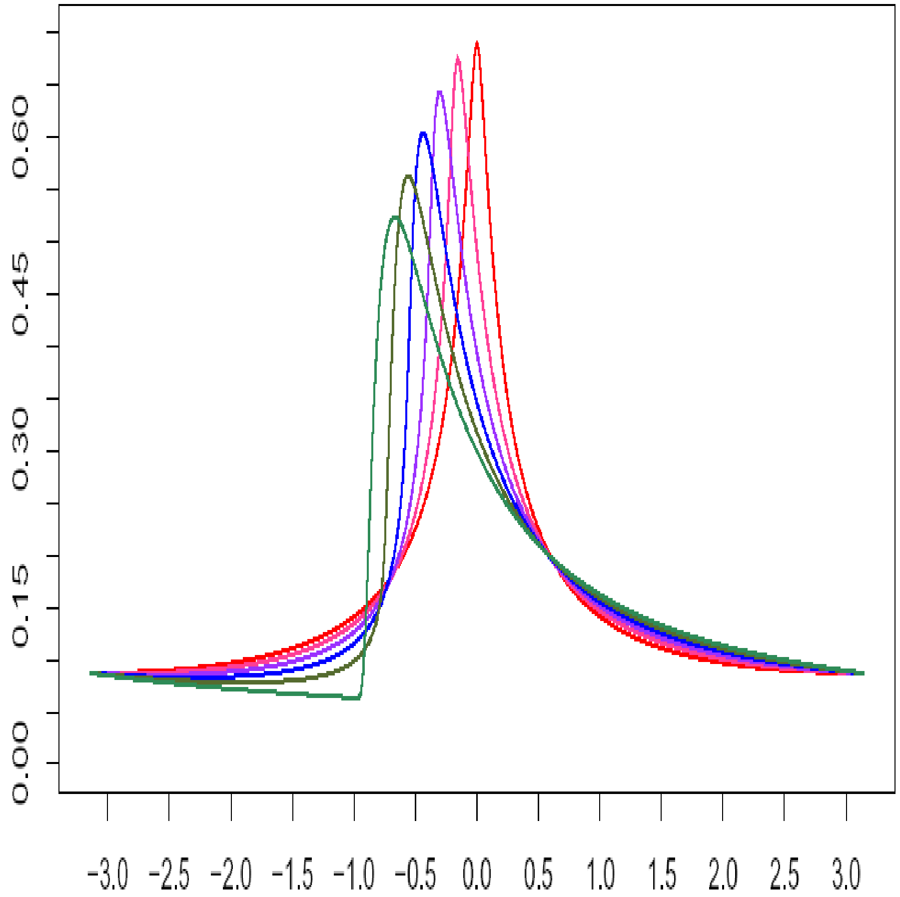
$$\rho_p = \exp\left\{-(\gamma p)^{\alpha}\right\} \in [0,1]$$

is the pth mean resultant length,

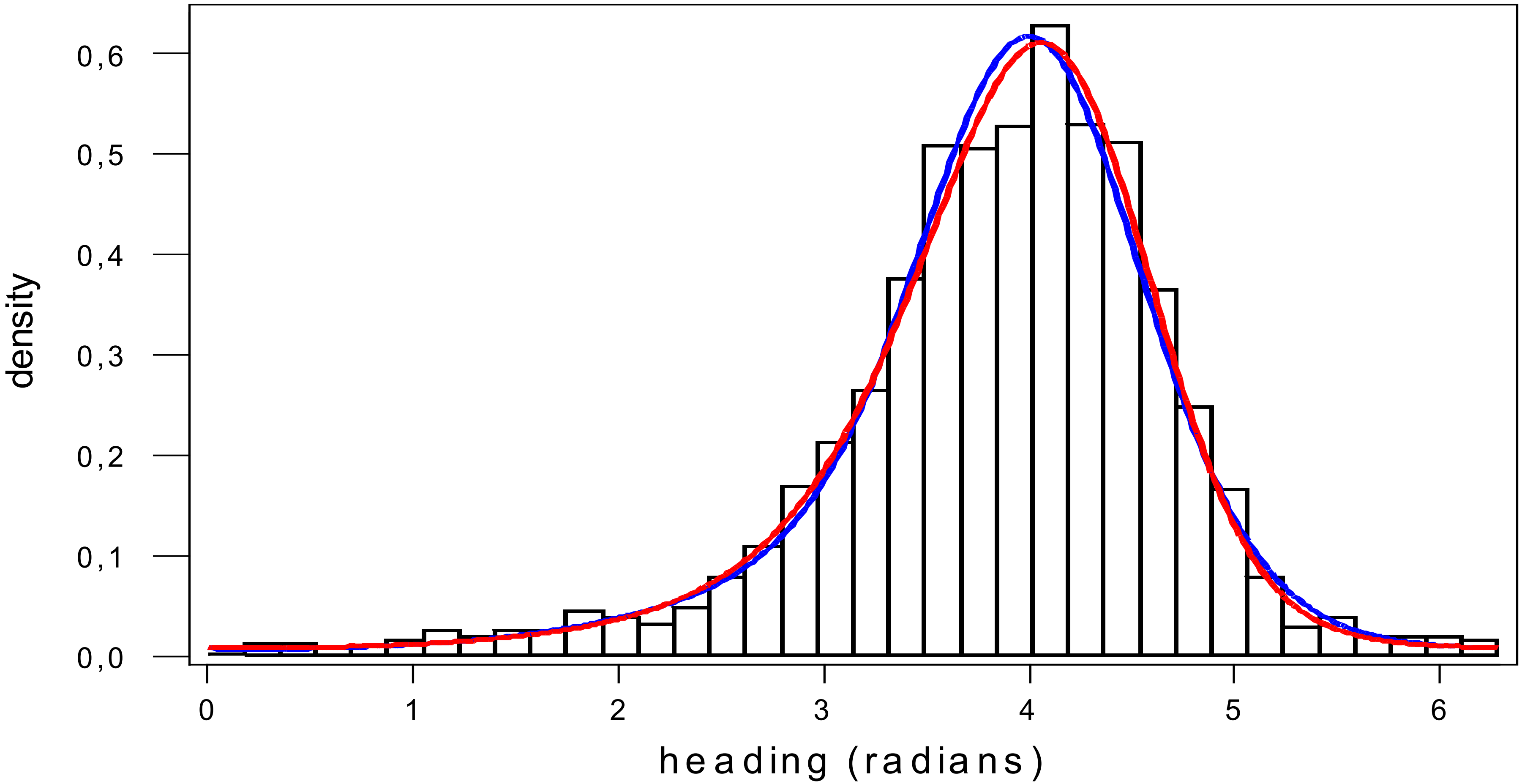
$$\mu_p = \begin{cases} \delta_0 p + \beta \tan\left(\dfrac{\pi\alpha}{2}\right)\left\{(\gamma p)^{\alpha} - \gamma p\right\} \ (\mathrm{mod}\, 2\pi), & \alpha \neq 1, \\ \delta_0 p - \beta \dfrac{2}{\pi} \gamma p \log(\gamma p) \ (\mathrm{mod}\, 2\pi), & \alpha = 1, \end{cases} \qquad \in [0, 2\pi)$$

is the pth mean direction, and $\delta_0 = \delta(\mathrm{mod}\, 2\pi)$.





$\boxed{\alpha = 1}$ $\beta = 0\ (0.2)\ 1$ $\boxed{\alpha = ½}$



Distribution	*No. of Parameters*	ℓ
von Mises mixture	5	-2131.10
wrapped stable	4	-2127.73

7.4 Testing for reflective symmetry (revisited)

For a parametric family of distributions containing asymmetric as well as symmetric cases, testing

H_0: Underlying distribution is reflectively symmetric

v

H_1: Underlying distribution is not reflectively symmetric

can be conducted using the usual likelihood ratio test based on the values of the log-likelihood for the best fitting member of the family and the best fitting symmetric member of the family.

7.5 Modelling multimodality using mixtures with skew components

1. Even more flexible (than using mixtures with just symmetric components).

2. Interpretation generally not so simple. Due to the lack of symmetry, the median and the percentiles of the components will generally be of greater interest than the mean or mode.

3. If there is asymmetry present, can generally get away with using fewer parameters than those for mixture models made up of symmetric components alone.

References

Gatto, R. & Jammalamadaka, S.R. (2003) Inference for wrapped symmetric alpha-stable circular models. *Sanhkya*, 65, 333-355.

Jammalamadaka, S.R. & Kozubowski, T.J. (2004) New families of wrapped distributions for modelling skew circular data. *Communications in Statistics – Theory and Methods*, 33, 2059-2074.

Jones, M.C. & Pewsey, A. (2005) A family of symmetric distributions on the circle. *Journal of the American Statistical Association*, 100, 1422-1428.

Jupp, P.E. & Spurr, B.D. (1983) Sobolev tests for independence of directions. *Annals of Statistics*, 13, 1225-1231.

Mardia, K.V. (1972) *Statistics of Directional Data*. London: Academic Press.

Pewsey, A. (2000) The wrapped skew-normal distribution on the circle. *Communications in Statistics – Theory and Methods*, 29, 2459-2472.

Pewsey, A. (2002) Testing circular symmetry. *The Canadian Journal of Statistics*, 30, 591-600.

Pewsey, A. (2004) The large-sample joint distribution of key circular statistics. *Metrika*, 60, 25-32.

Pewsey, A. (2006) The wrapped stable family of distributions as a flexible model for circular data. Submitted.

Pewsey, A., Lewis, T. & Jones, M.C. (2007) The wrapped *t* family of circular distributions. *Australian & New Zealand Journal of Statistics*, 49, 79-91.