# Several classes of infinitely divisible distributions and examples

Makoto Maejima
(Keio University, Yokohama)

Cherry Bud Workshop 2007
March 16, 2007

**How can the director of baseball team use data for the team? (Especially, if he is a probabilist.)**

**In US, many statisticians and computer scientists moved from "Wall Street" to "Major League" as the next market to apply their ideas, which they used in the stock market.**

SABERMETRICS
**( = Statistical analysis of baseball)**

Billy Beane, the GM of a poor team The Oakland Athletics, leveled up his team to be similar to a rich team The New York Yankees, by using SABERMETRICS.

Bill James joined The Boston Red Sox in 2002, and used SABERMETRICS to make the team the champion!

But, there is a limitation.

There are two waves for the victory in Major league.

1. Sabermetrics

2. Traditional strategy

Bondesson (1992)
"Since a lot of the standard distributions now as known to be infinitely divisible, the class of infinitely divisible distributions has perhaps partly lost its interest. Smaller classes should be more in focus."

## I. Introduction

(1) Class $B$ (Goldie–Steutel–Bondesson class on $\mathbb{R}_+$)

Bondesson (1981) studied generalized convolution of mixtures of exponential distributions on $\mathbb{R}_+$. (The smallest class that contains all mixtures of exponential distributions and that is closed under convolution and weak convergence on $\mathbb{R}_+$.)

(2) Class $T$ (Thorin class on $\mathbb{R}_+$)

Thorin (1977a, 1977b, 1978) studied generalized $\Gamma$-convolutions on $\mathbb{R}_+$ and $\mathbb{R}$. (The smallest class that contains all $\Gamma$-distributions and that is closed under convolution or making linear combinations of independent gamma random variables, and weak convergence on $\mathbb{R}_+$ and $\mathbb{R}$.)

(3) Class $G$ (Class of type $G$ distributions, which are *symmetric* on $\mathbb{R}$)

$V^{1/2}Z$, where $V > 0$, $\mathcal{L}(V) \in I(\mathbb{R})$, $Z$ is the standard normal random variable, and $V$ and $Z$ are independent.

(4) Class $L$ (Class of selfdecomposable distributions on $\mathbb{R}$)

For any $0 < c < 1$, $\widehat{\mu}(z) = \widehat{\mu}(cz)\widehat{\rho}_c(z)$ for some $\rho_c \in I(\mathbb{R}^d)$.
$X \stackrel{\mathrm{d}}{=} cX + Y_c$, where $X$ and $Y_c$ are independent.

## II. Notation

$I(\mathbb{R}^d) = \{$all infinitely divisible distributions on $\mathbb{R}^d\}$

$I_{sym}(\mathbb{R}^d) = \{\mu \in I(\mathbb{R}^d) : \mu$ is symmetric $\}$

$I_{\log}(\mathbb{R}^d) = \{\mu \in I(\mathbb{R}^d) : \int_{\mathbb{R}^d} \log^+ |x| \mu(dx) < \infty\}$

$I_{\log^m}(\mathbb{R}^d) = \{\mu \in I(\mathbb{R}^d) : \int_{\mathbb{R}^d} (\log^+ |x|)^m \mu(dx) < \infty\}$

$\mathcal{L}(X)$ : the law of a random variable $X$

Lévy-Khintchine representation of $\mu \in I(\mathbb{R}^d)$

$$\widehat{\mu}(z) = \exp\left\{ i\langle z, \gamma\rangle - \frac{1}{2}\langle z, Az\rangle \right.$$
$$\left. + \int_{\mathbb{R}^d} \left( e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle 1_{\{|x|<1\}}(x) \right) \nu(dx) \right\},$$

where

$A$ : $d \times d$ nonnegative-definite matrix,

$\gamma \in \mathbb{R}^d$,

$\nu$ : Lévy measure satisfying

$$\nu(\{0\}) = 0, \quad \int_{\mathbb{R}^d} (\|x\|^2 \wedge 1)\nu(dx) < \infty.$$

$(A, \nu, \gamma)$ is unique and called the generating triplet of $\mu$ or $\widehat{\mu}$.

Polar decomposition of Lévy measures :

$$\nu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi)\nu_\xi(dr),$$

$$B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}),$$

where

$\lambda$ : a measure on $S = \{\xi \in \mathbb{R}^d : |\xi| = 1\}$ with $0 < \lambda(S) \leq \infty$

$\{\nu_\xi : \xi \in S\}$ : a family of measures on $(0, \infty)$ such that $\nu_\xi(B)$ is measurable in $\xi$ for each $B \in \mathcal{B}((0, \infty))$, $0 < \nu_\xi((0, \infty)) \leq \infty$ for each $\xi \in S$.

Here $\lambda$ and $\{\nu_\xi\}$ are uniquely in some sense.

$\{X_t, t \geq 0\}$ on $\mathbb{R}^d$ is *Lévy process* (in law), if

(i) $X_0 = 0$ a.s.,

(ii) independent increments,

(iii) stationary increments,

(iv) stochastically continuous.

(Examples : Brownian motion, Poisson process, compound Poisson process)

(i) $\{X_t, t \geq 0\}$ on $\mathbb{R}^d$ is Lévy process
$\Rightarrow \mathcal{L}(X_t) \in I(\mathbb{R}^d)$

(ii) $\mu \in I(\mathbb{R}^d) \Rightarrow$
there exists uniquely a Lévy process
$\{X_t, t \geq 0\}$ on $\mathbb{R}^d$ such that $\mathcal{L}(X_1) = \mu$.

Notation:
$\{X_t^{(\mu)}, t \geq 0\}$ is a Lévy process with $\mathcal{L}(X_1) = \mu$.

**III. Characterization of $B, T, G, L$ in terms of Lévy measures**

(1) Class $B(\mathbb{R}^d)$ (Goldie–Steutel–Bondesson class)

$$\nu_\xi(dr) = \ell_\xi(r)dr,$$

where $\ell_\xi(r)$ is measurable in $\xi \in S$ and completely monotone on $(0, \infty)$.

(2) Class $T(\mathbb{R}^d)$ (Thorin class)

$$\nu_\xi(dr) = \frac{k_\xi(r)}{r}dr,$$

where $k_\xi(r)$ is measurable in $\xi \in S$ and completely monotone on $(0, \infty)$.

(3) Class $G(\mathbb{R}^d)$ (Class of type $G$ distributions)

$$\mu \in I_{sym}(\mathbb{R}^d) \text{ and } \nu_\xi(dr) = g_\xi(r^2)dr,$$

where $g_\xi(r)$ is measurable in $\xi \in S$ and completely monotone on $(0, \infty)$.

(4) Class $L(\mathbb{R}^d)$ (Class of selfdecomposable distributions)

$$\nu_\xi(dr) = \frac{k_\xi(r)}{r}dr,$$

where $k_\xi(r)$ is measurable in $\xi \in S$ and nonincreasing on $(0, \infty)$.

**IV. Relationships among classes**

(1) $T(\mathbb{R}^d) \subsetneqq B(\mathbb{R}^d) \cap L(\mathbb{R}^d)$ (by definition)

(2) $B(\mathbb{R}^d) \cap I_{sym}(\mathbb{R}^d) \subsetneqq G(\mathbb{R}^d)$
(Aoyama-M-Rosiński (2006))

## 3. Mappings

$\{X_t^{(\mu)}\}$ : a Lévy process on $\mathbb{R}^d$ with $\mathcal{L}(X_1) = \mu$.

**Definition 1** *($\Upsilon$-mapping)*
(Barndorff-Nielsen+M+Sato (2006))
*For* $\mu \in I(\mathbb{R}^d)$,

$$\Upsilon(\mu) = \mathcal{L}\left(\int_0^1 \log \frac{1}{t} dX_t^{(\mu)}\right).$$

For $d = 1$, this $\Upsilon$-mapping was introduced by Barndorff-Nielsen+Thorbjørnsen (2002).

**Definition 2** *($\Phi$-mapping)*
*For* $\mu \in I_{\log}(\mathbb{R}^d)$,

$$\Phi(\mu) = \mathcal{L}\left(\int_0^{\infty-} e^{-t} dX_t^{(\mu)}\right).$$

**Definition 3** *($\Psi$-mapping)*
*Let* $e(x) = \int_x^\infty e^{-u}u^{-1}du$ *and denote its inverse function by* $e^*(t)$.

*For* $\mu \in I_{\log}(\mathbb{R}^d)$,

$$\Psi(\mu) = \mathcal{L}\left(\int_0^{\infty-} e^*(t)dX_t^{(\mu)}\right).$$

**Result**(Barndorff-Nielsen+M+Sato (2006))

$$\Psi = \Upsilon \circ \Phi = \Phi \circ \Upsilon$$

**Definition 4** *($\mathcal{G}$-mapping) (Aoyama+M (2007))* *Let* $\varphi(u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2}$ *and* $h(x) = \int_x^\infty \varphi(u) du$, $x \in \mathbb{R}$, *and denote its inverse function by* $h^*(t)$.

*For* $\mu \in I(\mathbb{R}^d)$,

$$\mathcal{G}(\mu) = \mathcal{L}\left(\int_0^1 h^*(t) dX_t^{(\mu)}\right).$$

**Results**

(1) $B(\mathbb{R}^d) = \Upsilon(I(\mathbb{R}^d))$
(Barndorff-Nielsen+M+Sato (2006))

(2) $L(\mathbb{R}^d) = \Phi(I_{\mathsf{log}}(\mathbb{R}^d))$
(Wolfe (1982) and others)

(3) $T(\mathbb{R}^d) = \Psi(I_{\mathsf{log}}(\mathbb{R}^d))$
(Barndorff-Nielsen+M+Sato (2006))

(4) $G(\mathbb{R}^d) = \mathcal{G}(I(\mathbb{R}^d))$
(Aoyama+M (2007))

## 4. Decreasing subclasses

$B_0(\mathbb{R}^d) = B(\mathbb{R}^d)$,
$L_0(\mathbb{R}^d) = L(\mathbb{R}^d)$,
$T_0(\mathbb{R}^d) = T(\mathbb{R}^d)$,
$G_0(\mathbb{R}^d) = G(\mathbb{R}^d)$

For $m = 0, 1, 2, ...,$

$$B_m(\mathbb{R}^d) = \Upsilon^{m+1}(I(\mathbb{R}^d))$$

$$L_m(\mathbb{R}^d) = \Phi^{m+1}(I_{\log^{m+1}}(\mathbb{R}^d))$$

$$\begin{aligned} T_m(\mathbb{R}^d) &= (\Upsilon \circ \Phi^{m+1})(I_{\log^{m+1}}(\mathbb{R}^d)) \\ &= \Upsilon(L_m(\mathbb{R}^d)) \end{aligned}$$

$$G_m(\mathbb{R}^d) = \mathcal{G}^{m+1}(I(\mathbb{R}^d))$$

They are decreasing as $m$ increases.

**Fact** $T_\infty(\mathbb{R}^d) = L_\infty(\mathbb{R}^d)(= \overline{S(\mathbb{R}^d)})$
(Barndorff-Nielsen+M+Sato (2006))

($S(\mathbb{R}^d)$ = all stable distributions on $\mathbb{R}^d$)

## IV. Examples on $\mathbb{R}$

$(S = \{-1, 1\})$

**(1)** gamma distribution

$\gamma_{c,\lambda}$ : gamma random variable with parameter $c > 0, \lambda > 0$

$\circ$ $P(\gamma_{c.\lambda} \in B) = \frac{\lambda^c}{\Gamma(c)} \int_{B \cap (0,\infty)} x^{c-1} e^{-\lambda x} dx.$

(If $c = 1$, it is exponential.)

$\circ$ Lévy measure: $\nu(dx) = ce^{-\lambda x} \frac{1}{x} 1_{(0,\infty)}(x) dx$. (See Jurek (1997).)

$\mathcal{L}(\gamma_{c,\lambda}) \in T(\mathbb{R}_+)$, (from the form of the Lévy measure of $\mathcal{L}(\gamma_{c,\lambda})$).

$\mathcal{L}(\gamma_{c,\lambda}) \notin L_1(\mathbb{R})$.
(Barndorff-Nielsen+Pedersen+Sato (2001).)

**(2)** logarithm of gamma random variable $\gamma_{c,\lambda}$

∘ Lévy measure: $\nu(dx) = \frac{e^{cx}}{|x|(1-e^x)} 1_{(-\infty,0)}(x)dx$.

(a) $\mathcal{L}(\log \gamma_{c,\lambda}) \in L(\mathbb{R})$
(Shanbhag+Sreehari (1977).)

(b) $\mathcal{L}(\log \gamma_{c,\lambda}) \in L_1(\mathbb{R})$ if $c \geq \frac{1}{2}$
(Akita+M (2002))

(c) $\mathcal{L}(\log \gamma_{c,\lambda}) \in L_2(\mathbb{R})$ if $c \geq 1$
(Akita+M (2002))

(a') $\Upsilon(\mathcal{L}(\log \gamma_{c,\lambda})) \in T(\mathbb{R})$
(Barndorff-Nielsen+M+Sato (2006))

(b') $\Upsilon(\mathcal{L}(\log \gamma_{c,\lambda})) \in T_1(\mathbb{R})$ if $c \geq \frac{1}{2}$
(Barndorff-Nielsen+M+Sato (2006))

(c') $\Upsilon(\mathcal{L}(\log \gamma_{c,\lambda})) \in T_2(\mathbb{R})$ if $c \geq 1$
(Barndorff-Nielsen+M+Sato (2006))

**(3)** symmetrized gamma distribution (with parameter $c > 0, \lambda > 0$) (sym-gamma $(c, \lambda)$). (See Steutel+van Harn (2004), p.142.)

$\circ$ ch.f.: $\varphi_c(z) = \left(\frac{\lambda^2}{\lambda^2+z^2}\right)^c$.

$\circ$ Lévy measure: $\nu(dr) = \frac{c}{|r|}e^{-\lambda|r|}dr$, $(r \neq 0)$. (See Steutel+van Barn (2004), p.279.)

$\circ$ When $c = 1$: Laplace distribution

(a) sym-gamma $(c, \lambda) \in T(\mathbb{R})$.
(from the form of the Lévy measure above.)

Thus
(b) sym-gamma $(c, \lambda) \in G(\mathbb{R})$.
(See Rosinski (1991), p.29.)

**(4)** tempered stable distribution (Rosiński (2004))

**Definition** $(0 < \alpha < 2)$
$T_\alpha$ : tempered $\alpha$-stable random variable, if

$$\nu_\xi(dr) = r^{-\alpha-1} q_\xi(r)dr, \ r > 0,$$

where $q_\xi$ is completely monotone and
$q_\xi(0+) = 1$, $q_\xi(+\infty) = 0$.
(*not* stable)

(Barndorff-Nielsen+M+Sato (2006))
(a) $\mathcal{L}(T_\alpha) \in T(\mathbb{R})$
(b) $\mathcal{L}(T_\alpha) \in T_1(\mathbb{R})$ if $1 \leq \alpha < 2$
(c) $\mathcal{L}(T_\alpha) \in L_1(\mathbb{R})$ if $\frac{1}{4} \leq \alpha < 2$
(d) $\mathcal{L}(T_\alpha) \in L_2(\mathbb{R})$ if $\frac{2}{3} \leq \alpha < 2$
(e) $\mathcal{L}(T_\alpha) \notin L_1(\mathbb{R})$ if $0 < \alpha < \frac{1}{4}$
and $q_\xi = c(\xi)e^{-b(\xi)r}$

**(5)** Examples in $T(\mathbb{R})$ (MANY!!)

**1.** $\mathcal{L}(\chi^2(r)) \in T(\mathbb{R}_+)$, $r \in \mathbb{N}$,
(since $\chi^2(r) = \gamma_{r/2,1/2}$).

**2.** Generalized inverse Gaussian distributions belong to $T(\mathbb{R})$.

**3.** Let $X_\alpha$ be a positive stable random variable with $0 < \alpha < 1$.
Then $\mathcal{L}(\log Y_\alpha) \in T(\mathbb{R})$.
(See Bondesson (1992), p.114.)

**(6)** Examples in $L(\mathbb{R})$. (MANY!!) The following are some of them.

**1.** Let $\mathcal{L}(Z)$ be the standard normal, $\mathcal{L}(t)$ the student $t$ distribution and $\mathcal{L}(F)$ the $F$ distribution. Then $\mathcal{L}(\log|Z|) \in L(\mathbb{R})$, $\quad \mathcal{L}(\log|t|) \in L(\mathbb{R})$, $\quad \mathcal{L}(\log F) \in L(\mathbb{R})$.
(Shanbhag+Sreehari (1977).)

**2.** Let $G_1(x) = 1 - e^{-e^x}, x \in \mathbb{R}$, and $G_2(x) = e^{-e^{-x}}, x \in \mathbb{R}$. $G_1$ (resp. $G_2$) is the distribution of the plus (resp. minus) of logarithm of the standard exponential random variable. They are in $L(\mathbb{R})$. (See Steutel+van Harn (2004).)

**3.** Hyperbolic sine and cosine distributions belong to $L(\mathbb{R})$. (See Jurek (1998).)

$\circ$ ch.f. of hyperbolic sine distribution: $\varphi(z) = \pi z(\sinh \pi z)^{-1}$.

○ ch.f. of hyperbolic cosine distribution: $\varphi(z) = \pi z(\cosh(\pi z/2))^{-1}$.

**4.** Generalized hyperbolic distributions belong to $L(\mathbb{R})$.

**5.** Let $Y$ be a beta random variable. Then $\mathcal{L}\left(\log \frac{Y}{1-Y}\right) \in L(\mathbb{R})$.
(Barndorff-Nielsen et al. (1982).)

**6.** (The stochastic area of two-dimensional Brownian motion by Lévy.)
The density function is

$$f(x) = \frac{1}{\pi \cosh x} = \frac{2}{\pi(e^x + e^{-x})}$$

and it belongs to $L(\mathbb{R})$. In this case, $k_\xi(r)$ in (2.3) is $|2 \sinh x|^{-1}$.
(See Sato (1999), p. 98 Example 15.15.)

**(7)** Limits of generalized Ornstein-Uhlenbeck processes

(a) Let $\{(X_t, Y_t), t \geq 0\}$ be a 2-dimensional Lévy process. Suppose that $\{X_t\}$ does not have positive jumps, $0 < E[X_1] < \infty$ and $\mathcal{L}(Y_1) \in I_{sym}(\mathbb{R})$, where $\nu_Y$ is the Lévy measure of $\{Y_t\}$. Then

$$\mathcal{L}\left(\int_0^\infty e^{-X_{t-}} dY_t\right) \in L(\mathbb{R}).$$

(Bertoin+Lindner+Maller (2006).)

(b) Let $\{N_t\}$ be a Poisson process, and let $\{Y_t\}$ be a strictly stable Lévy process or a Brownian motion with drift. Then

$$\mathcal{L}\left(\int_0^\infty e^{-N_{t-}} dY_t\right) \in L(\mathbb{R}).$$

(Kondo+M+Sato (2006).)

(c) Let $X_t = 2t - N_t$, where $\{N_t\}$ is a standard Poisson process and $Y_t = t$. Then

$$\mathcal{L}\left(\int_0^\infty e^{-(2t-N_{t-})} dt\right) \notin L_1(\mathbb{R}).$$

(Lindner+M (2007).)

**(8)** type $S$

Let $0 < \alpha < 1$, $X \stackrel{\mathrm{d}}{=} Y^{1/\alpha} X_\alpha$, where $Y$ and $X_\alpha$ are independent, and where $Y \stackrel{\mathrm{d}}{=} \gamma_{c,1}$ and $X_\alpha \stackrel{\mathrm{d}}{=}$ strictly $\alpha$-stable.

(a) $\mathcal{L}(X) \in \mathcal{G}(T(\mathbb{R})) \subset T(\mathbb{R})$
(See Bondesson (1992), p.38.)

(b) Suppose $X_\alpha$ is symmetirc.
$\mathcal{L}(X)$ is of type $S_\alpha$, thus it belongs to $G(\mathbb{R})$.
(See Kondo+M+Sato (2006).)

**(9)** Convolution of stable distributions of different indeces.

ch.f.: $\phi(z) = \exp\left\{\int_{(0,2)} -|z|^{\alpha} m(d\alpha)\right\}$, where $m$ is a measure on the interval $(0, 2)$.

(a) $\in L_{\infty}(\mathbb{R})$
(See, e.g. Roch-Arteaga+Sato (2003).)

Thus

(b) $\in G_{\infty}(\mathbb{R}) \subset G(\mathbb{R})$
(Rosinski (2001).)

**(10)** Product of independent standard normal random variables

$Z_1$, $Z_2$ : independent standard normal random variables

(a) $\mathcal{L}(Z_1 Z_2) \in G_1(\mathbb{R})$. (M+Rosinski (2001).)

Since $\mathcal{L}(Z_1 Z_2) = \mathcal{L}(\text{sym-gamma}(\frac{1}{2}, 1))$
(see Steutel+Van Harn (2004), p.504),

(b) $\mathcal{L}(Z_1 Z_2) \in T(\mathbb{R})$

(c) $\mathcal{L}(Z_1 Z_2) \in \mathcal{G}(L(\mathbb{R}))$
*Proof.* $Z_1 Z_2 \overset{\mathrm{d}}{=} (Z_1^2)^{1/2} Z_2$ and $\mathcal{L}(Z_1^2)$ is $\chi^2$-distribution, which is known to be selfdecomposable (see, e.g. Jurek (1997), p.98).

**(11)** Examples related to gamma random variables

(a) (Product of independent gamma random variables)
$X_1, X_2, ...X_n$ : independent gamma random variables
$q_1, q_2, ..., q_n \in \mathbb{R}$ with $|q_j| \geq 1$
Then $\mathcal{L}(X_1^{q_1} X_2^{q_2} \cdots X_n^{q_n}) \in L(\mathbb{R}_+)$.
(Steutel+Van Barn (2004), p.360.)

(b) (exponential function of gamma random variable)
$X$ : denumerable convolution of gamma random variables $\gamma_{c_j,\lambda_j}$ with $c_j \geq 1$
Then $\mathcal{L}(e^X) \in T(\mathbb{R}_+)$.
(Bondesson (1992), p.94.)

**(12)** log-normal distribution

$Z$ : standard normal
Then log-normal distribution $\mathcal{L}(e^Z) \in T(\mathbb{R}_+)$.
(Bondesson (1992), p.59.)

**(13)** Examples in $T(\mathbb{R})$. (random excursion of Bessel processes)

The following is from Bertoin+Fujita+ Roynette+Yor (2006).

Let $\{R_t, t \geq 0\}$ be a Bessel process with $R_0 = 0$, with dimension $d = 2(1-\alpha)$. $(0 < \alpha < 1$, equivalently $0 < d < 2.)$ When $\alpha = \frac{1}{2}$, $\{R_t\}$ is a Brownian motion. Let

$$g_t^{(\alpha)} := \sup\{s \leq t : R_s = 0\},$$

$$d_t^{(\alpha)} := \inf\{s \geq t : R_s = 0\}$$

and

$$\Delta_t^{(\alpha)} := d_t^{(\alpha)} - g_t^{(\alpha)},$$

which is the length of the excursion above 0, straddling $t$, for the process $\{R_u, u \geq 0\}$, and let $\varepsilon$ be a standard exponential variable independent from $\{R_u, u \geq 0\}$. Let $\Delta_\alpha := \Delta_\varepsilon^{(\alpha)}$. Then

$$\mathcal{L}(\Delta_\alpha) \in T(\mathbb{R}_+)(\subset L(\mathbb{R}_+)).$$

In Bertoin+Fujita+Roynette+Yor (2006), only "$\in L(\mathbb{R})$" is mentioned. However, they actually showed that

$$E\left[e^{-\lambda\Delta_\alpha}\right]$$
$$= \exp\left\{-(1-\alpha)\int_0^\infty\left(1-e^{-\lambda x}\right)\frac{E[e^{-xG_\alpha}]}{x}dx\right\},\ \lambda>0,$$

with a random variable $G_\alpha$. (The density function of $G_\alpha$ is explicitly known.) Since $k(x) := E[e^{-xG_\alpha}]$ is completely monotone by Bernstein theorem, $\mathcal{L}(\Delta_\alpha)$ belongs to not only $L(\mathbb{R}_+)$ but also $T(\mathbb{R}_+)$.

**(14)** $PCP(\mathbb{R}) = \{\mu \in I(\mathbb{R}):$

$\mu$ is positive compound Poisson distribution$\}$.

(a) $\mathcal{L}(\Delta_\alpha) \in \Phi(PCP(\mathbb{R}))$.
(Bertoin+Fujita+Roynette+Yor (2006).)

(b) $\mathcal{L}(\gamma_{c,\lambda}) \in \Phi(PCP(\mathbb{R}))$. (Jurek (1997).)

*Question.* Characterize the class $\Phi(PCP(\mathbb{R}))$.

**(15)** Examples in $B(\mathbb{R})$

(a) Compound Poisson $X = \sum_{j=1}^{N} Y_j$, where $\{Y_j\}$ are i.i.d. exponential.
Then $\mathcal{L}(X) \in B(\mathbb{R}_+)$.
(Bondesson (1992), p.143.)

(b) $X = -\log Y$, $Y = Y(\alpha, \beta)$ is a beta random variable.
(b1) $\mathcal{L}(X) \in B(\mathbb{R}_+)$
(b2) $\mathcal{L}(X) \in L(\mathbb{R}_+)$ iff $2\alpha + \beta \geq 1$
(Bondesson (1992), pp.143-144.)

**(16).** From the observation above, we see the following. We know that the compound Poisson distribution is not selfdecomposable. Hence,

$$\mathcal{L}(\Delta_\alpha) \notin \Phi^2(I(\mathbb{R})) = L_1(\mathbb{R}).$$

**(17)** Examples in $T(\mathbb{R}) \cap L_1(\mathbb{R})^c$. (Revisit.)

(a) $\mathcal{L}(\gamma_{c,\lambda})$. (See **(1)**.)

(b) $\mathcal{L}(T_\alpha)$ if $0 < \alpha < \frac{1}{4}$. (See **(4)**, (a) and (e).)

(c) $\mathcal{L}\left(\int_0^\infty e^{-(2t-N_{t-})}dt\right)$. (See **(7)**, (a) and (c).)

(d) $\mathcal{L}(\Delta_\alpha)$. (See **(12)** and **(14)**.)