Generalized Additive Modelling for Sample Extremes: An Environmental Example

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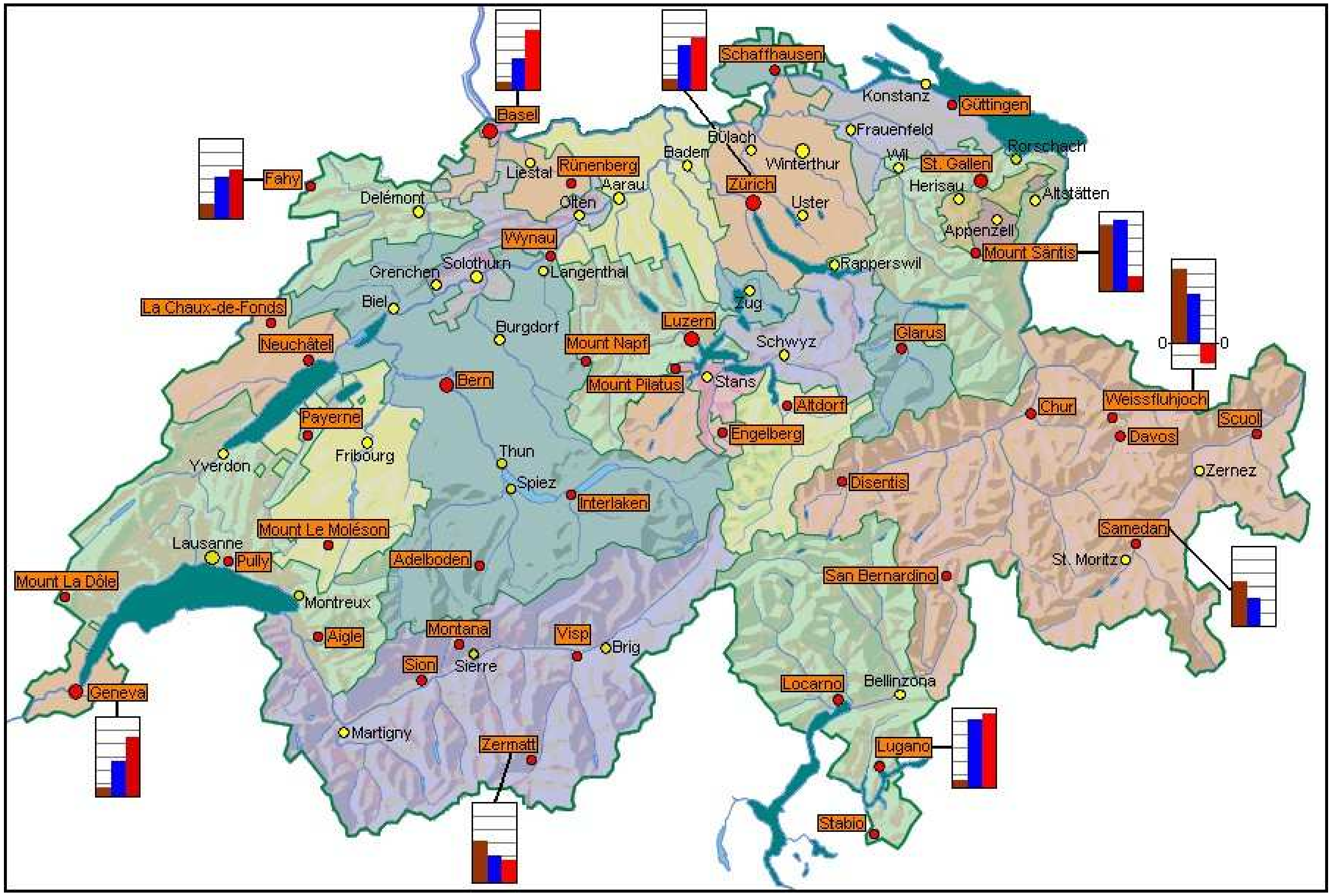
Tokyo, March 2007

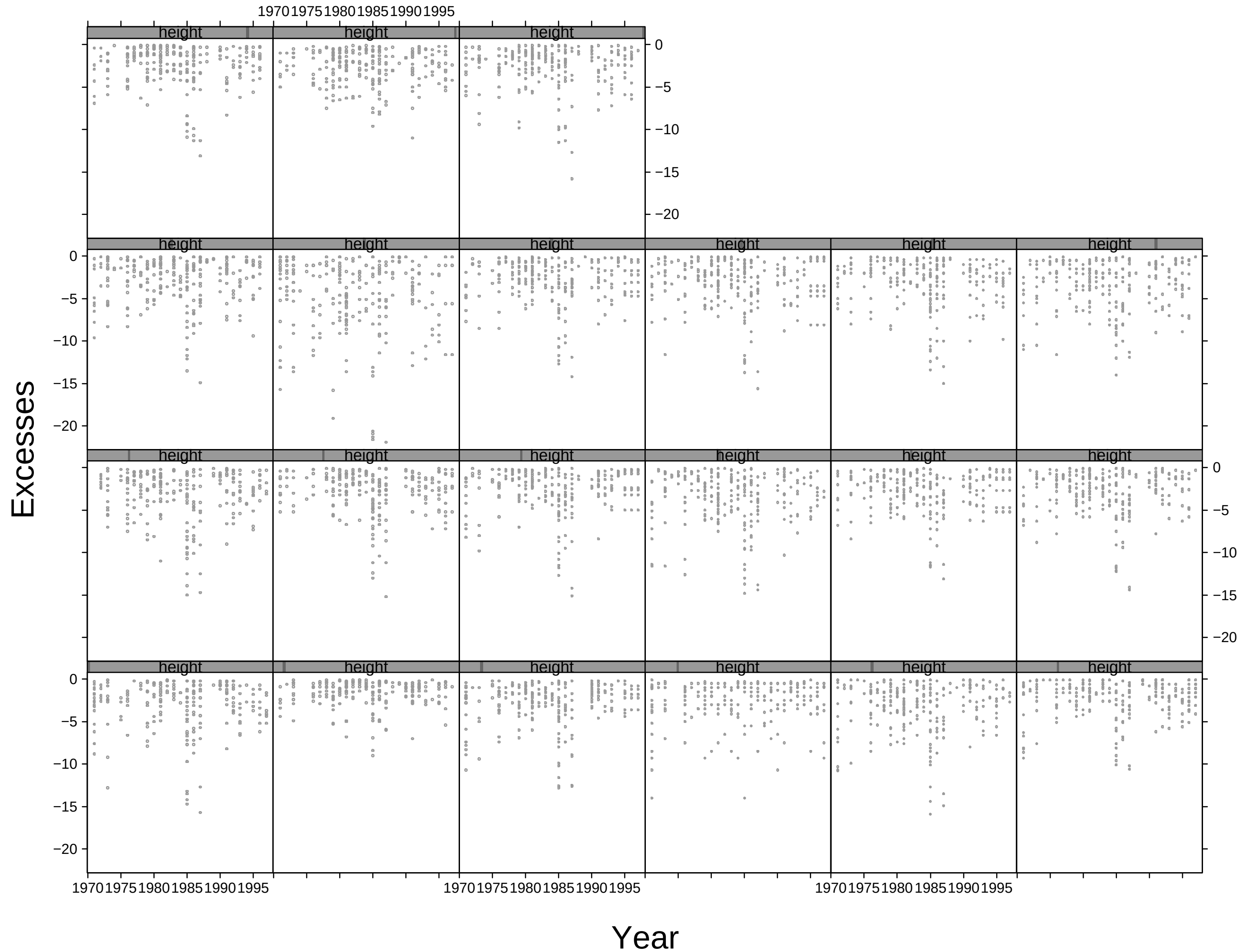
Changes in extremes?

- Likely to be slow in environmental applications
- May be difficult to detect because of noise
- Aim to combine the point process approach to exceedances with smoothing methods to give a flexible exploratory approach to modelling changes in extremes

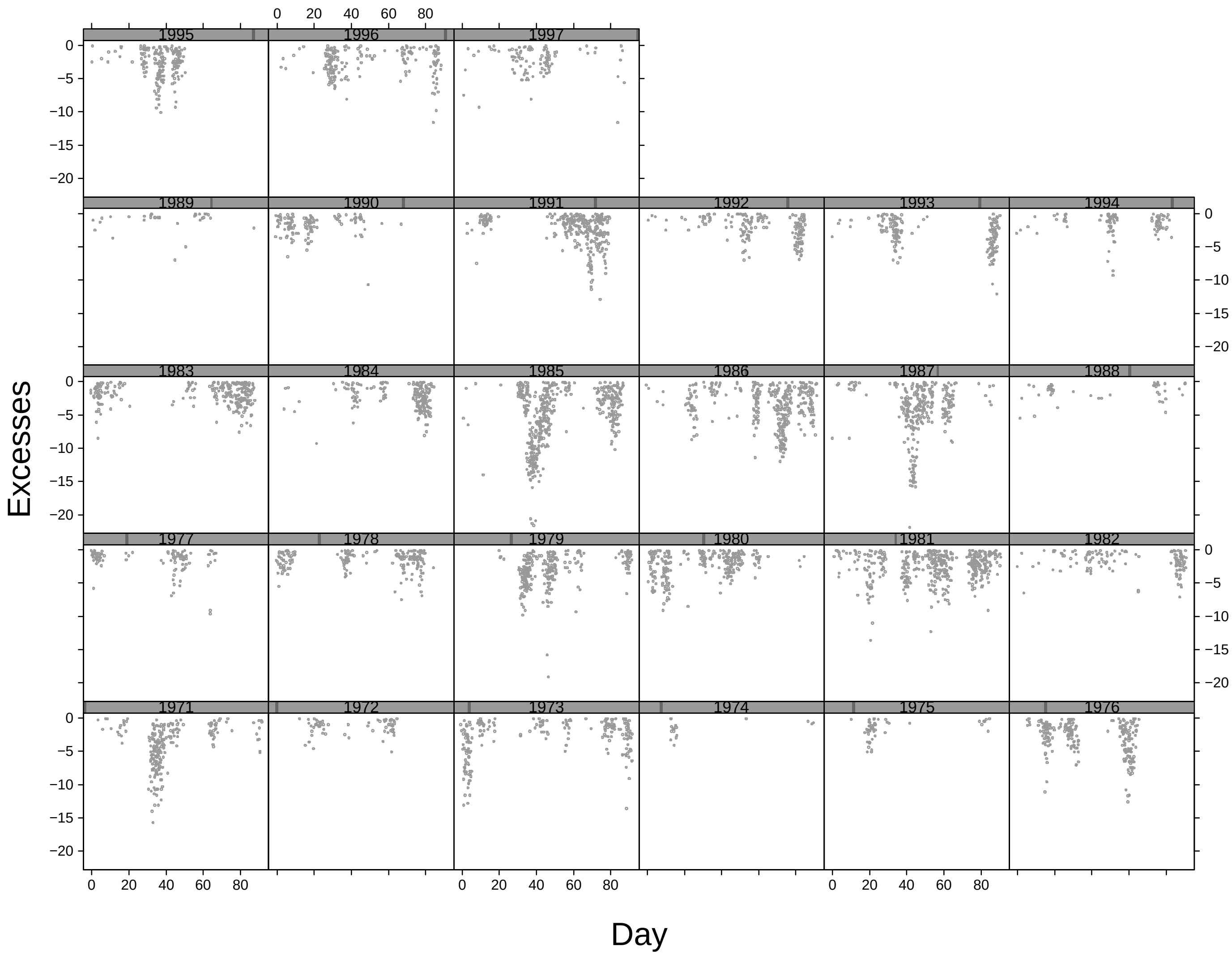
Stations in Swiss Alps



Winter temperatures at 21 Swiss stations



Swiss winter temperatures by year



Summary

- Climate change and extremes? Need flexible models
- Mix threshold approach to extremal modelling, semiparametric smoothing, and bootstrap
- Brief description of the threshold method
- Implementation of spline smoothers
- Application to the Swiss Alps data
- Discussion

Traditional Method

- The mathematical foundation of EVT is the class of extreme value limit laws
- $X_1, X_2, \ldots,$ are independent random variables with common distribution function F and $M_n = \max\{X_1, \ldots, X_n\}$
- for suitable normalising constants $a_n > 0$ and b_n, we seek a limit law satisfying $P\left\{\frac{M_n - b_n}{a_n} \leq x\right\} = F^n(a_n x + b_n) \to G(x)$

- There are only 3 fundamental types of extreme value limit laws that can be combined into a simple GEV distribution

$$H(x) = \exp\left\{ -\left(1 + \kappa \frac{x-\mu}{\psi}\right)_+^{-1/\kappa} \right\}$$

The parameters $-\infty < \mu < \infty$, $\psi > 0$ and $-\infty < \kappa < \infty$ are resp. the location, scale and shape parameters

r-largest Extremes

- $M_{-}^{n}, \ldots, M_r^n$: the r-largest observations among $X_1, \ldots, X_n$ to get more information about the extremes than the max alone

- The asymptotic joint distribution of $M_1^n, \ldots, M_r^n$ at $m_1^n, \ldots, m_r^n$ is
$\exp\left\{-\left(1+\kappa\frac{m_n^r-\mu}{\psi}\right)^{-1/\kappa}\right\} \times$
$\prod_{j=1}^{r} \frac{1}{\psi}\left(1+\kappa\frac{m_n^i-\mu}{\psi}\right)_+^{-1/\kappa-1}$
which forms a likelihood for the parameters

- In case m years of data are available, the likelihood is constructed from the r-largest values in each year, considering data for different years as independent, an overall likelihood is simply the product of such terms , for all years
- Choice of r; bias if r is too large

Threshold method

- Treat occurrences of events over (or under) threshold u as Poisson process
- Number of exceedances N over u follows homogeneous Poisson process, rate λ
- Exceedance sizes $W_j = Y_j - u$ are random sample from GPD

$$G(w) = \begin{cases} 1 - (1 + \kappa w/\sigma)_+^{-1/\kappa} & \text{if} \quad \kappa \neq 0 \\ 1 - \exp(-w/\sigma) & \text{if} \quad \kappa = 0 \end{cases}$$

where σ and κ are scale and shape parameters

- Use orthogonal parametrization $\kappa, \nu = \sigma(1+\kappa)$ below
- Log likelihood for data splits into two parts

$$l(\lambda, \kappa, \sigma) = l_N(\lambda) + l_W(\kappa, \nu)$$

Semiparametric model

- Generalize previous approach

- Take λ to be time-varying, where

$$\lambda = \exp\left\{x^T\alpha + f(t)\right\}$$

- Take exceedances to be GPD with

$$\kappa = x^T\beta + g(t), \quad \nu = \exp\left\{x^T\eta + s(t)\right\}$$

- f, g and s are smooth functions of time t, and parameters can also depend on ordinary covariates
- Penalize roughness of f, g and s through second derivatives
- Other link functions possible

Penalized log likelihoods

- For rate of exceedances λ, maximize

$$l_N(\lambda) - \frac{1}{2}\gamma_\lambda \int f''(t)^2 dt,$$

equivalent to fitting standard generalized additive model

- For sizes of exceedances, maximize

$$l_W\left\{\kappa(\beta, g), \nu(\eta, s)\right\} - \frac{1}{2}\gamma_\kappa \int g''(t)^2 dt - \frac{1}{2}\gamma_\nu \int s''(t)^2 dt$$

If g, s are cubic splines, equivalent to maximizing

$$l_W \{\kappa(\beta, g), \nu(\eta, s)\} - \frac{1}{2}\gamma_\kappa g^T K g - \frac{1}{2}\gamma_\nu s^T K s$$

over β, η, g, s and leads to generalized ridge regression

- Parameters γ_λ, γ_κ and γ_ν control smoothness of f, g and s

Methodology

- Choose forms for λ, κ and ν and fit

- Choose smoothing parameters γ_λ etc using AIC

- Use likelihood ratio statistics/AIC for model comparisons

- When model correct, residuals

$$R_j = -\hat{\kappa}_j^{-1} \log\{1 - \hat{\kappa}_j W_j (1 - \hat{\kappa}_j)\hat{\nu}_j\}$$

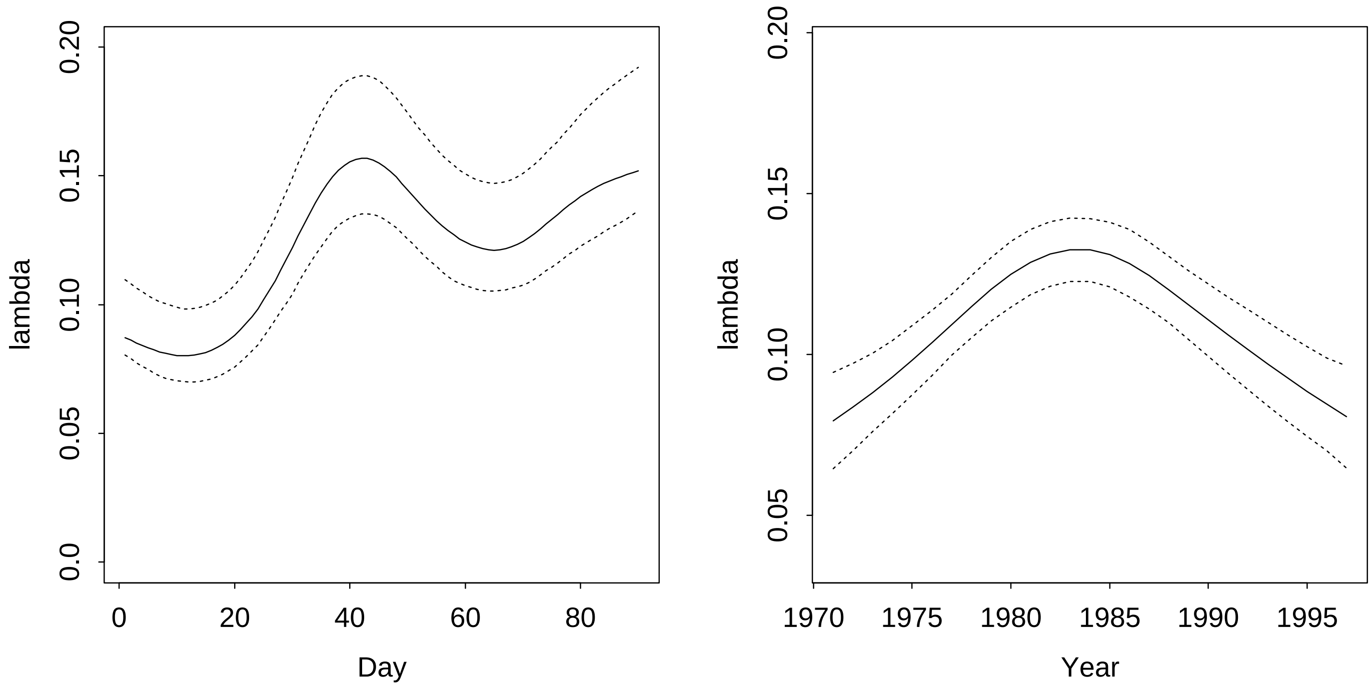
are approximately independent unit exponential variables

Bootstrap uncertainty assessment

- Need model-robust assessment of uncertainty
- Clustering across stations must be taken into account
- Use bootstrap, either resampling the R_j
  - computed from undersmoothed curves
  - added to oversmoothed curves
- or resample seasons within blocks
- Either yields percentile confidence intervals/pointwise bands

Alpine winter temperatures

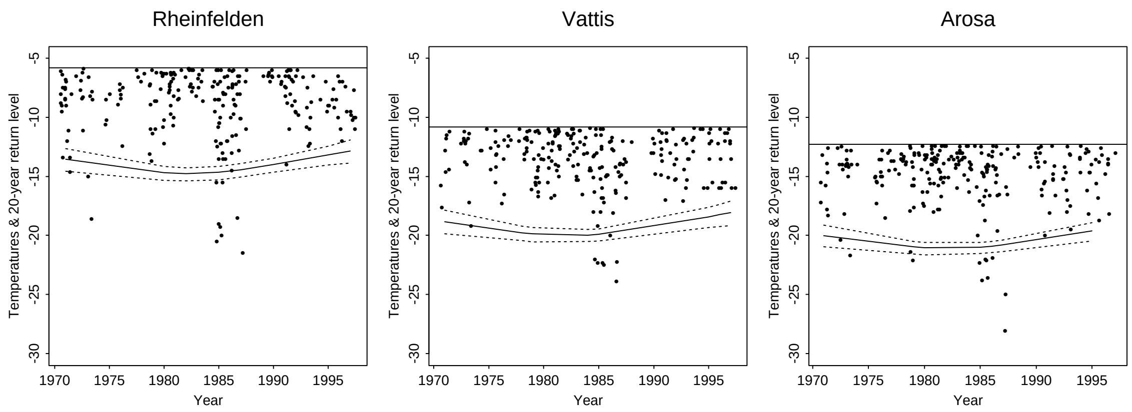
- Fitted intensity $\log\hat{\lambda} = \hat{\alpha_0} + \hat{f}(d,4) + \hat{q}(t,2)$ at Vattis for 1984–5 (left) and for January 1 from 1971–95 (right)



Fitted model and 20-year return level

$$\log\hat{\lambda} = \hat{\alpha}_0 + \hat{f}(d,4) + \hat{q}(t,2),$$

$$\hat{\kappa} = \hat{\beta}_0 + 10^{-2}(h - 1000)\hat{\beta}_1, \qquad \log\hat{\nu} = \hat{\eta}_0 + \hat{\eta}_2 t + \hat{s}(d,4)$$



Discussion

- Inhomogeneous Poisson process λ depends on time but not location
- Shape parameter κ varies with altitude — exceedances at higher stations have shorter tails
- 'Scale' parameter ν depends on time but not on altitude
- Increase since 1985 is consistent with the supposed effect of climate change but also with short-term fluctuations (decrease from 1970–85!)

Conclusion

- Exceedances over/under thresholds
  - widely-used approach with natural interpretation
  - exceedance times modelled using existing code (GAM)
- Smoothing extremes by penalized log likelihood
  - convenient and rapid exploration technique
  - highlights features of underlying distribution

References

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