Unstable periodic orbits of a chaotic growth cycle model

Yoshitaka Saiki

Department of Mathematical Sciences
University of Tokyo

(with Ken-ichi Ishiyama)

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Business cycle theory [2]

- Important topic in macro-economics
- Periodic fluctuation is essential characteristic
- Low dimensional dynamical systems
  - Stable model (Periodic point)

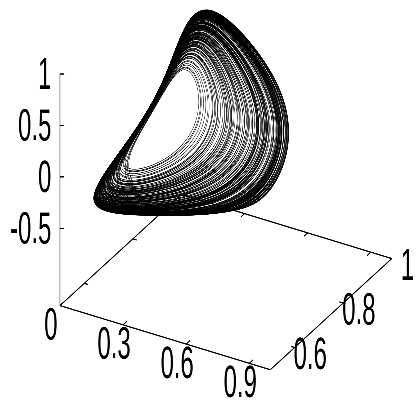
    [Gap exists, even if dynamics look similar]
  - Chaotic model

$$\Downarrow$$

How to bridge the gap ?

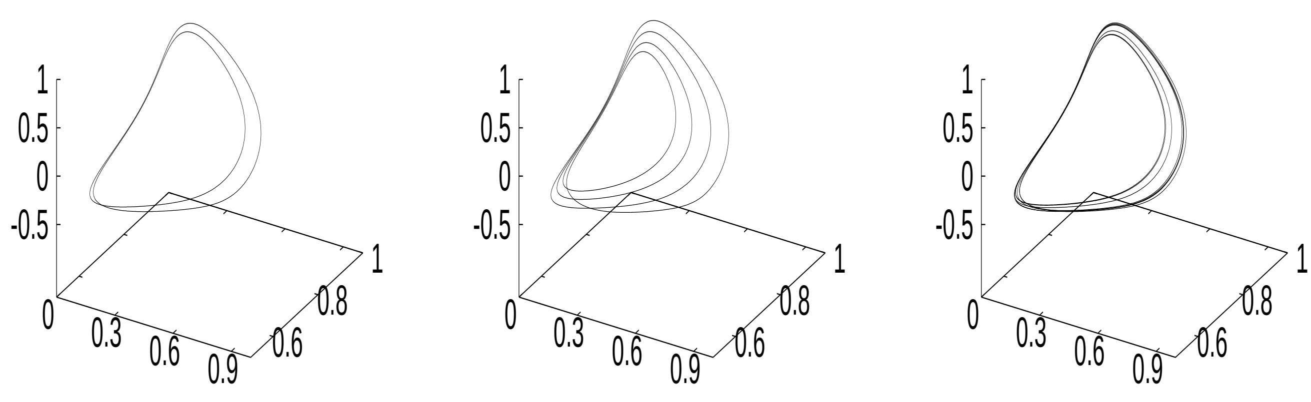
Purpose [3]

Clearly recognize chaotic behavior by unstable periodic orbits (UPOs)



Chaotic behavior (Complicated object)

⇑



UPOs (Simple objects)

Preparation [4]

'Chaos': No single definition

Deterministic behavior with two properties;

- Instability
- Reccurence

$\longrightarrow$ Chaotic

$\Downarrow$ Special solution

Unstable periodic solution

'Chaos'$\leftrightarrow$UPOs

- Chaotic attractor: Infinite number of UPOs are (densely) embedded
- Chaotic orbit: 'UPO of infinite period'

Chaotic Analysis by UPOs [5]

Detection of UPOs: Difficult $\Rightarrow$ A few studies

Background

UPO analysis of fluid turbulence
(Kawahara & Kida(2001), Kato & Yamada(2003))

Small number of UPOs (Only one or two)

$\Downarrow$

Coherent structure, Turbulent statistics

Our study

Two country chaotic business cycle model

$\Downarrow$

Numerically detect 1000 sorts of UPOs [using PC cluster (40 CPUs)]

$\Downarrow$

Characterize various chaotic behaviors by UPOs

Plan [6]

1. Chaotic business cycle model
2. **Chaotic Analysis based on UPOs**
   - **Regime and Regime transitions**
   - **Statistical properties**
   - **Growth rate of the number of UPOs**
3. Summary

Two country business cycle model [7]

Skeleton of the model

- Three agents in each country
  - Capitalists
  - Workers
  - Government with Keynesian fiscal policy
- Investment interaction between two countries
- The only difference between two countries is their Keynesian fiscal policies

$$\Downarrow$$

Six dimensional ODEs ($i = 1, 2$)

- Labor share rate u_i
- Employment rate v_i
- Expected rate of inflation π_i^e

Mutual action parameter between countries: η

($\eta = 3.5$: usual)

Model (Keyenes·Goodwin type) [8]

The first country [weak Keynesian policy]

$$\frac{du_1}{dt} = 0.5\left(\frac{0.1}{1-v_1} + \pi_1^e - 0.5\right)u_1$$

$$\begin{aligned}\frac{dv_1}{dt} = 0.1(1.5(1-u_1)^5 + {} & \underline{\underline{\eta(u_2-u_1)^3}} \\ & + 0.5u_1 \underline{\underline{\underline{-0.875v_1 - 0.1}}})v_1\end{aligned}$$

$$\frac{d\pi_1^e}{dt} = \frac{v_1(0.4\pi_1^e + 0.2) - 0.4\pi_1^e - 0.16}{1-v_1}$$

The second country [strong Keynesian policy]

$$\frac{du_2}{dt} = 0.5\left(\frac{0.1}{1-v_2} + \pi_2^e - 0.5\right)u_2$$

$$\begin{aligned}\frac{dv_2}{dt} = 0.1(1.5(1-u_2)^5 + {} & \underline{\underline{\eta(u_1-u_2)^3}} \\ & + 0.5u_2 \underline{\underline{\underline{-4.2v_2 + 2.56}}})v_2\end{aligned}$$

$$\frac{d\pi_2^e}{dt} = \frac{v_2(0.4\pi_2^e + 0.2) - 0.4\pi_2^e - 0.16}{1-v_2}$$

(Parameters： Yoshida & Asada (2001))

Chaotic attractor [9]

Bounded & Positive Lyapunov exponent(0.099)

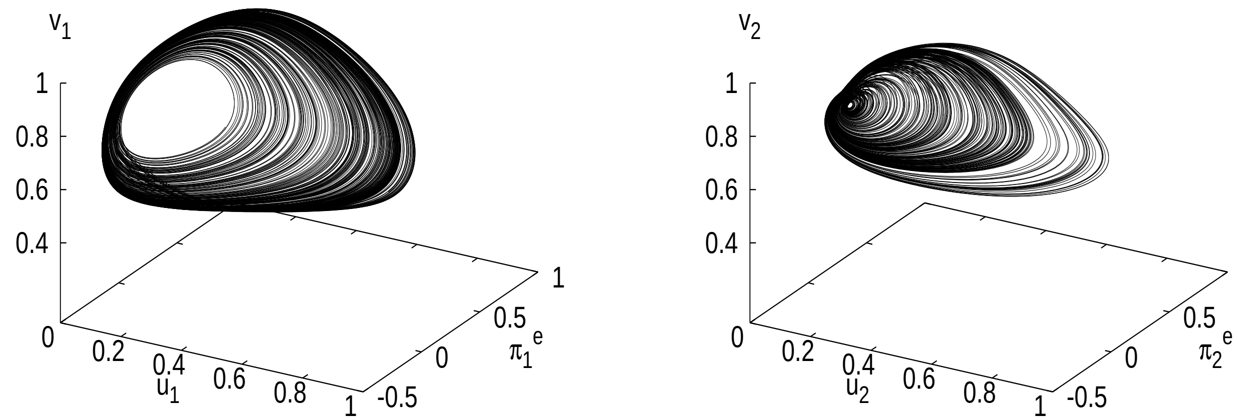
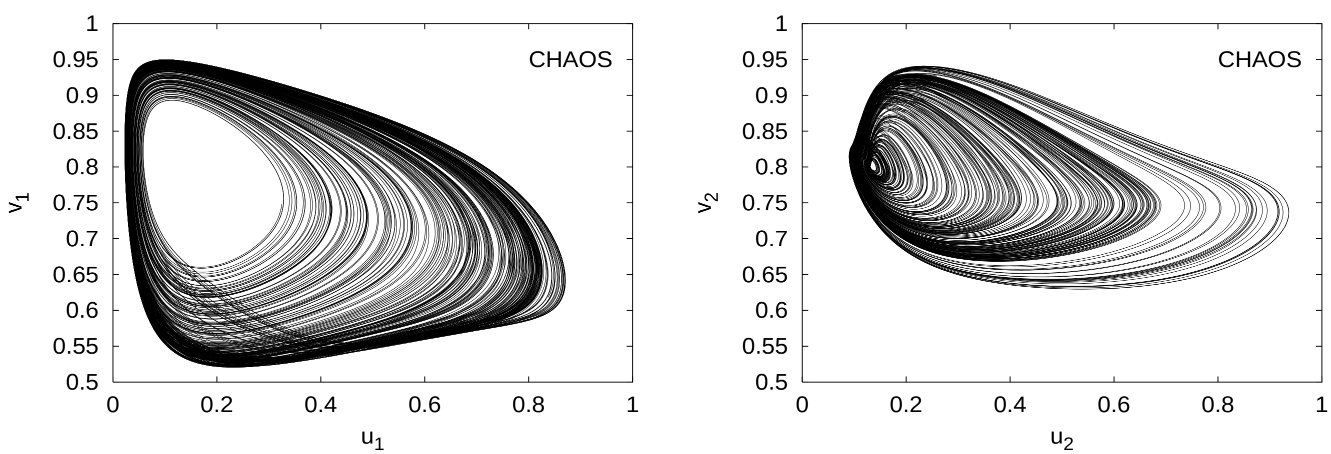


Figure: Projections to each country (u, v, π^e)
(left： 1st country, right：2nd country)



Figure：Projections onto (u, v)-plane
(left： 1st country, right： 2nd country)

- Consistent with the empirical research by Havie(2000)

Classification of typical dynamics [10]

Regime: Time period in which local maxima in u_1 are monotone increasing Regime n: Regime which has n oscillations
Regime transition $m \to n$: Regime $m \to$ Regime n

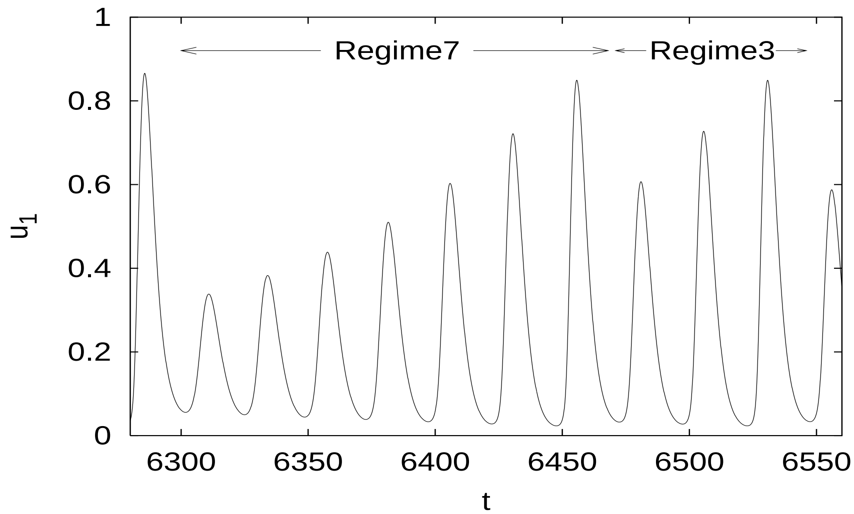


Figure: Classification of Regime in the time development of u_1
(Regime7, Regime3 and Regime transition 7$\to$ 3)

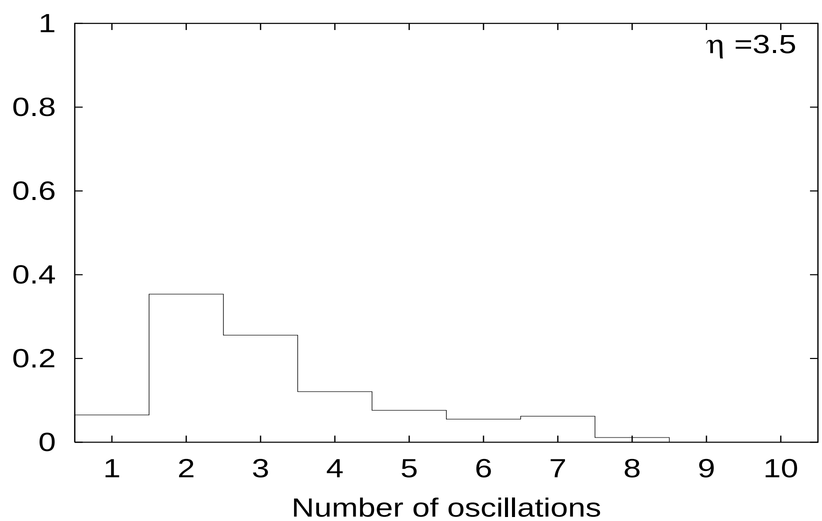


Figure: Regime appearance in a chaotic orbit
(99% of the Regimes are classified into Regime 1 to Regime 7)

Chaotic analysis by UPOs [11]

Detection

Newton-Raphson-Mees method with modification

9000 UPOs are numerically detected
$\Downarrow$ Remove indistinguishable UPOs
<u>1000 sorts of UPOs are identified</u>

Most of UPOs with short period are covered !

UPOs with only one Regime [12]

$UPOn$: UPO with n oscillations making a single Regime n

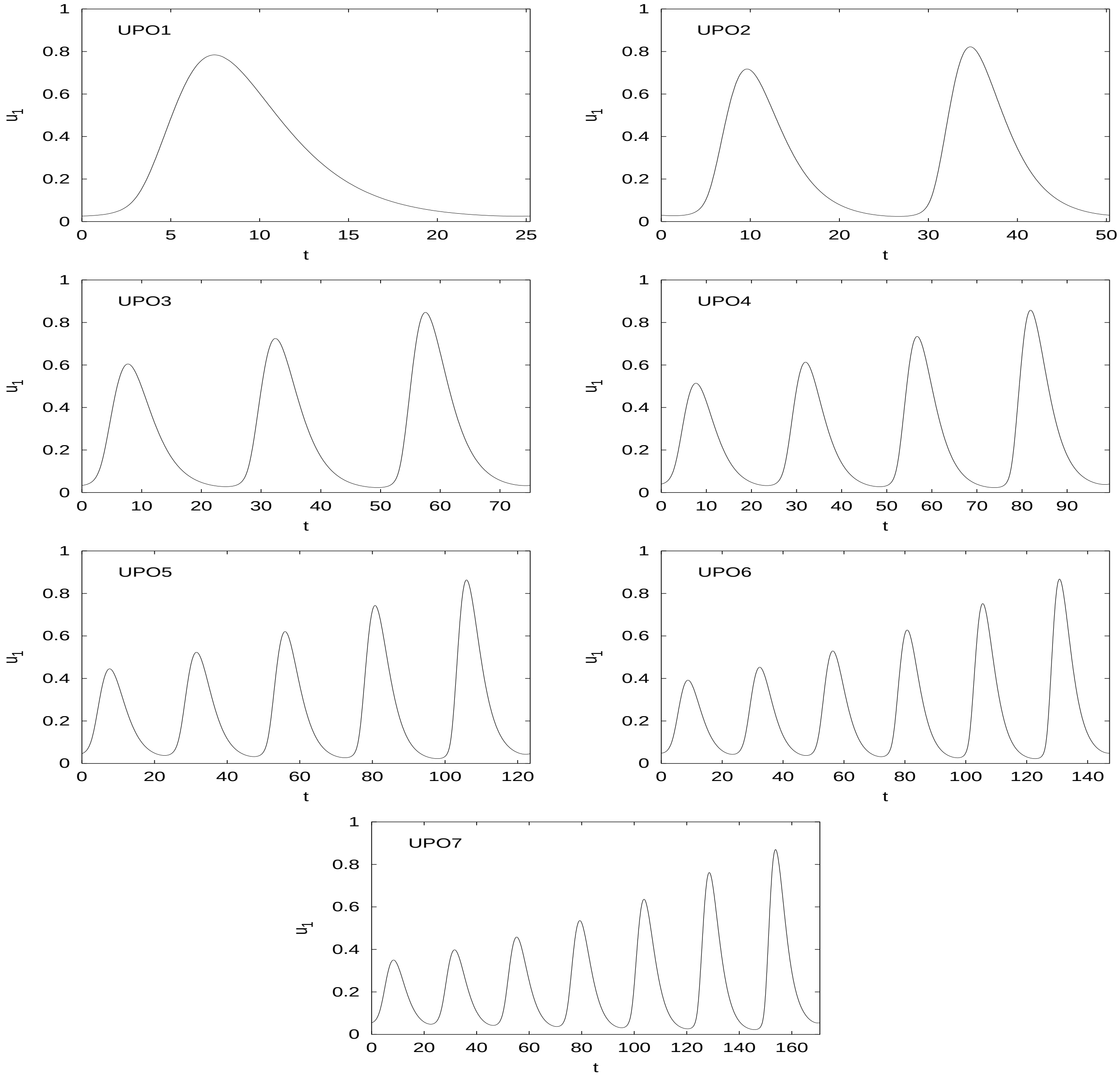


Figure: Time development of u_1 along $UPOn$ ($T \approx 25n$)

Seven types of UPOs ($UPO1, \cdots, UPO7$) are detected (NO $UPO8$ is found)

$\Downarrow$

<u>This corresponds to the main Regimes appearing in a chotic orbit</u>

UPOs with several Regimes [13]

Ex)$UPOn.m$ [composed of Regime n and Regime m]

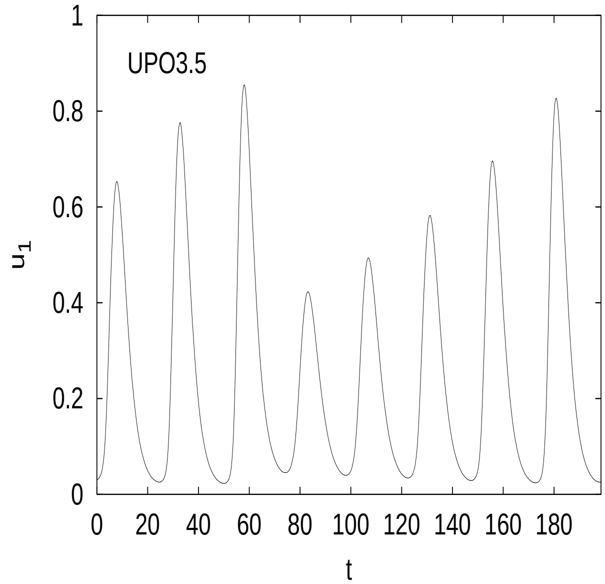


Figure: Time development of u_1 along $UPO3.5$
(Regime transition $3 \leftrightarrows 5$)

Ex)$UPOn.m.l$

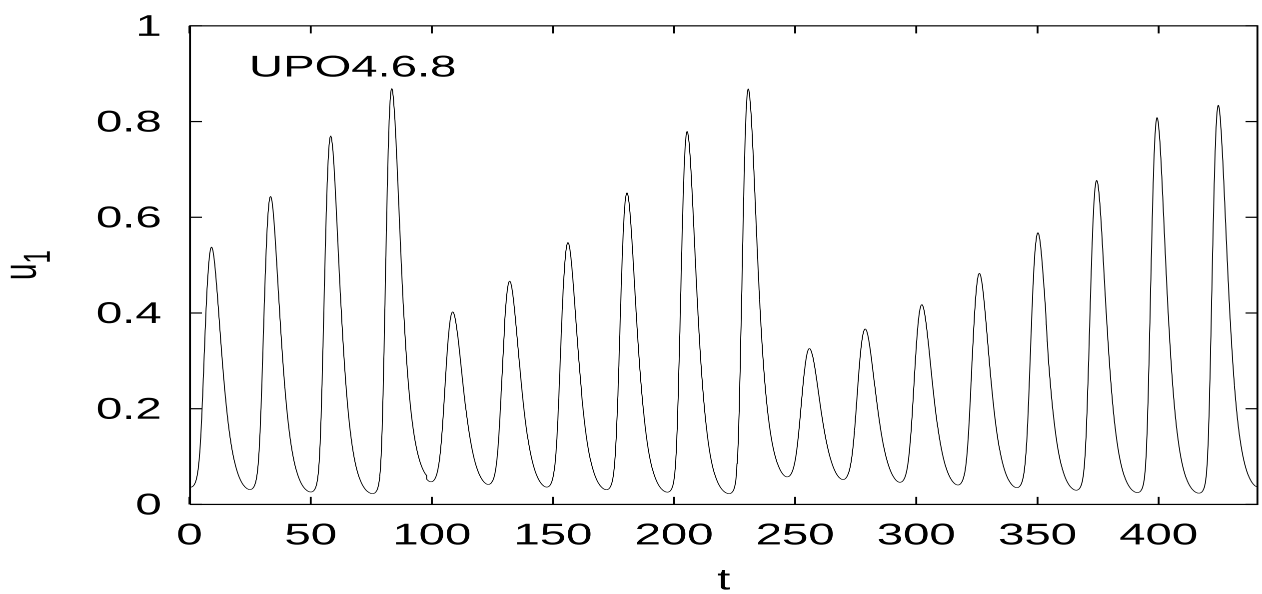


Figure: Time development of u_1 along $UPO4.6.8$
(Regime transition $4 \rightarrow 6$, $6 \rightarrow 8$, $8 \rightarrow 4$)

Regime transitions of chaos and UPOs [14]

From\To	1	2	3	4	5	6	7	8	9
1	C U	C U	C U						
2	C U	C U	C U						
3	C U	C U	C U	C U	C U	C U			
4	C U	C U	C U	C U	C U	C U	C U		
5	C U	C U	C U	C U	C U	C U	C U	C U	
6	C U	C U	C U	C U	C U	C U	C U	C U	
7	C U	C U	C U	C U	C U	C U	C U	C U	
8			C U	C U	C U				
9									

Table**:** Regime transitions observed in a chaotic orbit (C) and UPOs (U)

<u>All Regime transitions in a chaotic orbit are recognized by UPOs</u>

Ex) Chaos: Existence of Regime transition 4⟶6, Regime transition 6⟶8 and Regime transition 8⟶4
$\updownarrow$ consistent
UPO: Existence of $UPO4.6.8$

Ex) Chaos: Non-existence of Regime transition $8 \longrightarrow 8$
$\updownarrow$ consistent
UPO: Non-existence of $UPO8$

Bifurcations of periodic orbits [15]

Change of interaction between two countries (η)

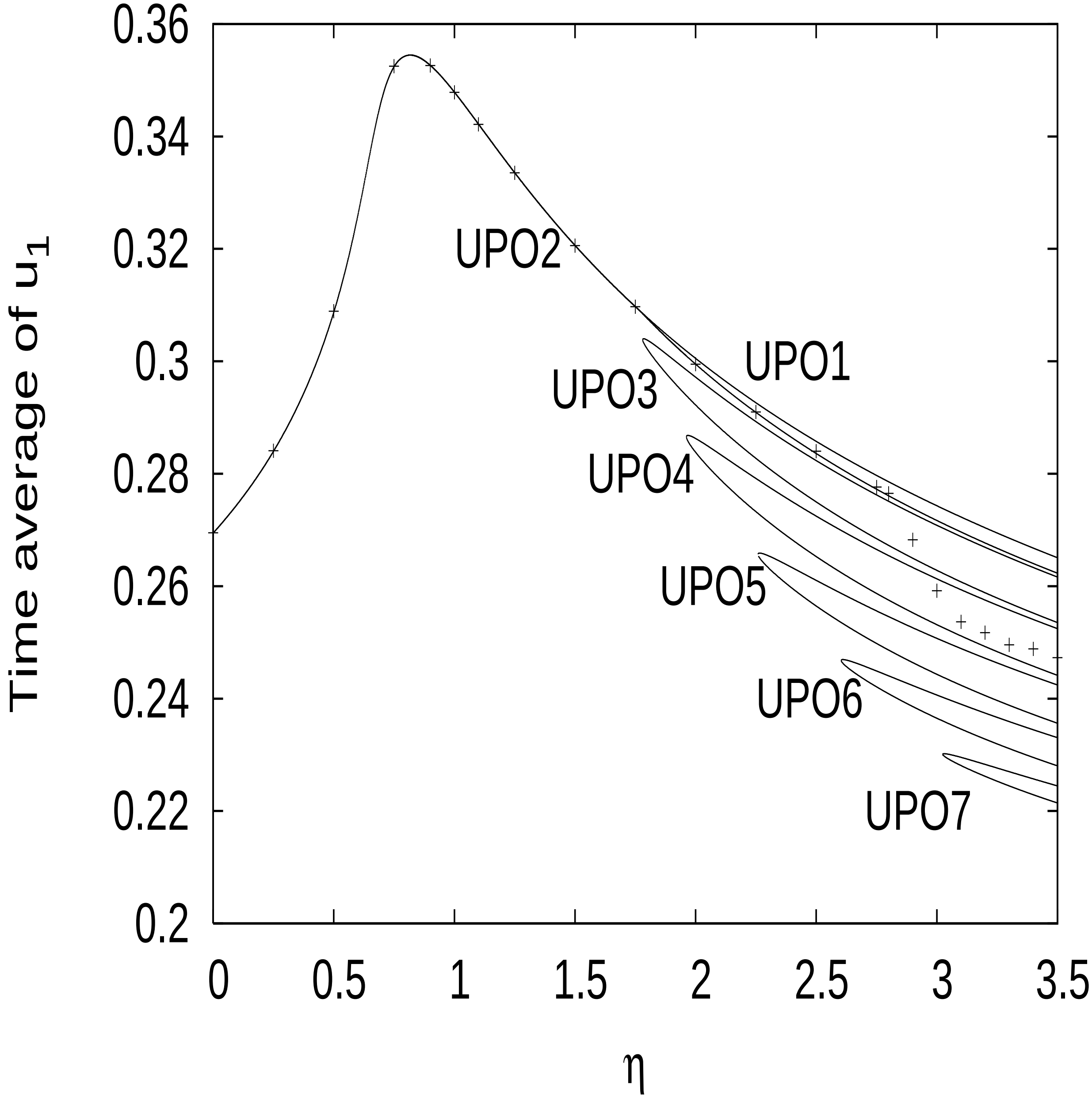


Figure: Time average of u_1 along UPOs at each η
(+ Time average along an orbit on the attractor)

$\eta = 3.5$ (usual) : $UPO1, \cdots, UPO7$ exist

$\eta = 2.6$: $UPO1, \cdots, UPO5$ exist

UPOs embedded in the attractor and chaos ($\eta = 2.6$) [16]

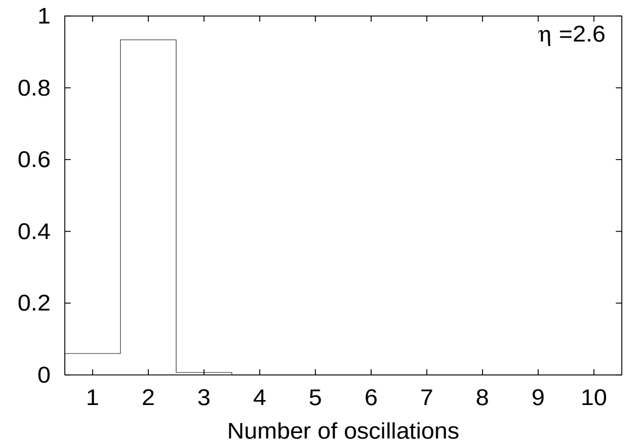


Figure: Regime appearance in a chaotic orbit($\eta = 2.6$)

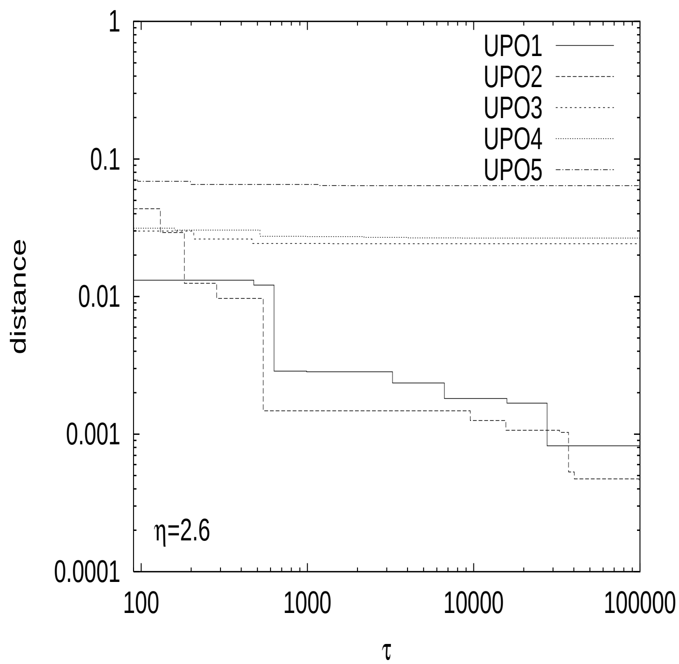


Figure: Distance between a point on the UPO ($\boldsymbol{x}_{UPO}$) and a chaotic orbit $\{\phi_t(\boldsymbol{X})\}_{t\geq 0}$ ($\mathrm{d}(\tau) = \min_{t\leq\tau} |\phi_t(\boldsymbol{X}) - \boldsymbol{x}_{UPO}|$)

- $\eta = 2.6$: Regime 1 and 2 are mainly observed $\leftrightarrow UPO1, \cdots, UPO5$ are detected. But only $UPO1, UPO2$ are embedded

UPOs embedded in the attractor and chaos ($\eta = 3.5$: Usual) [17]

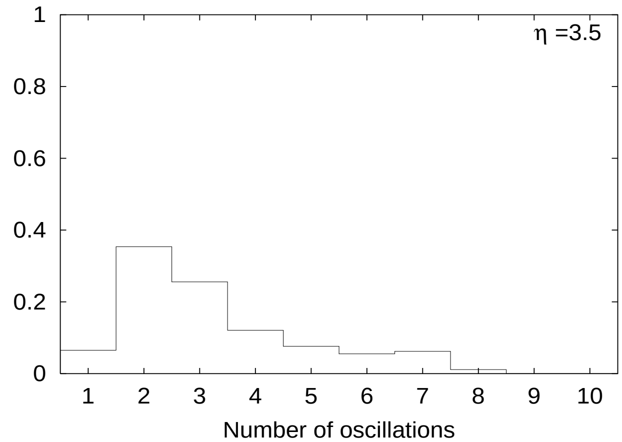


Figure: Regime appearance in a chaotic orbit

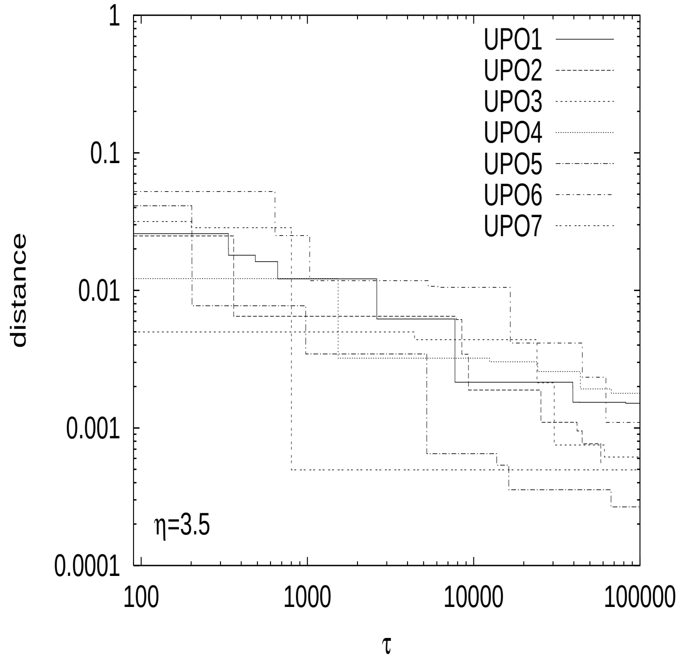


Figure: Distance between a point on the UPO ($\boldsymbol{x}_{UPO}$) and a chaotic orbit $\{\phi_t(\boldsymbol{X})\}_{t\geq 0}$

- $\eta = 3.5$: Regime1 $\cdots$ 7 are mainly observed $\leftrightarrow$ UPO1 $\cdots$ 7 are embedded in the attractor

<u>UPOs are to be embedded in the attractor for capturing chaotic behavior</u>

UPO2 is embedded in the attractor at various η [18]

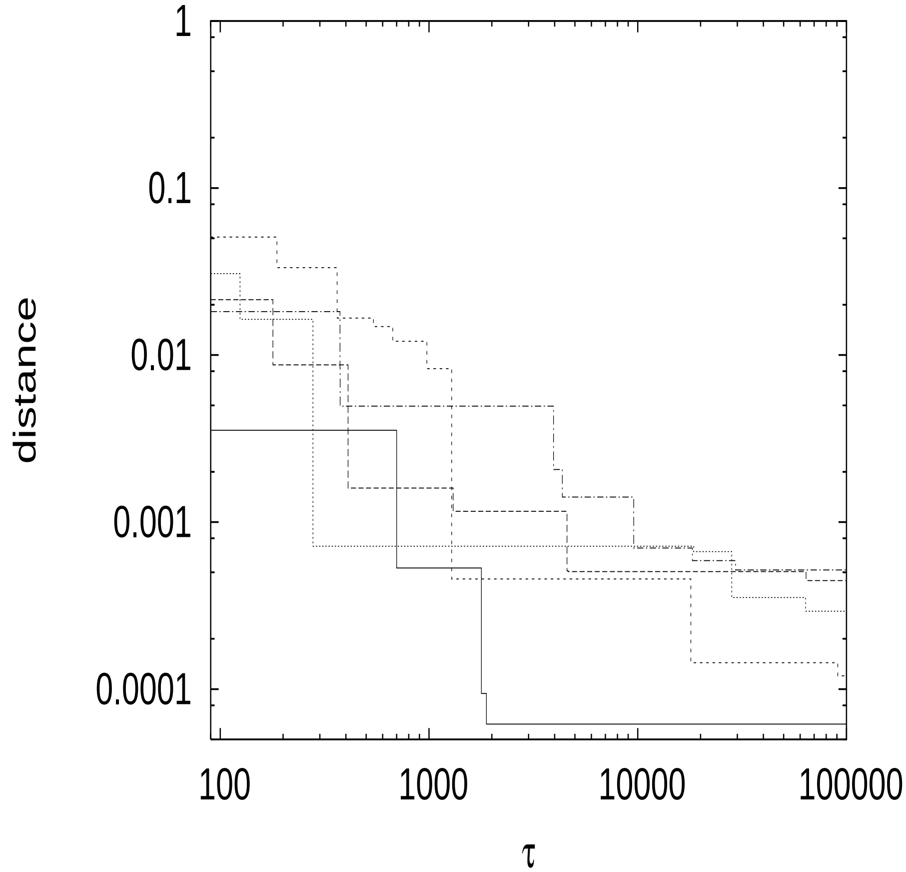


Figure : Distance between a point on the UPO2 and chaotic orbit (η =2.7, 2.9, 3.1, 3.3, 3.5)

$0 \leq \eta \lesssim 2.5$: UPO2 itself becomes the attractor

$\Downarrow$

UPO2 is embedded in the attractor at various η $(0 \leq \eta \leq 3.5)$

$\Downarrow$

<u>UPO2 constructs the skeleton of economic behavior at various η</u>

Statistical property (Time average along UPO) [19]

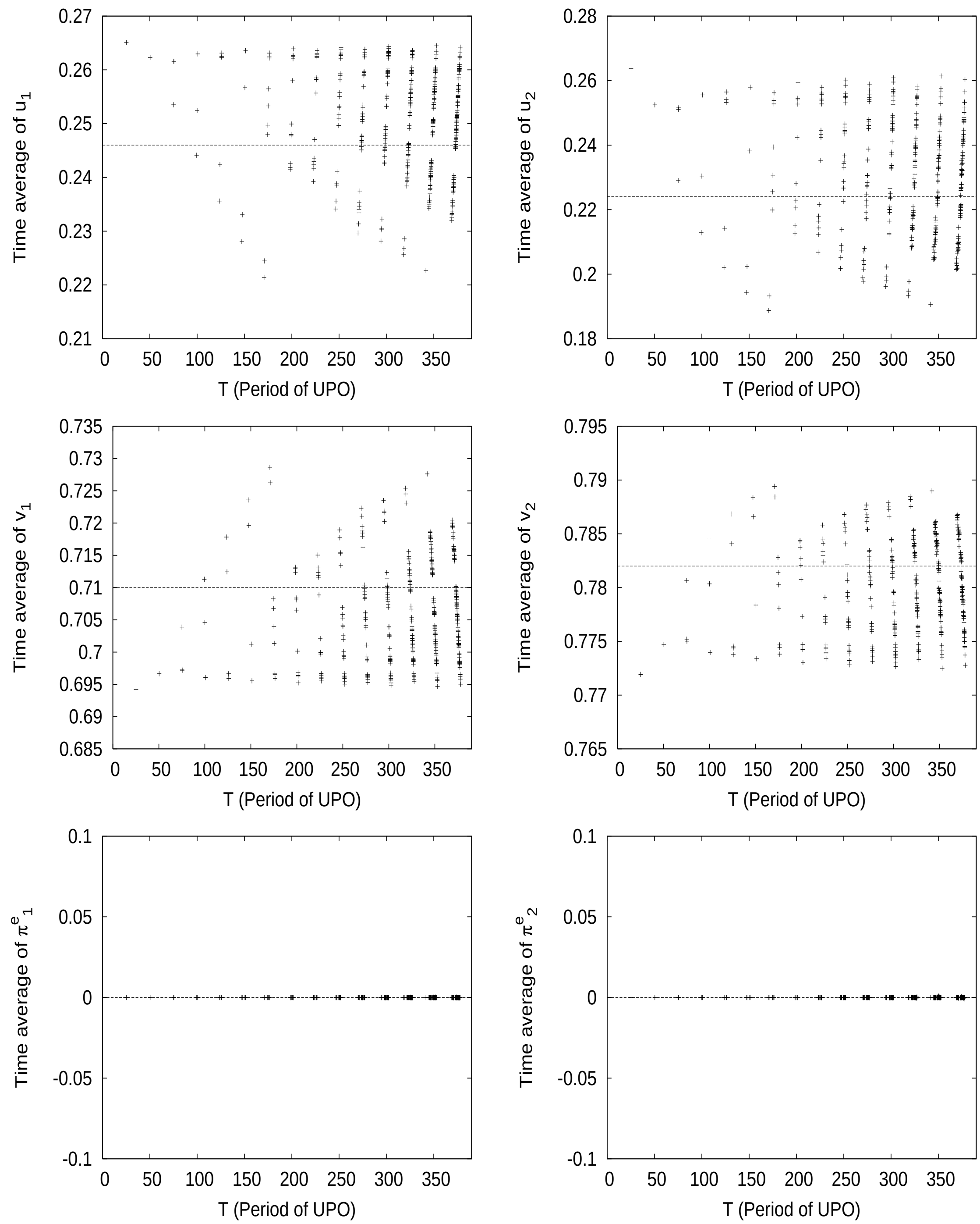


Figure : Time averages of six variables along UPOs with period T (Line: Chaotic orbit)

We can roughly estimate the time average of a chaotic orbit by one of any UPOs

Statistical correspondence between chaotic orbits and UPO2 [20]

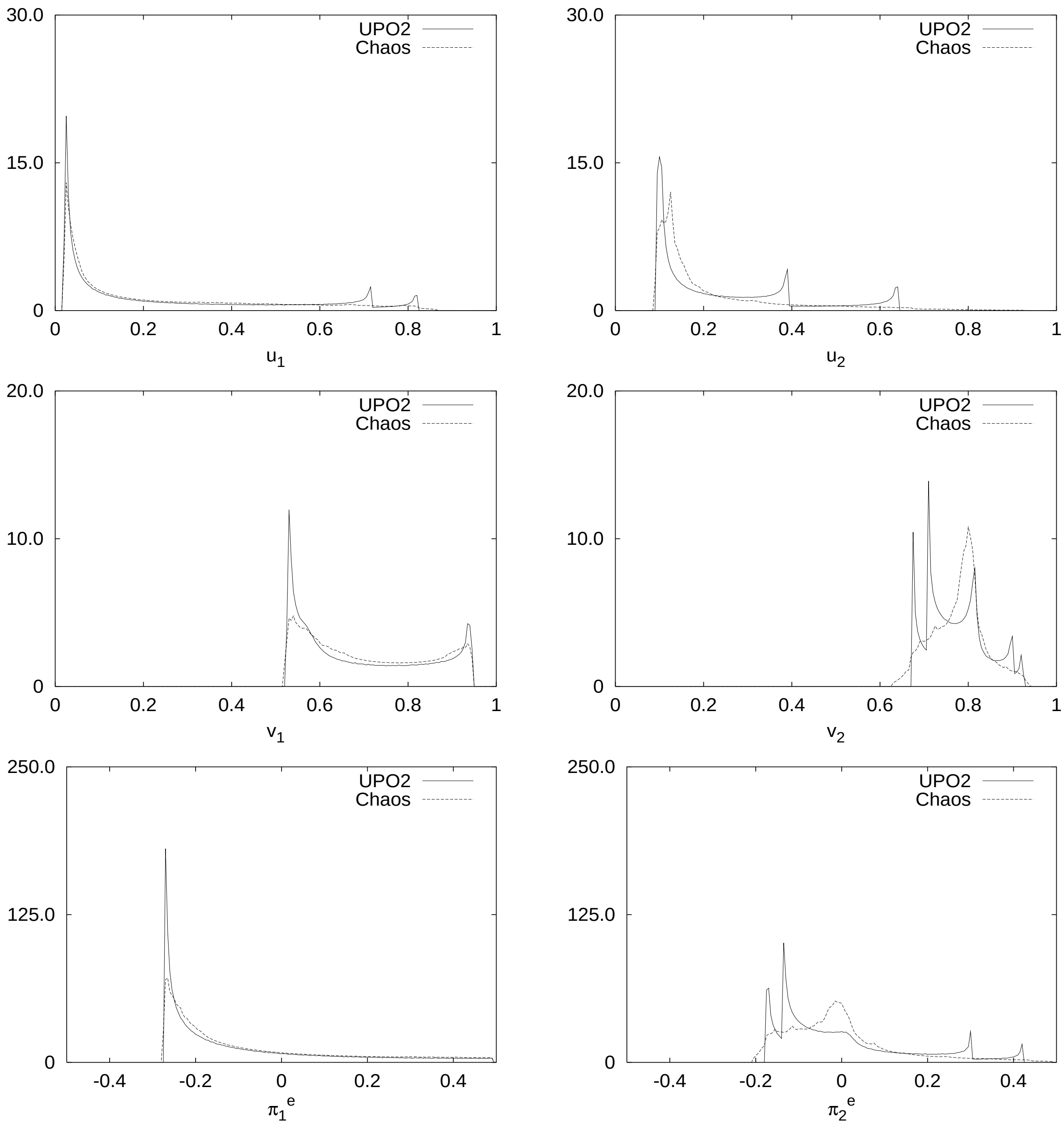


Figure: PDFs of six variables along a chaotic orbit (Dotted line) and UPO2 (Solid line) (left: 1st country, right: 2nd country)

<u>Statistics along a chaotic orbit and UPO2: Similar</u>

Number of UPOs [21]

Topological variations of chaotic orbits can be estimated by variations of UPOs

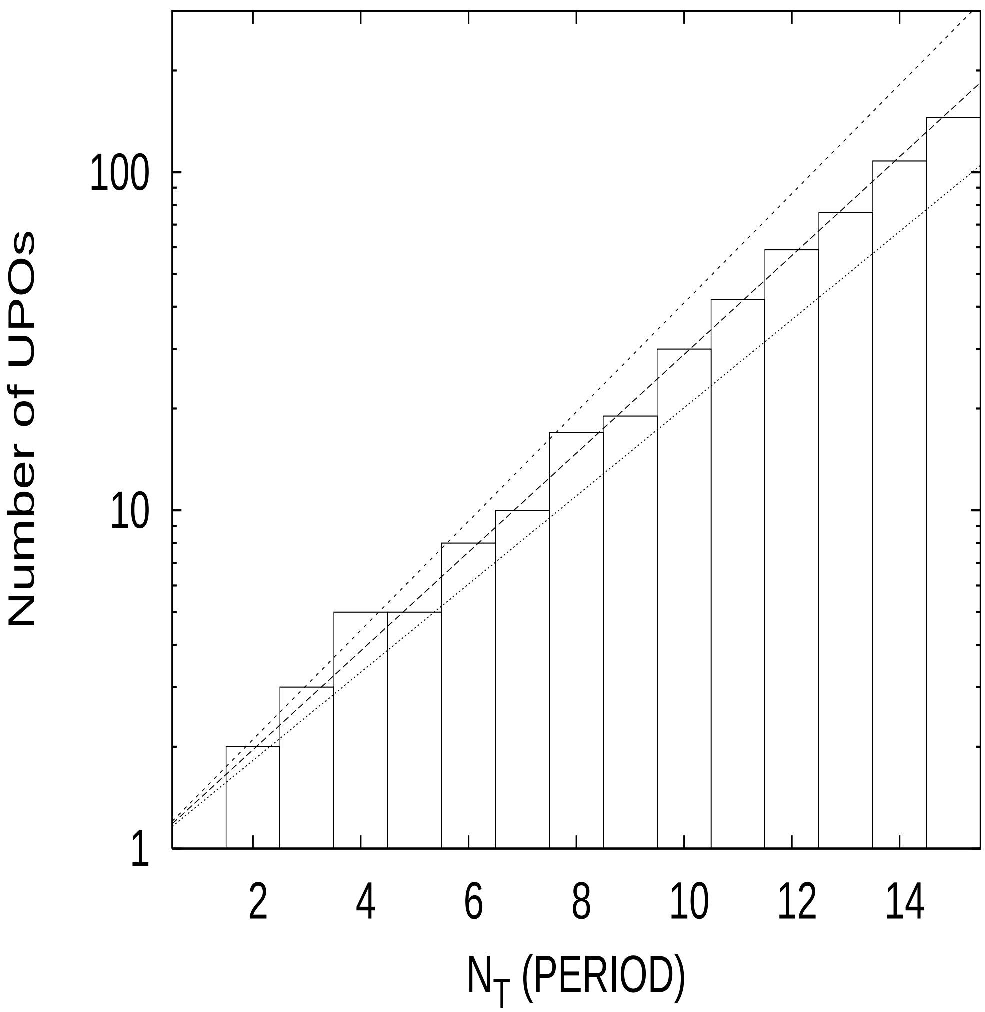


Figure: Number of UPOs against number of oscillations ($N_T \approx T/25$) and $1.35^{N_T}, 1.4^{N_T}, 1.45^{N_T}$

$$\#\{\text{UPOs with } N_T\text{-oscillations}\} \approx 1.4^{N_T}$$

$$\Downarrow$$

Topological entropy is $\log 1.4$

$$\Downarrow$$

<u>Complexity of the model: Low</u>

Conclusion [22]

Two-country chaotic business cycle model

- Several characteristics of the model are captured by various UPOs.
  - Typical structure (Regime, Regime transition) of a chaotic business cycle
  - Statistical property (Time average)
  - Complexity of chaos (Growth rate of number of UPOs)

- Macroscopic character is captured by only one of any UPOs (*e.g.* UPO2).

- UPO2 is still embedded in the attractor with weaker interactions between two countries

$$\Downarrow$$

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UPO2 is the skeleton of the attractor over a range of η

$\longrightarrow$ The macroscopic structure of business cycle is not affected by the change of interaction parameter.

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Remarks [23]

Business cycle theory

- Important topic in macro-economics
- Periodic fluctuation is essential characteristic
- Low dimensional dynamical systems
  - Stable model (Periodic point)
    [Gap exists, even if dynamics look similar]
  - Chaotic model

Our study

Chaotic business cycle model

$$\Downarrow$$

Characterized by unstable periodic orbits

$$\Downarrow$$

Bridge the gap through periodic orbit

[Details $\Rightarrow$ Please see our papers !]

Acknowledgements [24]

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Computer resource

- PC cluster at University of Tokyo

Future work [25]

Construct and analyze more realistic economic model by UPOs

- Non-hyperbolic structure [Quasi-stationary state]
- 'Multiple attractor'
- Chaotic no-attractor [Chaotic transient]
  (Chian et al. (2006))

Collaboration

Let's try UPO analyses about some chaotic model !
Please E-mail me ! ⇒saiki@ms.u-tokyo.ac.jp

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Validity of periodic orbits [A1]

Numerical error grows exponentially by time

rounding off error : $\delta = 10^{-16}$
stability exponent : $\lambda = 0.03$
permitted error : $err = 10^{-8}$

Maximum value of t (t_{max}) satisfying

$$\delta e^{\lambda t} \leq err \tag{1}$$

is

$$t_{max} \approx 613 \tag{2}$$

- Chaotic orbit : <u>Numerically invalid</u> in general
- Detected periodic orbit : <u>Numericaly valid</u>

$\Downarrow$

Big advantage

Numerical method for detecting unstable periodic orbits [A2]
(Newton·Raphson·Mees method + <u>damping</u>)

$\{\phi_t(\boldsymbol{X})\}_{t\in\boldsymbol{R}}$: the orbit passing through $\boldsymbol{X}(\in \mathbf{R}^n)$ at $t=0$ of

$$\frac{d\boldsymbol{x}}{dt} = f(\boldsymbol{x})\ (\boldsymbol{x}\in\mathbf{R}^n) \tag{3}$$

Periodic is identified by the zeros of

$$H(\boldsymbol{X},T) = \phi_T(\boldsymbol{X}) - \boldsymbol{X}.$$

$n+1$ unknowns: one point $\boldsymbol{X}^*$ + period T^*.

Algorithm (Newton method):

$$\Delta H(\boldsymbol{X},T) \approx D_{\boldsymbol{X}}H(\boldsymbol{X},T)\Delta\boldsymbol{X} + D_T H(\boldsymbol{X},T)\Delta T$$

Determine $\Delta\boldsymbol{X}$ and ΔT satisfying

$$\underline{H(\boldsymbol{X},T) + \Delta H(\boldsymbol{X},T) = 0.} \tag{4}$$

Additional constraint : Modified vector $\Delta\boldsymbol{X}$ is orthogonized by the orbit;

$$\underline{< f(\boldsymbol{X}), \Delta\boldsymbol{X} >= 0.}$$

$$\Downarrow n+1 \text{ constraints}$$

Introducing damping coefficient m

$$(\boldsymbol{X}', T') = (\boldsymbol{X},T) + 2^{-m}(\Delta\boldsymbol{X}, \Delta T)$$

$$(2^{-m} \sim 1/e^{\lambda T}\ (\lambda : \text{stability exponent}))$$

Criterion of convergence [A3]

- $\left|\phi_{T^{(i)}}(\boldsymbol{X}^{(i)}) - \boldsymbol{X}^{(i)}\right|$ (Practical error)
- $\left|(\Delta \boldsymbol{X}^{(i)}, \Delta T^{(i)})\right|$ (Absolute value of modified vecor)

are sufficiently small (order: $10^{-9}, 10^{-7}$)

Detecting unstable periodic orbit

More than 9000 periodic orbits are numerically detected

$\Downarrow$(remove overlapped orbits)

1000 sorts of UPOs

Damping coefficient [A4]

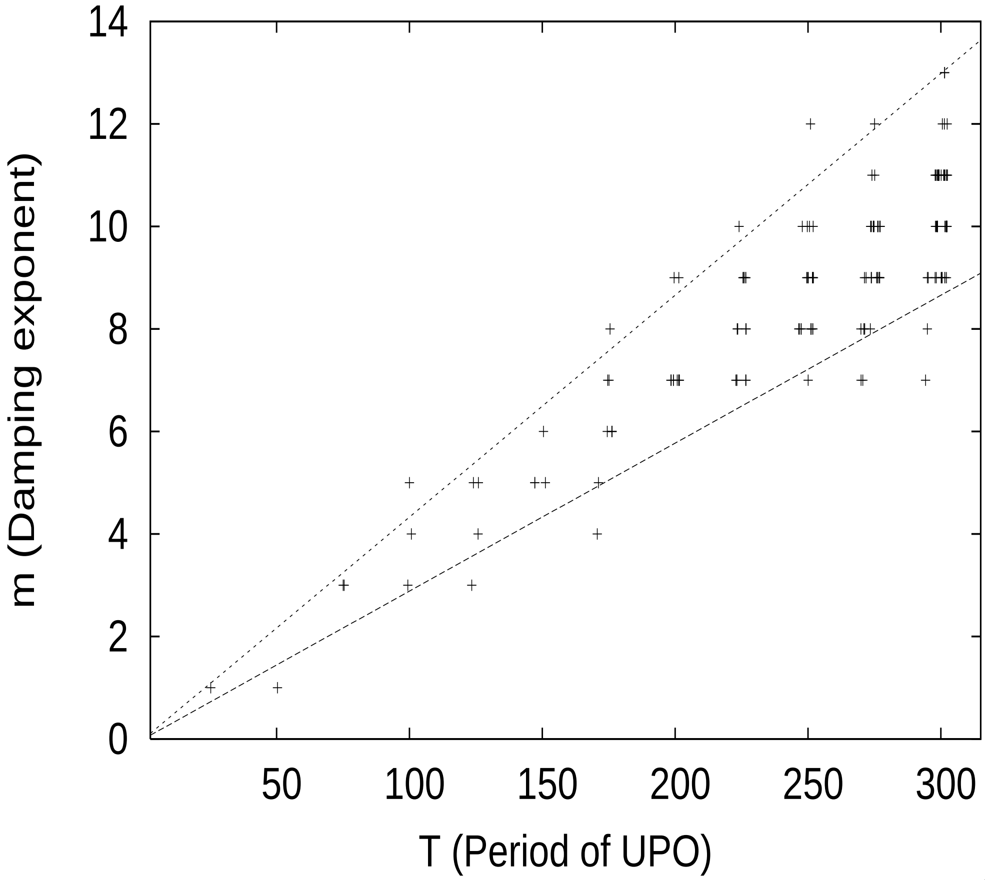

Figure : Minimal damping coefficient $m(\in \mathbf{Z})$ needed to detect each periodic orbit and $\frac{\lambda T}{\log 2}$ $(\lambda = 0.02, 0.03)$

This corresponds to the fact that Floquet exponent of periodic orbits are between 0.02 and 0.03.

Validity of periodic orbit (time step) [A5]

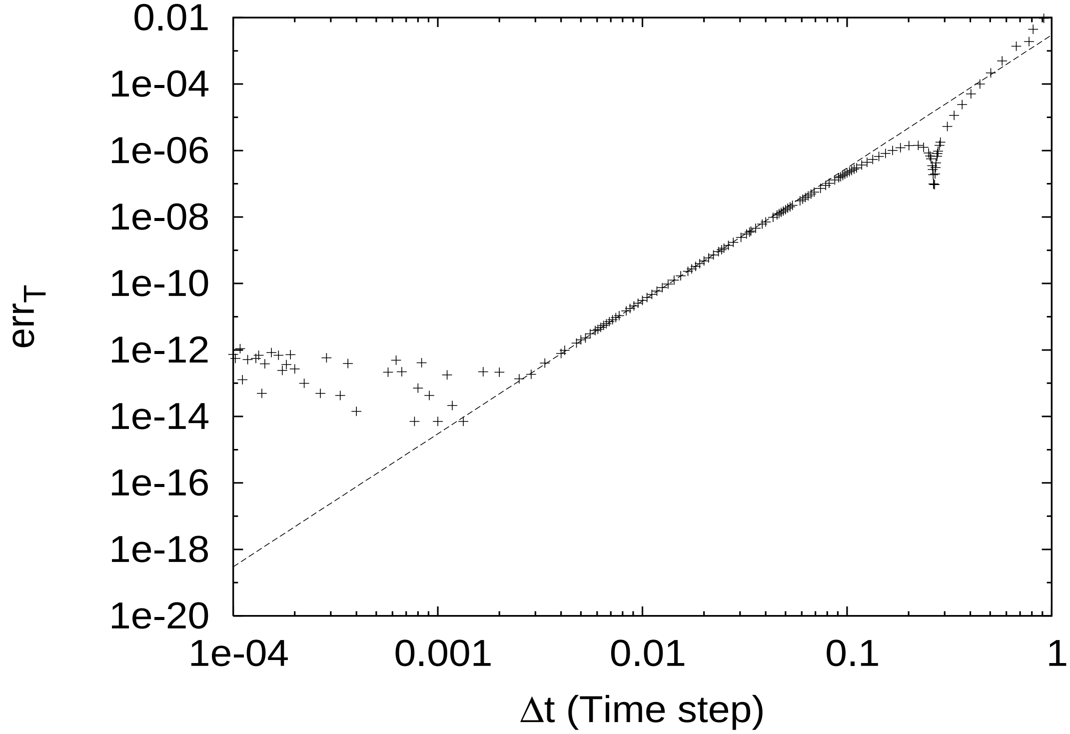

Figure : Error ($err_T(\Delta t) := |T - 50.344307687915|$) about the period of UPO2 numerically integrated by various time steps (Δt)

$$err_T(\Delta t) \propto (\Delta t)^4 \quad (0.003 \lesssim \Delta t \lesssim 0.1)$$

$\updownarrow$

Fourth order Runge-Kutta method

$\Downarrow$

Numerical time integration of UPO2: Valid

(Cancelling effect: ($\Delta t \lesssim 0.003$))

Goodwin type growth cycle model [A6]

Class conflict between capitalists and labors

- Goodwin(1967) : continuous, conservative (stable (center)) periodic orbit
- Desai(1973) : continuous, dissipative (limit cycle : Expected rate of inflation)
- Pohjola(1981) : discrete (chaos)
- Wolfstetter(1982) : continuous, dissipative (fixed point, limti cycle : Keynes type finantial policy)
- Sportelli(1995):continuous, dissipative (limit cycle: price rigidity)
- Yoshida & Asada(2001) : continuous, dissipative (Choas : time lag in policy)
- Ishiyama & Saiki(2005) : continuous, dissipative (Chaos : two countries)

Harvie(2000) : Empirical research (OECD ten countries)

Skelton of the model [A7]

(Agents：capitalists, labors， government)

Labor share rate $u \equiv \dfrac{wL}{pY} \left(= \dfrac{w}{pa}\right)$ $\left[\dfrac{du}{dt} = (\hat{w} - (\hat{p} + \hat{a}))u\right]$

Employment rate $v \equiv \dfrac{L}{N} \left(= \dfrac{Y}{aN}\right)$ $\left[\dfrac{dv}{dt} = (\hat{Y} - (\hat{a} + \hat{N}))v\right]$

Expected rate of inflation π^e $\left[\dfrac{d\pi^e}{dt} = \theta\left(\hat{p} - \pi^e\right)\right]$

Variable

w: Nomial wage rate
L: Labor level
p: Price level
Y: Gross national output
(=Gross national income=Gross national expenditure)
N: Labor supply
[$a(\equiv Y/L)$: Labor productivity]
[$\hat{p}(\equiv \dot{p}/p)$: Actual inflation rate]

Constant

$\alpha(\equiv \hat{a})$: Technical progress rate
$\beta(\equiv \hat{N})$: Growth rate of labors
θ: Adaptive speed of worker's expectation

- Y is effected by the mutual action between countries
- $\hat{x}$ means the growth rate of x $(\dot{x}/x)$

Details of the model [A8]

Wage bargaining $\hat{w}_i = f_i(v_i, \pi_i^e)$, $\dfrac{\partial f_i}{\partial v_i} > 0, \dfrac{\partial f_i}{\partial \pi_i^e} > 0$,

Price fluctuation $\hat{p}_i = \gamma_i(\hat{w}_i - \alpha_i)$,

(Elasticity coefficient γ_i=0.5, Technical progress rate α_i=0.02)

Output plan $\hat{Y}_i = \varepsilon_i \left(\dfrac{I_i + G_i + C_i - Y_i}{Y_i} \right)$,

(Output adjustment coefficient $\varepsilon_i = 0.1$)

Private investment $I_i = \underline{\underline{h_i(u_i, u_j)}}Y_i$,

$$\frac{\partial h_i}{\partial u_i} < 0,\ \frac{\partial h_i}{\partial u_j} > 0,\ \ j = 1, 2,\ j \neq i.$$

Government expenses $G_i = \delta_i Y_i + \mu_i(v^* - v_i)Y_i$

(Income tax rate δ_i=2/7, Reaction coefficient of the government $\mu_1 = 1.25, \mu_2 = 6.0$)

Consumption $C_i = c_i \left((1 - \delta_i)((1 - u_i)Y_i + r_i B_i) - q_i \dfrac{dB_i}{dt} \right)$

$$+ (1 - \delta_i)u_i Y_i$$

(Consumption coefficient of capitalists c_i=0.3)

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Phillips curve $f_i(v_i, \pi_i^e) = 0.1 \left(\dfrac{1}{1 - v_i} - 4.8 \right) + \pi_i^e$

Investment function $h_i(u_i, v_j) = 1.5(1 - u_i)^5 + \eta(u_j - u_i)^3$,

$\eta > 0$

Statistical value of UPO [A9]

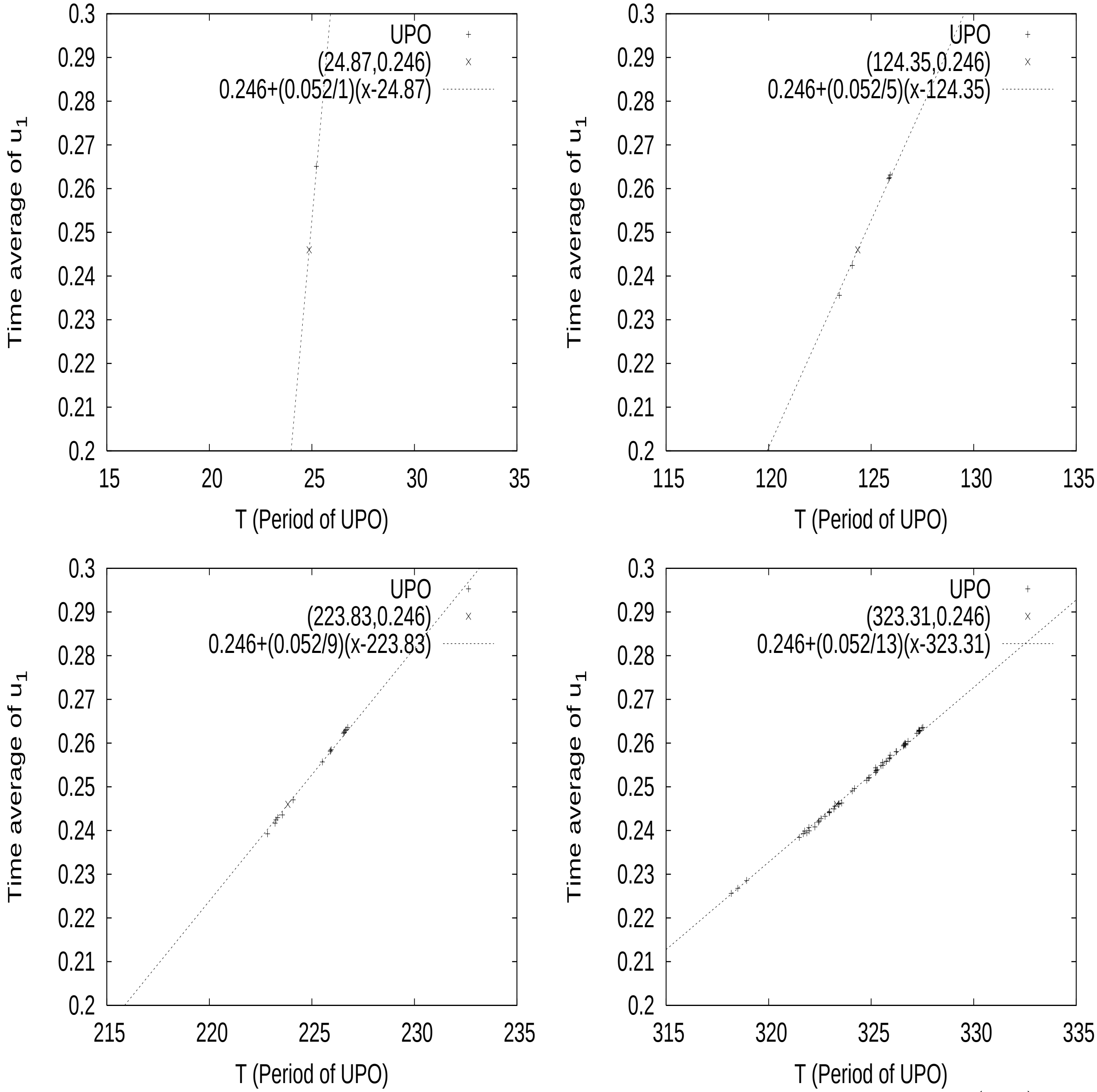

Figure : Time average of u_1 along UPO with period T ($\overline{u_1}$) and $(0.246 + (0.052/N_T)(T - 24.87N_T))$

Any UPOs seem to have macroscopically similar structure

New knowledges about the UPO analysis [A10]

- Detection of UPO: Damping coeffiiceint is important [General]
- Various chaotic behavior: classified by UPOs [General (Hyperbolic)]
- UPO sometimes becomes out of the attractor by the change of parameter [General]
- Deformation of chaotic attractor corresponds to the change of variations of UPOs embedded in the attractor [General]
- Exponential growth rate of UPOs can be estimated by detected UPOs [Hyperbolic ('Single global structure')]
- Statistics of UPOs with the same Poincare map periods have smooth dependence on the period of UPO [Hyperbolic ('Single global structure')]
- Statistical values of chaos are approximated by one of any UPOs [Hyperbolic ('Single global structure')]
- UPO with long period does not necesarily well approximate the statistical value of chaos [General]