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Rank Process and Stochastic Corridor
: Nonparametric Statistics of Lognormal Observations and Exotic Derivatives based on them

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Outline of my talk.

:1. Non-Parametric Statistics

discrete time	continuous time
: (1) Empirical Distribution Function	(Fixed level Corridor)
: (2) Order Statistics	Alpha-Quantile
: (3) Rank Statistic	Rank Process

: 2. Exotic Options based on them

- Corridor with Fixed Level K, and with Stochastic Level S at t.
- Derivatives. : Forward, Swap, Options based on the Two Types of Corridor

: (1) Spot Start Corridor

: (2) Forward Starting Corridor

: 1. Definition of Non-Parametric Statistics/Exotic options

We assume that the stock price S_u follow a Geometric Brownian motion;

$$S_u = S_0 e^{X_u} = S_0 e^{\mu u + \sigma W_u} \text{ , for u} \in \text{[o,T]}$$

through out this paper.

The following lemma plays a key role at several parts in this paper, where a calculation encounters for an expectation of our non-parametric statistics such as a fixed corridor and a stochastic corridor. The proof of the following lemma is referred to Fujita(1997), Fujita & Miura(2002,2004), and Borodin & Salminen(2002) (This handbook shows the formulae without proof)

Lemma 1.

$$P(W_t \in da, \int_0^t I\{W_s < 0\} ds \in du)$$
$$= \begin{Bmatrix} (\int_u^t \frac{a}{2\pi\sqrt{s^3(t-s)^3}} e^{\frac{-a^2}{2(t-s)}} ds) dadu \cdots fora > 0 \\ (\int_0^u \frac{-a}{2\pi\sqrt{s^3(t-s)^3}} e^{\frac{-a^2}{2s}} ds) dadu \cdots fora < 0 \end{Bmatrix}$$

: 1-1. Corridor function: Empirical Distribution Function

The probability distribution of the fixed level corridor was derived for a drift-less case in a brief manner in Miura(1992), where the fixed level corridor was discussed as the limit of an empirical distribution function of stock prices during the time interval [0,T]. The mathematical argument to derive its probability distribution can be reduced to that for the probability distribution of

$$\int_0^t I\{W_u \le 0\}du$$

, and then Cameron-Martin Theorem can be used to argue for the non-zero drift case.

Let μ =0 for the time being.

For any fixed constant K, the quantity

$$\int_0^T I\{S_u \le K\}du$$

$$= \int_0^T I\{S_0 e^{\sigma W_u} \le K\}du$$

$$= \int_0^T I\{W_u \le \frac{1}{\sigma}\log\left(\frac{K}{S_0}\right)\}du$$

(assume $S_0 < K$ and put $A=\frac{1}{\sigma}\log\left(\frac{K}{S_0}\right)$)

$$= \int_0^T I\{W_u \le A\}du$$

(define $\tau = \inf\{u : W_u \le A, 0 < u < T\}$)

$$= \tau + \int_\tau^T I\{W_u - W_\tau + W_\tau \le A\}du$$

(note that $W_\tau = A$)

$$= \tau + \int_\tau^T I\{Z_{u-\tau} \le 0\}du$$

(where we put $Z_{u-\tau} = W_u - W_\tau$)

$$= \tau + \int_0^{T-\tau} I\{Z_u \le 0\}du$$

This counts how much time during the time interval [0,T] the stock price stays below the given level K. It is thus a continuous time version of the empirical process for stock prices S_u during the time interval [0,T]. This is

called a corridor or a corridor option
In this paper, we call it a fixed level corridor or in short, a fixed corridor.
The probability distribution of

$$\int_0^T I\{S_u \le K\}du$$

where

$$S_u = S_0 e^{\sigma W_u}$$

, is given by

$$G(x, K : 0, \sigma) = P\{\int_0^T I\{S_u \le K\}du < x\}$$

$$= \int_0^x \frac{2}{\pi}\sin^{-1}((\frac{x-s}{1-s})^{\frac{1}{2}}) \cdot h_A(s)ds$$

where h_A is the probability density function for the stopping time τ;
$\tau = \inf\{u : W_u > A\}$

For the case with a drift $\mu \neq 0$, Cameron-Martin theorem can be used to obtain its probability distribution function ;
By Cameron-Martin theorem, we have

$$E[h(\int_0^T I\{S_0 e^{\mu u + \sigma W_u} \le K\}du)] = E[e^{\frac{\mu}{\sigma}W_T - (\frac{\mu}{\sigma})^2\frac{T}{2}} h(\int_0^T I\{W_u \le \frac{1}{\sigma}\log\frac{K}{S_0}\}du)]$$

for any integrable function h(.), and then,

: 1-2. α-quantile: Order Statistics

α-quantile is defined as a quantity m(α) for any given $\alpha \in [0, 1]$ such that

$$\alpha = \frac{1}{T}\int_0^T I\{S_u \leq m(\alpha)\}du$$

i.e. α-quantile is the level below which the stock price stays for 100α-percent of time during the time interval [0,T]. Thus it could be regarded as a continuous time version of "order statistics" of stock prices observed during the time interval [0,T]. This quantity depends on the path of stock price and it can be determined only at the end of the time interval. That is why the derivatives based on this are called the "look-back" options.

Based on the relation; for any y≧0,

$$\{\frac{1}{T}\int_0^T I\{S_u \leq y\}du < \alpha\} = \{m(\alpha) > y\}$$

the probability distribution for m(α) can be obtained as
P$\{m(\alpha) < y\} = 1 - G(\alpha, y : \mu, \sigma)$

Akahori(1995) rigorously derived the probability distribution of m(α), and Fujita(1997) derived the joint probability density function of

$$(S_T, m(\alpha))$$

in order to price a call option with payoff function; max{S_T - m(α),0}.

: 1-3. Stochastic Corridor:Rank Statistics

For any fixed time t in [0,T], the rank process can be defined and presented as follows.

$$
\begin{aligned}
T \cdot R_{t,T}^{*\mu,\sigma} &= \int_0^T I\{S_u \le S_t\} du \\
&= \int_0^T I\{S_0 e^{X_u} \le S_0 e^{X_t}\} du \\
&= \int_0^T I\{X_u \le X_t\} du \\
&= \int_0^T I\{\mu u + \sigma W_u \le \mu t + \sigma W_t\} du \\
&= \int_0^T I\{\frac{\mu}{\sigma} u + W_u \le \frac{\mu}{\sigma} t + W_t\} du
\end{aligned}
$$

Denote this by $R_{t,T}^{*\mu,\sigma}$ to indicate the parameters of the process X_t.

Fujita and Miura (2004) remarks that the rank statistics in the case $\mu=0$, $\sigma=1$, is decomposed to a weighted sum of two independent random variables A_1 and A_2, each of which follows an arcsine law. That is;

$$
\begin{aligned}
T \cdot R_{t,T}^{*0,1} &= \int_0^T I\{W_u \le W_t\} du \\
&= \int_0^t I\{W_u - W_t \le 0\} du + \int_t^T I\{W_u - W_t \le 0\} du \\
&= t \frac{1}{t} \int_0^t I\{Z_s \le 0\} ds + (T-t) \frac{1}{T-t} \int_0^{T-t} I\{Z_s^* \le 0\} du \\
&(\text{where } Z_s = W_{t-s} - W_t \text{ and } Z_s^* = W_{t+s} - W_t) \\
&= tA_1 + (T-t)A_2
\end{aligned}
$$

Then, Fujita and Miura (2004) showed that Cameron-Martin theorem helps us to reduce the expectation to the drift-less case

$$\mu = 0, \sigma = 1$$

That is, for any integrable function h(.),

$$E[h(T\square R_{t,T}^{*\mu,\sigma})] = E[e^{\frac{\mu}{\sigma}W_T - (\frac{\mu}{\sigma})^2\frac{T}{2}} h(T\square R_{t,T}^{*0,1})]$$

$$= \iint_{-\infty<x<\infty, 0<y<1} e^{\frac{\mu}{\sigma}x - (\frac{\mu}{\sigma})^2\frac{T}{2}} h(y) f_{(W_T, T\square R_{t,T}^{*0,1})}(x,y)dydx$$

More precisely,(Note that $W_T = Z_{T-t}^* - Z_t$)

$$= E[e^{\frac{\mu}{\sigma}(Z_{T-t}^* - Z_t) - (\frac{\mu}{\sigma})^2\frac{T}{2}} h(\int_0^t I\{Z_s \le 0\}ds + \int_0^{T-t} I\{Z_s^* \le 0\}ds)]$$

$$= \iint_{-\infty<x_1<\infty, 0<y_1<T-t} \iint_{-\infty<x_2<\infty, 0<y_2<t} e^{\frac{\mu}{\sigma}(x_1 - x_2) - (\frac{\mu}{\sigma})^2\frac{T}{2}} h(y_2 + y_1)$$

$$f_{(Z_{T-t}^*, \int_0^{T-t} I\{Z_s^* \le 0\}ds)}(x_1, y_1)dy_1dx_1 f_{(Z_t, \int_0^t I\{Z_s \le 0\}ds)}(x_2, y_2)dy_2dx_2$$

Thus, it was enough to derive a joint probability distribution function (or density function) of (W_T, $R_{t,T}^{*0,1}$) rather than that of (W_T, $R_{t,T}^{*\mu,\sigma}$) in order to calculate the above expectation. These joint densities of the decomposed variables;

$$f_{(Z_{T-t}^*, \int_0^{T-t} I\{Z_s^* \le 0\}ds)}(x_1, y_1), f_{(Z_t, \int_0^t I\{Z_s \le 0\}ds)}(x_2, y_2)$$

can be obtained from Lemma1.

: 2. Exotic Derivatives based

: 2-1. Fixed Level Corridor and Stochastic Level Corridor (a). Fixed Level Corridor

For any fixed constant K, the quantity

$$F_{K,T}^{\mu,\sigma} = \int_0^T I\{S_u \le K\}du$$

is called a fixed level corridor or in short, fixed corridor.

In some applications such as the currency exchange rate derivatives, this statistics counts how many days the exchange rate stays below the given fixed level K, and the pay-off (which is the value of the derivative at the time of exercise, or of expiration) of the derivative (contract) may promise to pay to the holder of the derivative the amount of money proportional to the statistics. This is called a corridor option.

These corridors could be used in principle for the weather derivatives, for example, to count the number of days where the temperature of the day stays below the prefixed or stochastic level.

(b). Stochastic Level Corridor

The rank process for any prefixed time point t in [0,T]

$$R_{t,T}^{\mu,\sigma} = \int_0^T I\{S_u \le S_t\}du$$

can be used in a similar interpretation to the fixed corridor, by replacing the K in the fixed corridor with St which is stochastic.

The fixed level corridor and the stochastic level corridor both can be used by itself as a payoff of derivatives.

The prices of them at time 0 are given respectively by

$$e^{-rT}E_0[F_{K,T}^{r,\sigma}]$$

$$e^{-rT}E_0[R_{t,T}^{r,\sigma}]$$

in a Black-Scholes market.

2-2 Swap of Stochastic and Fixed level corridor

Here we go further to define a “swap” or an exchange of the two derivatives which requires some thought of what the value of K should be. The payoff of the swap contract is

$$F_{K,T}^{\mu,\sigma} - R_{t,T}^{\mu,\sigma}$$

$$(= \int_0^T I\{S_u \le K\}du - \int_0^T I\{S_u \le S_t\}du).$$

The price of swap at the time of the contract is zero so that we have an equation as usual,

$$0 = e^{-rT} E_0[\int_0^T I\{S_u \le K\}du - \int_0^T I\{S_u \le S_t\}du]$$

Thus, the prefixed constant K has to satisfy the equation

$$E_0[\int_0^T I\{S_u \le K\}du] = E_0[\int_0^T I\{S_u \le S_t\}du]$$

In order to see the existence of such a constant K, we note that the right hand side is a non-negative bounded constant less than T, and that the left hand side is a strictly increasing continuous function of K ranging from zero to T, so that there must exist a constant K which satisfies the above equality.

2-3 Corridor Option

It is possible to define Put-type and Call-type options using the fixed and stochastic corridor. And their pricing can be done in a straight manner since it does not require any further distributional argument.

For a notational simplicity, let us set as follows;

$$F_{K,T}^{\mu,\sigma} = \int_0^T I\{S_u \le K\}du$$

$$R_{t,T}^{\mu,\sigma} = \int_0^T I\{S_u \le S_t\}du$$

We define a corridor call option on the stochastic corridor with the fixed level corridor as its exercise value. The pay-off of the corridor call option is

$$V_{C,T} = \max\{R_{t,T}^{\mu,\sigma} - F_{K,T}^{\mu,\sigma}, 0\}.$$

Also in a similar way, the pay-off of the corridor put option is

$$V_{P,T} = \max\{F_{K,T}^{\mu,\sigma} - R_{t,T}^{\mu,\sigma}, 0\}.$$

The price of these Call and Put at time zero in the Black-Scholes model are given as

$$V_{C,0} = e^{-rT}E_0[V_{C,T}]$$

and

$$V_{P,0} = e^{-rT}E_0[V_{P,T}]$$

respectively.

The expectation for Call option can be calculated in the similar way as in the above.

:2-4. Forward Starting Corridor

Let $[T_0, T_1]$, $0 < T_0 < T_1$, be a future time interval where a corridor option counts the amount of time that the stock prices stay below the level, either a prefixed one or a stochastic one. The payoffs of forward starting fixed corridor and stochastic corridor are respectively,

$$F^{\mu,\sigma}_{K,(T_0,T_1)} = \int_{T_0}^{T_1} I\{S_u \le K\} du$$

and

$$R^{\mu,\sigma}_{t,(T_0,T_1)} = \int_{T_0}^{T_1} I\{S_u \le S_t\} du .$$

A contract is made at time 0 and the payoff is paid to the holder at time T_1. Then the prices of these options in the Black-Scholes model are

$$e^{-rT_1} E_0[\int_{T_0}^{T_1} I\{S_u \le K\} du]$$

$$e^{-rT_1} E_0[\int_{T_0}^{T_1} I\{S_u \le S_t\} du]$$

respectively.
As we saw in the previous section, note that the probability distribution of the stochastic corridor is independent of the value

$$S_{T_0}$$

; the value of the initial stock price in the future time interval $[T_0, T_1]$. This independence property may be expected to be useful in practice when they set a level for the corridor. In order to decide a constant level K, it may be required in practice to have a certain idea or a prediction of overall level of stock prices during the future time interval $[T_0, T_1]$. Since it is not easy to make a prediction, it may be plausible sometimes to depend on a stochastic value to determine an overall level, for example S_{T_0}. Or there might be a special time point t during the future time interval $[T_0, T_1]$ that suits to make S_t a stochastic level for the stochastic corridor.

In the case that one wants to compensate the result from the ambiguity of a suitable value of K with the difference between the two forward starting corridors, a swap to exchange the forward starting fixed corridor with the forward starting stochastic corridor. The payoff of our swap to exchange the forward starting fixed corridor and the stochastic corridor is ;

$$\int_{T_0}^{T_1} I\{S_u \le S_t\}du - \int_{T_0}^{T_1} I\{S_u \le K\}du$$

or .

$$\int_{T_0}^{T_1} I\{S_u \le K\}du - \int_{T_0}^{T_1} I\{S_u \le S_t\}du$$

We need to be able to determine a proper theoretical value of K which makes the price of the swap contract be zero at the time of the contract; i.e. at time 0.

K has to satisfy the equation

$$0 = e^{-rT_1}E_0[\int_{T_0}^{T_1} I\{S_u \le K\}du - \int_{T_0}^{T_1} I\{S_u \le S_t\}du]$$

That is;

$$E_0[\int_{T_0}^{T_1} I\{S_u \le K\}du] = E_0[\int_{T_0}^{T_1} I\{S_u \le S_t\}du].$$

The above expectations are the conditional expectations taken under the condition that the value of S_0 is given. The existence of such a constant K is assured exactly in the same way as in the previous section.

The probability distribution of

$$\int_{T_0}^{T_1} I\{S_u \le S_t\}du$$

is S_{T_0} - independent and is the same as that of

$\int_0^{T_1-T_0} I\{S_u \le S_t\}du$, hence it can be seen in Fujita and Miura(2004).

However, the calculation for $E_0[\int_{T_0}^{T_1} I\{S_u \le K\}du]$ may require some note.

(See the Working Paper for its detail.)

References

:[1]. Akahori, J. (1995). “Some formulae for a new type of path-dependent option.” Ann. Appl. Probab. 5. 383-388.

:[2] M.Baxter and A.Rennie. (1996) Financial Calculus; An Introduction to Derivative Pricing. Cambridge University Press.

:[3]. A.N. Borodin and P. Salminen. (2002). Handbook of Brownian Motion - Facts and Formulae, 2nd. edition p.256. (the first edition was published in1996) 。Birkhauser.

:[4]. Dassios, A. (1995). “The distribution of the quantile of a Brownian motion with drift and the pricing of related path-dependent options. Ann. Appl. Probab. 5. 389-398.

:[5]. Embrecht, P., Rogers, L.C.G. and Yor, M. (1995). “A proof of Dassios’s representation of the α-quantile of Brownian motion with drift.” Annals of Applied Probability. 5,757-767.

:[6]. Fujita, T.(1997). ”On the price of α-percentile options.” Working Paper Series #24, Faculty of Commerce Hitotsubashi University.

:[7]. Fujita, T. (2000). “A note on the joint distribution of α, β percentiles and its applications to the option pricing.” Asia-Pacific Financial Markets. Vol.7(4), 339-344.

:[8]. Fujita, T. and Miura, R.(2002). “Edokko Options: A New Framework of Barrior Options.” Asia-Pacific Financial Markets. Vol.9(2),December,141-151.

:[9]. Fujita, T. and Miura, R. (2004). “ The distribution of Continuous Time Rank Process.” To appear in Advances in Mathematical Economics,Vol.9. 2006. Springer Verlag.

:[10] Goldman M. Barry, Howard B. Sosin and Marry Ann Gatto (1979) “Path Dependent Options ; Buy at the low, Sell at the high.” Journal of Finance, Vol.34, pp.1111-1127.

:[11]. Miura, R. (1992). “ A note on look-back option based on order statistics.” Hitotsubashi Journal of Commerce and Management. 27,15-28.

:[12] S.E.Shreve (2004). Stochastic Calculus for Finance II; Continuous-Time Models. Springer.

:[13]. Yor, M.(1995).”The distribution of Brownian quantiles.” Journal of Applied Probability. 2, 405-416.

:[14]. Miura, R. (2005). “Rank Process and Stochastic Corridor
: Nonparametric Statistics of Lognormal Observations and Exotic Derivatives based on them”. Working paper. Serial Number # FS-2005-E-01. Graduate School of International Corporate Strategy. Hitotsubashi University, Tokyo, Japan