The behavior of the general stable distributions and their Fisher information matrix near the normal distribution

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Abstract

Abstract

- Introduction and applications.
- Expressions of the characteristic function and the density of a general stable distribution.
- The density of a general stable distribution close to the normal distribution.
- Fisher information close to the normal distribution.
- Numerical study (symmetric case).

Heavy tailed data

Example

- Financial data.
- Network models.
- Telecommunication

How to deal with ?

- Extreme value theory
- Heavy tailed distribution (e.x. pareto distribution)
- Heavy tailed time series model (e.x. GARCH, SV)
- Copulas

Note that stable distributions can model very heavy tailed data only. (no second moments)

Definition and Properties

Definition

- The distribution R is stable (in the broad sense) if for each n there exist constants $c_n > 0,\ \gamma_n$ such that

$$S_n \stackrel{d}{=} c_n X + \gamma_n$$

and R is not concentrated at one point. R is stable in the strict sense with $\gamma_n = 0$.

Widely known distributions;
Normal distribution, Cauchy distribution and Lévy distribution.
Except these 3 distributions, Probability densities have
no analytic expression.
Tail is very tick;
Except Normal distribution, no second moment exists.

Definition and Properties

Domain of Attraction

- The distribution F of the independent random variables X_k belongs to the domain of attraction of a distribution R if there exist norming constants $a_n > 0,\ b_n$ such that the distribution of $a_n^{-1}(S_n - b_n)$ tends to R.

A distribution R possesses a domain of attraction iff it is stable.

Many applications
(see recently published book, Uchaikin and Zolotarev (1999)):
Chaos, fractal, physics, astrophysics, cosmology.
In economics financial applications are expected.

Characteristic function: $\Phi(t;\alpha,\beta)$

Zolotarev's (M) parameterization (see p.11 of Zolotarev (1986)).

$$\Phi(t;\mu,\sigma,\alpha,\beta) = \exp\bigl(-|\sigma t|^{\alpha}[1+i\beta(\tan\frac{2\alpha}{\pi})(\operatorname{sgn} t)(|\sigma t|^{1-\alpha})-1)]+i\mu t\bigr).$$

$0<\alpha\le 2, \quad -1\le\beta\le 1, \quad -\infty<\mu<\infty, \quad \sigma>0.$
location: μ, scale: σ, kurtosis: α, skewness: β.

We consider the standard case $(\mu,\sigma)=(0,1)$.

$$\Phi(t;\alpha,\beta) = \exp\{-|t|^{\alpha}[1+i\beta(\operatorname{sgn} t)(\tan\frac{2\alpha}{\pi})(|t|^{1-\alpha}-1)]\}.$$

Location scale family $f(x;\alpha,\beta) := f(x;0,1,\alpha,\beta)$:

$$f(x;\mu,\sigma,\alpha,\beta) = \frac{1}{\sigma}f(\frac{x-\mu}{\sigma};\alpha,\beta).$$

Density: $f(x, \alpha, \beta)$

Let

$$\zeta = -\beta \tan \frac{\pi\alpha}{2},$$

$$\varrho = \frac{2}{\pi\alpha} \arctan(\beta \tan \frac{\pi\alpha}{2}).$$

From (2.2.18) of Zolotarev (1986), for $\alpha \neq 1$ and $x > \zeta$,

$$f(x;\alpha,\beta) = \frac{\alpha(x-\zeta)^{1/(\alpha-1)}}{2|\alpha-1|} \int_{-\varrho}^{1} A(\varphi) \exp\left(-(x-\zeta)^{\alpha/(\alpha-1)} A(\varphi)\right) d\varphi.$$

$$A(\varphi;\alpha,\beta) = (\cos \frac{\pi}{2}\alpha\varrho)^{\frac{1}{\alpha-1}} \left(\frac{\cos \frac{\pi}{2}\varphi}{\sin \frac{\pi}{2}\alpha(\varphi+\varrho)} \right)^{\frac{\alpha}{\alpha-1}} \frac{\cos \frac{\pi}{2}(\alpha\varrho + (\alpha-1)\varphi)}{\cos \frac{\pi}{2}\varphi},$$

$f(x;\alpha,\beta) = f(-x;\alpha,-\beta)$ for $x < \zeta$.

Normal: $f(x;2)$

As $\alpha \uparrow 2$, ζ and $\varrho \to 0$.
Unusual representation of the normal distribution
$(\mu = 0, \sigma = 2)$,

$$f(x;2) = x \int_0^1 1/(2\sin\frac{\pi}{2}\varphi)^2 \exp(-x^2/(2\sin\frac{\pi}{2}\varphi)^2)\, d\varphi.$$

Definition of problem

MLE of stable distributions

- Brorsen ans Yang (1990),
- Nolan (2001),
- Matsui and Takemura (2004).

Behavior of the Fisher information matrix

- DuMouchel (1975, 1983) proved
  Fisher information $I_{\alpha\alpha}$ (w.r.t α) $\to \infty$ as $\alpha \to 2$.
- Nagaev and Shkol'nik (1988) derived asymptotic behavior of $I_{\alpha\alpha}$ as $\alpha \to 2$ in symmetric case.

$$I_{\alpha\alpha} = \frac{1}{4\Delta \log(1/\Delta)}(1 + o(1)), \qquad \Delta = 2 - \alpha.$$

Definition of problem

Nagaev and Shkol'nik (1988) stated
"We note the problems under study are as yet unresolved for non-symmetric stable distributions."

We clarify the limiting values of the 4×4 Fisher Information matrix with respect to μ, σ, α and β.

From here let

$$\Delta = 2 - \alpha,$$

$$\beta^* = \beta \,\mathrm{sgn}(x - \zeta).$$

and let $w(t) > 0$ be a function which satisfies the property

$$\lim_{t \to 0} w(t) = 0.$$

Theorem 2.1:Density near the normal distribution

Let $|\beta| \neq 1$ be fixed. We define

$$\begin{aligned} F_1(x;\alpha,\beta) &= f(|x-\zeta|;2), \\ F_2(x;\alpha,\beta) &= \Delta(1+\beta^*)|x-\zeta|^{\Delta-3}, \\ g(x;\alpha,\beta) &= F_1(x;\alpha,\beta)+F_2(x;\alpha,\beta). \end{aligned}$$

Then for an arbitrarily small $\epsilon > 0$ there exit Δ_0 and x_0 such that for all $\Delta < \Delta_0$ and $|x| > x_0$,

$$|f(x;\alpha,\beta)/g(x;\alpha,\beta) - 1| < \epsilon.$$

Furthermore, for an arbitrarily small constant $\delta > 0$,

$$g(x;\alpha,\beta) = \begin{cases} F_1(x;\alpha,\beta)(1+w(\Delta)) & \text{if } |x-\zeta| \leq (2-\delta)(\log 1/\Delta)^{1/2} \\ F_2(x;\alpha,\beta)(1+w(\Delta)) & \text{if } |x-\zeta| \geq (2+\delta)(\log 1/\Delta)^{1/2}. \end{cases}$$

Visualization: Integrand near Normal

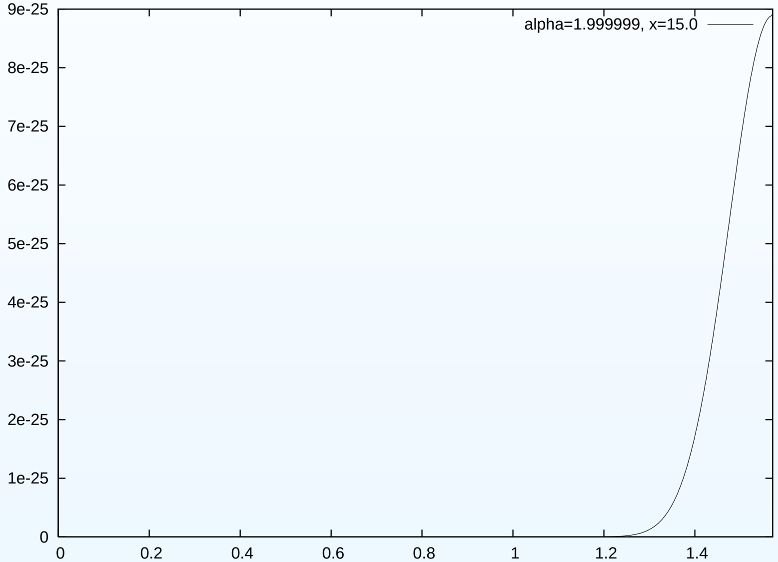

Figure 1: $\alpha = 1.999999,\ x = 15.0$

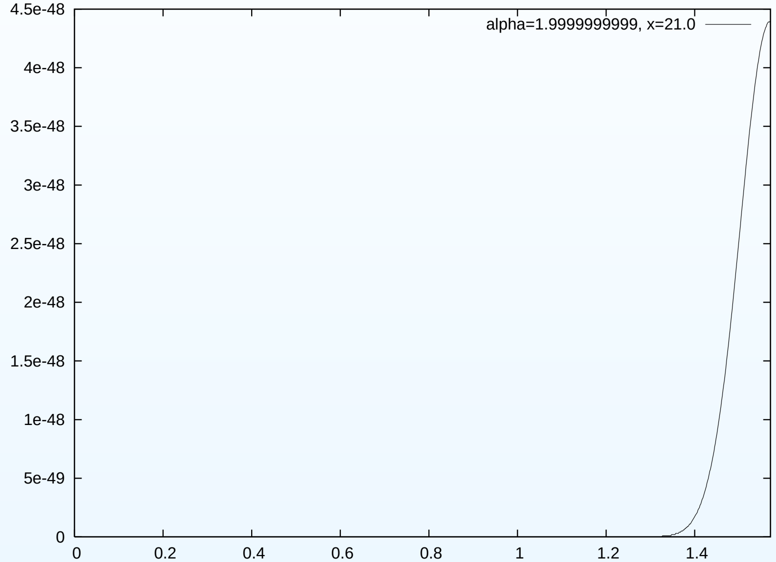

Figure 2: $\alpha = 1.9999999999,\ x = 21.0$

The integrand of density near Normal and $x \to \infty$.

$$f(x;\alpha,\beta) = \frac{\alpha(x-\zeta)^{1/(\alpha-1)}}{2|\alpha-1|}\int_{-\varrho}^{1} A(\varphi)\exp\big(-(x-\zeta)^{\alpha/(\alpha-1)}A(\varphi)\big)d\varphi.$$

Proof

Assume $x - \zeta > 0$.
Notations:

$$\begin{aligned}
\epsilon, \epsilon', \tau, \eta &:= \text{arbitrarily small positive numbers},\\
c &:= \text{any positive constant},\\
\lambda &:= 1 - \varphi,\\
\varphi_\Delta &:= 1 - \Delta^{1/2-\epsilon},\\
z &:= (x-\zeta)^{\alpha/(\alpha-1)} \quad := x^2,\\
\varphi_0 &:= \varphi_\Delta - z^{-1/2+\tau}.
\end{aligned}$$

$$f(x;\alpha,\beta) = \frac{\alpha(x-\zeta)^{1/(\alpha-1)}}{2|\alpha-1|} \times H.$$

Proof

We divide integral

$$H = \int_{-\varrho}^{1} A(\varphi;\alpha,\beta)\exp(-zA(\varphi;\alpha,\beta))d\varphi,$$

into

$$H = \sum_{k=1}^{6} H_k$$

where each H_k corresponds to the integration of H for k-th intervals of

$$\begin{array}{lll} [-\varrho, 1-\mu), & [1-\mu, \varphi_0), & [\varphi_0, \varphi_\Delta) \\ [\varphi_\Delta, 1-\Delta/\epsilon'), & [1-\Delta/\epsilon', 1-\Delta\epsilon'), & [1-\Delta\epsilon', 1]. \end{array}$$

Lemmas

Lemma 2.1 : $A(\varphi)$ on $[1-\Delta/\epsilon', 1-\Delta\epsilon')$

As $\Delta \to 0$ and for $0 \le \lambda \le \Delta/\epsilon'$,

$$A(1-\lambda) = \frac{(\lambda/\Delta)^{\frac{1}{1-\Delta}}(1+\beta+\lambda/\Delta)}{(1+\beta+2\lambda/\Delta)^2}(1+o(\Delta\log(1/\Delta))).$$

Lemma 2.2 : $A(\varphi)$ on $[\varphi_0, \varphi_\Delta)$ and $[\varphi_\Delta, 1-\Delta/\epsilon')$

As $\Delta \to 0$

$$\begin{aligned} A(\varphi_\Delta) &= \frac{1}{4} + \frac{\pi^2}{16}\Delta^{1-2\epsilon} + o(\Delta), \\ A'(\varphi_\Delta) &= -\frac{\pi^2}{8}\Delta^{1/2-\epsilon} - \frac{(1+\beta)^2}{8}\Delta^{1/2+3\epsilon} + o(\Delta). \end{aligned}$$

Lemmas

Lemma 2.3 : $A(\varphi)$ on $[\varphi_0, \varphi_\Delta)$

As $\Delta \to 0$ and $\lambda \to 0$
for $\varphi = 1 - \lambda < \varphi_\Delta = 1 - \Delta^{1/2-\epsilon}$,

$$
\begin{aligned}
A(1-\lambda) &= \frac{1}{4} + \frac{\pi^2}{16}\lambda^2 + o(\lambda^2), \\
A'(1-\lambda) &= -\frac{\pi^2}{8}\lambda - \frac{\Delta^2(1-\lambda+\beta)^2}{8\lambda^3} + o(\lambda^2), \\
A''(1-\lambda) &= \frac{\pi^2}{8} + \frac{\pi^2}{8}\lambda^2 - \frac{3}{8}\frac{\Delta^2(1-\lambda+\beta)^2}{\lambda^4} \\
&\quad + \frac{3}{4}\frac{\Delta^2(1-\lambda+\beta)}{\lambda^3}\left(1 - \frac{\Delta(1-\lambda+\beta)^2}{\lambda^2}\right) \\
&\quad + o(\lambda^2).
\end{aligned}
$$

Lemmas

Note since
$0 \leq \varphi \leq \varphi_{\Delta} \; \Leftrightarrow \; \Delta/\epsilon \leq \Delta^{1/2-\epsilon} \leq \lambda \leq 1 \; \Rightarrow \; \Delta = o(\lambda^2)$,
we have to consider terms like Δ^2/λ^4 or Δ^2/λ^3 in Lemma 2.3.

Lemma 2.4 :
If Δ is sufficiently small, $A(\varphi)$ is a monotonically decreasing function on $(-\varrho, 1)$ (integral range).

Calculation of H_1

Lemma 2.1 $\Rightarrow A(\varphi_\Delta) > \frac{1}{4}$.
Lemma 2.4 and $1-\mu < \varphi_\Delta$
$\Rightarrow \exists\rho \in (0,1)$ such that $\rho A(1-\mu) > \frac{1}{4}$.

$$\begin{aligned} H_1 &= \frac{1}{z(1-\rho)} \int_{-\varrho^*}^{1-\mu} z(1-\rho)A(\varphi)\exp\{-z\rho A(\varphi) - z(1-\rho)A(\varphi)\}d\varphi \\ &\leq \frac{1}{z(1-\rho)} \exp(-z\rho A(1-\mu)) \\ &= O(\exp(-\gamma z)/z), \quad \text{for } \gamma > \frac{1}{4}. \end{aligned}$$

Calculation of H_2

Lemma 4 $\Rightarrow$

$$\begin{aligned} H_2 &= \int_{1-\mu}^{\varphi_0} A(\varphi)\exp(-zA(\varphi))d\varphi \\ &= \mu A(1-\mu)\exp(-zA(\varphi_0)) \\ &= O(\exp(-zA(\varphi_0))). \end{aligned}$$

$$A(\varphi_0) = A(\varphi_\Delta) + A'(\varphi_\Delta)(\varphi_0 - \varphi_\Delta) + \frac{1}{2}A''(\xi)(\varphi_0 - \varphi_\Delta)^2,$$

Lemma 2.2, Lemma 2.3 and $\varphi_0 - \varphi_\Delta = -z^{-1/2+\tau} \Rightarrow$

$$H_2 = O\left(\exp\left(-\frac{z}{4} - R_1(\Delta, z, \epsilon) - R_2(\Delta, z, \epsilon) - cz^{2\tau}\right)\right),$$

$$R_1(\Delta, z, \epsilon) = O(z\Delta^{1-2\epsilon}) > 0, \quad R_2(\Delta, z, \epsilon) = O(\Delta^{1/2-\epsilon}z^{1/2+\tau}) > 0.$$

Calculation of H_3

$$
\begin{aligned}
H_3 &= \exp(-zA(\varphi_\Delta)) \times \\
&\int_{\varphi_0}^{\varphi_\Delta} A(\varphi) \exp\left(-zA'(\varphi_\Delta)(\varphi - \varphi_\Delta) - \frac{z}{2} A''(\xi)(\varphi - \varphi_\Delta)^2\right) d\varphi.
\end{aligned}
$$

Lemma 2.2 and Lemma 2.3 $\Rightarrow$

$$
\begin{aligned}
A(\varphi) &= \frac{1}{4} + O(\lambda^2), \\
zA(\varphi_\Delta) &= \frac{z}{4} + R_1(\Delta, z, \epsilon), \\
zA'(\varphi_\Delta)(\varphi - \varphi_\Delta) &= R_3(\Delta, z, \epsilon), \quad 0 < R_3(\Delta, z, \epsilon) \le R_2(\Delta, z, \epsilon), \\
\frac{z}{2} A''(\xi)(\varphi - \varphi_\Delta)^2 &= \frac{z}{2}\frac{\pi^2}{8}(\varphi - \varphi_0)^2 + O(z^{-1+4\tau}) + O(\Delta^{4\epsilon} z^{2\tau}).
\end{aligned}
$$

Calculation of H_3

$$H_3 = R(\lambda, \Delta, z, \epsilon)\frac{1}{4}\exp\left(-\frac{z}{4}\right)\int_{\varphi_0}^{\varphi_\Delta}\exp\left(-\frac{z}{2}\frac{\pi^2}{8}(\varphi-\varphi_\Delta)^2\right)d\varphi,$$

$$R(\lambda, \Delta, z, \epsilon) = (1+o(1))\exp\left(-R_1(\Delta, z, \epsilon) - R_3(\Delta, z, \epsilon)\right).$$

$$\begin{aligned}
&\int_{\varphi_0}^{\varphi_\Delta}\exp\left(-\frac{z}{2}\frac{\pi^2}{8}(\varphi-\varphi_0)^2\right)d\varphi \\
&= \frac{1}{\sqrt{z}}\int_0^{\infty}\exp\left(-\frac{z}{2}\frac{\pi^2}{8}\varphi^2\right)d\varphi - \frac{1}{\sqrt{z}}\int_{z^\tau}^{\infty}\exp\left(-\frac{z}{2}\frac{\pi^2}{8}\varphi^2\right)d\varphi \\
&= \frac{2}{\sqrt{\pi z}} - O\left(\frac{1}{z^{1/2+\tau}}\exp\left(\frac{-z^{2\tau}}{2}\right)\right).
\end{aligned}$$

$$H_3 = \frac{1}{2\sqrt{\pi z}}\exp\left(-\frac{z}{4} - R_1(\Delta, z, \epsilon) - R_3(\Delta, z, \epsilon)\right)(1+o(1)).$$

Calculation of H_4

Lemma 2.4 $\Rightarrow$

$$\begin{aligned} H_4 &\le \int_{\varphi_\Delta}^{1-\Delta/\epsilon'} A(\varphi_\Delta)\exp(-zA(1-\Delta/\epsilon'))d\varphi \\ &\le \Delta^{1/2-\epsilon} A(\varphi_\Delta)\exp(-zA(1-\Delta/\epsilon')). \end{aligned}$$

$\lambda/\Delta \le 1/\epsilon'$, Lemma 2.1 and 2.4 $\Rightarrow$

$$A(1-\Delta/\epsilon') = \frac{1}{4}(1-w(\epsilon')), \quad \text{as } \Delta \to 0.$$

$\Rightarrow$

$$H_4 \le O\left(\Delta^{1/2-\epsilon}\exp\left(-\frac{z}{4}(1-w(\epsilon'))\right)\right).$$

Calculation of H_5

From Lemma 1 for $\epsilon' \le \lambda/\Delta \le 1/\epsilon'$, $A(\varphi)$ is bounded. Then

$$\begin{aligned} H_5 &= \int_{1-\Delta/\epsilon'}^{1-\Delta\epsilon'} A(\varphi)\exp(-zA(\varphi))d\varphi \\ &\le (\Delta/\epsilon' - \Delta\epsilon') \sup_{\varphi\in(1-\Delta/\epsilon', 1-\Delta\epsilon)} A(\varphi)\exp(-zA(\varphi)) \\ &\le O(\Delta\exp(-cz)). \end{aligned}$$

Calculation of H_6

Lemma 2.1 $\Rightarrow$ for $\lambda/\Delta \le \epsilon'$

$$A(\varphi;\alpha,\beta) \quad = \quad \frac{1+R_4(\Delta,\epsilon')}{1+\beta}(\lambda/\Delta)^{1/(1-\Delta)},$$

$$R_4(\Delta,\epsilon') = O(\epsilon') + O(\Delta\log(1/\Delta)).$$

$\varphi = 1-\lambda \to \lambda$ in H_6

$$H_6 = \left(\frac{1+R_4}{1+\beta}\right)\int_0^{\Delta\epsilon'}(\lambda/\Delta)^{1/(1-\Delta)}\exp\left(-z\left(\frac{1+R_4}{1+\beta}\right)(\lambda/\Delta)^{1/(1-\Delta)}\right)d\lambda.$$

$$\lambda = \Delta\left(\frac{1+\beta}{1+R_4}\frac{x}{z}\right)^{1-\Delta} \to x, \quad g(z) = {\epsilon'}^{1/(1-\Delta)}\left(\frac{1+R_4}{1+\beta}\right)z,$$

$$H_6 = \frac{\Delta(1+\beta)^{1-\Delta}}{z^{2-\Delta}}\int_0^{g(z)} x\exp(-x)dx \times (1+o(1)).$$

Calculation of H_6

$$
\begin{aligned}
\int_0^{g(z)} x \exp(-x) dx &= \left[-xe^{-x} - e^{-x}\right]_0^{g(z)} \\
&= 1 + O\left(ge^{-g}\right),
\end{aligned}
$$

$g(z) \to \infty$ as $z \to \infty$. Therefore

$$
H_6 = \Delta(1+\beta) z^{\Delta - 2}(1 + o(1)).
$$

Review

$$f(x;\alpha,\beta) = \frac{\alpha(x-\zeta)^{1/(\alpha-1)}}{2|\alpha-1|} \times H.$$

$$H = \int_{-\varrho}^{1} A(\varphi;\alpha,\beta)\exp(-zA(\varphi;\alpha,\beta))d\varphi,$$

$$H = \sum_{k=1}^{6} H_k$$

where each H_k corresponds to the integration of H for k-th intervals of

$$[-\varrho, 1-\mu), \quad [1-\mu, \varphi_0), \quad [\varphi_0, \varphi_\Delta)$$
$$[\varphi_\Delta, 1-\Delta/\epsilon'), \quad [1-\Delta/\epsilon', 1-\Delta\epsilon'), \quad [1-\Delta\epsilon', 1].$$

Calculation of H

$$
\begin{aligned}
H_1 &= O(\exp(-\gamma z)/z), \quad \text{for } \gamma > \frac{1}{4} \\
H_2 &= O\left(\exp\left(-\frac{z}{4} - R_1(\Delta, z, \epsilon) - R_2(\Delta, z, \epsilon) - cz^{2\tau}\right)\right) \\
H_3 &= \frac{1}{2\sqrt{\pi z}} \exp\left(-\frac{z}{4} - R_1(\Delta, z, \epsilon) - R_3(\Delta, z, \epsilon)\right)(1 + o(1)). \\
H_4 &\leq O\left(\Delta^{1/2-\epsilon} \exp\left(-\frac{z}{4}(1 - w(\epsilon'))\right)\right) \\
H_5 &\leq O(\Delta \exp(-cz)) \\
H_6 &= \Delta(1+\beta)z^{\Delta-2}(1+o(1)).
\end{aligned}
$$

$\Rightarrow$

$$
\max(H_1, H_2, H_4, H_5) = o(\max(H_3, H_6)).
$$

Calculation of H

For an arbitrarily small $\delta > 0$,

$$H_6 = o(H_3) \quad \text{if } z \le (4-\delta)\log 1/\Delta,$$
$$H_3 = o(H_6) \quad \text{if } z \ge (4+\delta)\log 1/\Delta.$$

If $z \le (4-\delta)\log 1/\Delta$, $R_1(\Delta, z, \epsilon) \downarrow 0$, $R_3(\Delta, z, \epsilon) \downarrow 0$,

$$H_3 = \frac{1}{2\sqrt{\pi z}} \exp\left(-\frac{z}{4}\right)(1+o(1)).$$

$\square$

$f_\alpha(x;\alpha,\beta)$

Lemma 3.1:

As $\Delta = 2-\alpha \to 0$, there exists x_0 and for all $|x| \geq x_0$,

$$f_\alpha(x;\alpha,\beta) =$$
$$-\frac{1}{|y|^{1+\alpha}}\left(1+\beta^* + \Delta(M_1 + M_2\log|y|) + \frac{M_3}{|y|} + \frac{M_4 + M_5\log|y|}{|y|^\alpha}\right)$$
$$+o\left(\frac{1}{|y|^{1+2\alpha}}\right),$$

where $y = x-\zeta$ and $M_i's$ are some constants.

$\square$

Lemma 3.1 covers the case of $\alpha = 2$.

Information matrix

Theorem 4.1:

As $\Delta = 2 - \alpha \to 0$, Behavior of Fisher information matrix of general stable distributions when $\mu = 0$, $\sigma = 1$, $\beta \neq \pm 1$ is given as follows.

$$\begin{bmatrix} I_{\mu\mu} & I_{\mu\sigma} & I_{\mu\alpha} & I_{\mu\beta} \\ * & I_{\sigma\sigma} & I_{\sigma\alpha} & I_{\sigma\beta} \\ * & * & I_{\alpha\alpha} & I_{\alpha\beta} \\ * & * & * & I_{\beta\beta} \end{bmatrix} = \begin{bmatrix} 0.5 + o(1) & o(1) & o(1) & O(\Delta) \\ * & 2.0 + o(1) & -\frac{1}{2}\log\log 1/\Delta & o(\Delta \log\log 1/\Delta) \\ * & * & \frac{1}{4\Delta \log 1/\Delta} & o\left(\frac{1}{\log 1/\Delta}\right) \\ * & * & * & \frac{\Delta}{4(1-\beta^2)\log 1/\Delta} \end{bmatrix}.$$

Information matrix

Table 1: Limit of information matrix at $\alpha = 2$

$I_{\theta\theta}$	μ	σ	α	β
μ	0.5	0	0	0
σ	0	2.0	$-\infty$	0
α	0	$-\infty$	∞	0
β	0	0	0	0

Visualization: Density and Score in Symmetric Case

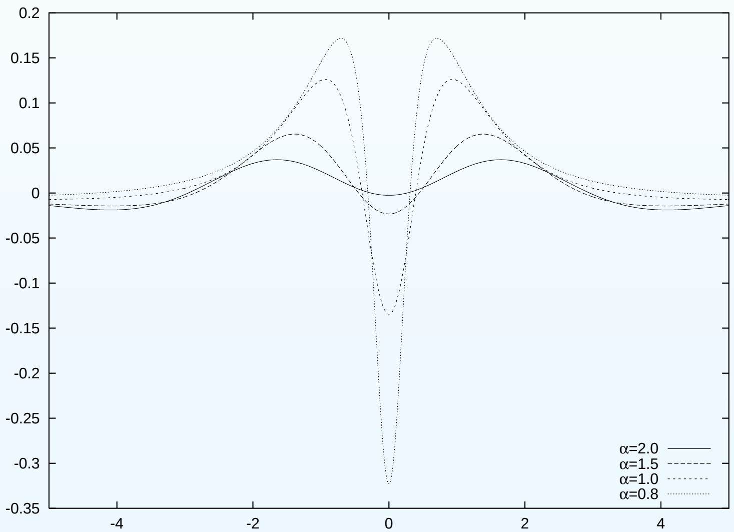

Figure 3: $f_\alpha(x;\alpha,0)$

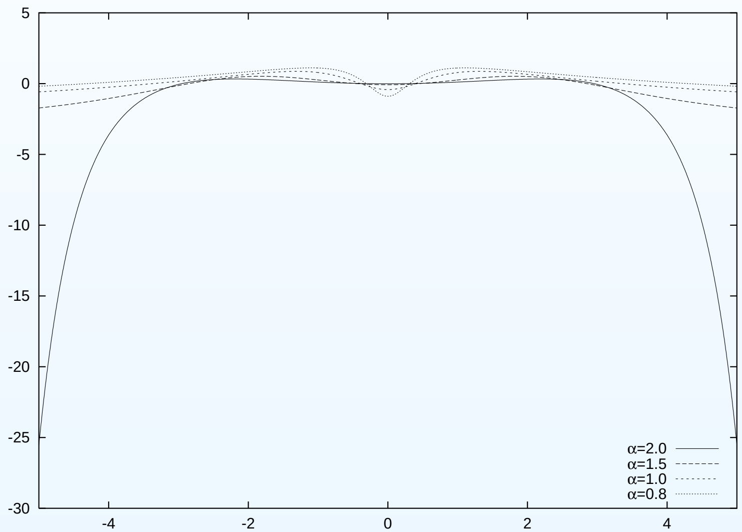

Figure 4: $f_\alpha(x;\alpha,0)/f(x;\alpha,0)$

While $f(x;\alpha,0)$ is exponential order as $x \to \infty$ at $\alpha = 2$, $f_\alpha(x;\alpha,0)$ is polynomial order as $x \to \infty$ at $\alpha = 2$. Therefore Fisher information $I_{\alpha\alpha} \uparrow \infty$.

Numerical work

Numerical work

- Maximum likelihood estimation.
- Observed Fisher information matrix.
- Numerical confirmation of $I_{\alpha\alpha}$ as $\alpha \to 2$.

All numerical works are done in symmetric case.

Maximum Likelihood Estimation (Symmetric Case)

n	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{I}_{\mu\mu}$	$\hat{I}_{\sigma\sigma}$	$\hat{I}_{\alpha\alpha}$
	0	1.0	2.0	0.5	2.0	∞
50	0.00014	0.977	1.976	0.607	1.875	5.647
100	0.00017	0.975	1.990	0.868	1.402	9.445
200	0.00094	0.977	1.994	0.784	1.239	12.77
	0	1.0	1.8	0.4552	1.3898	0.5937
50	0.0012	0.991	1.818	0.487	1.231	0.676
100	0.0033	1.002	1.822	0.482	1.356	0.584
200	-0.0000	1.000	1.810	0.450	1.399	0.603
	0	1.0	1.5	0.4281	0.9556	0.4737
50	-0.0022	1.012	1.548	0.3161	0.5796	0.4252
100	-0.0000	1.000	1.524	0.3914	0.9138	0.4278
200	-0.0003	1.000	1.510	0.4028	0.9474	0.4683
	0	1.0	1.0	0.5	0.5	0.8590
50	-0.0029	0.996	1.026	0.4243	0.4877	0.670
100	-0.0041	0.988	1.001	0.4438	0.4205	0.746
200	-0.0039	1.001	1.006	0.4929	0.5013	0.845
	0	1.0	0.8	0.6800	0.3586	1.3928
50	-0.0016	1.005	0.815	0.5434	0.3243	1.111
100	-0.0001	1.003	0.811	0.6015	0.3459	1.171
200	-0.0009	1.000	0.805	0.6232	0.3708	1.303

Observed Fisher information

Observed Fisher information w.r.t. α

$$\begin{aligned}\hat{I}_{\alpha\alpha}(x_1,\ldots,x_n) &= -\frac{1}{n}\sum_{i=1}^{n}\frac{\partial^2 \log f(x_i;\hat{\alpha})}{\partial\alpha^2} \\ &= \frac{1}{n}\sum_{i=1}^{n}\left(\frac{f_\alpha(x_i;\hat{\alpha})}{f(x_i;\hat{\alpha})}\right)^2 - \frac{1}{n}\sum_{i=1}^{n}\frac{f_{\alpha\alpha}(x_i;\hat{\alpha})}{f(x_i;\hat{\alpha})}\end{aligned}$$

We define this as $\hat{I}_{\alpha\alpha}(2)$. Observed Fisher information only use the first derivatives

$$\hat{I}_{\alpha\alpha}(1) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{f_\alpha(x_i;\hat{\alpha})}{f(x_i;\hat{\alpha})}\right)^2$$

Observed Fisher information

Simulated observed Fisher information (Sample size $n = 50$, 1000 iterations)
In table, $\hat{I}_{\alpha\alpha}(i),\ i = 1, 2$ are means of 1000 iterations. and the variances of $\hat{I}_{\alpha\alpha}(i),\ i = 1, 2$ are in ().

α	$\bar{\alpha}$	$\hat{I}_{\alpha\alpha}(1)$		$\hat{I}_{\alpha\alpha}(2)$		$I_{\alpha\alpha}$
1.5	1.531	0.4863	(0.033)	0.5145	(0.018)	0.4737
1.0	1.025	0.8541	(0.157)	0.9001	(0.099)	0.8590
0.5	0.509	4.2819	(3.517)	4.4660	(2.755)	4.2748

$\hat{I}_{\alpha\alpha}(1)$: small bias and large variance.
$\hat{I}_{\alpha\alpha}(2)$: small variance and large bias.
Sometimes $\hat{I}_{\alpha\alpha}(1)$ (only use the first derivatives) is useful.

Numerical confirmation of $I_{\alpha\alpha}$ as $\alpha \to 2$

In symmetric case we numerically examine Nagaev and Shkol'nik (1988) results.

$$I_{\alpha\alpha} = \frac{1}{4\Delta \log(1/\Delta)}(1 + o(1)), \qquad \Delta = 2 - \alpha.$$

α	$I_{\alpha\alpha}(1)$	$I_{\alpha\alpha}(2)$	N&S	$I_{\sigma\alpha}(1)$	$I_{\sigma\alpha}(2)$
$2.0 - 10^{-10}$	106860414	92384764	108573620	−1.3482	−1.3427
$2.0 - 10^{-9}$	10810787	10167389	12063736	−1.3482	−1.3395
$2.0 - 10^{-8}$	1144778	1131645	1357170	−1.3482	−1.3123
$2.0 - 10^{-7}$	127953	127802	155105.2	−1.3478	−1.2553
$2.0 - 10^{-6}$	14724	14722	18095.60	−1.2094	−1.1923
1.99999	1750	1750	2171.472	−1.1100	−1.1100
1.9999	217	217	271.4341	−1.0069	−1.0069

$I_{\alpha\alpha}(1)$ near $x \to \infty$ Taylor expansion at $\alpha = 2$ w.r.t. α.
$I_{\alpha\alpha}(2)$ near $x \to \infty$ use asymptotic expansion.

Future work

What can do ? with stable

- Very heavy tail model.
- Distributions with finite order moments.

Future work

- Tail is not so heavy (have second moment)
- Non i.i.d. dependent data.
- Multivariate data.

Copulas ? Resampling method?

important $=$ why, for what purpose, how you use this model.

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Proof of $I_{\alpha\alpha}$

We divide integral of Fisher information matrix into tree parts.

$$
\begin{aligned}
I_{\alpha\alpha} &= \int_0^\infty \frac{\{f_\alpha(x+\zeta;\alpha,\beta)\}^2}{f(x+\zeta;\alpha,\beta)}dx + \int_0^\infty \frac{\{f_\alpha(x+\zeta;\alpha,-\beta)\}^2}{f(x+\zeta;\alpha,-\beta)}dx \\
&\quad + \int_{-\zeta}^{\zeta} \frac{\{f_\alpha(x;\alpha,\beta)\}}{f(x;\alpha,\beta)}dx \\
&= I^1_{\alpha\alpha} + I^2_{\alpha\alpha} + I^3_{\alpha\alpha}.
\end{aligned}
$$

This is obtained by the relation $f(x;\alpha,\beta) = f(-x;\alpha,-\beta)$ for $x - \zeta < 0$.

Proof of $I_{\alpha\alpha}$

$I_{\alpha\alpha}^1$ into five subintegrals,

$$I_{\alpha\alpha}^1 = \sum_{k=1}^{5} I_{\alpha\alpha}(k)$$

where each $I_{\alpha\alpha}(k)$ corresponds to the k-th intervals of $[0, T)$, $[T, x_1(\Delta))$, $[x_1(\Delta), x_2(\Delta))$, $[x_2(\Delta), x_3(\Delta))$, $[x_3(\Delta), \infty)$.

$$x_1(\Delta) = (2-\delta)(\log 1/\Delta)^{1/2}, \quad x_2(\Delta) = (2+\delta)(\log 1/\Delta)^{1/2},$$
$$x_3(\Delta) = \exp(\Delta^{-1/2}).$$

Theorem 2.1 and Lemma 3.1

$$I_{\alpha\alpha}(1) < \infty.$$

Proof of $I_{\alpha\alpha}$

For $I_{\alpha\alpha}(2)$,

$$f(x+\zeta;\alpha,\beta) = f(x;2)(1+o(1)), \quad f_\alpha(x+\zeta;\alpha,\beta) = \text{const}\times x^{\Delta-3}.$$

$$\begin{aligned} I_{\alpha\alpha}(2) &= \text{const}\times \int_T^{x_1(\Delta)} x^{2\Delta-6}\exp\left(\frac{x^2}{4}\right)dx \\ &= \text{const}\times \frac{1}{\Delta^{1-\delta}(\log 1/\Delta)^{5/2-\Delta}}. \end{aligned}$$

For $I_{\alpha\alpha}(3)$,

$$f(x+\zeta;\beta,\alpha) = \text{const}\times\Delta x^{\Delta-3}, \quad f_\alpha(x+\zeta;\alpha,\beta) = \text{const}\times x^{\Delta-3}.$$

$$\begin{aligned} I_{\alpha\alpha}(3) &= \text{const}\times \frac{1}{\Delta}\int_{x_1(\Delta)}^{x_2(\Delta)} x^{\Delta-3}dx \\ &= \text{const}\times \frac{\delta}{\Delta\log 1/\Delta}. \end{aligned}$$

Proof of $I_{\alpha\alpha}$

For $I_{\alpha\alpha}(4)$,

$$\begin{aligned} f(x+\zeta;\alpha,\beta) &= (1+\beta)\Delta x^{\Delta-3}(1+o(1)), \\ f(x+\zeta;\alpha,\beta) &= -(1+\beta)x^{\Delta-3}(1+o(1)). \end{aligned}$$

$$\begin{aligned} I_{\alpha\alpha}(4) &= \frac{1+\beta}{\Delta}(1+o(1))\int_{x_2(\Delta)}^{x_3(\Delta)} x^{\Delta-3}dx \\ &= \frac{1+\beta}{8\Delta\log 1/\Delta}(1+o(1)). \end{aligned}$$

For $I_{\alpha\alpha}(5)$,

$$\begin{aligned} f(x+\zeta;\alpha,\beta) &= (1+\beta)\Delta x^{\Delta-3}(1+o(1)), \\ f_\alpha(x+\zeta;\alpha,\beta) &= -(1+\beta+\Delta\log x)x^{\Delta-3}(1+o(1)). \end{aligned}$$

Proof of $I_{\alpha\alpha}$

$$
\begin{aligned}
I_{\alpha\alpha}(5) &= \text{const} \times \frac{1}{\Delta} \int_{x_3(\Delta)}^{\infty} x^{\Delta-3} \{\max(1+\beta, \Delta \log x)\}^2 dx \\
&\leq \text{const} \times \Delta \int_{x_3(\Delta)}^{e^{(1+\beta)/\Delta}} x^{\Delta-3} (\log x)^2 dx \\
&\quad + \text{const} \times \frac{1}{\Delta} \int_{e^{(1+\beta)/\Delta}}^{\infty} x^{\Delta-3} dx \\
&\leq \text{const} \times \frac{1}{\Delta} \int_{x_3(\Delta)}^{\infty} x^{\Delta-3} dx \\
&= O(e^{-2/\Delta^{1/2}}/\Delta) \to 0, \text{ as } \Delta \to 0.
\end{aligned}
$$

Proof of $I_{\alpha\alpha}$

$$\boldsymbol{I^1_{\alpha\alpha} = \frac{1+\beta}{8\Delta \log 1/\Delta}(1+o(1)).}$$

Setting $\beta \to -\beta$ in $\boldsymbol{I^1_{\alpha\alpha}}$, we obtain $\boldsymbol{I^2_{\alpha\alpha}}$. By $\zeta = \boldsymbol{O(\Delta)}$ and finiteness of the integrand of $\boldsymbol{I^3_{\alpha\alpha}}$, $\boldsymbol{I^3_{\alpha\alpha} = O(\Delta)}$.

$\square$